# SCHATTEN CLASS MEMBERSHIP OF HANKEL OPERATORS ON THE UNIT SPHERE

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**Abstract.** Let $H^2(S)$ be the Hardy space on the unit sphere $S$ in $\mathbf{C}^n$, $n \geq 2$. Consider the Hankel operator $H_f = (1-P)M_f|H^2(S)$, where the symbol function $f$ is allowed to be arbitrary in $L^2(S, d\sigma)$. We show that for $p > 2n$, $H_f$ is in the Schatten class $\mathcal{C}_p$ if and only if $f - Pf$ belongs to the Besov space $\mathcal{B}_p$. To be more precise, the "if" part of this statement is easy. The main result of the paper is the "only if " part. We also show that the membership $H_f \in \mathcal{C}_{2n}$ implies $f - Pf = 0$, i.e., $H_f = 0$.

## 1. Introduction

Let $S$ denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Throughout the paper, we assume that the complex dimension $n$ is greater than or equal to 2. Let $\sigma$ be the positive, regular Borel measure on $S$ which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. We take the usual normalization $\sigma(S) = 1$.

Recall that the orthogonal projection $P$ from $L^2(S, d\sigma)$ onto the Hardy space $H^2(S)$ is given by the Cauchy integral formula

$$(Pf)(w) = \int \frac{f(\zeta)}{(1 - \langle w, \zeta\rangle)^n} d\sigma(\zeta), \qquad |w| < 1,$$

[9,page 39]. As usual, the Hankel operator $H_f : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$ is defined by the formula

$$H_f = (1-P)M_f|H^2(S).$$

Here we are interested in the so-called "one-sided" theory of Hankel operators, as explained on page 27 in [12]. The challenge of the "one-sided" theory is to deal with Hankel operators $H_f$ which cannot be expressed in the form of a commutator $[M_g, P]$, $g \in L^2(S, d\sigma)$. As it turns out, "one-sided" problems can usually be interpreted as concrete versions of this simple question: if $H_f$ has a certain property, does $H_{\overline{f-Pf}}$ have the same property?

Since the boundedness and compactness of $H_f$ were characterized in [12], in this paper we will take up the task of determining when $H_f$ belongs to a Schatten class. Recall that for each $1 \leq p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators $A$ satisfying the condition $\|A\|_p < \infty$, where the $p$-norm is given by the formula

$$\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}. \tag{1.1}$$

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In terms of the $s$-numbers $s_1(A)$, $s_2(A)$, ..., $s_j(A)$, ... of $A$ (see [5,Section II.7]), we have $\|A\|_p = (\sum_{j=1}^{\infty}\{s_j(A)\}^p)^{1/p}$. For convenience, we adopt the convention that $\|X\|_p = \infty$ if the operator $X$ is unbounded.

The motivation for this investigation mainly came from the following sources:

(1) In the unit circle case, the classic result of Peller [7,8] completely determines the Schatten class membership of the Hankel operator $H_f$, $f \in L^2(\mathbf{T})$. But this result is really about commutators, for on $L^2(\mathbf{T})$ we always have $H_f = [M_{f-Pf}, P]$.

(2) The result of Janson and Wolff on the Schatten-class membership of commutators of singular integral operators on $\mathbf{R}^n$ [6].

(3) In [3] , Feldman and Rochberg showed that if $h \in H^2(S)$ and if $p > 2n$, then $H_{\bar{h}} \in \mathcal{C}_p$ if and only if $h \in \mathcal{B}_p$. But the assumption that $h \in H^2(S)$ leads to the identity $H_{\bar{h}} = [M_{\bar{h}}, P]$. So, again, this is a result about commutators.

(4) In [3], Feldman and Rochberg also showed that if $h \in H^2(S)$ and if $H_{\bar{h}} \in \mathcal{C}_{2n}$, then $h$ is a constant.

Although these results are all about commutators, they do provide hints as to what we should expect for $H_f$. To state our results, let us introduce the Besov spaces on $S$.

**Definition 1.1.** (a) For each $1 \le p < \infty$ and each $g \in L^2(S, d\sigma)$, denote

$$\mathcal{I}_p(g) = \iint \frac{|g(\zeta) - g(\xi)|^p}{|1 - \langle \zeta, \xi \rangle|^{2n}} d\sigma(\zeta) d\sigma(\xi).$$

(b) For each $1 \le p < \infty$, the Besov space $\mathcal{B}_p$ consists of those $g \in L^2(S, d\sigma)$ which satisfy the condition $\mathcal{I}_p(g) < \infty$.

Using interpolation techniques [1,6], it is easy to prove

**Proposition 1.2.** *In the case $2n < p < \infty$, if $f \in \mathcal{B}_p$, then $[M_f, P] \in \mathcal{C}_p$.*

Since $H_f = H_{f-Pf}$, from this proposition we immediately obtain

**Corollary 1.3.** *Let $2n < p < \infty$. For any $f \in L^2(S, d\sigma)$, if $f - Pf \in \mathcal{B}_p$, then $H_f \in \mathcal{C}_p$.*

The main result of the paper is the converse to Corollary 1.3:

**Theorem 1.4.** *Let $2n < p < \infty$. Then there exists a constant $0 < C < \infty$ which depends only on $n$ and $p$ such that the inequality*

$$\mathcal{I}_p(f - Pf) \le C\|H_f\|_p^p \tag{1.2}$$

*holds for every $f \in L^2(S, d\sigma)$, where $\|.\|_p$ is the Schatten $p$-norm defined by* (1.1).

Unlike Peller's classic result on the unit circle, in the case $n \ge 2$ there is a complete "cutoff line" for Schatten class Hankel operators at $p = 2n$:

**Theorem 1.5.** *Let $f \in L^2(S, d\sigma)$. If $H_f$ belongs to the Schatten class $\mathcal{C}_{2n}$, then $H_f = 0$.*

In fact, we have a more quantitative result in terms of $s$-numbers:

**Theorem 1.6.** *Let $f \in L^2(S, d\sigma)$. If $H_f$ is bounded and if $H_f \neq 0$, then there exists an $\epsilon = \epsilon(f) > 0$ such that*

$$s_1(H_f) + ... + s_k(H_f) \geq \epsilon k^{(2n-1)/2n}$$

*for every $k \in \mathbf{N}$.*

It is elementary that, for any $1 < p < \infty$, if $\{a_k\} \in \ell^p_+$, then $k^{-(p-1)/p} \sum_{j=1}^k a_j \to 0$ as $k \to \infty$. Thus Theorem 1.5 is an immediate consequence of Theorem 1.6.

The rest of the paper contains the proofs of these results. Section 2 deals with various estimates of mean oscillation. The culmination of these estimates is an inequality (Lemma 2.4) which tells us how mean oscillation behaves under the combined action of $P$ and Möbius transform. In Section 3 we derive a "quasi-resolution" of the Cauchy projection $P$, which is perhaps the key to the proof of Theorem 1.4. This "quasi-resolution" is what allows $\|H_f\|_p$ to get into the action. In Section 4 we introduce a gadget called $\mathcal{J}_p$, and we show that it dominates the $\mathcal{I}_p$ defined in Definition 1.1. Roughly speaking, $\mathcal{J}_p$ "takes the exponent $p$ outside the integral", and the fact that $\mathcal{J}_p$ dominates $\mathcal{I}_p$ is a kind of "reverse Hölder's inequality". We should mention that the proof of Proposition 4.2 is based on ideas adapted from [6].

In Section 5 we bring together the estimates in the above-mentioned three sections to show that there is a $C$ such that inequality (1.2) holds for every $f \in L^2(S, d\sigma)$ satisfying the condition $\mathcal{I}_p(f - Pf) < \infty$. The reason that we need this intermediate step is that our proof uses cancellation (twice). Finally, in Section 6 we use a technique called "smoothing" to remove the *a priori* condition $\mathcal{I}_p(f - Pf) < \infty$, completing the proof of Theorem 1.4.

In Section 7 we give an easy proof of Proposition 1.2. Although Proposition 1.2 can be proved by using the conventional interpolation techniques in [1,6], the proof given here can perhaps best be described as a "hybrid" proof. That is, we combine the *idea* behind the Marcinkiewicz interpolation with the fact that we are dealing with a commutator, which offers nice cancellation properties. By taking full advantage of cancellation, we are able to find a rather explicit bound for $\|[M_f, P]\|_p^p$. This more explicit version of the result will be established as Proposition 7.1.

The technique in the proof of Proposition 7.1 can be further exploited. In Proposition 7.2, we use the same technique to show that, if $f$ is Lipschitz on $S$, then the commutator $[M_f, P]$ belongs to the Lorentz-like ideal $\mathcal{C}^+_{2n}$ [5]. This provides an interesting contrast to Theorem 1.5: while there are no nonzero Hankel operators in the Schatten class $\mathcal{C}_{2n}$, there are plenty of nonzero Hankel operators in the slightly larger ideal $\mathcal{C}^+_{2n}$. The significance of Proposition 7.2 extends beyond curiosity; it will be needed in the proof of Theorem 1.6.

Section 8, the longest in the paper, is devoted to the proof of Theorem 1.6. The length of the section is a reflection of the fact that the proof is really technical. The proof involves functions of a very specific type and hinges on obtaining the lower bound given in Lemma 8.14. A moment of reflection on the lower bound tells us that this is a natural approach for proving Theorem 1.6.

In Section 9 we derive two more conditions which are equivalent to the membership $H_f \in \mathcal{C}_p$, $p > 2n$. Then we determine the distribution of the $s$-numbers of $H_f$ in the case where $f$ is Lipschtz on $S$. The final result of the section is a re-interpretation of Theorem 1.6 in the language of norm ideals [5].

Since the paper is full of estimates, there are many constants involved. Constants which appear in the statement of a proposition or lemma usually carry the same enumeration as that proposition or lemma. For example, $C_{2.1}$ is the constant that appears in Proposition 2.1. The reason for this is that they will be cited in later proofs. For constants which occur in proofs, we label them sequentially as $C_1, C_2, \cdots$, and so on.

## 2. Estimates of Mean Oscillation

We begin with the basics. It is elementary that if $c$ is a complex number with $|c| \leq 1$ and if $0 \leq \rho \leq 1$, then

$$2|1 - \rho c| \geq |1 - c|.$$

In the sequel this fact will be used frequently. For the rest of the paper, we write $\mathbf{B}$ for the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. For each $z \in \mathbf{B}$, we denote

$$k_z(w) = \frac{(1 - |z|^2)^{n/2}}{(1 - \langle w, z\rangle)^n}, \qquad |w| \leq 1.$$

It is well known that the formula

$$d(\zeta, \xi) = |1 - \langle \zeta, \xi\rangle|^{1/2}, \qquad \zeta, \xi \in S, \tag{2.1}$$

defines a metric on $S$ [9,page 66]. Throughout the paper, we denote

$$B(\zeta, r) = \{x \in S : |1 - \langle x, \zeta\rangle|^{1/2} < r\}$$

for $\zeta \in S$ and $r > 0$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \leq \sigma(B(\zeta, r)) \leq A_0 r^{2n} \tag{2.2}$$

for all $\zeta \in S$ and $0 < r \leq \sqrt{2}$ [9,Proposition 5.1.4]. Note that the upper bound actually holds when $r > \sqrt{2}$. For any $f \in L^2(S, d\sigma)$, define

$$\mathrm{SD}(f; \zeta, r) = \left(\frac{1}{\sigma(B(\zeta, r))} \int_{B(\zeta,r)} |f - f_{B(\zeta,r)}|^2 d\sigma\right)^{1/2},$$

where

$$f_{B(\zeta,r)} = \frac{1}{\sigma(B(\zeta, r))} \int_{B(\zeta,r)} f d\sigma.$$

It is easy to see if $\zeta, \xi \in S$ and $r, \rho \in (0, \infty)$ satisfy the relation $B(\xi, \rho) \subset B(\zeta, r)$, then

$$\mathrm{SD}(f; \xi, \rho) \leq \left\{\frac{\sigma(B(\zeta, r))}{\sigma(B(\xi, \rho))}\right\}^{1/2} \mathrm{SD}(f; \zeta, r). \tag{2.3}$$

Using the newly introduced notation SD, we can restate [12,Proposition 2.2] as

**Proposition 2.1.** *There exists a constant* $0 < C_{2.1} < \infty$ *such that the inequality*

$$\mathrm{SD}(Pf;\zeta,a) \le C_{2.1}\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{SD}(f;\zeta,2^k a)$$

*holds for all* $f \in L^2(S,d\sigma)$, $\zeta \in S$ *and* $a > 0$.

**Lemma 2.2.** *There exists a constant* $0 < C_{2.2} < \infty$ *such that for all* $f \in L^2(S,d\sigma)$ *and* $z \in \mathbf{B}\backslash\{0\}$, *we have*

$$\|(f - \langle fk_z, k_z\rangle)k_z\| \le C_{2.2}\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{SD}(f;\zeta,2^k a),$$

*where* $a = (1-|z|^2)^{1/2}$ *and* $\zeta = z/|z|$.

*Proof.* Let $f$, $z$, $\zeta$ and $a$ be as above. Write $B_k = B(\zeta, 2^k a)$ for every $k \in \mathbf{N}$. Then

$$\|(f - \langle fk_z, k_z\rangle)k_z\|^2 \le \int_{B_1}|f - f_{B_1}|^2|k_z|^2 d\sigma + \sum_{k=2}^{\infty}\int_{B_k\backslash B_{k-1}}|f - f_{B_1}|^2|k_z|^2 d\sigma. \tag{2.4}$$

For $x \in B_1$, we have $|1 - \langle x, z\rangle| \ge 1 - |z| \ge a^2/2$. Thus we have

$$|k_z(x)|^2 \le \left\{\frac{a^2}{(a^2/2)^2}\right\}^n = \frac{2^{2n}}{a^{2n}} \le \frac{2^{4n}A_0}{\sigma(B_1)} \quad \text{if} \;\; x \in B_1. \tag{2.5}$$

If $x \in S\backslash B_{k-1}$, $k \ge 2$, then $|1 - \langle x, z\rangle| \ge (1/2)|1 - \langle x, \zeta\rangle| \ge 2^{2k-3}a^2$. Hence

$$|k_z(x)|^2 \le \left\{\frac{a^2}{(2^{2k-3}a^2)^2}\right\}^n = \frac{1}{2^{(4k-6)n}a^{2n}} \le \frac{2^{6n}A_0}{2^{2nk}\sigma(B_k)} \quad \text{if} \;\; x \in B\backslash B_{k-1}. \tag{2.6}$$

Write $C_1 = 2^{6n}A_0$. Then by (2.4-6) we have

$$\|(f - \langle fk_z, k_z\rangle)k_z\|^2 \le C_1\sum_{k=1}^{\infty}\frac{1}{2^{2nk}\sigma(B_k)}\int_{B_k}|f - f_{B_1}|^2 d\sigma. \tag{2.7}$$

For any integer $k \ge 2$,

$$\begin{aligned}|f - f_{B_1}|^2 &\le 2|f - f_{B_k}|^2 + 2|f_{B_k} - f_{B_1}|^2 \le 2|f - f_{B_k}|^2 + 2(k-1)\sum_{j=2}^{k}|f_{B_{j-1}} - f_{B_j}|^2\\ &\le 2|f - f_{B_k}|^2 + 2(k-1)\sum_{j=2}^{k}\frac{1}{\sigma(B_{j-1})}\int_{B_{j-1}}|f - f_{B_j}|^2 d\sigma\\ &\le 2|f - f_{B_k}|^2 + C_2(k-1)\sum_{j=2}^{k}\frac{1}{\sigma(B_j)}\int_{B_j}|f - f_{B_j}|^2 d\sigma,\end{aligned}$$

where $C_2 = 2^{3n+1}A_0$. Let $C_3 = C_1(2+C_2)$. Combining this with (2.7), we see that

$$\begin{aligned}
\|(f-\langle fk_z,k_z\rangle)k_z\|^2 &\le \sum_{k=1}^{\infty}\frac{C_3k}{2^{2nk}}\sum_{j=1}^{k}\frac{1}{\sigma(B_j)}\int_{B_j}|f-f_{B_j}|^2d\sigma \\
&= \sum_{j=1}^{\infty}\{\mathrm{SD}(f;\zeta,2^ja)\}^2\sum_{k=j}^{\infty}\frac{C_3k}{2^{2nk}} \le \sum_{j=1}^{\infty}\frac{1}{2^{2j}}\{\mathrm{SD}(f;\zeta,2^ja)\}^2\sum_{k=1}^{\infty}\frac{C_3k}{2^{2(n-1)k}}.
\end{aligned}$$

If $t_j \ge 0$ for every $j \ge 1$, then $(\sum_j t_j)^{1/2} \le \sum_j t_j^{1/2}$. Hence the above yields

$$\|(f-\langle fk_z,k_z\rangle)k_z\| \le \left\{\sum_{k=1}^{\infty}\frac{C_3k}{2^{2(n-1)k}}\right\}^{1/2}\sum_{j=1}^{\infty}\frac{1}{2^j}\mathrm{SD}(f;\zeta,2^ja).$$

This completes the proof. □

For each $z \in \mathbf{B}\backslash\{0\}$, define the Möbius transform

$$\varphi_z(w) = \frac{1}{1-\langle w,z\rangle}\left\{z - \frac{\langle w,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(w - \frac{\langle w,z\rangle}{|z|^2}z\right)\right\}, \qquad |w| \le 1.$$

Then $\varphi_z$ is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$ [9,Theorem 2.2.2]. Recall that the formula

$$(U_zf)(\zeta) = f(\varphi_z(\zeta))k_z(\zeta), \qquad \zeta \in S \text{ and } f \in L^2(S,d\sigma), \tag{2.8}$$

defines a unitary operator with the property $[U_z, P] = 0$ [11,Section 6].

**Lemma 2.3.** *There is a constant* $C_{2.3}$ *such that the following estimate holds*: *Let* $0 < a < 1$ *and* $\zeta \in S$, *and set* $z = (1-a^2)^{1/2}\zeta$. *Let* $f \in L^2(S,d\sigma)$. *Then for each* $a \le b \le 4$,

$$\mathrm{SD}(f\circ\varphi_z;\zeta,b) \le C_{2.3}\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{SD}(f;\zeta,2^{k+2}(a/b)). \tag{2.9}$$

*Proof.* Let $\zeta, a$ and $b$ be given as described above. Denote $G = B(\zeta,b)$. Then for any $f \in L^2(S,d\sigma)$ and any $c \in \mathbf{C}$ we have

$$\{\mathrm{SD}(f\circ\varphi_z;\zeta,b)\}^2 \le \frac{1}{\sigma(G)}\int_G|f\circ\varphi_z - c|^2d\sigma = \int_{\varphi_z(G)}|f-c|^2\frac{|k_z|^2}{\sigma(G)}d\sigma. \tag{2.10}$$

Note that $4(a/b) \ge a$ under our assumption. Thus if $b \ge 2^{-3}$, then (2.9) follows from (2.10), Lemma 2.2, (2.3) and (2.2). For the rest of the proof we assume $b < 2^{-3}$. Then there exist an integer $\ell \ge 3$ and an $R \in [1/2,1)$ such that

$$b = 2^{-\ell}R.$$

To complete the proof, we first show that

$$\varphi_z(G) \subset S \backslash B(\zeta, 2^{\ell-1}a). \tag{2.11}$$

To verify (2.11), consider any $y \in G = B(\zeta, b)$. We have $|1-\langle y, z\rangle| \leq 1-|z|+|z||1-\langle y, \zeta\rangle| \leq a^2 + b^2 \leq 2b^2$. Note that for the last $\leq$ we used the assumption $b \geq a$. It follows from [9, Theorem 2.2.2(iii)] that $k_z \circ \varphi_z = 1/k_z$. Thus for $y \in G$ we have

$$|k_z(\varphi_z(y))|^2 = |k_z(y)|^{-2} = \{|1-\langle y, z\rangle|^2 a^{-2}\}^n \leq \{4b^4 a^{-2}\}^n.$$

In other words, if $x \in \varphi_z(G)$, then $\{a^2/|1-\langle x, z\rangle|^2\}^n = |k_z(x)|^2 \leq \{4b^4a^{-2}\}^n$. Hence

$$|1-\langle x, z\rangle| \geq (2b^2)^{-1}a^2 = (2R^2)^{-1}(2^\ell a)^2 \geq (1/2)(2^\ell a)^2 \quad \text{if } \ x \in \varphi_z(G). \tag{2.12}$$

On the other hand, if $w \in B(\zeta, 2^{\ell-1}a)$, then

$$|1-\langle w, z\rangle| \leq 1-|z|+|z||1-\langle w, \zeta\rangle| \leq a^2 + 2^{2\ell-2}a^2 \leq (5/16)(2^\ell a)^2. \tag{2.13}$$

Thus (2.11) follows from a comparison between (2.12) and (2.13).

Denote $B_k = B(\zeta, 2^k a)$ for $k \geq \ell - 1$. If $x \in B_{k+1}\backslash B_k$, then $|1-\langle x, z\rangle| \geq (1/2)|1-\langle x, \zeta\rangle| \geq 2^{2k-1}a^2$. Recalling (2.2), for $x \in B_{k+1}\backslash B_k$ we have

$$\frac{|k_z(x)|^2}{\sigma(G)} \leq \left(\frac{a^2}{\{2^{2k-1}a^2\}^2}\right)^n \cdot \frac{1}{2^{-n}(2^{-\ell}R)^{2n}} \leq \frac{C_1 2^{2n(\ell-k)}}{\sigma(B_{k+1})},$$

where $C_1 = 2^{7n}A_0$. Combining this with (2.10) and (2.11), we have

$$\begin{aligned}\{\mathrm{SD}(f\circ\varphi_z;\zeta,b)\}^2 &\leq \sum_{k=\ell-1}^{\infty}\int_{B_{k+1}\backslash B_k}|f-c|^2\frac{|k_z|^2}{\sigma(G)}d\sigma \\ &\leq \sum_{k=\ell-1}^{\infty}\frac{C_1 2^{2n(\ell-k)}}{\sigma(B_{k+1})}\int_{B_{k+1}\backslash B_k}|f-c|^2 d\sigma \leq \sum_{j=1}^{\infty}\frac{C_2 2^{-2nj}}{\sigma(B_{j+L})}\int_{B_{j+L}}|f-c|^2 d\sigma,\end{aligned} \tag{2.14}$$

where $L = \ell-1$, $C_2 = 2^{4n}C_1$, and $c$ is any complex number. The rest of the proof resembles the proof of Lemma 2.2, as it should. For any integer $j \geq 1$,

$$\begin{aligned}|f-f_{B_L}|^2 &\leq 2|f-f_{B_{j+L}}|^2 + 2|f_{B_L}-f_{B_{j+L}}|^2 \\ &\leq 2|f-f_{B_{j+L}}|^2 + C_3 j\sum_{k=1}^{j}\frac{1}{\sigma(B_{k+L})}\int_{B_{k+L}}|f-f_{B_{k+L}}|^2 d\sigma,\end{aligned}$$

where $C_3 = 2^{3n+1}A_0$. Let $C_4 = C_2(2+C_3)$. Setting $c = f_{B_L}$ in (2.14), we find that

$$\begin{aligned}\{\mathrm{SD}(f\circ\varphi_z;\zeta,b)\}^2 &\leq \sum_{j=1}^{\infty}\frac{C_4}{2^{2nj}}j\sum_{k=1}^{j}\frac{1}{\sigma(B_{k+L})}\int_{B_{k+L}}|f-f_{B_{k+L}}|^2 d\sigma \\ &= \sum_{k=1}^{\infty}\{\mathrm{SD}(f;\zeta,2^{k+L}a)\}^2\sum_{j=k}^{\infty}\frac{C_4 j}{2^{2nj}} \leq \sum_{k=1}^{\infty}\frac{1}{2^{2k}}\{\mathrm{SD}(f;\zeta,2^{k+L}a)\}^2\sum_{j=1}^{\infty}\frac{C_5 j}{2^{2(n-1)j}}.\end{aligned}$$

If $t_k \geq 0$ for every $k \geq 1$, then $(\sum_k t_k)^{1/2} \leq \sum_k t_k^{1/2}$. Hence the above yields

$$\mathrm{SD}(f \circ \varphi_z; \zeta, b) \leq \left\{ \sum_{j=1}^{\infty} \frac{C_5 j}{2^{2(n-1)j}} \right\}^{1/2} \sum_{k=1}^{\infty} \frac{1}{2^k} \mathrm{SD}(f; \zeta, 2^{k+L} a).$$

Since $(1/4)(a/b) \leq 2^L a \leq a/b$, the lemma follows from this inequality and (2.3). $\square$

**Lemma 2.4.** *There is a constant* $C_{2.4}$ *such that the following estimate holds: Let* $0 < a < 1$ *and* $\zeta \in S$. *Set* $z = (1-a^2)^{1/2}\zeta$. *If* $N \in \mathbf{N}$ *satisfies the condition* $2^N a \leq 4$, *then*

$$\sum_{k=N}^{\infty} \frac{1}{2^k} \mathrm{SD}((P(f \circ \varphi_z)) \circ \varphi_z; \zeta, 2^k a) \leq C_{2.4} \frac{1}{2^{\epsilon N}} \sum_{j=1}^{\infty} \frac{j}{2^{(1-\epsilon)j}} \mathrm{SD}(f; \zeta, 2^j a)$$

*for all* $f \in L^2(S, d\sigma)$ *and* $0 < \epsilon \leq 1/2$.

*Proof.* First note that for any $\xi \in S$ and $r > 0$, if $\nu \geq 0$ is such that $2^\nu r \geq 2$, then

$$\sum_{j=\nu}^{\infty} \frac{1}{2^j} \mathrm{SD}(f; \xi, 2^j r) = 2 \frac{1}{2^\nu} \mathrm{SD}(f; \xi, 2^\nu r). \tag{2.15}$$

Consider any $k \geq N$ such that $2^k a \leq 4$. Applying Lemma 2.3 and (2.15), we have

$$\mathrm{SD}((P(f \circ \varphi_z)) \circ \varphi_z; \zeta, 2^k a) \leq C_1 \sum_{m=0}^{k-1} \frac{1}{2^m} \mathrm{SD}(P(f \circ \varphi_z); \zeta, 2^{m-k+2}).$$

Then apply Proposition 2.1 to each term on the right hand side. We have

$$\mathrm{SD}((P(f \circ \varphi_z)) \circ \varphi_z; \zeta, 2^k a) \leq C_2 \sum_{m=0}^{k-1} \sum_{d=0}^{k-1-m} \frac{1}{2^{m+d}} \mathrm{SD}(f \circ \varphi_z; \zeta, 2^{d+m-k+2}).$$

By the condition $2^k a \leq 4$, we have $a \leq 2^{-k+2} \leq 2^{d+m-k+2}$. On the other hand, if $d \leq k-1-m$, then $2^{d+m-k+2} \leq 2$. Thus we can apply Lemma 2.3 to each term on the right hand side to obtain

$$\mathrm{SD}((P(f \circ \varphi_z)) \circ \varphi_z; \zeta, 2^k a) \leq C_3 \sum_{m=0}^{k-1} \sum_{d=0}^{k-1-m} \sum_{i=1}^{\infty} \frac{1}{2^{m+d+i}} \mathrm{SD}(f; \zeta, 2^{i+k-m-d} a).$$

Combining this inequality with (2.15), we have

$$\begin{aligned}
\sum_{k=N}^{\infty} \frac{1}{2^k} \mathrm{SD}((P(f \circ \varphi_z)) \circ \varphi_z; \zeta, 2^k a) &\leq 2 \sum_{2^N a \leq 2^k a \leq 4} \frac{1}{2^k} \mathrm{SD}(P(f \circ \varphi_z) \circ \varphi_z; \zeta, 2^k a) \\
&\leq 2C_3 \sum_{k=N}^{\infty} \sum_{m=0}^{k-1} \sum_{d=0}^{k-1-m} \sum_{i=1}^{\infty} \frac{1}{2^{k+m+d+i}} \mathrm{SD}(f; \zeta, 2^{i+k-m-d} a) \\
&\leq 2C_3 \sum_{j=1}^{\infty} \mathrm{SD}(f; \zeta, 2^j a) \sum_{C(N,j;k,m,d,i)} \frac{1}{2^{k+m+d+i}}, \qquad (2.16)
\end{aligned}$$

where $C(N,j;k,m,d,i)$ represents the following set of constraints: $k \geq N$, $m \geq 0$, $d \geq 0$, $i \geq 1$, and $i+k-m-d=j$. For any $0 < \epsilon \leq 1/2$, we have

$$\begin{aligned}\sum_{C(N,j;k,m,d,i)} \frac{1}{2^{k+m+d+i}} &\leq \frac{1}{2^{\epsilon N}} \sum_{C(N,j;k,m,d,i)} \frac{1}{2^{(1-\epsilon)(k+m+d+i)}} \\ &= \frac{1}{2^{\epsilon N}} \cdot \frac{1}{2^{(1-\epsilon)j}} \sum_{C(N,j;k,m,d,i)} \frac{1}{2^{2(1-\epsilon)(m+d)}}.\end{aligned} \tag{2.17}$$

Now we need to count the number of tuples $(i,k,m,d)$ satisfying $C(N,j;k,m,d,i)$ and the additional constraint $m+d=t$, $t \geq 0$. There are at most $t+1$ pairs of such $(m,d)$, and there are at most $j+t+1$ pairs of $(i,k)$ satisfying the condition $i+k-t=j$. Therefore the total number of such tuples $(i,k,m,d)$ does not exceed $(t+1)(j+t+1)$. Thus

$$\begin{aligned}\sum_{C(N,j;k,m,d,i)} \frac{1}{2^{2(1-\epsilon)(m+d)}} &\leq \sum_{t=0}^{\infty} \frac{1}{2^{2(1-\epsilon)t}}(t+1)(j+t+1) \\ &\leq j\sum_{t=0}^{\infty} \frac{1}{2^{t}}(t+1)(t+2) = C_4 \cdot j.\end{aligned} \tag{2.18}$$

Now we substitute (2.18) into (2.17), and then the new (2.17) into (2.16). This gives us the desired estimate. $\square$

Let $d\lambda$ be the Möbius invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}},$$

where $dv$ is the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$.

For each $z \in \mathbf{B}\backslash\{0\}$ and each integer $k \geq 0$, denote

$$B_k(z) = B(z/|z|, 2^k(1-|z|^2)^{1/2}). \tag{2.19}$$

Keep in mind that $B_k(z)$ is a ball with respect to the metric $d$ in $S$.

**Lemma 2.5.** *There is a constant* $C_{2.5}$ *such that the inequality*

$$\int \frac{\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi)}{\sigma^2(B_k(z))}\, d\lambda(z) \leq \frac{C_{2.5}k \cdot 2^{2nk}}{|1-\langle\zeta,\xi\rangle|^{2n}}.$$

*holds for all* $k \in \mathbf{N}$ *and* $\zeta \neq \xi$ *in* $S$.

*Proof.* Given a pair of $\zeta \neq \xi$ in $S$, we have $2^{-\ell} \leq d(\zeta,\xi) < 2^{-\ell+1}$ for some $\ell \geq 0$. If $\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi) \neq 0$, then $2^k(1-|z|^2)^{1/2} \geq 2^{-\ell-1}$, which implies that

$$(1-|z|^2)^{1/2} \geq 2^{-\ell-k-1}.$$

Define $G_j = \{z \in \mathbf{B} : \xi \in B_k(z), 2^{-j} \le (1-|z|^2)^{1/2} < 2^{-j+1}\}$ for $1 \le j \le \ell + k + 1$. Then

$$\int \frac{\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi)}{\sigma^2(B_k(z))}\, d\lambda(z) \le \sum_{j=1}^{\ell+k+1} \int_{G_j} \frac{1}{\sigma^2(B_k(z))}\, d\lambda(z). \tag{2.20}$$

Recalling (2.2), if $z \in G_j$ with $j \ge k$, then

$$\sigma(B_k(z)) \ge C_1(2^k(1-|z|^2)^{1/2})^{2n} \ge C_1(2^k 2^{-j})^{2n} = C_1 2^{2n(k-j)}, \tag{2.21}$$

where $C_1 = 2^{-2n}$. If $z \in G_j$ with $j \le k-1$, then $\sigma(B_k(z)) = \sigma(S) = 1$. Note that

$$\lambda(G_j) \le \frac{v(G_j)}{(2^{-j})^{2(n+1)}} \le C_2 \frac{2^{-2j}(2^{k-j})^{2n}}{(2^{-j})^{2(n+1)}} = C_2 2^{2nk} \tag{2.22}$$

for every $1 \le j \le \ell + k + 1$. Combining (2.20), (2.21) and (2.22), we get

$$\begin{aligned}\int \frac{\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi)}{\sigma^2(B_k(z))}\, d\lambda(z) &\le \sum_{j=0}^{k-1} C_2 2^{2nk} + \sum_{j=k}^{\ell+k+1} \frac{C_2 2^{2nk}}{C_1^2 2^{4n(k-j)}} \\ &\le C_3\left(k \cdot 2^{2nk} + \frac{2^{4n(\ell+k+1)}}{2^{2nk}}\right) \le C_4 \frac{k \cdot 2^{2nk}}{|1-\langle \zeta, \xi\rangle|^{2n}},\end{aligned}$$

where the last $\le$ holds because $d(\zeta, \xi) < 2^{-\ell+1}$. This completes the proof. $\square$

**Proposition 2.6.** *Let $2n < p < \infty$. There is a constant $C_{2.6}(p)$ which depends only on $p$ and $n$ such that the inequality*

$$\int \|(f - \langle f k_z, k_z\rangle)k_z\|^p\, d\lambda(z) \le C_{2.6}(p)\mathcal{I}_p(f)$$

*holds for every $f \in L^2(S, d\sigma)$.*

*Proof.* From Lemma 2.2, for each $z \in \mathbf{B}\backslash\{0\}$ we have

$$\|(f - \langle f k_z, k_z\rangle)k_z\| \le C_{2.2} \sum_{k=1}^{\infty} \frac{1}{2^k} \mathrm{SD}(f; z/|z|, 2^k(1-|z|^2)^{1/2}).$$

Since $p > 2n$, we can write $p = 2n + 2\epsilon$ with some $\epsilon > 0$. Splitting $2^{-k}$ as $2^{-\epsilon k/p} \cdot 2^{-(2n+\epsilon)k/p}$ and applying Hölder's inequality to the above, we find that

$$\|(f - \langle f k_z, k_z\rangle)k_z\|^p \le C_1 \sum_{k=1}^{\infty} \frac{1}{2^{(2n+\epsilon)k}} \{\mathrm{SD}(f; z/|z|, 2^k(1-|z|^2)^{1/2})\}^p.$$

By Hölder's inequality,

$$\{\mathrm{SD}(f; z/|z|, 2^k(1-|z|^2)^{1/2})\}^p \le \frac{1}{\sigma(B_k(z))} \int_{B_k(z)} |f - f_{B_k(z)}|^p d\sigma. \tag{2.23}$$

On the other hand, for each $\zeta \in S$ we have

$$|f(\zeta) - f_{B_k(z)}|^p \le \frac{1}{\sigma(B_k(z))} \int_{B_k(z)} |f(\zeta) - f(\xi)|^p d\sigma(\xi). \tag{2.24}$$

Thus the combination of the above three inequalities yields

$$\|(f - \langle f k_z, k_z\rangle)k_z\|^p \le C_1 \sum_{k=1}^{\infty} \frac{2^{-(2n+\epsilon)k}}{\sigma^2(B_k(z))} \iint_{B_k(z)\times B_k(z)} |f(\zeta) - f(\xi)|^p d\sigma(\zeta) d\sigma(\xi).$$

Integrating both sides with respect to the measure $d\lambda$, we obtain

$$\int \|(f - \langle f k_z, k_z\rangle)k_z\|^p \, d\lambda(z) \le C_1 \iint \mathcal{G}(\zeta, \xi)|f(\zeta) - f(\xi)|^p d\sigma(\zeta) d\sigma(\xi), \tag{2.25}$$

where

$$\mathcal{G}(\zeta, \xi) = \sum_{k=1}^{\infty} \frac{1}{2^{(2n+\epsilon)k}} \int \frac{\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi)}{\sigma^2(B_k(z))} \, d\lambda(z).$$

It follows from Lemma 2.5 that

$$\mathcal{G}(\zeta, \xi) \le C_{2.5} \left( \sum_{k=1}^{\infty} \frac{k \cdot 2^{2nk}}{2^{(2n+\epsilon)k}} \right) \frac{1}{|1 - \langle \zeta, \xi\rangle|^{2n}} = C_{2.5} \left( \sum_{k=1}^{\infty} \frac{k}{2^{\epsilon k}} \right) \frac{1}{|1 - \langle \zeta, \xi\rangle|^{2n}}.$$

Substituting this in (2.25), the proof is complete. $\square$

## 3. A Quasi-resolution of the Identity Operator

Let $t$ be a positive real number. For each $z \in \mathbf{B}$, define the function

$$\psi_{z,t}(w) = \frac{(1 - |z|^2)^{(n/2)+t}}{(1 - \langle w, z\rangle)^{n+t}},$$

$|w| \le 1$. We also define the Schur multiplier

$$m_z(w) = \frac{1 - |z|}{1 - \langle w, z\rangle}, \tag{3.1}$$

$|w| \le 1$. Then we have the relation

$$\psi_{z,t} = (1 + |z|)^t m_z^t k_z. \tag{3.2}$$

Given a $t > 0$, we need a crude asymptotic formula for $t(t + 1)...(t + k)$, which is derived in the same way as Stirling's formula for factorial. We have the identity

$$\frac{1}{2}\{f(1) + f(0)\} = \int_0^1 f(x)dx - \frac{1}{2}\int_0^1 (x^2 - x)f''(x)dx$$

for any $C^2$-function $f$ on any neighborhood of $[0,1]$. From this it follows that

$$\sum_{j=0}^{k}\log(t+j)=\frac{1}{2}\{\log t+\log(t+k)\}+\int_0^k\log(t+x)dx+\frac{1}{2}\sum_{j=0}^{k-1}\int_0^1\frac{x^2-x}{(t+j+x)^2}dx,$$

$k\in\mathbf{N}$. Evaluating the integral $\int_0^k$ and then exponentiating both sides, we find that

$$\prod_{j=0}^{k}(t+j)=(t+k)^{t+k+(1/2)}e^{-k}e^{c(t;k)}, \tag{3.3}$$

where $c(t;k)$ has a finite limit (which depends on $t$) as $k\to\infty$.

**Proposition 3.1.** *For each* $t>0$, *the self-adjoint operator*

$$R_t=\int\psi_{z,t}\otimes\psi_{z,t}d\lambda(z)$$

*is bounded on the Hardy space* $H^2(S)$. *In other words, for any given* $t>0$, *there exists a constant* $0<\beta(t)<\infty$ *which depends only on* $t$ *and the complex dimension* $n$ *such that*

$$\langle R_th,h\rangle\leq\beta(t)\|h\|^2$$

*for every* $h\in H^2(S)$.

*Proof.* Write $C_k^m$ for the binomial coefficient $m!/(k!(m-k)!)$. We first show that for all $w,w'\in\mathbf{B}$ and integer $k\geq 0$,

$$C_k^{n-1+k}\int\langle w,u\rangle^k\langle u,w'\rangle^k d\sigma(u)=\langle w,w'\rangle^k. \tag{3.4}$$

Since any two monomials of different degrees in $H^2(S)$ are orthogonal to each other, for every $0\leq r<1$ we have

$$\begin{aligned}r^k\langle w,w'\rangle^k&=\int\frac{\langle u,w'\rangle^k}{(1-r\langle w,u\rangle)^n}d\sigma(u)=\sum_{j=0}^{\infty}C_j^{n-1+j}r^j\int\langle u,w'\rangle^k\langle w,u\rangle^j d\sigma(u)\\&=C_k^{n-1+k}r^k\int\langle w,u\rangle^k\langle u,w'\rangle^k d\sigma(u),\end{aligned}$$

proving (3.4).

Given any $t>0$, we have the power series expansion

$$\frac{1}{(1-v)^{n+t}}=\sum_{k=0}^{\infty}a_{k,t}v^j$$

on the open unit disc $\{v \in \mathbf{C} : |v| < 1\}$, where $a_{0,t} = 1$ and

$$a_{k,t} = \frac{1}{k!} \prod_{j=0}^{k-1} (n + t + j) \tag{3.5}$$

for $k \geq 1$. Thus for $w, w' \in \mathbf{B}$ and $0 \leq r < 1$, we have

$$\begin{aligned}
\int \psi_{ru,t}(w)\overline{\psi_{ru,t}(w')}d\sigma(u) &= \int \frac{(1-r^2)^{n+2t}}{(1-r\langle w,u\rangle)^{n+t}(1-r\langle u,w'\rangle)^{n+t}} d\sigma(u) \\
&= \sum_{k=0}^{\infty} a_{k,t}^2 (1-r^2)^{n+2t} r^{2k} \int \langle w,u\rangle^k \langle u,w'\rangle^k d\sigma(u) \\
&= \sum_{k=0}^{\infty} \frac{a_{k,t}^2}{C_k^{n-1+k}} (1-r^2)^{n+2t} r^{2k} \langle w,w'\rangle^k,
\end{aligned}$$

where the last $=$ follows from (3.4). Therefore

$$\begin{aligned}
\int \psi_{z,t}(w)\overline{\psi_{z,t}(w')}d\lambda(z) &= \int_0^1 \int \psi_{ru,t}(w)\overline{\psi_{ru,t}(w')}d\sigma(u) \frac{2nr^{2n-1}dr}{(1-r^2)^{n+1}} \\
&= 2n \sum_{k=0}^{\infty} \frac{a_{k,t}^2}{C_k^{n-1+k}} \int_0^1 \frac{(1-r^2)^{n+2t} r^{2k} r^{2n-1}}{(1-r^2)^{n+1}} dr \langle w,w'\rangle^k \\
&= 2n \sum_{k=0}^{\infty} \frac{a_{k,t}^2}{C_k^{n-1+k}} \int_0^1 (1-r^2)^{2t-1} r^{2k+2n-1} dr \langle w,w'\rangle^k.
\end{aligned}$$

Since $2t - 1 > -1$, we can integrate by parts to obtain

$$2\int_0^1 (1-r^2)^{2t-1} r^{2k+2n-1} dr = \int_0^1 (1-x)^{2t-1} x^{n-1+k} dx = \frac{(n-1+k)!}{\prod_{j=0}^{n-1+k}(2t+j)}.$$

Hence

$$\int \psi_{z,t}(w)\overline{\psi_{z,t}(w')}d\lambda(z) = \sum_{k=0}^{\infty} b_{k,t} C_k^{n-1+k} \langle w,w'\rangle^k, \tag{3.6}$$

where

$$b_{k,t} = n \left( \frac{a_{k,t}}{C_k^{n-1+k}} \right)^2 \frac{(n-1+k)!}{\prod_{j=0}^{n-1+k}(2t+j)}.$$

Using (3.3) and (3.5), it is straightforward to verify that there exists a $0 < \beta(t) < \infty$ which depends only on $t$ and $n$ such that

$$b_{k,t} \leq \beta(t) \tag{3.7}$$

for every $k \geq 0$.

For each $k \geq 0$, let $T_k$ be the integral operator with the kernel function $\langle \zeta, \zeta' \rangle^k$ on $H^2(S)$. In other words,

$$(T_k h)(\zeta) = \int h(\zeta')\langle \zeta, \zeta' \rangle^k d\sigma(\zeta'),$$

$h \in H^2(S)$. Then obviously each $T_k$ is a positive operator. For each $0 < \rho < 1$, define

$$\psi_{z,t,\rho}(\zeta) = \psi_{z,t}(\rho\zeta), \qquad \zeta \in S.$$

Applying (3.6) and (3.7), for any $h \in H^2(S)$ we have

$$\begin{aligned}
\int |\langle h, \psi_{z,t,\rho}\rangle|^2 d\lambda(z) &= \int\int h(\zeta')\overline{h(\zeta)}\left\{\int \psi_{z,t}(\rho\zeta)\overline{\psi_{z,t}(\rho\zeta')}d\lambda(z)\right\} d\sigma(\zeta')d\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} b_{k,t}C_k^{n-1+k}\rho^{2k}\langle T_k h, h\rangle \leq \beta(t)\sum_{k=0}^{\infty} C_k^{n-1+k}\rho^{2k}\langle T_k h, h\rangle \\
&= \beta(t)\int h(\rho^2\zeta)\overline{h(\zeta)}d\sigma(\zeta) \leq \beta(t)\|h\|^2.
\end{aligned}$$

Clearly, for each $z$ we have $\|\psi_{z,t,\rho} - \psi_{z,t}\| \to 0$ as $\rho \uparrow 1$. Thus, by Fatou's lemma,

$$\langle R_t h, h\rangle = \int |\langle h, \psi_{z,t}\rangle|^2 d\lambda(z) \leq \liminf_{\rho\uparrow 1}\int |\langle h, \psi_{z,t,\rho}\rangle|^2 d\lambda(z) \leq \beta(t)\|h\|^2,$$

establishing the bound for $R_t$. $\square$

**Corollary 3.2.** *Let $t > 0$. Then for any positive operator $A$ on $H^2(S)$ we have*

$$\int \langle A\psi_{z,t}, \psi_{z,t}\rangle d\lambda(z) \leq \beta(t)\mathrm{tr}(A), \tag{3.8}$$

*where $\beta(t)$ is the constant provided by Proposition* 3.1.

*Proof.* If $\mathrm{rank}(A) < \infty$, then the left hand side of (3.8) is just $\mathrm{tr}(AR_t) = \mathrm{tr}(A^{1/2}R_tA^{1/2})$. Hence (3.8) follows from Proposition 3.1 in the case $\mathrm{rank}(A) < \infty$. For an arbitrary $A$, consider an increasing sequence of finite-rank orthogonal projections $\{E_k\}$ which converges to 1 strongly on $H^2(S)$. Since (3.8) holds for each $A_k = A^{1/2}E_kA^{1/2}$, applying the monotone convergence theorem to both sides, the general case follows. $\square$

**Lemma 3.3.** *There exists a constant $0 < C_{3.3} < \infty$ which depends only on the complex dimension $n$ such that the inequality $\|[P, M_{m_z^t}]\| \leq C_{3.3}t$ holds for all $z \in \mathbf{B}$ and $t > 0$.*

*Proof.* It is well known [2] that there is a constant $C$ which depends only on $n$ such that

$$\|[P, M_f]\| \leq C\|f\|_{\mathrm{BMO}} \tag{3.9}$$

for every $f \in \mathrm{BMO}$ (also see [13]). By (3.9), it suffices to find a $C_1$ which depends only on $n$ such that

$$\|m_z^t\|_{\mathrm{BMO}} \le C_1 t \tag{3.10}$$

for all $z \in \mathbf{B}$ and $t > 0$.

To prove (3.10), let $\eta$ be the function on the unit circle $\mathbf{T} = \{e^{ix} : 0 < x \le 2\pi\}$ such that $\eta(e^{ix}) = \pi - x$ for $0 < x \le 2\pi$. Then $\eta(e^{ix}) = -i\sum_{k=1}^{\infty}(1/k)(e^{ikx} - e^{-ikx})$. Integrating this against the Poisson kernel on $\mathbf{T}$, we conclude that the inequality

$$\left|\log\frac{1}{1-v} - \overline{\log\frac{1}{1-v}}\right| \le \pi \tag{3.11}$$

holds on the open unit disc $\{v \in \mathbf{C} : |v| < 1\}$. For each $z \in \mathbf{B}$, define the functions

$$\Omega_z(\zeta) = \log\frac{1}{1-\langle\zeta,z\rangle} - \overline{\log\frac{1}{1-\langle\zeta,z\rangle}} \quad \text{and} \quad L_z(\zeta) = \log\frac{1}{1-\langle\zeta,z\rangle},$$

$\zeta \in S$. Then (3.11) tells us that $\|\Omega_z\|_\infty \le \pi$ for every $z \in \mathbf{B}$. Since $P\Omega_z = L_z$, it follows from Proposition 2.1 that

$$\|L_z\|_{\mathrm{BMO}} = \|P\Omega_z\|_{\mathrm{BMO}} \le C_{2.1} \cdot 2\|\Omega_z\|_\infty \le 2\pi C_{2.1}, \tag{3.12}$$

$z \in \mathbf{B}$. Let

$$J_z(\zeta) = \log\frac{1-|z|}{1-\langle\zeta,z\rangle}.$$

Since $\log(1-|z|)$ is a constant on $S$, from (3.12) we obtain

$$\|J_z\|_{\mathrm{BMO}} \le 2\pi C_{2.1}. \tag{3.13}$$

For each $z \in \mathbf{B}$, let $X_z$ and $Y_z$ be the real part and imaginary part of $J_z$ respectively. Because $e^{X_z(\zeta)} = |m_z(\zeta)| \le 1$ for every $\zeta \in S$, we conclude that $X_z \le 0$ on $S$.

Let an arbitrary $B = B(\xi, r)$ be given, where $\xi \in S$ and $r > 0$. Obviously, $(X_z)_B \le 0$. Since the inequality $|e^x - e^y| \le |x-y|$ holds for all $x, y \in (-\infty, 0]$, for $t > 0$ we have

$$\frac{1}{\sigma(B)}\int_B |e^{tX_z} - e^{t(X_z)_B}|d\sigma \le \frac{1}{\sigma(B)}\int_B |tX_z - t(X_z)_B|d\sigma \le 2\pi C_{2.1}t, \tag{3.14}$$

where the second $\le$ follows from (3.13). Also, we have $|e^{ix} - e^{iy}| \le |x-y|$ for all $x, y \in \mathbf{R}$. Since $Y_z$ is a real-valued function, we have

$$\frac{1}{\sigma(B)}\int_B |e^{itY_z} - e^{it(Y_z)_B}|d\sigma \le \frac{1}{\sigma(B)}\int_B |tY_z - t(Y_z)_B|d\sigma \le \pi t, \tag{3.15}$$

where the second $\leq$ follows from the facts that $Y_z = \Omega_z/2i$ and that $\|\Omega_z\|_\infty \leq \pi$. Since $X_z + iY_z = J_z = \log m_z$, we have

$$|m_z^t - e^{t(X_z)_B}e^{it(Y_z)_B}| = |e^{tX_z}e^{itY_z} - e^{t(X_z)_B}e^{it(Y_z)_B}| \leq |e^{tX_z} - e^{t(X_z)_B}| + |e^{itY_z} - e^{it(Y_z)_B}|.$$

Thus from (3.14) and (3.15) we obtain

$$\frac{1}{\sigma(B)}\int_B |m_z^t - e^{t(X_z)_B}e^{it(Y_z)_B}|d\sigma \leq \pi(2C_{2.1}+1)t.$$

Since $B = B(\xi, r)$ is arbitrary, this implies $\|m_z^t\|_{\mathrm{BMO}} \leq 2\pi(2C_{2.1}+1)t$, verifying (3.10). $\square$

**Lemma 3.4.** *Let $f \in L^2(S, d\sigma)$ and write $g = f - Pf$. Then for every $z \in \mathbf{B}\backslash\{0\}$ we have $H_f k_z = v_z k_z$, where $v_z = g - (P(g \circ \varphi_z)) \circ \varphi_z$.*

*Proof.* We use the $U_z$ defined by (2.8). Since $[U_z, P] = 0$ and $\varphi_z \circ \varphi_z = \mathrm{id}$, we have

$$v_z k_z = gk_z - U_z P(g \circ \varphi_z) = gk_z - PU_z(g\circ\varphi_z) = gk_z - P(gk_z) = H_g k_z = H_f k_z. \quad \square$$

**Lemma 3.5.** *Let $p \geq 2$. Then for all $0 < t \leq 1$ and $f \in L^2(S, d\sigma)$ we have*

$$\int \|M_{m_z} H_f k_z\|^p d\lambda(z) \leq 2^{2p}\beta(t)\|H_f\|_p^p + 2^{p-1}(C_{3.3}t)^p \int \|H_f k_z\|^p d\lambda(z), \tag{3.16}$$

*where $\beta(t)$ and $C_{3.3}$ are the constants given by Proposition* 3.1 *and Lemma* 3.3 *respectively.*

*Proof.* We may assume $\|H_f\|_p < \infty$, for otherwise (3.16) holds trivially. By Corollary 3.2,

$$\int \langle (H_f^* H_f)^{p/2}\psi_{z,t}, \psi_{z,t}\rangle d\lambda(z) \leq \beta(t)\mathrm{tr}((H_f^*H_f)^{p/2}) = \beta(t)\|H_f\|_p^p.$$

Since $p/2 \geq 1$, by the spectral decomopsition of $H_f^* H_f$ and Hölder's inequality,

$$\|H_f\psi_{z,t}\|^p = \langle H_f^* H_f \psi_{z,t}, \psi_{z,t}\rangle^{p/2} \leq \langle (H_f^*H_f)^{p/2}\psi_{z,t}, \psi_{z,t}\rangle \|\psi_{z,t}\|^{p-2}.$$

We have $\|\psi_{z,t}\| \leq 2^t$ by (3.1-2), and $2^t \leq 2$ since we assume $0 < t \leq 1$. Thus the combination of the above two inequalities gives us

$$\int \|H_f \psi_{z,t}\|^p \lambda(z) \leq 2^{p-2}\beta(t)\|H_f\|_p^p. \tag{3.17}$$

For the given $f$, let $g$ and $v_z$ be the same as in Lemma 3.4. Then $f - v_z = Pf + (P(g\circ\varphi_z))\circ\varphi_z \in H^2(S)$. Recalling (3.2), we have

$$\begin{aligned} \|H_f\psi_{z,t}\| &\geq \|H_f(m_z^t k_z)\| = \|H_{v_z}(m_z^t k_z)\| = \|(1-P)M_{m_z^t} v_z k_z\| \\ &\geq \|M_{m_z^t}(1-P)v_z k_z\| - \|[1-P, M_{m_z^t}]\|\|v_z k_z\|. \end{aligned} \tag{3.18}$$

By Lemma 3.4, $v_z k_z = H_f k_z$. And by Lemma 3.3, $\|[1-P, M_{m_z^t}]\| = \|[P, M_{m_z^t}]\| \le C_{3.3}t$. Also, since we now assume $0 < t \le 1$ and since $|m_z| \le 1$ on $S$, we have $\|M_{m_z^t}u\| \ge \|M_{m_z}u\|$ for every $u \in L^2(S, d\sigma)$. Bringing these facts into (3.18), we find that

$$\|M_{m_z}H_f k_z\| \le \|H_f \psi_{z,t}\| + C_{3.3}t\|H_f k_z\|.$$

Since $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for all $a, b \in [0, \infty)$, this leads to

$$\int \|M_{m_z}H_f k_z\|^p d\lambda(z) \le 2^{p-1}\int \|H_f\psi_{z,t}\|^p \lambda(z) + 2^{p-1}(C_{3.3}t)^p \int \|H_f k_z\|^p \lambda(z).$$

Substituting (3.17) in the above, (3.16) follows. □

## 4. Spherical Decomposition

For each $k \ge 0$, let $\{\xi_{k,1}, ..., \xi_{k,\nu(k)}\}$ be a subset of $S$ which is *maximal* with respect to the property

$$B(\xi_{k,i}, 2^{-k+1}) \cap B(\xi_{k,j}, 2^{-k+1}) = \emptyset \quad \text{if} \ \ i \ne j. \tag{4.1}$$

Denote

$$A_{k,j} = B(\xi_{k,j}, 2^{-k+3}), \qquad B_{k,j} = B(\xi_{k,j}, 2^{-k+4}) \qquad \text{and} \qquad C_{k,j} = B(\xi_{k,j}, 2^{-k+5}), \tag{4.2}$$

$k \ge 0$, $1 \le j \le \nu(k)$. The maximality of $\{\xi_{k,1}, ..., \xi_{k,\nu(k)}\}$ implies that

$$\cup_{j=1}^{\nu(k)} A_{k,j} = S. \tag{4.3}$$

**Definition 4.1.** For $p \ge 1$ and $g \in L^2(S, d\sigma)$, write

$$\mathcal{J}_p(g) = \sum_{k=0}^{\infty}\sum_{j=1}^{\nu(k)} \left( \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma \right)^p.$$

**Proposition 4.2.** *Given any $p > 1$, there exists a constant $0 < C_{4.2}(p) < \infty$ which depends only on $p$ and $n$ such that*

$$\mathcal{I}_p(g) \le C_{4.2}(p)\mathcal{J}_p(g)$$

*for every $g \in L^2(S, d\sigma)$.*

*Proof.* Let $g \in L^2(S, d\sigma)$ be given. We may assume $\mathcal{J}_p(g) < \infty$, for otherwise the desired inequality holds trivially. For every integer $k \ge 0$ define the function

$$g_k(\zeta) = \frac{1}{\sigma(B(\zeta, 2^{-k}))} \int_{B(\zeta, 2^{-k})} g d\sigma, \qquad \zeta \in S.$$

In other words, $g_k(\zeta)$ is the mean value of $g$ on $B(\zeta, 2^{-k})$. For each $k \geq 0$, define

$$E_k = \{(\zeta, \xi) \in S \times S : 2^{-k} \leq d(\zeta, \xi) < 2^{-k+1}\},$$

where $d$ was given by (2.1). We have

$$\begin{aligned} \mathcal{I}_p(g) &\leq \sum_{k=0}^{\infty} 2^{4nk} \iint_{E_k} |g(\zeta) - g(\xi)|^p d\sigma(\zeta) d\sigma(\xi) \\ &\leq \sum_{k=0}^{\infty} 2^{4nk} \iint_{E_k} 3^{p-1}(|g(\zeta) - g_k(\zeta)|^p + |g_k(\zeta) - g_k(\xi)|^p + |g_k(\xi) - g(\xi)|^p) d\sigma(\zeta) d\sigma(\xi). \end{aligned} \tag{4.4}$$

Applying Fubini's theorem and (2.2), we have

$$\begin{aligned} \iint_{E_k} |g(\zeta) - g_k(\zeta)|^p d\sigma(\zeta) d\sigma(\xi) &\leq \int |g(\zeta) - g_k(\zeta)|^p \sigma(B(\zeta, 2^{-k+1})) d\sigma(\zeta) \\ &\leq A_0 2^{-2n(k-1)} \int |g - g_k|^p d\sigma. \end{aligned}$$

Substituting this in (4.4), we see that

$$\mathcal{I}_p(g) \leq 3^{p-1}\{2^{2n+1} A_0 I_1 + I_2\}, \tag{4.5}$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} 2^{2nk} \int |g - g_k|^p d\sigma \quad \text{and} \\ I_2 &= \sum_{k=0}^{\infty} 2^{4nk} \iint_{E_k} |g_k(\zeta) - g_k(\xi)|^p d\sigma(\zeta) d\sigma(\xi). \end{aligned}$$

We will estimate $I_1$ and $I_2$ separately.

For $I_1$, note that by (4.3) and the fact that $\sigma(A_{k,j}) \leq A_0 2^{2n(-k+3)}$, we have

$$\begin{aligned} 2^{2nk} \int |g_k - g_{k+1}|^p d\sigma &\leq C \sum_{j=1}^{\nu(k)} \frac{1}{\sigma(A_{k,j})} \int_{A_{k,j}} |g_k - g_{k+1}|^p d\sigma \\ &\leq C_1 \sum_{j=1}^{\nu(k)} \frac{1}{\sigma(A_{k,j})} \int_{A_{k,j}} (|g_k - g_{C_{k,j}}|^p + |g_{C_{k,j}} - g_{k+1}|^p) d\sigma. \end{aligned} \tag{4.6}$$

But for any $\zeta \in A_{k,j}$ we have $B(\zeta, 2^{-k}) \subset C_{k,j}$ and

$$|g_k(\zeta) - g_{C_{k,j}}| \leq \frac{1}{\sigma(B(\zeta, 2^{-k}))} \int_{B(\zeta, 2^{-k})} |g - g_{C_{k,j}}| d\sigma \leq \frac{C_2}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma. \tag{4.7}$$

A similar inequality holds for $|g_{C_{k,j}} - g_{k+1}(\zeta)|$, $\zeta \in A_{k,j}$. Therefore from (4.6) we obtain

$$2^{2nk} \int |g_k - g_{k+1}|^p d\sigma \le C_3 \sum_{j=1}^{\nu(k)} \left( \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma \right)^p, \tag{4.8}$$

$k \ge 0$. Now for any $L \in \mathbf{N}$, it follows from Hölder's inequality that

$$|g_k - g_{k+L}|^p = \left| \sum_{i=0}^{L-1} \frac{2^{i/p}}{2^{i/p}} (g_{k+i} - g_{k+i+1}) \right|^p \le \left( \sum_{i=0}^{\infty} \frac{1}{2^{i/(p-1)}} \right)^{p-1} \sum_{i=0}^{L-1} 2^i |g_{k+i} - g_{k+i+1}|^p.$$

Combining this with (4.8), we see that

$$2^{2nk} \int |g_k - g_{k+L}|^p d\sigma \le C_4 \sum_{\ell=k}^{\infty} \sum_{j=1}^{\nu(\ell)} \left( \frac{1}{\sigma(C_{\ell,j})} \int_{C_{\ell,j}} |g - g_{C_{\ell,j}}| d\sigma \right)^p \tag{4.9}$$

for all $k \ge 0$ and $L \ge 1$. But for each $k$, we have $g_{k+L}(\xi) \to g(\xi)$ as $L \to \infty$ if $\xi$ is a Lebesgue point for $g$. Applying this fact and Fatou's lemma to (4.9), we find that

$$2^{2nk} \int |g_k - g|^p d\sigma \le C_4 \mathcal{J}_p(g) \tag{4.10}$$

for every $k \ge 0$.

For each $m \in \mathbf{N}$, write

$$I_{1,m} = \sum_{k=0}^{m} 2^{2nk} \int |g - g_k|^p d\sigma.$$

Let $N$ be the smallest natural number such that $2^{p-1} 2^{-2nN} \le 1/2$. If $m > N$, then

$$\begin{aligned} I_{1,m} &\le 2^{p-1} \sum_{k=0}^{m} 2^{2nk} \int |g - g_{k+N}|^p d\sigma + 2^{p-1} \sum_{k=0}^{m} 2^{2nk} \int |g_{k+N} - g_k|^p d\sigma \\ &\le 2^{p-1} 2^{-2nN} \sum_{k=0}^{m-N} 2^{2n(k+N)} \int |g - g_{k+N}|^p d\sigma + 2^{p-1} \sum_{k=m-N+1}^{m} 2^{2nk} \int |g - g_{k+N}|^p d\sigma \\ &+ (2N)^{p-1} \sum_{i=0}^{N-1} \sum_{k=0}^{m} 2^{2nk} \int |g_{k+i+1} - g_{k+i}|^p d\sigma. \end{aligned}$$

Taking (4.10) and (4.8) into account, we see that

$$I_{1,m} \le 2^{p-1} 2^{-2nN} I_{1,m} + 2^{p-1} N C_4 \mathcal{J}_p(g) + (2N)^p C_3 \mathcal{J}_p(g).$$

By the assumption $\mathcal{J}_p(g) < \infty$ and (4.10), we have $I_{1,m} < \infty$ for every $m \in \mathbf{N}$. Since $2^{p-1}2^{-2nN} \leq 1/2$, we can cancel out $2^{p-1}2^{-2nN}I_{1,m}$ from both sides to obtain

$$(1/2)I_{1,m} \leq \{2^{p-1}NC_4 + (2N)^p C_3\}\mathcal{J}_p(g).$$

Letting $m \to \infty$, we have

$$I_1 \leq 2\{2^{p-1}NC_4 + (2N)^p C_3\}\mathcal{J}_p(g), \tag{4.11}$$

where $N$ is the smallest natural number such that $2^{p-1}2^{-2nN} \leq 1/2$.

To estimate $I_2$, note that (4.3) implies $E_k \subset \cup_{j=1}^{\nu(k)}(B_{k,j} \times B_{k,j})$. Therefore

$$\begin{aligned} 2^{4nk} \iint_{E_k} |g_k(\zeta) - g_k(\xi)|^p d\sigma(\zeta)d\sigma(\xi) &\leq \sum_{j=1}^{\nu(k)} 2^{4nk} \iint_{B_{k,j}\times B_{k,j}} |g_k(\zeta) - g_k(\xi)|^p d\sigma(\zeta)d\sigma(\xi) \\ &\leq 2^{p-1}\sum_{j=1}^{\nu(k)} 2^{4nk}\sigma(B_{k,j}) \int_{B_{k,j}} |g_k - g_{C_{k,j}}|^p d\sigma \\ &\leq C_5 \sum_{j=1}^{\nu(k)} \frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |g_k - g_{C_{k,j}}|^p d\sigma. \end{aligned}$$

If $\zeta \in B_{k,j}$, then $B(\zeta, 2^{-k}) \subset C_{k,j}$. Thus (4.7) still holds if $\zeta \in B_{k,j}$. Substituting (4.7) in the above inequality, we have

$$2^{4nk} \iint_{E_k} |g_k(\zeta) - g_k(\xi)|^p d\sigma(\zeta)d\sigma(\xi) \leq C_5 C_2^p \sum_{j=1}^{\nu(k)} \left(\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma\right)^p.$$

Summing over all $k \geq 0$, we obtain $I_2 \leq C_5 C_2^p \mathcal{J}_p(g)$. Combining this with (4.5) and (4.11), the proposition is proved. □

## 5. Cancellation

In this section, we will show that there is a $C$ such that inequality (1.2) holds for every $f \in L^2(S, d\sigma)$ satisfying the condition $\mathcal{I}_p(f - Pf) < \infty$.

**Lemma 5.1.** *For each $k \geq 0$, there is a $C_{5.1}(k)$ which depends only on $k$ and $n$ such that*

$$\mathrm{SD}(v_z; z/|z|, 2^k(1 - |z|^2)^{1/2}) \leq C_{5.1}(k)\|M_{m_z} H_f k_z\|$$

*for all $f \in L^2(S, d\sigma)$ and $z \in \mathbf{B}\backslash\{0\}$, where the relation between $f$ and $v_z$ is the same as in Lemma* 3.4.

*Proof.* Let $z \in \mathbf{B}\backslash\{0\}$. If $\xi \in B_k(z)$ (see (2.19)), then

$$|1 - \langle\xi, z\rangle| \leq 1 - |z| + |1 - \langle\xi, z/|z|\rangle| \leq 1 - |z|^2 + 2^{2k}(1 - |z|^2) \leq 2^{2k+1}(1 - |z|^2).$$

Therefore for each $\xi \in B_k(z)$ we have

$$|m_z(\xi)k_z(\xi)|^2 \geq \frac{(1-|z|^2)^{n+2}}{4|1-\langle \xi, z\rangle|^{2n+2}} \geq \frac{2^{-(2n+2)(2k+1)-2}}{(1-|z|^2)^n} \geq \frac{c(n;k)}{\sigma(B_0(z))} \geq \frac{c(n;k)}{\sigma(B_k(z))}.$$

Recall from Lemma 3.4 that $H_f k_z = v_z k_z$. Therefore

$$\begin{aligned}\|M_{m_z} H_f k_z\|^2 = \|m_z v_z k_z\|^2 &\geq \frac{c(n;k)}{\sigma(B_k(z))}\int_{B_k(z)} |v_z|^2 d\sigma \\ &\geq c(n;k)\{\mathrm{SD}(v_z; z/|z|, 2^k(1-|z|^2)^{1/2})\}^2.\end{aligned}$$

This completes the proof. □

**Lemma 5.2.** *Suppose $p > 2n$. Let $\gamma > 0$ be given. Then there is a constant $C_{5.2}(\gamma)$ which depends only on $n, p$ and $\gamma$ such that for any $f \in L^2(S, d\sigma)$,*

$$\mathcal{J}_p(f - Pf) \leq C_{5.2}(\gamma)\int \|M_{m_z} H_f k_z\|^p d\lambda(z) + \gamma \mathcal{I}_p(f - Pf).$$

*Proof.* Let $f \in L^2(S, d\sigma)$ be given and write

$$g = f - Pf.$$

For each pair of $k \geq 7$ and $1 \leq j \leq \nu(k)$, define

$$F_{k,j} = \{z \in \mathbf{B} : 2^{-k+5} \leq (1-|z|^2)^{1/2} < 2^{-k+6}, z/|z| \in B(\xi_{k,j}, 2^{-k+1})\},$$

where $\{\xi_{k,1}, \cdots \xi_{k,\nu(k)}\}$ were given at the beginning of Section 4. It is easy to see that

$$B_1(z) \supset C_{k,j} \quad \text{if} \quad z \in F_{k,j}. \tag{5.1}$$

And, it is easy to verify that there is a $c > 0$ such that

$$\lambda(F_{k,j}) \geq c \tag{5.2}$$

for all $k \geq 7$ and $1 \leq j \leq \nu(k)$. The condition $(1-|z|^2)^{1/2} \leq 2^{-k+6}$ for $z \in F_{k,j}$ guarantees that there is a $0 < C_1 < \infty$ such that $\sigma(B_1(z)) \leq C_1 \sigma(C_{k,j})$ if $z \in F_{k,j}$. Therefore for $z \in F_{k,j}$ we have

$$\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma \leq \frac{2C_1}{\sigma(B_1(z))}\int_{B_1(z)} |g - g_{B_1(z)}| d\sigma. \tag{5.3}$$

Recall from Lemma 3.4 that we have the decomposition $g = h_z + v_z$ where $v_z = g - (P(g \circ \varphi_z)) \circ \varphi_z$ and $h_z = (P(g \circ \varphi_z)) \circ \varphi_z$. Since $h_z = -Pv_z$, we have

$$\mathrm{SD}(g; \xi, r) \leq \mathrm{SD}(v_z; \xi, r) + \mathrm{SD}(Pv_z; \xi, r) \leq (1 + C_{2.1})\sum_{k=0}^{\infty} \frac{1}{2^k}\mathrm{SD}(v_z; \xi, 2^k r),$$

where the second $\leq$ follows from Proposition 2.1. Combing this with (5.3), we see that

$$\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma \leq C_2\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{SD}(v_z;z/|z|,2^k(1-|z|^2)^{1/2}) \tag{5.4}$$

if $z \in F_{k,j}, k \geq 7$, where $C_2 = 4C_1(1+C_{2.1})$.

Now let $N \geq 8$ be given. We define

$$T_N^{(1)}(z) = \sum_{k=1}^{N-1}\frac{1}{2^k}\mathrm{SD}(v_z;z/|z|,2^k(1-|z|^2)^{1/2}),$$

$$T_N^{(2)}(z) = \sum_{k=N}^{\infty}\frac{1}{2^k}\mathrm{SD}(v_z;z/|z|,2^k(1-|z|^2)^{1/2}), \tag{5.5}$$

$z \in \mathbf{B}\backslash\{0\}$. Then (5.4) yields

$$\left(\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma\right)^p \leq 2^{p-1}C_2^p((T_N^{(1)}(z))^p+(T_N^{(2)}(z))^p)$$

if $z \in F_{k,j}$. Combining this with (5.2), we find that

$$\left(\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma\right)^p \leq \frac{2^{p-1}C_2^p}{c}\int_{F_{k,j}}((T_N^{(1)}(z))^p+(T_N^{(2)}(z))^p)d\lambda(z), \tag{5.6}$$

$k \geq 7, 1 \leq j \leq \nu(k)$. Next we estimate $T_N^{(1)}(z)$ and $T_N^{(2)}(z)$.

By Lemma 5.1, there is a constant $C_3(N)$ such that

$$T_N^{(1)}(z) \leq C_3(N)\|M_{m_z}H_fk_z\| \quad \text{for all} \quad z \in \mathbf{B}\backslash\{0\}. \tag{5.7}$$

Let us consider $T_N^{(2)}(z)$. Since $v_z = g-(P(g\circ\varphi_z))\circ\varphi_z$, we have $T_N^{(2)}(z) \leq T_N^{(3)}(z)+T_N^{(4)}(z)$, where

$$T_N^{(3)}(z) = \sum_{k=N}^{\infty}\frac{1}{2^k}\mathrm{SD}(g;z/|z|,2^k(1-|z|^2)^{1/2}),$$

$$T_N^{(4)}(z) = \sum_{k=N}^{\infty}\frac{1}{2^k}\mathrm{SD}((P(g\circ\varphi_z))\circ\varphi_z;z/|z|,2^k(1-|z|^2)^{1/2}).$$

Since $p > 2n$, there exist $\delta > 0$ and $0 < \epsilon < 1/2$ such that $p(1-\epsilon) = 2n+2\delta$. Define

$$F_N = \bigcup_{k=N+6}^{\infty}\bigcup_{j=1}^{\nu(k)}F_{k,j}.$$

If $z \in F_N$, then $2^N(1-|z|^2)^{1/2} \le 4$. By Lemma 2.4,

$$T_N^{(4)}(z) \le C_{2.4}\frac{1}{2^{\epsilon N}}\sum_{k=1}^{\infty}\frac{k}{2^{(1-\epsilon)k}}\mathrm{SD}(g; z/|z|, 2^k(1-|z|^2)^{1/2})$$

for $z \in F_N$. Obviously,

$$T_N^{(3)}(z) \le \frac{1}{2^{\epsilon N}}\sum_{k=N}^{\infty}\frac{k}{2^{(1-\epsilon)k}}\mathrm{SD}(g; z/|z|, 2^k(1-|z|^2)^{1/2}).$$

Therefore, if we set $C_5 = 1 + C_{2.4}$, then

$$T_N^{(2)}(z) \le C_5\frac{1}{2^{\epsilon N}}\sum_{k=1}^{\infty}\frac{k}{2^{(1-\epsilon)k}}\mathrm{SD}(g; z/|z|, 2^k(1-|z|^2)^{1/2}) \quad \text{for } z \in F_N.$$

Since $1-\epsilon = (2n+2\delta)/p$, we can split $2^{-(1-\epsilon)k}$ as $2^{-\delta k/p} \cdot 2^{-(2n+\delta)k/p}$ and apply Hölder's inequality to the above. The result of this is

$$(T_N^{(2)}(z))^p \le \frac{C_6}{2^{\epsilon Np}}\sum_{k=1}^{\infty}\frac{k^p}{2^{(2n+\delta)k}}\{\mathrm{SD}(g; z/|z|, 2^k(1-|z|^2)^{1/2})\}^p \quad \text{for } z \in F_N.$$

From (2.23) and (2.24) we see that

$$\{\mathrm{SD}(g; z/|z|, 2^k(1-|z|^2)^{1/2})\}^p \le \frac{1}{\sigma^2(B_k(z))}\iint_{B_k(z)\times B_k(z)}|g(\zeta)-g(\xi)|^p d\sigma(\zeta)d\sigma(\xi).$$

Therefore for each $z \in F_N$, we have

$$(T_N^{(2)}(z))^p \le \frac{C_6}{2^{\epsilon Np}}\sum_{k=1}^{\infty}\frac{k^p}{2^{(2n+\delta)k}}\cdot\frac{1}{\sigma^2(B_k(z))}\iint_{B_k(z)\times B_k(z)}|g(\zeta)-g(\xi)|^p d\sigma(\zeta)d\sigma(\xi).$$

Integrating the above against $d\lambda$ over $F_N$, we find that

$$\begin{aligned}
&\int_{F_N}(T_N^{(2)}(z))^p d\lambda(z) \\
&\le \frac{C_6}{2^{\epsilon Np}}\int\sum_{k=1}^{\infty}\frac{k^p}{2^{(2n+\delta)k}}\cdot\frac{1}{\sigma^2(B_k(z))}\iint_{B_k(z)\times B_k(z)}|g(\zeta)-g(\xi)|^p d\sigma(\zeta)d\sigma(\xi)d\lambda(z) \\
&= \frac{C_6}{2^{\epsilon Np}}\iint\mathcal{F}(\zeta,\xi)|g(\zeta)-g(\xi)|^p d\sigma(\zeta)d\sigma(\xi),
\end{aligned}$$

where

$$\mathcal{F}(\zeta,\xi) = \sum_{k=1}^{\infty}\frac{k^p}{2^{(2n+\delta)k}}\int\frac{\chi_{B_k(z)}(\zeta)\chi_{B_k(z)}(\xi)}{\sigma^2(B_k(z))}d\lambda(z).$$

By Lemma 2.5,

$$\mathcal{F}(\zeta,\xi) \le C_{2.5}\left(\sum_{k=1}^{\infty}\frac{k^{p+1}}{2^{\delta k}}\right)\frac{1}{|1-\langle\zeta,\xi\rangle|^{2n}} = \frac{C_7}{|1-\langle\zeta,\xi\rangle|^{2n}}.$$

Consequently

$$\int_{F_N}(T_N^{(2)}(z))^p d\lambda(z) \le \frac{C_8}{2^{\epsilon Np}}\mathcal{I}_p(g). \tag{5.8}$$

From the definition of $F_{k,j}$ and (4.1) we see that $F_{k,j}\cap F_{k',j'} = \emptyset$ if either $k \neq k'$ or $j \neq j'$. Therefore it follows from (5.6),(5.7) and (5.8) that

$$\sum_{k=N+6}^{\infty}\sum_{j=1}^{\nu(k)}\left(\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma\right)^p \le C_9(N)\int\|M_{m_z}H_f k_z\|^p d\lambda(z) + \frac{C_{10}}{2^{\epsilon Np}}\mathcal{I}_p(g).$$

Suppose now $\gamma > 0$ is given. We pick an $N = N(\gamma) \ge 8$ such that $C_{10}/2^{\epsilon Np} \le \gamma$. This determines the value of $N$ in terms of $\gamma$ and converts $C_9(N)$ to $C_{11}(\gamma)$. We can write the above inequality as

$$\sum_{k=N+6}^{\infty}\sum_{j=1}^{\nu(k)}\left(\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma\right)^p \le C_{11}(\gamma)\int\|M_{m_z}H_f k_z\|^p d\lambda(z)+\gamma\mathcal{I}_p(g). \tag{5.9}$$

Next we consider the terms in $\mathcal{J}_p(g)$ corresponding to $0 \le k \le N+5$.

First all, there is a $c_{12}(\gamma)$ such that $\sigma(C_{k,j}) \ge c_{12}(\gamma)$ when $k \le N+5$. Therefore

$$\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|g-g_{C_{k,j}}|d\sigma \le C_{13}(\gamma)\|g\|$$

if $0 \le k \le N+5$ and $1 \le j \le \nu(k)$. Combining this inequality with (5.9), we obtain

$$\mathcal{J}_p(g) \le C_{11}(\gamma)\int\|M_{m_z}H_f k_z\|^p d\lambda(z) + C_{14}(\gamma)\|g\|^p + \gamma\mathcal{I}_p(g).$$

Thus the proof of the lemma will be complete if we can find a constant $C_{15}$ such that

$$\|g\|^p \le C_{15}\int\|M_{m_z}H_f k_z\|^p d\lambda(z). \tag{5.10}$$

Note that $M_{m_z}H_f k_z = M_{m_z}H_g k_z = M_{m_z}(1-P)M_{k_z}g$. Also note that

$$M_{m_z}(1-P)M_{k_z}g - g = \{M_{m_z}(1-P)M_{k_z} - (1-P)\}g.$$

It is obvious that there is an $a \in (0,1)$ such that $\|M_{m_z}(1-P)M_{k_z} - (1-P)\| \le 1/2$ if $|z| \le a$. This means that $\|M_{m_z}H_f k_z\| = \|M_{m_z}(1-P)M_{k_z}g\| \ge (1/2)\|g\|$ when $|z| \le a$. Thus if we let $\Omega = \{z : |z| \le a\}$, then

$$\|g\|^p \le \frac{2^p}{\lambda(\Omega)} \int \|M_{m_z}H_f k_z\|^p d\lambda(z).$$

This establishes (5.10) and completes the proof of the lemma. □

**Proposition 5.3.** *Let $2n < p < \infty$. Then there exists a constant $0 < C_{5.3}(p) < \infty$ which depends only on $n$ and $p$ such that the inequality*

$$\mathcal{I}_p(f - Pf) \le C_{5.3}(p)\|H_f\|_p^p$$

*holds for every $f \in L^2(S, d\sigma)$ satisfying the condition $\mathcal{I}_p(f - Pf) < \infty$.*

*Proof.* Let $f \in L^2(S, d\sigma)$ and suppose $\mathcal{I}_p(f - Pf) < \infty$. Denote

$$g = f - Pf$$

as before. Let $\gamma > 0$. It follows from Proposition 4.2 and Lemma 5.2 that

$$\mathcal{I}_p(g) \le C_{4.2}(p)C_{5.2}(\gamma) \int \|M_{m_z}H_f k_z\|^p d\lambda(z) + C_{4.2}(p)\gamma\mathcal{I}_p(g).$$

Pick a $\gamma$ such that $C_{4.2}(p)\gamma \le 1/2$. Then since $\mathcal{I}_p(g) < \infty$, we can cancel out $(1/2)\mathcal{I}_p(g)$ from both sides to obtain

$$(1/2)\mathcal{I}_p(g) \le C_{4.2}(p)C_{5.2}(\gamma) \int \|M_{m_z}H_f k_z\|^p d\lambda(z).$$

Now apply Lemma 3.5 with $0 < t \le 1$ to the right hand side of the above. This gives us

$$(5.11) \quad (1/2)\mathcal{I}_p(g) \le C_{4.2}(p)C_{5.2}(\gamma)\left(2^{2p}\beta(t)\|H_f\|_p^p + 2^{p-1}(C_{3.3}t)^p \int \|H_f k_z\|^p d\lambda(z)\right).$$

Since $\|H_f k_z\| = \|H_{g-c}k_z\| \le \|(g-c)k_z\|$, $c \in \mathbf{C}$, it follows from Proposition 2.6 that

$$\int \|H_f k_z\|^p d\lambda(z) \le C_{2.6}(p)\mathcal{I}_p(g).$$

Substituting this into (5.11), we see that

$$(1/2)\mathcal{I}_p(g) \le C_{4.2}(p)C_{5.2}(\gamma)\{2^{2p}\beta(t)\|H_f\|_p^p + 2^{p-1}(C_{3.3}t)^p C_{2.6}(p)\mathcal{I}_p(g)\}.$$

Now set $t$ to be such that $C_{4.2}(p)C_{5.2}(\gamma) \cdot 2^{p-1}(C_{3.3}t)^p C_{2.6}(p) \le 1/4$. Then, since $\mathcal{I}_p(g) < \infty$, we can cancel out $(1/4)\mathcal{I}_p(g)$ from both sides to obtain

$$(1/4)\mathcal{I}_p(g) \le C_{4.2}(p)C_{5.2}(\gamma)2^{2p}\beta(t)\|H_f\|_p^p.$$

This completes the proof. □

## 6. Smoothing

Obviously, our goal here is to remove the *a priori* condition $\mathcal{I}_p(f - Pf) < \infty$ in Proposition 5.3. This is the soft part of the proof of Theorem 1.4, but it is a part of the proof nonetheless. To carry out this part of the proof, we need to have available a sufficiently large class of functions for which the desired inequality holds.

Many of the facts established in this section will also be needed in Section 8. For the rest of the paper, let $\mathrm{Lip}(S)$ denote the collection of functions which are Lipschitz with respect to the *Euclidian* metric on $S$. For any $\zeta, \xi \in S$, we have $|\zeta - \xi|^2 = 2 - 2\mathrm{Re}\langle \zeta, \xi\rangle$, which implies $|\zeta - \xi| \le \sqrt{2}|1 - \langle \zeta, \xi\rangle|^{1/2}$. Thus each $g \in \mathrm{Lip}(S)$ is also Lipschitz with respect to the metric $d$ defined by (2.1).

**Proposition 6.1.** *If $g \in \mathrm{Lip}(S)$, then $\mathcal{I}_p(g) < \infty$ for every $p > 2n$.*

*Proof.* Let $g \in \mathrm{Lip}(S)$. Then there is an $L$ such that $|g(\zeta) - g(\xi)| \le L|1 - \langle \zeta, \xi\rangle|^{1/2}$ for all $\zeta, \xi \in S$. If $p > 2n$, then $p/2 = n + s$ for some $s > 0$. Therefore

$$\begin{aligned}\iint \frac{|g(\zeta) - g(\xi)|^p}{|1 - \langle \zeta, \xi\rangle|^{2n}} d\sigma(\zeta)d\sigma(\xi) &\le \iint \frac{L^p|1 - \langle \zeta, \xi\rangle|^{p/2}}{|1 - \langle \zeta, \xi\rangle|^{2n}} d\sigma(\zeta)d\sigma(\xi) \\ &= \iint \frac{L^p}{|1 - \langle \zeta, \xi\rangle|^{n-s}} d\sigma(\zeta)d\sigma(\xi).\end{aligned}$$

By [9,Proposition 1.4.10], this quantity is finite. □

Let $\mathcal{U} = \mathcal{U}(n)$ denote the collection of unitary transformations on $\mathbf{C}^n$. For each $U \in \mathcal{U}$, define the operator $W_U : L^2(S, d\sigma) \to L^2(S, d\sigma)$ by the formula

$$(W_U g)(\zeta) = g(U\zeta),$$

$g \in L^2(S, d\sigma)$. By the invariance of $\sigma$, $W_U$ is a unitary operator on $L^2(S, d\sigma)$.

**Lemma 6.2.** *Let $\varphi \in C(S)$. If there exists a positive number $L$ such that*

$$\|\varphi - W_U\varphi\|_\infty \le L\|1 - U\|$$

*for every $U \in \mathcal{U}$, then $\varphi \in \mathrm{Lip}(S)$.*

*Proof.* Clearly, the conclusion of the lemma follows from the following basic fact: Given a pair of $\zeta, \xi \in S$, there is a $U = U_{\zeta,\xi} \in \mathcal{U}$ which has the properties that $U\zeta = \xi$ and that $\|1-U\| \le \sqrt{2}|\zeta-\xi|$. This can be easily proved by considering the orthogonal decomposition $\mathbf{C}^n = \mathcal{E} \oplus (\mathbf{C}^n \ominus \mathcal{E})$, where $\mathcal{E} = \mathrm{span}\{\zeta, \xi\}$. We omit the details. □

Next we recall the smoothing technique in [12]. With the usual multiplication and the operator-norm topology, $\mathcal{U}$ is a compact group. We write $dU$ for the Haar measure on $\mathcal{U}$

as in [9,12]. For each $g \in L^2(S, d\sigma)$, the map $U \mapsto W_U g$ is continuous with respect to the norm topology of $L^2(S, d\sigma)$. Let $\Phi \in C(\mathcal{U})$. For each $g \in L^2(S, d\sigma)$ we define

$$Y_\Phi g = \int \Phi(U) W_U g dU$$

in the sense that

$$\langle Y_\Phi g, f\rangle = \int \Phi(U)\langle W_U g, f\rangle dU$$

for every $f \in L^2(S, d\sigma)$.

**Lemma 6.3.** *If $\Psi$ is Lipschitz with respect to the operator norm on $\mathcal{U}$, then $Y_\Psi g \in \mathrm{Lip}(S)$ for every $g \in L^2(S, d\sigma)$.*

*Proof.* First recall that the inequality

$$|\langle Y_\Phi g, f\rangle| \le \|\Phi\|_\infty \int |g| d\sigma \int |f| d\sigma$$

holds for all $g, f \in L^2(S, d\sigma)$ and $\Phi \in C(\mathcal{U})$ [12,page 43]. This obviously means that

$$\|Y_\Phi g\|_\infty \le \|\Phi\|_\infty \|g\| \quad \text{for all} \;\; g \in L^2(S, d\sigma) \;\; \text{and} \;\; \Phi \in C(\mathcal{U}). \tag{6.1}$$

Using Fubini's theorem it is easy to see that if $\varphi \in C(S)$, then

$$(Y_\Phi \varphi)(\zeta) = \int \Phi(U)\varphi(U\zeta) dU, \qquad \zeta \in S.$$

From this we draw the conclusion that if $\varphi \in C(S)$, then $Y_\Phi \varphi \in C(S)$. But for any $f \in L^2(S, d\sigma)$, there is a sequence $\{f_k\} \subset C(S)$ such that $\|f - f_k\| \to 0$ as $k \to \infty$. Since $Y_\Phi f_k \in C(S)$, by (6.1) we also have $Y_\Phi f \in C(S)$.

Now let $\Psi$ be given as in the statement of the lemma, and let $g \in L^2(S, d\sigma)$ also be given. By the preceding paragraph and Lemma 6.2, to prove that $Y_\Psi g \in \mathrm{Lip}(S)$, it suffices to find a $C$ such that

$$\|Y_\Psi g - W_U Y_\Psi g\|_\infty \le C\|1 - U\| \tag{6.2}$$

for every $U \in \mathcal{U}$. To prove this, note that for any $U \in \mathcal{U}$ and $f \in L^2(S, d\sigma)$ we have

$$\begin{aligned} \langle W_U Y_\Psi g, f\rangle = \langle Y_\Psi g, W_U^* f\rangle &= \int \Psi(V)\langle W_V g, W_U^* f\rangle dV = \int \Psi(V)\langle W_U W_V g, f\rangle dV \\ &= \int \Psi(V)\langle W_{VU} g, f\rangle dV = \int \Psi(VU^*)\langle W_V g, f\rangle dV, \end{aligned} \tag{6.3}$$

where the last step uses the invariance of the Haar measure. Define the function

$$D_U(V) = \Psi(V) - \Psi(VU^*), \qquad V \in \mathcal{U},$$

for each $U \in \mathcal{U}$. Then, by (6.3), $Y_\Psi g - W_U Y_\Psi g = Y_{D_U} g$. Applying (6.1), we have

$$\|Y_\Psi g - W_U Y_\Psi g\|_\infty \le \|D_U\|_\infty \|g\|. \tag{6.4}$$

Since $\Psi$ is Lipschitz with respect to $\|.\|$, there is an $L$ such that

$$\|D_U\|_\infty = \sup_{V \in \mathcal{U}} |\Psi(V) - \Psi(VU^*)| \le L \sup_{V \in \mathcal{U}} \|V - VU^*\| = L\|1 - U\|$$

for every $U \in \mathcal{U}$. Obviously, (6.2) follows from (6.4) and this inequality. □

**Lemma 6.4.** *Let $f \in L^2(S, d\sigma)$, $\Phi \in C(\mathcal{U})$, $h \in H^\infty(S)$ and $\psi \in L^2(S, d\sigma)$. Then*

$$\langle H_{Y_\Phi f} h, \psi\rangle = \int \Phi(U)\langle W_U H_f W_U^* h, \psi\rangle dU$$

*Proof.* Let $f$, $\Phi$, $h$ and $\psi$ be given as above, and let $g = (1 - P)\psi$. Then

$$\begin{aligned}\langle H_{Y_\Phi f} h, \psi\rangle &= \langle (Y_\Phi f) \cdot h, g\rangle = \langle Y_\Phi f, \bar{h} g\rangle = \int \Phi(U)\langle W_U f, \bar{h} g\rangle dU \\ &= \int \Phi(U)\langle h \cdot W_U f, g\rangle dU = \int \Phi(U)\langle W_U M_f W_U^* h, g\rangle dU \\ &= \int \Phi(U)\langle W_U H_f W_U^* h, \psi\rangle dU,\end{aligned}$$

where the last step uses the fact that $[W_U, 1 - P] = 0$. □

**Lemma 6.5.** *Let $f \in L^2(S, d\sigma)$ and $\Phi \in C(\mathcal{U})$. If $H_f$ is bounded, then*

$$s_1(H_{Y_\Phi f}) + ... + s_k(H_{Y_\Phi f}) \le \|\Phi\|_1\{s_1(H_f) + ... + s_k(H_f)\}$$

*for every $k \in \mathbf{N}$, where $\|\Phi\|_1$ is the $L^1$-norm of $\Phi$ with respect to the Haar measure $dU$.*

*Proof.* Let $k \in \mathbf{N}$ be given. Consider any operator $E$ such that $\|E\| = 1$ and $\text{rank}(E) = k$. Recall that $s_j(ABC) \le \|A\| s_j(B)\|C\|$ [5,page 61]. Thus for each $U \in \mathcal{U}$, we have

$$|\text{tr}(W_U H_f W_U^* E)| \le \sum_{j=1}^k s_j(W_U H_f W_U^* E) \le \sum_{j=1}^k \|W_U\| s_j(H_f)\|W_U^* E\| = \sum_{j=1}^k s_j(H_f).$$

Combining this with Lemma 6.4, we find that

$$\begin{aligned}|\text{tr}(H_{Y_\Phi f} E)| &= \left|\int \Phi(U)\text{tr}(W_U H_f W_U^* E) dU\right| \le \int |\Phi(U)||\text{tr}(W_U H_f W_U^* E)| dU \\ &\le \|\Phi\|_1\{s_1(H_f) + ... + s_k(H_f)\}.\end{aligned}$$

Since $s_1(H_{Y_\Phi f}) + ... + s_k(H_{Y_\Phi f})$ is the supremum of $|\text{tr}(H_{Y_\Phi f} E)|$ over all possible $E$'s with $\|E\| = 1$ and $\text{rank}(E) = k$, the lemma follows. □

**Corollary 6.6.** *Let $\Phi \in C(\mathcal{U})$ be such that $\|\Phi\|_1 \neq 0$. Then the inequality $\|H_{Y_\Phi f}\|_p \leq \|\Phi\|_1 \|H_f\|_p$ holds for all $f \in L^2(S, d\sigma)$ and $1 \leq p < \infty$.*

*Proof.* This follows from Lemma 6.5 and the following easy exercise: If $a_1 \geq ... \geq a_k \geq ...$ and $b_1 \geq ... \geq b_k \geq ...$ are non-increasing sequences of non-negative numbers such that $a_1 + ... + a_k \leq b_1 + ... + b_k$ for every $k \in \mathbf{N}$, then $\sum_{j=1}^\infty a_j^p \leq \sum_{j=1}^\infty b_j^p$, $1 \leq p < \infty$. For a more general version of this exercise, see Lemma III.3.1 in [5]. $\square$

Let $\eta : [0, \infty) \to [0, 1]$ be the function such that $\eta = 1$ on $[0, 1]$, $\eta = 0$ on $[2, \infty)$, and $\eta(x) = 2 - x$ on $[1, 2]$. Of course, $\eta$ is Lipschitz on $[0, \infty)$. For each $j \in \mathbf{N}$, define

$$\Phi_j(U) = \frac{\eta(j\|1 - U\|)}{\int \eta(j\|1 - V\|)dV},$$

$U \in \mathcal{U}$. Then we have the following properties:

(1) $\Phi_j \geq 0$ on $\mathcal{U}$.
(2) $\int \Phi_j(U)dU = 1$.
(3) $\Phi_j$ is Lipschitz on $\mathcal{U}$ with respect to the operator norm.
(4) The sequence of operators $\{Y_{\Phi_j}\}$ converges to 1 strongly on $L^2(S, d\sigma)$.

In the above (1) and (2) are obvious, (3) can be easily deduced from the fact that $\eta$ is Lipschitz on $[0, \infty)$, and (4) was established on page 45 of [12].

*Proof of Theorem* 1.4. Let $f \in L^2(S, d\sigma)$ be given and write

$$g = f - Pf.$$

Furthermore, for each $j \geq 1$ let

$$f_j = Y_{\Phi_j} f \quad \text{and} \quad g_j = f_j - Pf_j.$$

Because $[P, W_U] = 0$ for every $U \in \mathcal{U}$, we have $[P, Y_{\Phi_j}] = 0$. Therefore

$$g_j = Y_{\Phi_j} g \tag{6.5}$$

for every $j \geq 1$. Let $2n < p < \infty$ also be given.

By (6.5), (3) and Lemma 6.3, we have $g_j \in \mathrm{Lip}(S)$. By Proposition 6.1, this means $\mathcal{I}_p(g_j) < \infty$. Therefore it follows from Proposition 5.3 that $\mathcal{I}_p(g_j) \leq C_{5.3}(p)\|H_{f_j}\|_p^p$. But by (1), (2) and Corollary 6.6, we have $\|H_{f_j}\|_p^p \leq \|H_f\|_p^p$. Thus we conclude that

$$\mathcal{I}_p(g_j) \leq C_{5.3}(p)\|H_f\|_p^p \quad \text{for every} \quad j \geq 1. \tag{6.6}$$

By (4) and (6.5), there is a subsequence $\{g_{j_\nu}\}$ of $\{g_j\}$ such that

$$\lim_{\nu \to \infty} g_{j_\nu}(\zeta) = g(\zeta) \quad \text{for} \quad \sigma\text{-a.e.} \quad \zeta \in S. \tag{6.7}$$

Applying Fatou's lemma, from (6.7) and (6.6) we obtain

$$\mathcal{I}_p(g) \leq \liminf_{\nu \to \infty} \mathcal{I}_p(g_{j_\nu}) \leq C_{5.3}(p)\|H_f\|_p^p.$$

This completes the proof of Theorem 1.4. □

## 7. Estimates for Commutators

Recall that the $s$-numbers of a bounded operator $A$ are denoted by $s_1(A)$, $s_2(A), \cdots$, $s_j(A), \cdots$ [5,Section II.7]. For each $t > 0$, define

$$N_A(t) = \text{card}\{j \in \mathbf{N} : s_j(A) > t\}.$$

It follows from [5,Theorem II.7.1] that $s_{j+k+1}(A+B) \le s_{j+1}(A)+s_{k+1}(B)$ for any bounded operators $A$, $B$ and any $j \ge 0$, $k \ge 0$. A consequence of this is that

$$N_{A+B}(t) \le N_A(t/2) + N_B(t/2). \tag{7.1}$$

To see this, suppose that $N_A(t/2) = j(t)$ and $N_B(t/2) = k(t)$. Then by the definition of $N$ we have $s_{j(t)+1}(A) \le t/2$ and $s_{k(t)+1}(B) \le t/2$. Therefore

$$s_{j(t)+k(t)+1}(A+B) \le s_{j(t)+1}(A) + s_{k(t)+1}(B) \le t,$$

which implies $N_{A+B}(t) \le j(t) + k(t)$. It is well known [4,Lemma I.4.1] that

$$\sum_{j=1}^{\infty}(s_j(A))^p = p\int_0^{\infty} t^{p-1}N_A(t)dt, \qquad 1 \le p < \infty. \tag{7.2}$$

**Proposition 7.1.** *Let $2 < p < \infty$ and $f \in L^2(S, d\sigma)$. Then*

$$\|[M_f, P]\|_p^p \le \frac{(36)^p p}{p-2}\iint \frac{|f(x) - f(y)|^p}{|1 - \langle x, y\rangle|^{2n}} d\sigma(x)d\sigma(y).$$

*Proof.* Consider a *real-valued* $f \in L^2(S, d\sigma)$. For any $t > 0$, define

$$E_{t,k} = \{x \in S : kt \le f(x) < (k+1)t\}, \tag{7.3}$$

$k \in \mathbf{Z}$. For each pair of $k \in \mathbf{Z}$ and $i \in \{-1, 0, 1\}$, define

$$T_{k,i}^{(t)} = M_{\chi_{E_{t,k}}}[M_f, P]M_{\chi_{E_{t,k+i}}}.$$

Taking advantage of the commutator, we can rewrite it as

$$T_{k,i}^{(t)} = M_{\chi_{E_{t,k}}}[M_{f-kt}, P]M_{\chi_{E_{t,k+i}}} = M_{(f-kt)\chi_{E_{t,k}}}PM_{\chi_{E_{t,k+i}}} - M_{\chi_{E_{t,k}}}PM_{(f-kt)\chi_{E_{t,k+i}}}.$$

By (7.3), we have $\|(f-kt)\chi_{E_{t,k}}\|_\infty \le t$ and $\|(f-kt)\chi_{E_{t,k+i}}\|_\infty \le (1+|i|)t$. Therefore for each pair of $k \in \mathbf{Z}$ and $i \in \{-1, 0, 1\}$ we have

$$\|T_{k,i}^{(t)}\| \le 3t. \tag{7.4}$$

Now for each $i \in \{-1, 0, 1\}$ define

$$T_i^{(t)} = \sum_{k \in \mathbf{Z}} T_{k,i}^{(t)}.$$

Since $\chi_{E_{t,k}} L^2(S, d\sigma) \perp \chi_{E_{t,\ell}} L^2(S, d\sigma)$ whenever $k \neq \ell$, (7.4) implies $\|T_i^{(t)}\| \leq 3t$, $i \in \{-1, 0, 1\}$. Write

$$T^{(t)} = T_{-1}^{(t)} + T_0^{(t)} + T_1^{(t)}.$$

Then $\|T^{(t)}\| \leq 9t$, which means

$$N_{T^{(t)}}(9t) = 0. \tag{7.5}$$

For each $i \in \{-1, 0, 1\}$, write $G_i^{(t)} = \cup_{k \in \mathbf{Z}} (E_{t,k} \times E_{t,k+i})$. Note that $G_{-1}^{(t)}$, $G_0^{(t)}$ and $G_1^{(t)}$ are mutually disjoint subsets of $S \times S$. Define

$$B^{(t)} = (S \times S) \backslash (G_{-1}^{(t)} \cup G_0^{(t)} \cup G_1^{(t)}).$$

If $(x, y) \in B^{(t)}$, $x \in E_{t,k}$ and $y \in E_{t,\ell}$, then $|k - \ell| \geq 2$. By (7.3), this means

$$B^{(t)} \subset \{(x, y) \in S \times S : |f(x) - f(y)| > t\}. \tag{7.6}$$

Now define

$$Y^{(t)} = [M_f, P] - T^{(t)}.$$

It is easy to estimate the Hilbert-Schmidt norm of $Y^{(t)}$. Indeed from the previous two paragraphs we see that $Y^{(t)}$ is the operator on $L^2(S, d\sigma)$ which has the function

$$\frac{f(x) - f(y)}{(1 - \langle x, y \rangle)^n} \chi_{B^{(t)}}(x, y)$$

as its integral kernel. This and (7.6) lead to the bound

$$\|Y^{(t)}\|_2^2 = \iint_{B^{(t)}} \frac{|f(x) - f(y)|^2}{|1 - \langle x, y \rangle|^{2n}} d\sigma(x) d\sigma(y) \leq \iint_{|f(x) - f(y)| > t} \frac{|f(x) - f(y)|^2}{|1 - \langle x, y \rangle|^{2n}} d\sigma(x) d\sigma(y). \tag{7.7}$$

Combining the identity $[M_f, P] = Y^{(t)} + T^{(t)}$ with (7.1) and (7.5), we have

$$N_{[M_f,P]}(18t) \leq N_{Y^{(t)}}(9t) + N_{T^{(t)}}(9t) = N_{Y^{(t)}}(9t) \leq N_{Y^{(t)}}(t) \leq \frac{1}{t^2} \|Y^{(t)}\|_2^2. \tag{7.8}$$

Therefore

$$\begin{aligned}
\int_0^\infty t^{p-1} N_{[M_f,P]}(18t) dt &\leq \int_0^\infty \frac{t^{p-1}}{t^2} \iint_{|f(x) - f(y)| > t} \frac{|f(x) - f(y)|^2}{|1 - \langle x, y \rangle|^{2n}} d\sigma(x) d\sigma(y) dt \\
&= \iint \left( \int_0^{|f(x) - f(y)|} t^{p-3} dt \right) \frac{|f(x) - f(y)|^2}{|1 - \langle x, y \rangle|^{2n}} d\sigma(x) d\sigma(y) \\
&= \frac{1}{p-2} \iint |f(x) - f(y)|^{p-2} \frac{|f(x) - f(y)|^2}{|1 - \langle x, y \rangle|^{2n}} d\sigma(x) d\sigma(y).
\end{aligned}$$

Making the substitution $s = 18t$, we have

$$\int_0^\infty s^{p-1} N_{[M_f,P]}(s)ds \le \frac{(18)^p}{p-2} \iint \frac{|f(x)-f(y)|^p}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y).$$

By (7.2), the proposition follows. $\square$

Recall that, for each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k\ge1} \frac{s_1(A)+s_2(A)+\dots+s_k(A)}{1^{-1/p}+2^{-1/p}+\dots+k^{-1/p}} \tag{7.9}$$

defines a symmetric norm for operators [5,Section III.14]. On any Hilbert space $\mathcal{H}$, the set $\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$ is a norm ideal [5,Section III.2] of compact operators. It is well known that $\mathcal{C}_p^+ \supset \mathcal{C}_p$ and that $\mathcal{C}_p^+ \ne \mathcal{C}_p$. An interesting property of $\mathcal{C}_p^+$ is that it is not separable with respect to the norm $\|.\|_p^+$.

**Proposition 7.2.** *There is a $0 < C < \infty$ which depends only on $n$ such that the inequality*

$$\|[M_f,P]\|_{2n}^+ \le CL(f)$$

*holds for every $f \in \mathrm{Lip}(S)$, where $L(f) = \sup_{x\ne y} |f(x)-f(y)|/|x-y|$.*

*Proof.* Recall that $|x-y| \le \sqrt{2}|1-\langle x,y\rangle|^{1/2}$, $x,y \in S$. Thus it suffices to consider a *real-valued* $f \in \mathrm{Lip}(S)$ with the property that $|f(x)-f(y)| \le d(x,y)$, $x,y \in S$. Consider any $t > 0$ and let $[M_f,P] = Y^{(t)} + T^{(t)}$ be the decomposition given in the proof of Proposition 7.1. It follows from (7.8) and (7.7) that

$$\begin{aligned} N_{[M_f,P]}(18t) &\le \frac{1}{t^2} \iint_{d(x,y)\ge t} \frac{1}{|1-\langle x,y\rangle|^{2n-1}} d\sigma(x)d\sigma(y) \\ &= \frac{1}{t^2} \int \sum_{k=0}^\infty \int_{B(x,2^{k+1}t)\setminus B(x,2^kt)} \frac{1}{|1-\langle x,y\rangle|^{2n-1}} d\sigma(y)d\sigma(x) \\ &\le \frac{1}{t^2} \int \sum_{k=0}^\infty \frac{\sigma(B(x,2^{k+1}t))}{(2^kt)^{4n-2}} d\sigma(x). \end{aligned}$$

Since $\sigma(B(x,2^{k+1}t)) \le A_0(2^{k+1}t)^{2n}$, we see that there is a $C_1$ which depends only on $n$ $(\ge 2)$ such that $N_{[M_f,P]}(18t) \le C_1t^{-2n}$. Thus if we set $C_2 = (18)^{2n}C_1$, then $N_{[M_f,P]}(t) \le C_2t^{-2n}$ for every $t > 0$. For each $k \in \mathbf{N}$, let $t_k > 0$ be such that $C_2t_k^{-2n} = k$. Then $N_{[M_f,P]}(t_k) \le C_2t_k^{-2n} = k$, which implies

$$s_{k+1}([M_f,P]) \le t_k = C_2^{1/2n}k^{-1/2n} \le 2C_2^{1/2n}(k+1)^{-1/2n}. \tag{7.10}$$

The condition $|f(x)-f(y)| \le d(x,y)$ implies $\|[M_f,P]\| \le 2\sqrt{2}$, i.e., $s_1([M_f,P]) \le 2\sqrt{2}$. This plus (7.10) gives us $s_k([M_f,P]) \le 2\max\{C_2^{1/2n},\sqrt{2}\}k^{-1/2n}$ for every $k \in \mathbf{N}$. By (7.9), this means $\|[M_f,P]\|_{2n}^+ \le 2\max\{C_2^{1/2n},\sqrt{2}\}$. $\square$

## 8. Lower Bound for $s$-Numbers

The proof of Theorem 1.6 is a long journey. We begin with the action of the $n$-dimensional torus on $S$. Let $\mathbf{T}^n = \{(\tau_1, ..., \tau_n) \in \mathbf{C}^n : |\tau_1| = ... = |\tau_n| = 1\}$. For each $\tau = (\tau_1, ..., \tau_n) \in \mathbf{T}^n$, define the unitary transformation $U_\tau$ on $\mathbf{C}^n$ by the formula

$$U_\tau(z_1, ..., z_n) = (\tau_1 z_1, ..., \tau_n z_n).$$

We will follow the usual multi-index convention given on page 3 in [9].

**Definition 8.1.** A function $f \in L^2(S, d\sigma)$ is said to be $\mathbf{T}^n$-invariant if $f \circ U_\tau = f$ for every $\tau \in \mathbf{T}^n$.

**Lemma 8.2.** *If $f$ is a $\mathbf{T}^n$-invariant function in $L^\infty(S, d\sigma)$, then $\|H_{\bar{f}}h\| = \|H_f h\|$ for every $h \in H^2(S)$.*

*Proof.* Let $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ be the standard orthonomal basis in $H^2(S)$. That is, $e_\alpha(\zeta) = c_\alpha \zeta^\alpha$, where $c_\alpha > 0$ is such that $\|e_\alpha\| = 1$. If $f$ is $\mathbf{T}^n$-invariant, then it is well known (and easy to verify) that the Toeplitz operator $T_f = PM_f|H^2(S)$ is a diagonal operator with respect to the orthonomal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$. Therefore $[T_f^*, T_f] = 0$ and, consequently,

$$PM_{\bar{f}}(1-P)M_f P = T_{|f|^2} - T_f^* T_f = T_{|f|^2} - T_f T_f^* = PM_f(1-P)M_{\bar{f}}P.$$

That is, $H_f^* H_f = H_{\bar{f}}^* H_{\bar{f}}$, which implies $\|H_{\bar{f}}h\| = \|H_f h\|$ for every $h \in H^2(S)$. $\square$

Next we consider functions of a very specific kind. For each $j \in \{1, ..., n\}$, let $e_j \in S$ be the vector whose $j$-th component is 1 and whose other components are 0. For each pair of $i, j \in \{1, ..., n\}$, define the function $p_{i,j}$ on $\mathcal{U}$ by the formula

$$p_{i,j}(U) = \langle Ue_i, e_j \rangle, \quad U \in \mathcal{U}.$$

For the rest of the section, let

$$F(\zeta) = \int m(U)\psi(U\zeta)dU, \quad \zeta \in S, \tag{8.1}$$

where $\psi \in C(S)$ and $m$ is a monomial in $p_{i,j}$ and $\bar{p}_{i',j'}$, $i, j, i', j' \in \{1, ..., n\}$.

**Lemma 8.3.** *For the $F$ given by* (8.1)*, if $H_F \neq 0$, then there is an $\epsilon_1 > 0$ such that $s_k(H_F) \geq \epsilon_1 k^{-1/2n}$ for every $k \in \mathbf{N}$.*

This lemma, whose proof will be given after we state Lemma 8.5, is one of the reduction steps in the proof of Theorem 1.6.

**Lemma 8.4.** *There exists a pair of $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$ in $\mathbf{Z}_+^n$ with the property that $\alpha_j \beta_j = 0$ for every $j \in \{1, ..., n\}$ such that*

$$F \circ U_\tau = \bar{\tau}^\alpha \tau^\beta F$$

*for every $\tau \in \mathbf{T}^n$.*

*Proof.* By the invariance of the Haar measure $dU$, we have

$$F(U_\tau\zeta) = \int m(U)\psi(UU_\tau\zeta)dU = \int m(UU_\tau^*)\psi(U\zeta)dU \tag{8.2}$$

for all $\zeta \in S$ and $\tau \in \mathbf{T}^n$. But for any $i, j \in \{1, ..., n\}$,

$$p_{i,j}(UU_\tau^*) = \langle UU_\tau^* e_i, e_j\rangle = \bar{\tau}_i\langle Ue_i, e_j\rangle = \bar{\tau}_i p_{i,j}(U)$$

if $\tau = (\tau_1, ..., \tau_n)$. Since $m$ is a monomial in $p_{i,j}$ and $\bar{p}_{i',j'}$, it is easy to see that there exists a pair of $\alpha$, $\beta$ as described in the statement of the lemma such that

$$m(UU_\tau^*) = \bar{\tau}^\alpha\tau^\beta m(U)$$

for all $U \in \mathcal{U}$ and $\tau \in \mathbf{T}^n$. Substituting this in (8.2), the lemma follows. $\square$

With the $\alpha$ given by Lemma 8.4, we define the function

$$G(\zeta) = \zeta^\alpha F(\zeta), \quad \zeta \in S. \tag{8.3}$$

**Lemma 8.5.** *For the $G$ given by* (8.3), *if $H_G \neq 0$, then there is an $\epsilon_2 > 0$ such that $s_k(H_G) \geq \epsilon_2 k^{-1/2n}$ for every $k \in \mathbf{N}$.*

Before embarking on the long proof of Lemma 8.5, let us first show that it implies Lemma 8.3.

*Proof of Lemma* 8.3. If $\alpha = (0, ..., 0)$, then $F = G$, and in this case Lemma 8.3 just duplicates Lemma 8.5. Now suppose that there is a $j_0 \in \{1, ..., n\}$ such that $\alpha_{j_0} > 0$.

We first show that the assumption $H_F \neq 0$ implies $H_G \neq 0$. For if it were true that $H_G = 0$, then we would have $G \in H^2(S)$. By (8.3) and Lemma 8.4,

$$G(U_\tau\zeta) = \tau^\alpha\zeta^\alpha\bar{\tau}^\alpha\tau^\beta F(\zeta) = \tau^\beta\zeta^\alpha F(\zeta) = \tau^\beta G(\zeta)$$

for all $\zeta \in S$ and $\tau \in \mathbf{T}^n$. The only functions in $H^2(S)$ which have this property are multiples of the monomial $\zeta^\beta$. That is, there is a $c \in \mathbf{C}$ such that

$$G(\zeta) = c\zeta^\beta, \quad \zeta \in S. \tag{8.4}$$

Since $\alpha_{j_0} > 0$, by Lemma 8.4 we have $\beta_{j_0} = 0$. Now let $\zeta_0$ be the vector whose $j_0$-th component is 0 and whose other components are $(n-1)^{-1/2}$. Then $\zeta_0^\alpha = 0$ and $\zeta_0^\beta \neq 0$. Combining (8.3) and (8.4), we have

$$0 = \zeta_0^\alpha F(\zeta_0) = G(\zeta_0) = c\zeta_0^\beta.$$

Since $\zeta_0^\beta \neq 0$, this means $c = 0$. By (8.4), we then have $G = 0$. Since the zero set of $\zeta^\alpha$ is nowhere dense in $S$, (8.3) and the continuity of $F$ lead to the conclusion $F = 0$ on $S$, which contradicts the assumption $H_F \neq 0$.

Hence if $H_F \neq 0$, then $H_G \neq 0$. By Lemma 8.5, this implies $s_k(H_G) \geq \epsilon_2 k^{-1/2n}$, $k \in \mathbf{N}$. Obviously, $H_G = H_F T_{\zeta^\alpha}$, where $T_{\zeta^\alpha} = PM_{\zeta^\alpha}|H^2(S)$. Since $\|T_{\zeta^\alpha}\| \leq 1$, we have $s_k(H_G) \leq s_k(H_F)$ [5,page 61]. Hence $s_k(H_F) \geq \epsilon_2 k^{-1/2n}$, $k \in \mathbf{N}$. $\square$

We now turn to the proof of Lemma 8.5. With the $\beta$ given in Lemma 8.4, we write

$$b(\zeta) = \zeta^\beta, \quad \zeta \in S. \tag{8.5}$$

Note that the assumption $H_G \neq 0$ in Lemma 8.5 in particular implies

$$G \text{ is not a multiple of } b \text{ on } S. \tag{8.6}$$

The basic idea for the proof of Lemma 8.5 is to show that (8.6) implies the lower bound given in Lemma 8.14 below. This involves many technical steps, and a major hurdle among these is the zero set of $b$. Due to the technicalities, it may be advisable for the reader to first read Lemma 8.14 and beyond, and then come back for the proofs.

Define $Q_j = \{(z_1, ..., z_n) \in \mathbf{C}^n : z_j = 0\}$ for each $j \in \{1, ..., n\}$. Furthermore, define

$$\mathcal{Z} = (S \cap Q_1) \cup ... \cup (S \cap Q_n).$$

An obvious property of $\mathcal{Z}$ is that it is invariant under $\{U_\tau : \tau \in \mathbf{T}^n\}$. The key step on our way to Lemma 8.14 is the following improvement of (8.6):

**Lemma 8.6.** *There exist $x, z \in S$ and $0 \leq r < s \leq \pi/2$ such that the following are true*:
(1) $\langle x, z\rangle = 0$.
(2) $\{\cos tx + \sin tz : t \in [r, s]\} \cap \mathcal{Z} = \emptyset$.
(3) *On the interval $[r, s]$, the function $t \mapsto G(\cos tx + \sin tz)$ is not a multiple of the function* $t \mapsto b(\cos tx + \sin tz)$.

*Proof.* Define the vector $u_0 = (n^{-1/2}, ..., n^{-1/2})$. We then define the linear subspaces $\mathcal{E}_1 = \text{span}\{u_0\}$ and $\mathcal{E}_2 = \mathbf{C}^n \ominus \mathcal{E}_1$ of $\mathbf{C}^n$. Furthermore, let

$$S_i = S \cap \mathcal{E}_i, \quad i = 1, 2.$$

The definition of $u_0$ guarantees that for each $j \in \{1, ..., n\}$, $Q_j$ contains vectors which are not orthogonal to $u_0$. Thus $Q_j \cap \mathcal{E}_2$ is a proper linear subspace of $Q_j$. Since $\dim(Q_j) = n-1$, we have $\dim(Q_j \cap \mathcal{E}_2) < n - 1$. Since $\dim(\mathcal{E}_2) = n - 1$, for each $j \in \{1, ..., n\}$ the set

$$B_j = Q_j \cap S_2$$

is nowhere dense in $S_2$. Consequently, the set $B_1 \cup ... \cup B_n$ is also nowhere dense in $S_2$. Hence the set

$$\Gamma = S_2 \backslash (B_1 \cup ... \cup B_n)$$

is dense in $S_2$. We have, of course, $\Gamma \cap \mathcal{Z} = \emptyset$. We first show that there exist an $x \in S_1$ and a $z \in \Gamma$ such that, on the entire interval $[0, \pi/2]$, the function $t \mapsto G(\cos tx + \sin tz)$ is not a multiple of the function $t \mapsto b(\cos tx + \sin tz)$.

If this assertion were false, then for each pair of $x \in S_1$ and $z \in \Gamma$ there would be a $c_{x,z} \in \mathbf{C}$ such that

$$G(\cos tx + \sin tz) = c_{x,z} b(\cos tx + \sin tz) \quad \text{for every} \ \ t \in [0, \pi/2].$$

But since $b(x) \neq 0$ and $b(z) \neq 0$, setting $t = 0$ and $t = \pi/2$ in the above, we have

$$G(x)/b(x) = c_{x,z} = G(z)/b(z).$$

If $z'$ is any other point in $\Gamma$, then we also have

$$c_{x,z'} = G(x)/b(x) = c_{x,z}.$$

Thus $c_{x,z}$ is independent of $z \in \Gamma$. A similar argument shows that $c_{x,z}$ is also independent of $x \in S_1$. Hence there is a $c \in \mathbf{C}$ such that

$$G(\cos tx + \sin tz) = cb(\cos tx + \sin tz) \quad \text{for all} \ \ x \in S_1, z \in \Gamma \ \ \text{and} \ \ t \in [0, \pi/2].$$

Since $\Gamma$ is dense in $S_2$ and since $G$, $b$ are continuous, the above implies

$$G(\cos tx + \sin tz) = cb(\cos tx + \sin tz) \quad \text{for all} \ \ x \in S_1, z \in S_2 \ \ \text{and} \ \ t \in [0, \pi/2].$$

Since $\{\cos tx + \sin tz : x \in S_1, z \in S_2, t \in [0, \pi/2]\} = S$, this contradicts (8.6).

Thus there exists a pair of $x \in S_1$ and $z \in \Gamma$ such that on the whole interval $[0, \pi/2]$, the function $t \mapsto G(\cos tx + \sin tz)$ is not a multiple of the function $t \mapsto b(\cos tx + \sin tz)$. Next we will show that for such a pair of $x, z$, there exist $0 \leq r < s \leq \pi/2$ such that the interval $[r, s]$ satisfies requirements (2) and (3). To do this, we note that since $x$, $z$ have no zero components and since $\tan t$ is strictly increasing on $[0, \pi/2)$, the set

$$\{t \in [0, \pi/2] : \cos tx + \sin tz \in \mathcal{Z}\}$$

is finite. If $\{t \in [0, \pi/2] : \cos tx + \sin tz \in \mathcal{Z}\} = \emptyset$, then $[r, s] = [0, \pi/2]$ will do. Otherwise, we enumerate the set $\{t \in [0, \pi/2] : \cos tx + \sin tz \in \mathcal{Z}\}$ in the ascending order as

$$t_1 < \dots < t_k,$$

$1 \leq k < \infty$. Since $t_1 > 0$ and $t_k < \pi/2$, we can define $t_0 = 0$ and $t_{k+1} = \pi/2$. If there is an $i \in \{1, \dots, k+1\}$ such that the function $t \mapsto G(\cos tx + \sin tz)$ is not a multiple of the function $t \mapsto b(\cos tx + \sin tz)$ on the interval $(t_{i-1}, t_i)$, then there is a non-trivial subinterval $[r, s]$ in $(t_{i-1}, t_i)$ for which (3) holds true. Such an $[r, s]$ also satisfies (2) because $\{t \in [0, \pi/2] : \cos tx + \sin tz \in \mathcal{Z}\} = \{t_1, \dots, t_k\}$.

Hence what remains for the proof is to rule out the possibility that, for each $1 \leq i \leq k+1$, there is a $c_i \in \mathbf{C}$ such that

$$G(\cos tx + \sin tz) = c_i b(\cos tx + \sin tz) \quad \text{for every} \ \ t \in (t_{i-1}, t_i). \tag{8.7}$$

First of all, the choice of $x, z$ does not allow the possibility $c_1 = c_2 = ... = c_{k+1}$. Thus if (8.7) were true for every $i \in \{1, ..., k+1\}$, then there would be a $\nu \in \{1, ..., k\}$ such that $c_\nu \neq c_{\nu+1}$. We will show that this leads to a contradiction.

By (8.5), the function $t \mapsto b(\cos tx + \sin tz)$ is a polynomial in $\cos t$ and $\sin t$. Since $\cos t$ and $\sin t$ have analytic extensions to $\mathbf{C}$, there is an analytic function $\tilde{b}$ on $\mathbf{C}$ such that

$$\tilde{b}(t) = b(\cos tx + \sin tz) \quad \text{for every} \quad t \in \mathbf{R}. \tag{8.8}$$

We claim that there is an analytic function $\tilde{G}$ on $\mathbf{C}$ such that

$$\tilde{G}(t) = G(\cos tx + \sin tz) \quad \text{for every} \quad t \in \mathbf{R}. \tag{8.9}$$

Postponing the proof of this claim for a moment, we first show that this leads to the contradiction promised in the preceding paragraph. This is because the combination of (8.8), (8.9) and (8.7) gives us

$$\begin{aligned} &\tilde{G}(t) = c_\nu \tilde{b}(t) \quad \text{for} \quad t \in (t_{\nu-1}, t_\nu) \quad \text{and} \\ &\tilde{G}(t) = c_{\nu+1}\tilde{b}(t) \quad \text{for} \quad t \in (t_\nu, t_{\nu+1}). \end{aligned}$$

The analyticity of $\tilde{G}$ and $\tilde{b}$ then leads to $\tilde{G} = c_\nu \tilde{b}$ on $\mathbf{C}$ and $\tilde{G} = c_{\nu+1}\tilde{b}$ on $\mathbf{C}$. This implies that $(c_{\nu+1} - c_\nu)\tilde{b} = \tilde{G} - \tilde{G} = 0$. Since $c_\nu \neq c_{\nu+1}$, this forces $\tilde{b} = 0$ on $\mathbf{C}$. By (8.8), this contradicts the fact that the function $t \mapsto b(\cos tx + \sin tz)$ is not identically zero.

We now turn to the proof that there is an analytic function $\tilde{G}$ on $\mathbf{C}$ such that (8.9) holds. For this we revert back to the function $F$. By (8.3) and the reasoning at the beginning of previous paragraph, it suffices to show that the function

$$t \mapsto F(\cos tx + \sin tz) \tag{8.10}$$

on $\mathbf{R}$ is a polynomial in $\cos t$ and $\sin t$. For this we need to introduce a one-parameter subgroup of $\mathcal{U}$, which will be used beyond this proof. Denote $\mathcal{E} = \text{span}\{x, z\}$. For each $t \in \mathbf{R}$, let $V_t$ be the unitary transformation on $\mathbf{C}^n$ such that

$$\begin{cases} V_t x &= \quad \cos tx + \sin tz \\ V_t z &= \ -\sin tx + \cos tz \\ \ V_t &= \quad 1 \ \text{ on } \ \mathbf{C}^n \ominus \mathcal{E} \end{cases}. \tag{8.11}$$

By (8.1) and the invariance of the Haar measure $dU$, we have

$$F(\cos tx + \sin tz) = F(V_t x) = \int m(U)\psi(UV_t x)dU = \int m(UV_t^*)\psi(Ux)dU. \tag{8.12}$$

Let $Q : \mathbf{C}^n \to \mathbf{C}^n \ominus \mathcal{E}$ be the orthogonal projection. For each pair of $i, j \in \{1, ..., n\}$,

$$\begin{aligned} p_{i,j}(UV_t^*) &= \langle V_t^* e_i, U^* e_j\rangle = \langle V_t^*(\langle e_i, x\rangle x + \langle e_i, z\rangle z + Qe_i), U^* e_j\rangle \\ &= \langle e_i, x\rangle\langle \cos tx - \sin tz, U^* e_j\rangle + \langle e_i, z\rangle\langle \sin tx + \cos tz, U^* e_j\rangle + \langle Qe_i, U^* e_j\rangle \\ &= (\langle e_i, x\rangle\langle Ux, e_j\rangle + \langle e_i, z\rangle\langle Uz, e_j\rangle)\cos t \\ &+ (\langle e_i, z\rangle\langle Ux, e_j\rangle - \langle e_i, x\rangle\langle Uz, e_j\rangle)\sin t + \langle UQe_i, e_j\rangle. \end{aligned}$$

Combining this with (8.12) and with the fact that $m$ is a monomial in $p_{i,j}$ and $\bar{p}_{i',j'}$, $i, j, i', j' \in \{1, ..., n\}$, we see that (8.10) is indeed a polynomial in $\cos t$ and $\sin t$. This completes the proof of the lemma. □

Now consider the consequence of Lemma 8.6. A byproduct of the above proof is that the function $t \mapsto G(V_t x)/b(V_t x)$ is smooth on the interval $(r, s)$. Since Lemma 8.6 tells us that this function is not a constant on $(r, s)$, there is a $\theta \in (r, s)$ such that

$$\frac{d}{dt}\left(\frac{G(V_t x)}{b(V_t x)}\right)\bigg|_{t=\theta} \neq 0.$$

Because $V_{t+\theta} = V_\theta V_t$, we can rewrite the above as

$$\frac{d}{dt}\left(\frac{G(V_\theta V_t x)}{b(V_\theta V_t x)}\right)\bigg|_{t=0} \neq 0.$$

Define

$$y = V_\theta x \quad \text{and} \quad y^\perp = V_\theta z. \tag{8.13}$$

Then, of course, $\langle y, y^\perp\rangle = \langle x, z\rangle = 0$. Since $\theta \in (r, s)$, $y = \cos\theta x + \sin\theta z \notin \mathcal{Z}$. Therefore

$$d(y, \mathcal{Z}) = \inf\{d(y, \xi) : \xi \in \mathcal{Z}\} = \rho > 0. \tag{8.14}$$

Since $V_\theta V_t x = V_\theta(\cos tx + \sin tz) = \cos ty + \sin ty^\perp$, from the above we obtain

**Corollary 8.7.** *For the $y$ and $y^\perp$ defined by* (8.13), *we have*

$$\frac{d}{dt}\left(\frac{G(\cos ty + \sin ty^\perp)}{b(\cos ty + \sin ty^\perp)}\right)\bigg|_{t=0} \neq 0.$$

Let $\eta : \mathbf{R} \to [0, 1]$ be a $C^\infty$-function such that $\eta = 0$ on $(-\infty, 1/2]$ and $\eta = 1$ on $[1, \infty)$. There is a sufficiently large number $R > 1$ such that if we define

$$\mu(w) = \prod_{j=1}^{n} \eta(R|w_j|), \quad \text{where } w = (w_1, ..., w_n), \tag{8.15}$$

then

$$\mu(u) = 1 \quad \text{for every } u \in B(y, \rho/2). \tag{8.16}$$

With this $\mu$ we define the functions $G_1$, $G_2$ on $S$ by the formulas

$$G_1 = \mu G \quad \text{and} \quad G_2 = (1 - \mu)G. \tag{8.17}$$

From the definition of $\mu$, it is clear that the function $\mu(w)/b(w) = \mu(w)/w^\beta$ has a natural smooth extension to $\mathbf{C}^n$. In other words, there is a $C^\infty$-function $g$ on the entire space $\mathbf{C}^n$ such that

$$g(w) = \begin{cases} \mu(w)/b(w) & \text{if} \quad \mu(w) \neq 0 \\ \\ 0 & \text{if} \quad \mu(w) = 0 \end{cases}. \tag{8.18}$$

For the rest of the section, let $\varphi$ denote the function given by the formula

$$\varphi(\zeta) = g(\zeta)G(\zeta), \qquad \zeta \in S. \tag{8.19}$$

By (8.18), the identity $b(\zeta)g(\zeta) = \mu(\zeta)$ holds on $S$. Hence

$$G_1(\zeta) = b(\zeta)\varphi(\zeta), \qquad \zeta \in S. \tag{8.20}$$

**Lemma 8.8.** *The function $\varphi$ is $\mathbf{T}^n$-invariant.*

*Proof.* Obviously, $\mu$ is $\mathbf{T}^n$-invariant. By (8.3), (8.5), (8.18) and Lemma 8.4, if $\zeta \in S$ satisfies the condition $\mu(\zeta) \neq 0$, then

$$\varphi(U_\tau\zeta) = \frac{\mu(U_\tau\zeta)}{(U_\tau\zeta)^\beta}(U_\tau\zeta)^\alpha F(U_\tau\zeta) = \frac{\mu(\zeta)}{\tau^\beta\zeta^\beta}\tau^\alpha\zeta^\alpha\bar{\tau}^\alpha\tau^\beta F(\zeta) = \frac{\mu(\zeta)}{\zeta^\beta}\zeta^\alpha F(\zeta) = \varphi(\zeta)$$

for every $\tau \in \mathbf{T}^n$. If $\zeta \in S$ is such that $\mu(\zeta) = 0$, then clearly $\varphi(U_\tau\zeta) = 0 = \varphi(\zeta)$ for every $\tau \in \mathbf{T}^n$. This completes the proof. $\square$

**Lemma 8.9.** *For each $V \in \mathcal{U}$, the derivative*

$$\frac{d}{dt}\varphi(\cos tVy + \sin tVy^\perp)\Big|_{t=0}$$

*exists. Moreover, as $t \to 0$, the convergence*

$$\frac{\varphi(\cos tVy + \sin tVy^\perp) - \varphi(Vy)}{t} \to \frac{d}{dt}\varphi(\cos tVy + \sin tVy^\perp)\Big|_{t=0}$$

*is uniform with respect to $V \in \mathcal{U}$. Finally, the map*

$$V \mapsto \frac{d}{dt}\varphi(\cos tVy + \sin tVy^\perp)\Big|_{t=0} \tag{8.21}$$

*is continuous with respect to the norm topology on $\mathcal{U}$.*

*Proof.* By (8.3) and (8.19), if we define $f(\zeta) = \zeta^\alpha g(\zeta)$, then $\varphi = fF$. Combining (8.13) and (8.11), we have $\cos tVy + \sin tVy^\perp = V(\cos ty + \sin ty^\perp) = VV_\theta V_t x$ for every $V \in \mathcal{U}$. Thus, recalling (8.1) and using the invariance of $dU$, we have

$$\begin{aligned} F(\cos tVy + \sin tVy^\perp) = F(VV_\theta V_t x) &= \int m(U)\psi(UVV_\theta V_t x)dU \\ &= \int m(UV_t^*V_\theta^*V^*)\psi(Ux)dU. \end{aligned}$$

By the nature of $m$ and the fact that $f$ is the restriction to $S$ of a $C^\infty$-function on $\mathbf{C}^n$, the desired conclusions follow immediately. □

**Lemma 8.10.** *We have*

$$\frac{d}{dt}\varphi(\cos ty + \sin ty^\perp)\Big|_{t=0} \neq 0.$$

*Proof.* By (8.19), (8.18) and (8.16), we have

$$\frac{d}{dt}\varphi(\cos ty + \sin ty^\perp)\Big|_{t=0} = \frac{d}{dt}\left(\frac{G(\cos ty + \sin ty^\perp)}{b(\cos ty + \sin ty^\perp)}\right)\Big|_{t=0}.$$

Corollary 8.7 tells us that this quantity is not 0. □

**Lemma 8.11.** *There exist $c > 0$, $0 < \delta < 1/2$ and $0 < \rho_0 < \rho/3$ such that if $u \in B(y, \rho_0)$ and $0 < t \le \delta$, then*

$$\sup_{v \in B(u,t)} |\varphi(v) - \varphi(u)| \ge ct.$$

*Proof.* By Lemma 8.10 and the continuity of the map (8.21), there exist a $c_0 > 0$ and an open neighborhood $\mathcal{N}$ of 1 in $\mathcal{U}$ such that the inequality

$$\left|\frac{d}{dt}\varphi(\cos tVy + \sin tVy^\perp)\Big|_{t=0}\right| \ge c_0$$

holds for every $V \in \mathcal{N}$. Combining this with the uniform convergence mentioned in Lemma 8.9, there is a $0 < \delta < 1/2$ such that

$$\left|\frac{\varphi(\cos tVy + \sin tVy^\perp) - \varphi(Vy)}{t}\right| \ge c_0/2 \tag{8.22}$$

if $0 < t \le \delta$ and $V \in \mathcal{N}$. Since $\mathcal{N}$ is an open set containing 1, there is a $0 < \rho_0 < \rho/3$ such that $\{Vy : V \in \mathcal{N}\} \supset B(y, \rho_0)$. Thus (8.22) tells us that for each $u \in B(y, \rho_0)$ and each $0 < t \le \delta$, there is a $u^\perp \in S$ with $\langle u, u^\perp\rangle = 0$ such that

$$|\varphi(\cos tu + \sin tu^\perp) - \varphi(u)| \ge c_0 t/2.$$

Since $\langle u, \cos tu + \sin tu^\perp\rangle = \cos t$, we have $d(u, \cos tu + \sin tu^\perp) = \sqrt{1 - \cos t} < \sin t < t$, i.e., $\cos tu + \sin tu^\perp \in B(u, t)$. Thus $c = c_0/2$ will do for the lemma. □

**Lemma 8.12.** *Let $\delta$ and $\rho_0$ be the same as in Lemma 8.11. There exists a $c_1 > 0$ such that if $u \in B(y, \rho_0)$ and $0 < t \le \delta$, and if we set*

$$w = (1 - t^2)^{1/2}u,$$

*then $\|H_\varphi k_w\| \ge c_1 t$.*

*Proof.* By (8.19), (8.3), (8.1) and Lemma 6.3, $\varphi$ is Lipschitz on $S$. Therefore there is an $L > c$ such that

$$|\varphi(\zeta) - \varphi(\xi)| \le Ld(\zeta, \xi) \quad \text{for all} \ \ \zeta, \xi \in S. \tag{8.23}$$

Let $u$, $t$ and $w$ be given as in the statement of the lemma. By Lemma 8.11, there is a $v \in B(u,t)$ such that $|\varphi(v) - \varphi(u)| \ge ct/2$. Combining this with (8.23), we have

$$|\varphi(\zeta) - \varphi(\xi)| \ge ct/6 \quad \text{if} \ \ \zeta \in B(v, ct/6L) \ \text{ and } \ \xi \in B(u, ct/6L).$$

Note that $B(v, ct/6L) \subset B(v,t) \subset B(u, 2t)$. Thus for any $\gamma \in \mathbf{C}$, we have

$$\begin{aligned}\sigma(\{\zeta \in B(u,2t) : |\varphi(\zeta) - \gamma| \ge ct/12\}) &\ge \min\{\sigma(B(u, ct/6L)), \sigma(B(v, ct/6L))\} \\ &= \sigma(B(u, ct/6L)).\end{aligned}$$

Consequently, there is an $a_1 > 0$ which depends only on $c$, $L$ and $n$ such that

$$\frac{1}{\sigma(B(u,2t))}\int_{B(u,2t)} |\varphi - \gamma|^2 d\sigma \ge \frac{\sigma(B(u, ct/6L))}{\sigma(B(u,2t))}(ct/12)^2 \ge a_1 t^2. \tag{8.24}$$

By Lemmas 8.8 and 8.2, $\|H_{\bar\varphi} k_w\|^2 = \|H_\varphi k_w\|^2$. Combining this with [11,(6.4)], we obtain

$$2\|H_\varphi k_w\|^2 = \|H_\varphi k_w\|^2 + \|H_{\bar\varphi} k_w\|^2 \ge \|(\varphi - \langle \varphi k_w, k_w\rangle)k_w\|^2. \tag{8.25}$$

If $\zeta \in B(u, 2t)$, then $|1 - \langle \zeta, w\rangle| \le 1 - |w| + |1 - \langle \zeta, u\rangle| \le t^2 + (2t)^2 = 5t^2$. Thus

$$|k_w(\zeta)|^2 \ge \frac{t^{2n}}{(5t^2)^{2n}} \ge \frac{a_2}{\sigma(B(u,2t))} \qquad \text{for} \ \ \zeta \in B(u, 2t),$$

where $a_2 > 0$ depends only on $n$. Combining this inequality with (8.25) and (8.24), we see that $2\|H_\varphi k_w\|^2 \ge a_2 a_1 t^2$, which proves the lemma. $\square$

**Lemma 8.13.** *There is a constant* $C_{8.13}$ *which depends only on* $n$ *such that the following estimate holds: Let* $u \in S$ *and* $0 < t < 1$, *and set*

$$w = (1 - t^2)^{1/2} u.$$

*Suppose that* $f_1, f_2$ *are functions on* $S$ *satisfying the condition*

$$|f_i(\zeta) - f_i(\xi)| \le L_i d(\zeta, \xi) \quad \text{for all} \ \ \zeta, \xi \in S,$$

$i = 1, 2$. *Then* $\|(f_1 - f_1(u))(f_2 - f_2(u))k_w\| \le C_{8.13} L_1 L_2 t^{3/2}$.

*Proof.* For any $\zeta \in S \backslash B(u, 2^{j-1}t)$, $j \ge 1$, we have $2|1 - \langle \zeta, w\rangle| \ge |1 - \langle \zeta, u\rangle| \ge (2^{j-1}t)^2$. Therefore, if $\zeta \in S\backslash B(u, 2^{j-1}t)$, then

$$|k_w(\zeta)|^2 = \frac{(1 - |w|^2)^n}{|1 - \langle \zeta, w\rangle|^{2n}} \le \frac{8^{2n} t^{2n}}{(2^j t)^{4n}} = \frac{1}{2^{2nj}} \cdot \frac{8^{2n}}{(2^j t)^{2n}} \le \frac{C_1}{2^{2nj}} \cdot \frac{1}{\sigma(B(u, 2^j t))}.$$

Also, for $\zeta \in B(u,t)$ we have

$$|k_w(\zeta)|^2 \le \frac{(1-|w|^2)^n}{(1-|w|)^{2n}} \le \frac{2^{2n}}{(1-|w|^2)^n} = \frac{2^{2n}}{t^{2n}} \le \frac{C_2}{\sigma(B(u,t))}.$$

For $\zeta \in B(u, 2^j t)$, $j \ge 0$, we have

$$|f_1(\zeta)-f_1(u)|^2|f_2(\zeta)-f_2(u)|^2 \le 2L_1|f_1(\zeta)-f_1(u)||f_2(\zeta)-f_2(u)|^2 \le 2L_1 \cdot L_1 L_2^2 (2^j t)^3.$$

Combining the above, we find that

$$\begin{aligned}\|(f_1-f_1(u))(f_2-f_2(u))k_w\|^2 &= \int_{B(u,t)} |f_1-f_1(u)|^2|f_2-f_2(u)|^2|k_w|^2 d\sigma \\ &+ \sum_{j=1}^{\infty}\int_{B(u,2^j t)\backslash B(u,2^{j-1}t)} |f_1-f_2(u)|^2|f_2-f_2(u)|^2|k_w|^2 d\sigma \\ &\le 2C_2L_1^2L_2^2t^3 + 2C_1L_1^2L_2^2\sum_{j=1}^{\infty}\frac{(2^j t)^3}{2^{2nj}}.\end{aligned}$$

By our standing assumption $n \ge 2$, we have $2n-3>0$. Thus the above inequality implies the desired estimate. □

**Lemma 8.14.** *Let $\delta$ and $\rho_0$ be the same as in Lemma 8.11. There exist a $0 < c_2 < c_1$ and a $0 < \delta_0 < \delta$ such that if $u \in B(y,\rho_0)$ and $0 < t \le \delta_0$, and if we set*

$$w = (1-t^2)^{1/2}u,$$

*then $\|H_G k_w\| \ge c_2 t$.*

*Proof.* Consider any $0 < t < \delta$ and $u \in B(y,\rho_0)$. For such a pair of $t$, $u$, define $w$ as above. Recall from (8.17) that $G = G_1 + G_2$. We first derive a lower bound for $\|H_{G_1}k_w\|$. By (8.14) and the fact that $\rho_0 < \rho/3$, we have

$$\inf_{v\in B(y,\rho_0)} |b(v)| = c_3 > 0. \tag{8.26}$$

Recall from (8.20) that $G_1 = b\varphi$. Therefore

$$H_{G_1}k_w = b(u)H_\varphi k_w + H_{(b-b(u))\varphi}k_w = b(u)H_\varphi k_w + H_{(b-b(u))(\varphi-\varphi(u))}k_w,$$

where the second $=$ is a crucial use of the fact that $H_{b-b(u)}k_w = 0$. There is an $M > 0$ such that $|b(\zeta)-b(\xi)| \le Md(\zeta,\xi)$ for all $\zeta, \xi \in S$. Applying Lemma 8.12, Lemma 8.13 and (8.26), we find that

$$\|H_{G_1}k_w\| \ge |b(u)|\|H_\varphi k_w\| - \|H_{(b-b(u))(\varphi-\varphi(u))}k_w\| \ge c_3c_1t - C_{8.13}LMt^{3/2},$$

where $L$ is the same as in (8.23). Now let $0 < \delta_1 < \delta$ be such that $C_{8.13}LM\delta_1^{1/2} \le c_3c_1/2$. The above yields

$$\|H_{G_1}k_w\| \ge c_3c_1t/2 \quad \text{if } 0 < t < \delta_1. \tag{8.27}$$

Next we give an upper bound for $\|H_{G_2}k_w\|$. By (8.16) and (8.17), $G_2 = 0$ on the set $B(y, \rho/2)$. Since $\rho_0 < \rho/3$, we see that there is a $0 < C < \infty$ which is independent of $u \in B(y, \rho_0)$ and certainly independent of $t$ such that

$$|G_2(\zeta)k_w(\zeta)| \le C(1 - |w|^2)^{n/2} = C(t^2)^{n/2} = Ct^n \quad \text{if } \zeta \in S\backslash B(y, \rho/2).$$

Therefore $\|H_{G_2}k_w\| \le Ct^n$. Since $n \ge 2$, there is a $0 < \delta_0 < \delta_1$ such that if $0 < t \le \delta_0$, then $\|H_{G_2}k_w\| \le c_3c_1t/4$. Combining this with (8.27), we see that

$$\|H_Gk_w\| \ge \|H_{G_1}k_w\| - \|H_{G_2}k_w\| \ge (c_3c_1t/2) - (c_3c_1t/4) = c_3c_1t/4$$

for such $t$ and $u$. Thus $c_2 = c_3c_1/4$ will do for the lemma. $\square$

**Lemma 8.15.** *There is a constant $C_{8.15}$ which depends only on $n$ such that the following estimate holds: Suppose that $0 < t < 1/2$ and that $\{u_j : j \in J\}$ is a subset of $S$ satisfying the condition*

$$B(u_i, t) \cap B(u_j, t) = \emptyset \quad \textit{for all } i \ne j. \tag{8.28}$$

*Define $z_j = (1 - t^2)^{1/2}u_j$, $j \in J$. Then the norm of the operator*

$$E = \sum_{j \in J} k_{z_j} \otimes k_{z_j}$$

*satisfies the inequality $\|E\| \le C_{8.15}$.*

*Proof.* Define $\mathcal{G} = \{w \in \mathbf{C}^n : |w| < 1/2\}$. We first show that

$$\varphi_{z_j}(\mathcal{G}) \subset \{ru : u \in B(u_j, 3t), (1 - (2t)^2)^{1/2} \le r \le (1 - (t/3)^2)^{1/2}\}, \tag{8.29}$$

$j \in J$. Indeed for any given $j \in J$ and $w \in \mathcal{G}$, write $\varphi_{z_j}(w) = ru$, where $u \in S$ and $0 \le r < 1$. By [9,page 26], we have $1 - \langle\varphi_{z_j}(w), z_j\rangle = (1 - |z_j|^2)/(1 - \langle w, z_j\rangle)$. Since $|w| < 1/2$, this gives us $|1 - \langle u, u_j\rangle| \le 2|1 - \langle\varphi_{z_j}(w), z_j\rangle| \le 2(t^2/2^{-1}) = 4t^2$. Thus $d(u, u_j) \le 2t < 3t$. To estimate $r$, note that

$$1 - |\varphi_{z_j}(w)|^2 = \frac{(1 - |z_j|^2)(1 - |w|^2)}{|1 - \langle w, z_j\rangle|^2} = \frac{1 - |w|^2}{|1 - \langle w, z_j\rangle|^2}t^2$$

(see [9,page 26]). Therefore

$$(t/3)^2 \le \frac{1 - (1/2)^2}{2^2}t^2 \le 1 - r^2 \le \frac{1}{(1/2)^2}t^2 = (2t)^2.$$

This completes the proof of (8.29). Set

$$W^{(t)} = \{ru : u \in S, (1-(2t)^2)^{1/2} \le r \le (1-(t/3)^2)^{1/2}\}.$$

By (8.28), there is a $C_1$ which depends only on $n$ such that $\mathrm{card}\{j \in J : u \in B(u_j, 3t)\} \le C_1$ for every $u \in S$. Combining this with (8.29), we see that

$$\sum_{j \in J} \chi_{\varphi_{z_j}(\mathcal{G})} \le C_1 \chi_{W^{(t)}} \quad \text{on} \quad \mathbf{B}. \tag{8.30}$$

Let $f$ be any function in $L^2(S, d\sigma)$ and denote $h = Pf$. Then $h \in H^2(S)$ and

$$\langle Ef, f\rangle = \sum_{j\in J} |\langle h, k_{z_j}\rangle|^2 = \sum_{j \in J}(1-|z_j|^2)^n |h(z_j)|^2 = t^{2n} \sum_{j \in J} |h^2(z_j)|. \tag{8.31}$$

By the Möbius invariance $d\lambda \circ \varphi_{z_j} = d\lambda$ [9,Theorem 2.2.6] and the fact $\varphi_{z_j}(0) = z_j$,

$$h^2(z_j) = h^2(\varphi_{z_j}(0)) = \frac{1}{\lambda(\mathcal{G})} \int_{\mathcal{G}} h^2 \circ \varphi_{z_j} d\lambda = \frac{1}{\lambda(\mathcal{G})} \int_{\varphi_{z_j}(\mathcal{G})} h^2 d\lambda \tag{8.32}$$

for each $j \in J$. Combining (8.31), (8.32) and (8.30), we have

$$\begin{aligned}
\langle Ef, f\rangle &\le t^{2n} \sum_{j \in J} \frac{1}{\lambda(\mathcal{G})} \int_{\varphi_{z_j}(\mathcal{G})} |h^2| d\lambda \le \frac{C_1}{\lambda(\mathcal{G})} t^{2n} \int_{W^{(t)}} |h|^2 d\lambda \\
&= \frac{C_1}{\lambda(\mathcal{G})} t^{2n} \int_{(1-(2t)^2)^{1/2}}^{(1-(t/3)^2)^{1/2}} \frac{2nr^{2n-1}}{(1-r^2)^{n+1}} \left( \int |h(ru)|^2 d\sigma(u) \right) dr \\
&\le \frac{C_1}{\lambda(\mathcal{G})} \|h\|^2 t^{2n} \int_{(1-(2t)^2)^{1/2}}^{(1-(t/3)^2)^{1/2}} \frac{2nr^{2n-1}}{(1-r^2)^{n+1}} dr.
\end{aligned}$$

But it is easy to see that there is a $C_2$ which depends only on $n$ such that

$$t^{2n} \int_{(1-(2t)^2)^{1/2}}^{(1-(t/3)^2)^{1/2}} \frac{2nr^{2n-1}}{(1-r^2)^{n+1}} dr \le C_2$$

for all $0 < t < 1/2$. This completes the proof. □

*Proof of Lemma* 8.5. Let $t \in (0, \delta_0)$ be given, where $\delta_0$ is the same as in Lemma 8.14. Then there is a subset $\{u_j : j \in J\}$ of $B(y, \rho_0)$ which is maximal with respect to the property

$$B(u_i, t) \cap B(u_j, t) = \emptyset \quad \text{if} \quad i \ne j.$$

The maximality of $\{u_j : j \in J\}$ implies $\cup_{j \in J} B(u_j, 2t) \supset B(y, \rho_0)$. Thus there are constants $0 < C_1 < C_2 < \infty$ which depend only on $\rho_0$ and $n$ such that

$$C_1 t^{-2n} \le \mathrm{card}(J) \le C_2 t^{-2n}. \tag{8.36}$$

For each $j \in J$, define $w_j = (1-t^2)^{1/2}u_j$. Then define the operator

$$E_t = \sum_{j\in J} k_{w_j} \otimes k_{w_j}.$$

Let $A = H_G^* H_G$. By Lemma 8.14, we have $\|H_G k_{w_j}\| \ge c_2 t$ for each $j \in J$. Combining this with the lower bound in (8.36), we obtain

$$\text{tr}(AE_t) = \sum_{j\in J} \|H_G k_{w_j}\|^2 \ge (c_2 t)^2 \cdot C_1 t^{-2n} = \epsilon t^{-2n+2}, \tag{8.37}$$

where $\epsilon = c_2^2 C_1$. We have $\|E_t\| \le C_{8.15}$ by Lemma 8.15 and $\text{rank}(E_t) \le C_2 t^{-2n}$ by the upper bound in (8.36). Also, $s_j(AE_t) \le s_j(A)\|E_t\|$ [5,page 61]. Hence

$$\text{tr}(AE_t) \le \|AE_t\|_1 = \sum_{j=1}^{\text{rank}(E_t)} s_j(AE_t) \le C_{8.15} \sum_{1\le j\le C_2 t^{-2n}} s_j(A). \tag{8.38}$$

Now suppose that an integer $k \ge C_2(\delta_0/2)^{-2n}$ is given. Let $t_k \in (0, \delta_0)$ be such that $C_2 t_k^{-2n} = k$. Then from (8.38) and (8.37) we obtain

$$C_{8.15}\{s_1(A) + ... + s_k(A)\} \ge \epsilon t_k^{-2n+2} = ak^{(n-1)/n},$$

where $a = \epsilon C_2^{(-n+1)/n}$. Since the above inequality holds for every $k \ge C_2(\delta_0/2)^{-2n}$, it is easy to see that there is an $a_1 > 0$ such that

$$s_1(A) + ... + s_k(A) \ge a_1 k^{(n-1)/n} \tag{8.39}$$

for every $k \in \mathbf{N}$.

On the other hand, Proposition 7.2 tells us that $\|H_G\|_{2n}^+ < \infty$. Observe that

$$ks_k(H_G) \le s_1(H_G) + ... + s_k(H_G) \le \|H_G\|_{2n}^+(1^{-1/2n} + ... + k^{-1/2n}) \le C_3 k^{1-(1/2n)}$$

for every $k \in \mathbf{N}$, where $C_3 = 3\|H_G\|_{2n}^+$. Hence $s_k(H_G) \le C_3 k^{-1/2n}$. Since $A = H_G^* H_G$, we have $s_k(A) = \{s_k(H_G)\}^2 \le (C_3)^2 k^{-1/n}$, $k \in \mathbf{N}$. Therefore

$$s_1(A) + ... + s_k(A) \le (C_3)^2(1^{-1/n} + ... + k^{-1/n}) \le 3(C_3)^2 k^{(n-1)/n} \tag{8.40}$$

for every $k \in \mathbf{N}$. Let $N \in \mathbf{N}$ be such that $a_1 N^{(n-1)/n} \ge 3(C_3)^2 + 1$. By (8.39) and (8.40),

$$Nks_k(A) \ge s_k(A) + ... + s_{Nk}(A) \ge a_1(Nk)^{(n-1)/n} - 3(C_3)^2 k^{(n-1)/n} \ge k^{(n-1)/n}$$

for each $k \in \mathbf{N}$. Thus if we set $a_2 = N^{-1}$, then $s_k(A) \ge a_2 k^{-1/n}$ for each $k \in \mathbf{N}$. Hence $s_k(H_G) = \{s_k(A)\}^{1/2} \ge \sqrt{a_2} k^{-1/2n}$. This completes the proof of Lemma 8.5. □

*Proof of Theorem* 1.6. Let $f \in L^2(S, d\sigma)$ and suppose that $H_f$ is bounded. If $H_f \neq 0$, then by using the sequence of approximate identity $\{\Phi_j\}$ in Section 6, we find that there is a $\Psi \in C(\mathcal{U})$ such that the function

$$\psi = Y_\Psi f = \int \Psi(U) W_U f dU$$

also has the property $H_\psi \neq 0$. Obviously, the functions $\{p_{i,j} : 1 \leq i, j \leq n\}$ separate points on $\mathcal{U}$. Thus, by the Stone-Weierstrass approximation theorem, the linear span of monomials in $p_{i,j}$ and $\bar{p}_{i',j'}$ is dense in $C(\mathcal{U})$ with respect to the norm $\|.\|_\infty$. Combining this fact with the sequence $\{\Phi_j\}$ in Section 6, we see that there is a monomial $m$ in $p_{i,j}$ and/or $\bar{p}_{i',j'}$, $i, j, i', j' \in \{1, ..., n\}$, such that the function

$$F = Y_m \psi = \int m(U) W_U \psi dU \tag{8.41}$$

also has the property $H_F \neq 0$. In the proof of Lemma 6.3 we showed that $\psi \in C(S)$. Hence from (8.41) we obtain the "pointwise" expression (8.1) for this $F$. Thus Lemma 8.3 is applicable. Since $H_F \neq 0$, Lemma 8.3 tells us that $s_k(H_F) \geq \epsilon_1 k^{-1/2n}$ for each $k \in \mathbf{N}$. Applying Lemma 6.5, we have

$$\begin{aligned}\epsilon_1 k^{(2n-1)/2n} &= k\epsilon_1 k^{-1/2n} \leq k s_k(H_F) \leq s_1(H_F) + ... + s_k(H_F) \\ &\leq \|m\|_1 \{s_1(H_\psi) + ... + s_k(H_\psi)\} \leq \|m\|_1 \|\Psi\|_1 \{s_1(H_f) + ... + s_k(H_f)\}\end{aligned}$$

for every $k \in \mathbf{N}$. Thus $\epsilon = \epsilon_1 \|m\|_1^{-1} \|\Psi\|_1^{-1}$ will do. $\square$

## 9. Further Results

In this section we first derive two more conditions (Corollary 9.3) which are equivalent to the membership $H_f \in \mathcal{C}_p$, $p > 2n$. Then we use Theorem 1.6 and Proposition 7.2 to describe the distribution of the $s$-numbers of $H_f$ in the case $f \in \mathrm{Lip}(S)$. The final result of the section is a re-interpretation of Theorem 1.6 in the language of norm ideals [5].

To obtain additional conditions equivalent to $H_f \in \mathcal{C}_p$, we begin with

**Lemma 9.1.** *Let $\Phi \in C(\mathcal{U})$ and suppose that $\Phi \geq 0$ on $\mathcal{U}$ and that $\int \Phi(U) dU = 1$. Then for all $f \in L^2(S, d\sigma)$ and $p \geq 2$ we have*

$$\int \|H_{Y_\Phi f} k_z\|^p d\lambda(z) \leq \int \|H_f k_z\|^p d\lambda(z).$$

*Proof.* Applying Lemma 6.4 twice, we obtain

$$\begin{aligned}\|H_{Y_\Phi f} k_z\|^2 &= \iint \Phi(U)\Phi(V) \langle W_U H_f W_U^* k_z, W_V H_f W_V^* k_z \rangle dU dV \\ &= \iint \Phi(U)\Phi(V) \langle W_U H_f k_{Uz}, W_V H_f k_{Vz} \rangle dU dV,\end{aligned}$$

$z \in \mathbf{B}$. Since $p/2 \geq 1$, Hölder's inequality yields

$$\begin{aligned}\|H_{Y_\Phi f}k_z\|^p &\leq \iint \Phi(U)\Phi(V)|\langle W_U H_f k_{Uz}, W_V H_f k_{Vz}\rangle|^{p/2} dUdV \\ &\leq \iint \Phi(U)\Phi(V)\|H_f k_{Uz}\|^{p/2}\|H_f k_{Vz}\|^{p/2} dUdV.\end{aligned}$$

Therefore

$$\begin{aligned}\int \|H_{Y_\Phi f}k_z\|^p d\lambda(z) &\leq \iint \Phi(U)\Phi(V) \int \|H_f k_{Uz}\|^{p/2}\|H_f k_{Vz}\|^{p/2} d\lambda(z) dUdV \\ &\leq \iint \Phi(U)\Phi(V) \left(\int \|H_f k_{Uz}\|^p d\lambda(z)\right)^{1/2} \left(\int \|H_f k_{Vz}\|^p d\lambda(z)\right)^{1/2} dUdV \\ &= \int \|H_f k_z\|^p d\lambda(z),\end{aligned}$$

where the $=$ follows from the $\mathcal{U}$-invariance of $d\lambda$ and the assumptions on $\Phi$. $\square$

**Proposition 9.2.** *Suppose that $p > 2n$. Then there exists a constant $0 < C_{9.2}(p) < \infty$ which depends only on $n$ and $p$ such that the inequality*

$$\mathcal{I}_p(f - Pf) \leq C_{9.2}(p) \int \|H_f k_z\|^p d\lambda(z) \tag{9.1}$$

*holds for every $f \in L^2(S, d\sigma)$.*

*Proof.* Let $f \in L^2(S, d\sigma)$ be given and write $g = f - Pf$ as before. Recall that $|m_z| \leq 1$ on $S$. Let $\gamma > 0$. Applying Propositions 4.2 and Lemma 5.2, we have

$$\mathcal{I}_p(g) \leq C_{4.2}(p)C_{5.2}(\gamma) \int \|H_f k_z\|^p d\lambda(z) + C_{4.2}(p)\gamma\mathcal{I}_p(g).$$

Again, we first prove (9.1) under the additional assumption $\mathcal{I}_p(g) < \infty$. Set $\gamma$ to be such that $\gamma C_{4.2}(p) \leq 1/2$. Subtracting $(1/2)\mathcal{I}_p(g)$ from both sides, we find that

$$(1/2)\mathcal{I}_p(g) \leq C_{4.2}(p)C_{5.2}(\gamma) \int \|H_f k_z\|^p d\lambda(z) \quad \text{if } \ \mathcal{I}_p(g) < \infty. \tag{9.2}$$

Next we drop the a priori assumption $\mathcal{I}_p(g) < \infty$. Let the sequence $\{\Phi_j\}$ be the same as in Section 6. For each $j \geq 1$, we set $f_j = Y_{\Phi_j} f$ and $g_j = f_j - Pf_j$ as in the proof of Theorem 1.4. Then (6.5) holds. Again, by Lemma 6.3 and Proposition 6.1, we have $\mathcal{I}_p(g_j) < \infty$ for each $j$. Applying (9.2) and Lemma 9.1, for each $j \geq 1$ we have

$$\mathcal{I}_p(g_j) \leq 2C_{4.2}(p)C_{5.2}(\gamma) \int \|H_{f_j} k_z\|^p d\lambda(z) \leq 2C_{4.2}(p)C_{5.2}(\gamma) \int \|H_f k_z\|^p d\lambda(z).$$

Then, as in the proof of Theorem 1.4, there is a subsequence $\{g_{j_\nu}\}$ such that

$$\mathcal{I}_p(g) \le \liminf_{\nu\to\infty} \mathcal{I}_p(g_{j_\nu}) \le 2C_{4.2}(p)C_{5.2}(\gamma)\int \|H_f k_z\|^p d\lambda(z).$$

Thus the constant $C_{9.2}(p) = 2C_{4.2}(p)C_{5.2}(\gamma)$ will do for the lemma. $\square$

For any $f \in L^2(S, d\sigma)$ and $c \in \mathbf{C}$, we have $H_f = H_{f-Pf-c}$. Thus, combining Propositions 9.2 and 2.6, we have

**Corollary 9.3.** *Let $p > 2n$. Then for every $f \in L^2(S, d\sigma)$ we have*

$$\begin{aligned}\mathcal{I}_p(f - Pf) &\le C_{9.2}(p)\int \|H_f k_z\|^p d\lambda(z)\\ &\le C_{9.2}(p)\int \|\{(f - Pf) - \langle (f - Pf)k_z, k_z\rangle\}k_z\|^p d\lambda(z)\\ &\le C_{9.2}(p)C_{2.6}(p)\mathcal{I}_p(f - Pf).\end{aligned}$$

For the distribution of $s$-numbers, we have

**Proposition 9.4.** *Let $f \in \mathrm{Lip}(S)$. If $H_f \ne 0$, then there exist $0 < a \le b < \infty$ such that*

$$ak^{-1/2n} \le s_k(H_f) \le bk^{-1/2n}$$

*for every $k \in \mathbf{N}$.*

*Proof.* Let $f \in \mathrm{Lip}(S)$. Then Proposition 7.2 tells us that $\|H_f\|_{2n}^+ < \infty$. For each $k \in \mathbf{N}$,

$$ks_k(H_f) \le s_1(H_f) + ... + s_k(H_f) \le \|H_f\|_{2n}^+(1^{-1/2n} + ... + k^{-1/2n}) \le 3\|H_f\|_{2n}^+ k^{(2n-1)/2n}.$$

Dividing both sides by $k$, we see that the desired upper bound holds with $b = 3\|H_f\|_{2n}^+$. Since $H_f \ne 0$, Theorem 1.6 provides an $\epsilon = \epsilon(f) > 0$ such that

$$s_1(H_f) + ... + s_k(H_f) \ge \epsilon k^{(2n-1)/2n}$$

for every $k \in \mathbf{N}$. Now we repeat the argument at the end of the proof of Lemma 8.5. Let $N \in \mathbf{N}$ be such that $\epsilon N^{(2n-1)/2n} \ge 3\|H_f\|_{2n}^+ + 1$. Then

$$\begin{aligned}Nks_k(H_f) &\ge s_k(H_f) + ... + s_{Nk}(H_f)\\ &\ge \epsilon(Nk)^{(2n-1)/2n} - 3\|H_f\|_{2n}^+ k^{(2n-1)/2n} \ge k^{(2n-1)/2n}\end{aligned}$$

for each $k \in \mathbf{N}$. Dividing both sides by $k$, we see that the desired lower bound holds with $a = N^{-1}$. $\square$

Finally, let us consider norm ideals. From now on the symbol $\Phi$ will be used to denote a *symmetric gauge function* (also called *symmetric norming function*) [5,page 71], whose definition we will now recall.

Let $\hat{c}$ be the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for each sequence $a_j \neq 0$ only for a finite number of $j$'s. A symmetric gauge function is a map $\Phi : \hat{c} \to [0, \infty)$ which has the following properties:

(a) $\Phi$ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, ..., 0, ...\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

Each symmetric gauge function $\Phi$ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{k\geq 1} \Phi(\{s_1(A), ..., s_k(A), 0, ..., 0, ...\})$$

for operators. On any Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\}$$

is a *norm ideal* [5,page 68]. This term refers to the following properties of $\mathcal{C}_\Phi$:

- For any $B, C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \leq \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \leq \|A\|_\Phi$, and the equality holds when $\text{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|.\|_\Phi$.

As an example, let us mention that for any $1 \leq p < \infty$, $\mathcal{C}_p^+$ is none other than the norm ideal $\mathcal{C}_{\Phi_p^+}$, where the symmetric gauge function $\Phi_p^+ : \hat{c} \to [0, \infty)$ is defined as follows. For any $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$,

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{k\geq 1} \frac{|a_{\pi(1)}| + ... + |a_{\pi(k)}|}{1^{-1/p} + ... + k^{-1/p}}, \tag{9.3}$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \geq |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$, which exists because $a_j = 0$ for all but a finite number of $j$'s.

Our final result asserts that $\mathcal{C}_{2n}^+$ is the smallest norm ideal of the form $\mathcal{C}_\Phi$ to which a nonzero Hankel operator $H_f$ can belong.

**Proposition 9.5.** *Let $\Phi$ be a symmetric gauge function. If there is an $f \in L^2(S, d\sigma)$ such that $0 < \|H_f\|_\Phi < \infty$, then $\mathcal{C}_\Phi \supset \mathcal{C}_{2n}^+$.*

*Proof.* Suppose that $0 < \|H_f\|_\Phi < \infty$. By Theorem 1.6, there is an $\epsilon = \epsilon(f) > 0$ such that

$$s_1(H_f) + ... + s_k(H_f) \geq \epsilon k^{(2n-1)/2n}$$

for every $k \in \mathbf{N}$. By (7.9) (or (9.3)), for each $A \in \mathcal{C}_{2n}^+$ we have

$$s_1(A) + ... + s_k(A) \leq \|A\|_{2n}^+(1^{-1/2n} + ... + k^{-1/2n}) \leq 3\|A\|_{2n}^+ k^{(2n-1)/2n},$$

$k \in \mathbf{N}$. The combination of the above two inequalities gives us

$$s_1(A) + ... + s_k(A) \leq (3\|A\|_{2n}^+/\epsilon)\{s_1(H_f) + ... + s_k(H_f)\},$$

$k \in \mathbf{N}$. By [5,Lemma III.3.1], this implies

$$\|A\|_\Phi \leq (3\|A\|_{2n}^+/\epsilon)\|H_f\|_\Phi.$$

Since $\|H_f\|_\Phi < \infty$, this means $\|A\|_\Phi < \infty$ for each $A \in \mathcal{C}_{2n}^+$. That is, $\mathcal{C}_\Phi \supset \mathcal{C}_{2n}^+$. $\square$

## References


1. J. Bergh and J. Löström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.
2. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611-635.
3. M. Feldman and R. Rochberg, *Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group*, Analysis and partial differential equations, 121-159, Lecture Notes in Pure and Appl. Math., **122**, Dekker, New York, 1990.
4. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
5. I. Gohberg and M. Krein, *Introduction to the theorey of linear nonselfadjoint operators*, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
6. S. Janson and T. Wolff, *Schatten classes and commutators of singular integral operators*, Ark. Mat. **20** (1982), 301-310.
7. V. Peller, *Hankel operators of class $S_p$ and their applications* (*rational approximation, Gaussian processes, the problem of majorization of operators*), Math. USSR Sbornik **41** (1982), 443-479.
8. V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
9. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
10. E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, 1993.
11. J. Xia, *Bounded functions of vanishing mean oscillation on compact metric spaces*, J. Funct. Anal. **209** (2004), 444-467.
12. J. Xia, *Boundedness and compactness of Hankel operators on the sphere*, J. Funct. Anal. **255** (2008), 25-45.
13. J. Xia, *Singular integral operators and essential commutativity on the sphere*, to appear in Canad. J. Math..
14. D. Zheng, *Toeplitz operators and Hankel operators on the Hardy space of the unit sphere*, J. Funct. Anal. **149** (1997), 1-24.
15. K. Zhu, *Operator theory in function spaces*, 2nd ed., Mathematical Surveys and Monographs **138**, American Mathematical Society, Providence, 2007.



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