ROOTS AND LOGARITHMS OF MULTIPLIERS

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Abstract. By now it is a well-known fact that if f is a multiplier for the Drury-Arveson space H^2_n, and if there is a $c > 0$ such that $|f(z)| \ge c$ for every $z \in \mathbf{B}$, then the reciprocal function $1/f$ is also a multiplier for H^2_n. We show that for such an f and for every $t \in \mathbf{R}$, f^t is also a multiplier for H^2_n. We do so by deriving a differentiation formula for $R^m(f^t h)$. Moreover, by this formula the same result holds for spaces $\mathcal{H}_{m,s}$ of the Besov-Dirichlet type. The same technique also gives us the result that for a non-vanishing multiplier f of H^2_n, $\log f$ is a multiplier of H^2_n if and only if $\log f$ is bounded on $\mathbf{B}$.

1. Introduction

Recall that the Drury-Arveson space H^2_n is the Hilbert space of analytic functions on the unit ball $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle}$$

as its reproducing kernel [1,4,9]. A function $f \in H^2_n$ is said to be a *multiplier* for H^2_n if it has the property that $fh \in H^2_n$ for every $h \in H^2_n$. Let $\mathcal{M}$ denote the collection of the multipliers for the Drury-Arveson space H^2_n. It is well known that for each $f \in \mathcal{M}$, the multiplication operator $M_f h = fh$, $h \in H^2_n$, is necessarily bounded on H^2_n [1].

In a remarkable paper [3], Costea, Sawyer and Wick proved the corona theorem for $\mathcal{M}$. This in particular implies the one-function corona theorem:

Theorem 1.1. *If f is a multiplier for H^2_n, and if there is a $c > 0$ such that $|f(z)| \ge c$ for every $z \in \mathbf{B}$, then the reciprocal function $1/f$ is also a multiplier for H^2_n.*

Shortly after [3] was published, interest arose to find a "short" proof of Theorem 1.1, a proof that does not rely on the heavy analytical tools in [3]. The first proof that meets the criterion of being "short" was given in [5]. While elementary, the proof in [5] of the one-function corona theorem still required one analytical tool, namely the von Neumann inequality for commuting tuples of row contractions. Later a second "short" proof of Theorem 1.1 was given in [10], but it also required an analytical tool in the form of [10, Lemma 5.3], which is similar in spirit to the von Neumann inequality for commuting row contractions. Also see [6, Corollary 5.1] and the discussion on this subject in [7].

Then in [2], Cao, He and Zhu gave a surprising proof of Theorem 1.1 by a formula. In addition to the elegance of proving a result by a formula, their approach has the advantage

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that it also works on spaces where the above-described analytical tools are not available. The main ingredient in [2] is the radial derivative.

Throughout the paper, we denote

$$R = z_1\partial_1 + \cdots + z_n\partial_n.$$

In other words, R denotes the usual radial derivative on $\mathbf{B}$.

Let $m \in \mathbf{N}$ and $-1 < s < \infty$. We define $\mathcal{H}_{m,s}$ to be the the collection of analytic functions h on $\mathbf{B}$ satisfying the condition $\|h\|_{m,s} < \infty$, where the norm $\|\cdot\|_{m,s}$ is defined by the formula

$$\|h\|_{m,s}^2 = |h(0)|^2 + \int_{\mathbf{B}} |(R^m h)(z)|^2 (1-|z|^2)^s dv(z).$$

Then $\mathcal{H}_{m,s}$ is a Hilbert space of analytic functions on $\mathbf{B}$. Obviously, $\mathcal{H}_{m,s}$ is a multi-variable, higher-derivative analogue of the Dirichlet space. See [12].

A function $f \in \mathcal{H}_{m,s}$ is said to be a *multiplier* for $\mathcal{H}_{m,s}$ if $fh \in \mathcal{H}_{m,s}$ for every $h \in \mathcal{H}_{m,s}$. If f is a multiplier for $\mathcal{H}_{m,s}$, then it follows easily from the closed-graph theorem that the multiplication operator $M_f h = fh$ is necessarily bounded on $\mathcal{H}_{m,s}$. The operator norm $\|M_f\|$ on $\mathcal{H}_{m,s}$ is generally called the multiplier norm of f.

We recall the following result of Cao, He and Zhu:

Theorem 1.2. [2] *Let $m \in \mathbf{N}$ and $-1 < s < \infty$, and let f be a multiplier for $\mathcal{H}_{m,s}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then $1/f$ is also a multiplier for $\mathcal{H}_{m,s}$.*

Given any $n \in \mathbf{N}$, let $m_0 \in \mathbf{N}$ and $k_0 \in \mathbf{Z}_+$ be such that $2m_0 - k_0 = n$. Then by a well-known estimate, there are constants $0 < c \leq C < \infty$ such that

$$c\|h\|_{m_0,k_0} \leq \|h\|_{H_n^2} \leq C\|h\|_{m_0,k_0} \tag{1.1}$$

for every $h \in H_n^2$. Therefore Theorem 1.2 implies Theorem 1.1. In [2], Theorem 1.2 was proved by using the following formula:

$$R^m(f^{-1}h) = \sum_{k=0}^{m} (-1)^{m-k} \frac{(m+1)!}{k!(m+1-k)!} \frac{1}{f^{m-k+1}} R^m(f^{m-k}h) \tag{1.2}$$

if f, h are analytic on $\mathbf{B}$ and if f does not vanish on $\mathbf{B}$.

Given Theorems 1.1 and 1.2, a natural thought that comes to mind is, what about the roots, or fractional powers, of f? For the unit ball, it is well know that if f is a non-vanishing analytic function on $\mathbf{B}$, then there is an analytic function g on $\mathbf{B}$ such that $f = e^g$, i.e., f has a logarithm. Thus if f is a non-vanishing analytic function on $\mathbf{B}$, then for every real number $t \in \mathbf{R}$, we have a well-defined analytic function f^t on $\mathbf{B}$.

Now, if f is a multiplier for H_n^2, and if there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then for a general $t \in \mathbf{R}$, is f^t a multiplier for H_n^2? Furthermore, what about the multipliers of $\mathcal{H}_{m,s}$? We have the following results:

Theorem 1.3. *Let $m \in \mathbf{N}$ and $-1 < s < \infty$. Let f be a multiplier for $\mathcal{H}_{m,s}$, and suppose that there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$. Then for every $t \in \mathbf{R}$, the function f^t is also a multiplier for $\mathcal{H}_{m,s}$.*

It is also natural to consider the logarithm of a non-vanishing multiplier:

Theorem 1.4. *Let $m \in \mathbf{N}$ and $-1 < s < \infty$. Let f be a multiplier for $\mathcal{H}_{m,s}$, and suppose that f does not vanish on $\mathbf{B}$. Then $\log f$ is a multiplier for $\mathcal{H}_{m,s}$ if and only if $\log f$ is bounded on $\mathbf{B}$.*

By (1.1), Theorem 1.3 immediately implies that if f is a multiplier for the Drury-Arveson space H^2_n and if there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then for every $t \in \mathbf{R}$, the function f^t is also a multiplier for H^2_n. Similarly, it follows from (1.1) and Theorem 1.4 that if f is a non-vanishing multiplier for the Drury-Arveson space H^2_n, then $\log f$ is a multiplier for H^2_n if and only if $\log f$ is bounded on $\mathbf{B}$.

Note that if f is a multiplier, then the sequence of functions $f, f^2, \ldots, f^k, \ldots$ are all multipliers. Our proofs of Theorems 1.3 and 1.4 are based on two differentiation formulas, which are the main results of the paper:

Proposition 1.5. *Given any $m \in \mathbf{N}$ and $t \in \mathbf{R}$, there are real numbers $\rho_1, \ldots, \rho_m$ such that for all analytic functions f, h on $\mathbf{B}$, if f does not vanish on $\mathbf{B}$, then*

$$R^m(f^t h) - f^t R^m h = f^t \sum_{k=1}^{m} \frac{\rho_k}{f^k}\{R^m(f^k h) - f^k R^m h\}. \tag{1.3}$$

Proposition 1.6. *Given any $m \in \mathbf{N}$, there are rational numbers $a_1, \ldots, a_m$ such that for all analytic functions f, h on $\mathbf{B}$, if f does not vanish on $\mathbf{B}$, then*

$$R^m((\log f)h) - (\log f)R^m h = \sum_{k=1}^{m} \frac{a_k}{f^k}\{R^m(f^k h) - f^k R^m h\}. \tag{1.4}$$

In Sections 3 and 4 we will see that the coefficients $\rho_1, \ldots, \rho_m$ and $a_1, \ldots, a_m$ in Propositions 1.5 and 1.6 are explicitly determined.

2. Proofs of Theorems 1.3 and 1.4

It is well known that if f is a multiplier for the Drury-Arveson space H^2_n, then f is necessarily bounded on the unit ball $\mathbf{B}$ [1], i.e., $\|f\|_\infty < \infty$. The same is true for the multipliers of the spaces $\mathcal{H}_{m,s}$ introduced in Section 1, although this fact is not as well known. For completeness, we first establish

Lemma 2.1. *Let $m \in \mathbf{N}$ and $-1 < s < \infty$. If f is a multiplier for $\mathcal{H}_{m,s}$, then $\|f\|_\infty < \infty$.*

Proof. Write $\|\cdot\|_s$ for the norm of the weighted Bergman space $L^2_a(\mathbf{B}, (1 - |z|^2)^s dv(z))$. By the definition of the norm $\|\cdot\|_{m,s}$, there is a $0 < C_s < \infty$ such that $\|h\|_s \leq C_s\|h\|_{m,s}$ for every $h \in \mathcal{H}_{m,s}$. Therefore for each given $w \in \mathbf{B}$, the map

$$h \mapsto h(w)$$

is a bounded linear functional on $\mathcal{H}_{m,s}$. Hence for each $w \in \mathbf{B}$, there is a $K_w^{(m,s)} \in \mathcal{H}_{m,s}$ such that

$$h(w) = \langle h, K_w^{(m,s)} \rangle_{m,s} \quad \text{for every} \quad h \in \mathcal{H}_{m,s}.$$

Let f be a multiplier for $\mathcal{H}_{m,s}$. For every pair of $w, z \in \mathbf{B}$, we have

$$\begin{aligned}\langle M_f^* K_w^{(m,s)}, K_z^{(m,s)} \rangle_{m,s} = \langle K_w^{(m,s)}, M_f K_z^{(m,s)} \rangle_{m,s} &= \overline{f(w) K_z^{(m,s)}(w)} \\ &= \overline{f(w)} \langle K_w^{(m,s)}, K_z^{(m,s)} \rangle_{m,s}.\end{aligned}$$

From this we deduce $M_f^* K_w^{(m,s)} = \overline{f(w)} K_w^{(m,s)}$ for every $w \in \mathbf{B}$. Thus $\|f\|_\infty \le \|M_f^*\| = \|M_f\| < \infty$. This completes the proof. □

Proof of Theorem 1.3. Let f be a multiplier for $\mathcal{H}_{m,s}$, and suppose that there is a $c > 0$ such that $|f(z)| \ge c$ for every $z \in \mathbf{B}$. Then $\|f^{t-k}\|_\infty < \infty$, whether $t - k \ge 0$ or $t - k < 0$.

Applying Proposition 1.5 to f and to each $h \in \mathcal{H}_{m,s}$, we obtain

$$R^m(f^t h) = \sum_{k=0}^m \rho_k f^{t-k} R^m(f^k h),$$

where $\rho_0 = 1 - \rho_1 - \cdots - \rho_m$. Write $\|\cdot\|_s$ for the norm of the weighted Bergman space $L_a^2(\mathbf{B}, (1 - |z|^2)^s dv(z))$. From the above identity we obtain

$$\|R^m(f^t h)\|_s \le \sum_{k=0}^m |\rho_k| \|f^{t-k}\|_\infty \|R^m(f^k h)\|_s \le \sum_{k=0}^m |\rho_k| \|f^{t-k}\|_\infty \|f^k h\|_{m,s}.$$

Combining the above with the obvious inequality

$$\|f^t h\|_{m,s} \le |f^t(0) h(0)| + \|R^m(f^t h)\|_s \le |f(0)|^t \|h\|_{m,s} + \|R^m(f^t h)\|_s,$$

we conclude that f^t is a multiplier for $\mathcal{H}_{m,s}$. This completes the proof. □

Proof of Theorem 1.4. Let f be a multiplier for $\mathcal{H}_{m,s}$, and suppose that f does not vanish on $\mathbf{B}$. Thus $\log f$ exists as an analytic function on $\mathbf{B}$.

Suppose that $\|\log f\|_\infty < \infty$. This implies that there is a $c > 0$ such that $|f(z)| \ge c$ for every $z \in \mathbf{B}$. Applying Proposition 1.6 to f and to each $h \in \mathcal{H}_{m,s}$, we obtain

$$R^m((\log f) h) = (\log f) R^m h + \sum_{k=0}^m a_k f^{-k} R^m(f^k h),$$

where $a_0 = -a_1 - \cdots - a_m$. Again, write $\|\cdot\|_s$ for the norm of the weighted Bergman space $L_a^2(\mathbf{B}, (1 - |z|^2)^s dv(z))$. From the above identity we obtain

$$\begin{aligned}\|R^m((\log f) h)\|_s &\le \|\log f\|_\infty \|R^m h\|_s + \sum_{k=0}^m |a_k| c^{-k} \|R^m(f^k h)\|_s \\ &\le \|\log f\|_\infty \|h\|_{m,s} + \sum_{k=0}^m |a_k| c^{-k} \|f^k h\|_{m,s}.\end{aligned}$$

Combining the above with the obvious inequality

$$\|(\log f)h\|_{m,s} \le |(\log f(0))h(0)| + \|R^m((\log f)h)\|_s \le |\log f(0)|\|h\|_{m,s} + \|R^m((\log f)h)\|_s,$$

we conclude that $\log f$ is a multiplier for $\mathcal{H}_{m,s}$.

On the other hand, if $\log f$ is a multiplier for $\mathcal{H}_{m,s}$, then by Lemma 2.1 we have $\|\log f\|_\infty < \infty$. This completes the proof. □

3. Differentiation formulas

We now turn to the proofs of Propositions 1.5 and 1.6. Our approach to the proofs of Propositions 1.5 and 1.6 is quite different from the approach to the proof of (1.2) in [2].

Our main idea for the proofs of Propositions 1.5 and 1.6 is to take advantage of the famous Faa di Bruno formula. But instead of the actual formula commonly attributed to Faa di Bruno (see, e.g., [8]), where all the coefficients are explicit, all we need is a very crude version of it. In fact, this is a point that we want to emphasize: the proofs of Propositions 1.5 and 1.6 do not need the delicate combinatorics that makes the coefficients in the Faa di Bruno formula explicit. We will see that the proofs of Propositions 1.5 and 1.6 involve nothing but linear algebra!

Given any $\nu \in \mathbf{N}$, we define A_ν to be the collection of $\alpha = (\alpha_1, \dots, \alpha_\nu) \in \mathbf{Z}_+^\nu$ satisfying the condition

$$1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots + \nu \cdot \alpha_\nu = \nu.$$

We follow the usual multi-index notation (see, e.g., [11, page 3]): for $\alpha = (\alpha_1, \dots, \alpha_\nu) \in \mathbf{Z}_+^\nu$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_\nu$ and $(z_1, \dots, z_\nu)^\alpha = z_1^{\alpha_1} \cdots z_\nu^{\alpha_\nu}$.

For each $\nu \in \mathbf{N}$, there exists a set of coefficients $\{b_\alpha : \alpha \in A_\nu\}$ such that the identity

$$\frac{d^\nu}{dx^\nu} F(G(x)) = \sum_{\alpha \in A_\nu} b_\alpha F^{(|\alpha|)}(G(x))(G^{(1)}(x), \dots, G^{(\nu)}(x))^\alpha \tag{3.1}$$

holds for all functions F, G that are sufficiently smooth.

In contrast to the coefficient-explicit version of the Faa di Bruno formula, (3.1) follows easily from an induction on ν. For the proofs of of Propositions 1.5 and 1.6, the actual values of the coefficients $\{b_\alpha : \alpha \in A_\nu\}$ are not relevant; the only thing that is relevant is the fact that such coefficients do exist.

To prove Propositions 1.5 and 1.6, for every pair of $1 \le i, k \le m$, we define

$$\beta_{i,k} = \begin{cases} k!/(k-i)! & \text{if} \quad i \le k \\ 0 & \text{if} \quad i > k \end{cases}. \tag{3.2}$$

Then the $m \times m$ upper-triangular matrix $[\beta_{i,k}]_{i,k=1}^m$ is invertible. Therefore there is an $m \times m$ matrix $[c_{k,r}]_{k,r=1}^m$ of rational entries such that for all $1 \le i, r \le m$,

$$\sum_{k=1}^m \beta_{i,k} c_{k,r} = \begin{cases} 1 & \text{if} \quad i = r \\ 0 & \text{if} \quad i \ne r \end{cases}. \tag{3.3}$$

Define the set $A_{\nu,i} = \{\alpha \in A_\nu : |\alpha| = i\}$ for each pair of $i \le \nu$ in $\{1, \dots, m\}$. We write C^m_ν for the binomial coefficient $\frac{m!}{\nu!(m-\nu)!}$.

Lemma 3.1. *Let f, h be analytic on $\mathbf{B}$. If f does not vanish on $\mathbf{B}$, we define*

$$X_{f,i}h = f^{-i} \sum_{\nu=i}^{m} C^m_\nu \sum_{\alpha \in A_{\nu,i}} b_\alpha (Rf, \dots, R^\nu f)^\alpha R^{m-\nu} h$$

for $1 \le i \le m$. Then for every $1 \le r \le m$, we have

$$\sum_{k=1}^{m} \frac{c_{k,r}}{f^k} \sum_{\nu=1}^{m} C^m_\nu (R^\nu f^k)(R^{m-\nu} h) = X_{f,r} h. \tag{3.4}$$

Proof. For each $1 \le \nu \le m$, if we apply (3.1) to the function $F(x) = x^k$, we obtain

$$\begin{aligned} R^\nu f^k &= \sum_{i=1}^{\min\{k,\nu\}} \sum_{\alpha \in A_{\nu,i}} b_\alpha \frac{k!}{(k-|\alpha|)!} f^{k-|\alpha|} (Rf, \dots, R^\nu f)^\alpha \\ &= \sum_{i=1}^{\nu} \beta_{i,k} f^{k-i} \sum_{\alpha \in A_{\nu,i}} b_\alpha (Rf, \dots, R^\nu f)^\alpha. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\nu=1}^{m} C^m_\nu (R^\nu f^k)(R^{m-\nu} h) &= \sum_{\nu=1}^{m} C^m_\nu \sum_{i=1}^{\nu} \beta_{i,k} f^{k-i} \sum_{\alpha \in A_{\nu,i}} b_\alpha (Rf, \dots, R^\nu f)^\alpha R^{m-\nu} h \\ &= f^k \sum_{i=1}^{m} \beta_{i,k} f^{-i} \sum_{\nu=i}^{m} C^m_\nu \sum_{\alpha \in A_{\nu,i}} b_\alpha (Rf, \dots, R^\nu f)^\alpha R^{m-\nu} h \\ &= f^k \sum_{i=1}^{m} \beta_{i,k} X_{f,i} h. \end{aligned}$$

Hence for each $1 \le r \le m$,

$$\sum_{k=1}^{m} \frac{c_{k,r}}{f^k} \sum_{\nu=1}^{m} C^m_\nu (R^\nu f^k)(R^{m-\nu} h) = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{k,r} \beta_{i,k} X_{f,i} h = X_{f,r} h,$$

where the second $=$ is obtained from (3.3). $\square$

Proof of Proposition 1.5. For each natural number $1 \le r \le m$, we define

$$u(r) = \prod_{j=0}^{r-1} (t-j) = (-1)^r \prod_{j=0}^{r-1} (j-t). \tag{3.5}$$

Thus $(x^t)^{(r)} = u(r)x^{t-r}$, $1 \le r \le m$. We then define $\rho_k = \sum_{r=1}^m u(r)c_{k,r}$ for each $1 \le k \le m$. Let us verify that (1.3) holds for the $\rho_1, \dots, \rho_m$ so defined.

Let f, h be analytic functions on $\mathbf{B}$, and suppose that f does not vanish on $\mathbf{B}$. By the Leibniz rule for differentiation, we have

$$R^m(f^t h) - f^t R^m h = \sum_{\nu=1}^m C_\nu^m (R^\nu f^t)(R^{m-\nu}h) \quad \text{and}$$
$$R^m(f^k h) - f^k R^m h = \sum_{\nu=1}^m C_\nu^m (R^\nu f^k)(R^{m-\nu}h), \quad 1 \le k \le m.$$

Thus (1.3) will hold for the $\rho_1, \dots, \rho_m$ defined above if we can show that

$$\sum_{\nu=1}^m C_\nu^m (R^\nu f^t)(R^{m-\nu}h) = f^t \sum_{r=1}^m u(r) \sum_{k=1}^m \frac{c_{k,r}}{f^k} \sum_{\nu=1}^m C_\nu^m (R^\nu f^k)(R^{m-\nu}h). \tag{3.6}$$

To prove (3.6), we apply (3.1) to the function $F(x) = x^t$. For each $1 \le \nu \le m$, (3.1) gives us

$$R^\nu f^t = \sum_{\alpha \in A_\nu} b_\alpha u(|\alpha|) f^{t-|\alpha|}(Rf, \dots, R^\nu f)^\alpha$$

(see (3.5)). Therefore

$$\begin{aligned}
\sum_{\nu=1}^m C_\nu^m (R^\nu f^t)(R^{m-\nu}h) &= \sum_{\nu=1}^m C_\nu^m \sum_{\alpha \in A_\nu} b_\alpha u(|\alpha|) f^{t-|\alpha|}(Rf, \dots, R^\nu f)^\alpha R^{m-\nu}h \\
&= \sum_{\nu=1}^m C_\nu^m \sum_{r=1}^\nu \sum_{\alpha \in A_{\nu,r}} b_\alpha u(|\alpha|) f^{t-|\alpha|}(Rf, \dots, R^\nu f)^\alpha R^{m-\nu}h \\
&= \sum_{r=1}^m u(r) f^{t-r} \sum_{\nu=r}^m C_\nu^m \sum_{\alpha \in A_{\nu,r}} b_\alpha (Rf, \dots, R^\nu f)^\alpha R^{m-\nu}h \\
&= f^t \sum_{r=1}^m u(r) X_{f,r} h. \qquad (3.7)
\end{aligned}$$

Applying Lemma 3.1 in (3.7), we obtain (3.6). This completes the proof. □

Proof of Proposition 1.6. For each natural number $1 \le r \le m$, we define

$$v(r) = (r-1)!(-1)^{r-1}. \tag{3.8}$$

Thus $(\log x)^{(r)} = v(r)x^{-r}$, $1 \le r \le m$. We then define $a_k = \sum_{r=1}^m v(r)c_{k,r}$ for each $1 \le k \le m$. Let us verify that (1.4) holds for the $a_1, \dots, a_m$ so defined.

Let f, h be analytic functions on $\mathbf{B}$, and suppose that f does not vanish on $\mathbf{B}$. By the Leibniz rule for differentiation, we have

$$R^m((\log f)h) - (\log f)R^m h = \sum_{\nu=1}^{m} C_\nu^m (R^\nu(\log f))(R^{m-\nu}h) \quad \text{and}$$
$$R^m(f^k h) - f^k R^m h = \sum_{\nu=1}^{m} C_\nu^m (R^\nu f^k)(R^{m-\nu}h), \quad 1 \le k \le m.$$

Thus (1.4) will hold for the $a_1, \ldots, a_m$ defined above if we can show that

$$\sum_{\nu=1}^{m} C_\nu^m (R^\nu(\log f))(R^{m-\nu}h) = \sum_{r=1}^{m} v(r) \sum_{k=1}^{m} \frac{c_{k,r}}{f^k} \sum_{\nu=1}^{m} C_\nu^m (R^\nu f^k)(R^{m-\nu}h). \tag{3.9}$$

To prove (3.9), we apply (3.1) to the function $F(x) = \log x$. For each $1 \le \nu \le m$, (3.1) gives us

$$R^\nu(\log f) = \sum_{\alpha \in A_\nu} b_\alpha v(|\alpha|) f^{-|\alpha|}(Rf, \ldots, R^\nu f)^\alpha$$

(see (3.8)). Therefore

$$\begin{aligned}
\sum_{\nu=1}^{m} C_\nu^m (R^\nu(\log f))(R^{m-\nu}h) &= \sum_{\nu=1}^{m} C_\nu^m \sum_{\alpha \in A_\nu} b_\alpha v(|\alpha|) f^{-|\alpha|}(Rf, \ldots, R^\nu f)^\alpha R^{m-\nu}h \\
&= \sum_{\nu=1}^{m} C_\nu^m \sum_{r=1}^{\nu} \sum_{\alpha \in A_{\nu,r}} b_\alpha v(|\alpha|) f^{-|\alpha|}(Rf, \ldots, R^\nu f)^\alpha R^{m-\nu}h \\
&= \sum_{r=1}^{m} v(r) f^{-r} \sum_{\nu=r}^{m} C_\nu^m \sum_{\alpha \in A_{\nu,r}} b_\alpha (Rf, \ldots, R^\nu f)^\alpha R^{m-\nu}h \\
&= \sum_{r=1}^{m} v(r) X_{f,r} h. \qquad (3.10)
\end{aligned}$$

Applying Lemma 3.1 in (3.10), we obtain (3.9). This completes the proof. □

4. More explicit formulas for the coefficients

Note that Propositions 1.5 and 1.6 were proved without any kind of computation. We will now add up the sums $\rho_k = \sum_{r=1}^m u(r)c_{k,r}$ and $a_k = \sum_{r=1}^m v(r)c_{k,r}$ to make these coefficients more explicit, which involves some elementary calculation.

First, note that $[\beta_{i,k}] = e^N F$, where F is the diagonal matrix whose diagonal entries are $1!, \ldots, m!$ and $N = [s_{i,k}]$, where $s_{i,i+1} = 1$ for $1 \le i \le m-1$ and $s_{i,k} = 0$ if $k \ne i+1$. Consequently, $[c_{k,r}] = F^{-1}e^{-N}$. That is, for all $k, r \in \{1, \ldots, m\}$ we have

$$c_{k,r} = \begin{cases} \frac{(-1)^{r-k}}{k!(r-k)!} & \text{if} \quad r \ge k \\ 0 & \text{if} \quad r < k \end{cases}. \tag{4.1}$$

Next, recall that for all non-negative integers $p \geq 0$ and $q \geq 0$, there is the identity

$$\sum_{j=0}^{q} \frac{(j+p)!}{j!} = \frac{(q+p+1)!}{q!(p+1)}, \tag{4.2}$$

which is proved by an easy induction on q.

By (3.8), (4.1) and (4.2), for each $1 \leq k \leq m$ we have

$$\begin{aligned} a_k &= \sum_{r=k}^{m} (-1)^{r-1+r-k} \frac{(r-1)!}{k!(r-k)!} = \frac{(-1)^{k+1}}{k!} \sum_{j=0}^{m-k} \frac{(j+k-1)!}{j!} \\ &= \frac{(-1)^{k+1}}{k!} \cdot \frac{(m-k+k-1+1)!}{(m-k)!k} = \frac{(-1)^{k+1}}{k} \cdot \frac{m!}{k!(m-k)!}. \end{aligned}$$

Similarly, by (4.1) and (3.5), for each $1 \leq k \leq m$,

$$\rho_k = \sum_{r=k}^{m} (-1)^{r+r-k} \frac{\prod_{i=0}^{r-1}(i-t)}{k!(r-k)!} = \frac{(-1)^k}{k!} \sum_{j=0}^{m-k} \frac{\prod_{i=0}^{j+k-1}(i-t)}{j!}. \tag{4.3}$$

In the case where t is a negative integer, by (4.2) we have

$$\begin{aligned} \sum_{j=0}^{m-k} \frac{\prod_{i=0}^{j+k-1}(i-t)}{j!} &= \sum_{j=0}^{m-k} \frac{(j+k-1-t)!}{(-t-1)!j!} = \frac{(m-k+k-1-t+1)!}{(-t-1)!(m-k)!(k-1-t+1)} \\ &= \frac{(m-t)!}{(-t-1)!(m-k)!(k-t)} = \frac{1}{(m-k)!} \prod_{i \in \{0,1,\ldots,m\} \backslash \{k\}} (i-t). \end{aligned} \tag{4.4}$$

If a and b are polynomials such that $a(t) = b(t)$ for every negative integer t, then $a(t) = b(t)$ for every $t \in \mathbf{R}$. Thus it follows from (4.3) and (4.4) that for every $1 \leq k \leq m$,

$$\rho_k = \frac{(-1)^k}{k!(m-k)!} \prod_{i \in \{0,1,\ldots,m\} \backslash \{k\}} (i-t).$$

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