GEOMETRIC ARVESON-DOUGLAS CONJECTURE FOR THE HARDY SPACE AND A RELATED COMPACTNESS CRITERION

Yi Wang[1] and Jingbo Xia

Abstract. We consider a class of analytic subsets $\tilde{M}$ of an open neighborhood of the closed unit ball in $\mathbf{C}^n$. Such an $\tilde{M}$ gives rise to a submodule $\mathcal{R}$ and a quotient module $\mathcal{Q}$ of the Hardy module $H^2(S)$ on the unit sphere $S \subset \mathbf{C}^n$. We show that, as predicted by the geometric Arveson-Douglas conjecture, the quotient module $\mathcal{Q}$ is p-essentially normal for $p > d = \dim_{\mathbf{C}} \tilde{M}$. We further show that, more interestingly, the quotient module $\mathcal{Q}$ exhibits a behavior that is only found on the Bergman space and the Fock space: an operator A in the Toeplitz algebra on $\mathcal{Q}$ is compact if and only if its Berezin transform vanishes near $\tilde{M} \cap S$.

1. Introduction

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. We write $d\sigma$ for the standard spherical measure on S, and we take the usual normalization $\sigma(S) = 1$. The simplest way to introduce the Hardy space $H^2(S)$ is to say that it is the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Nowadays, the Hardy space $H^2(S)$ is more commonly viewed as a Hilbert module over the ring of analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$, and the same is true for the other reproducing-kernel Hilbert spaces [7,11]. One of the reasons why we want to think of these spaces as modules over $\mathbf{C}[z_1, \dots, z_n]$ is that where there are modules, there are submodules and quotient modules, which can be sources of very interesting and challenging problems. A good example of such problems is the Arveson-Douglas conjecture, which in recent years has been a very active area of research [3,6,12-15,18,22,28].

Suppose that $\mathcal{N}$ is either a submodule or a quotient module of the Hardy module $H^2(S)$. Let $P_{\mathcal{N}} : H^2(S) \to \mathcal{N}$ be the orthogonal projection. Then we have the module operators

$$\mathcal{Z}_{\mathcal{N},j} = P_{\mathcal{N}} M_{z_j} | \mathcal{N}, \quad j = 1, \dots, n, \tag{1.1}$$

on $\mathcal{N}$. Recall that $\mathcal{N}$ is said to be p-essentially normal if all commutators $[\mathcal{Z}^*_{\mathcal{N},i}, \mathcal{Z}_{\mathcal{N},j}]$, $1 \le i, j \le n$, are in the Schatten class $\mathcal{C}_p$. The famous Arveson Conjecture [1,2] predicts that every graded submodule of the Drury-Arveson module is p-essentially normal for $p > n$. This was later refined by Douglas [10], who observed that in the case of the quotient module it should really be $p > d$, where d is the complex dimension of the variety involved. This conforms with the common view that quotient modules are rather "small".

Keywords: Quotient module, essential normality, compactness criterion.

[1]Supported in part by National Science Foundation grant DMS-1900076.

In this paper we consider a very specific class of submodules and quotient modules. Denote $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$, the open unit ball in $\mathbf{C}^n$. Let $\tilde{M}$ be an analytic subset [9] of an open neighborhood of $\overline{\mathbf{B}}$ with $1 \leq \dim_{\mathbf{C}} \tilde{M} \leq n - 1$. We will assume that $\tilde{M}$ has no singular points on S and that $\tilde{M}$ intersects S transversely. Denote $M = \mathbf{B} \cap \tilde{M}$. Then we have a submodule

$$\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$$

of $H^2(S)$. The corresponding quotient module is

$$\mathcal{Q} = H^2(S) \ominus \mathcal{R}.$$

Specialized to this particular setting, we have

Geometric Arveson-Douglas Conjecture. The quotient module $\mathcal{Q}$ is p-essentially normal for every $p > d = \dim_{\mathbf{C}} \tilde{M}$.

Since the Hardy module itself is p-essentially normal for $p > n$, the geometric Arveson-Douglas conjecture implies that the submodule $\mathcal{R}$ is p-essentially normal for $p > n$.

The analogous problem in the case of the Bergman module $L^2_a(\mathbf{B})$ was recently solved [14,28]. This gives us confidence that the geometric Arveson-Douglas conjecture for the Hardy module $H^2(S)$ can also be solved, although one should never take such things for granted. Our experience with the Bergman module $L^2_a(\mathbf{B})$ further tells us that it is the *quotient module* that holds the key to everything [16]. Therefore in this paper we will focus on $\mathcal{Q}$, which turns out to be the right decision.

Let us now discuss our results. First of all, the prediction of the geometric Arveson-Douglas conjecture is correct:

Theorem 1.1. *The quotient module $\mathcal{Q}$ is p-essentially normal for every $p > d = \dim_{\mathbf{C}} \tilde{M}$.*

Let Q denote the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{Q}$. As it turns out, everything we do in this paper depends on getting a good handle on the projection Q. Even though an explicit integral formula for Q is beyond reach, we manage to get the next best thing:

Theorem 1.2. *There is a measure μ on M such that the corresponding Toeplitz operator T_μ satisfies the operator inequality*

$$cQ \leq T_\mu \leq CQ \tag{1.2}$$

on $L^2(S, d\sigma)$ with coefficients $0 < c \leq C < \infty$.

We remind the reader that the Toeplitz operator T_μ is defined by the formula

$$(T_\mu h)(z) = \int_M \frac{h(w)}{(1 - \langle z, w \rangle)^n} d\mu(w),$$

$h \in H^2(S)$. Operator inequality (1.2) gives us enough control of the projection Q to prove Theorem 1.1 and, more important, to do more.

Recall that the normalized reproducing kernel for $H^2(S)$ is given by the formula

$$k_z(w) = \frac{(1-|z|^2)^{n/2}}{(1-\langle w, z\rangle)^n},$$

$z \in \mathbf{B}$ and $w \in \overline{\mathbf{B}}$. From the reproducing property of the kernel it is easy to see that $\mathcal{Q}$ is the closure of the linear span of $\{k_z : z \in M\}$.

Since we have a projection Q, we can mimic the definition of the standard Toeplitz operators to define "Toeplitz operators for the quotient module $\mathcal{Q}$". That is, for each $f \in L^\infty(S, d\sigma)$, we define

$$Q_f = QM_f|\mathcal{Q}.$$

We think of Q_f as a Toeplitz operator for the quotient module $\mathcal{Q}$. Let $\mathcal{TQ}$ be the C^*-algebra generated by $\{Q_f : f \in L^\infty(S, d\sigma)\}$. Obviously, $\mathcal{TQ}$ is the proper analogue on $\mathcal{Q}$ of the usual Toeplitz algebra. Our next result is at least somewhat unexpected:

Theorem 1.3. *Let $A \in \mathcal{TQ}$. If*

$$\lim_{\substack{z \in M \\ |z| \to 1}} \langle Ak_z, k_z\rangle = 0,$$

then A is a compact operator.

We say that this is "at least somewhat unexpected" because, previously, results of this genre have only been proven on the Bergman space and the Fock space [4,5,21,27,29,31]. What is more, this particular compactness criterion is known to fail for operators in the Toeplitz algebra $\mathcal{T}$ on the one-variable Hardy space H^2 [19, Section 2]. That notwithstanding, on the quotient module $\mathcal{Q}$ of the Hardy module $H^2(S)$, we have Theorem 1.3!

The original purpose of the Arveson-Douglas conjecture is to see how much of the operator theory on the standard reproducing kernel Hilbert spaces, such as the Bergman space, the Hardy space and the Drury-Arveson space, can be established on these submodules and quotient modules, *and* to explore what is new on these submodules and quotient modules. Thus Theorem 1.3 fits the context of the Arveson-Douglas conjecture very nicely.

The rest of the paper is organized as follows. In Section 2 we first record the precise definitions of $\tilde{M}$, M, $\mathcal{R}$, $\mathcal{Q}$ etc. We then introduce for each $z \in M$ near S the modified tangent space T_z^{mod}, which is a copy of $\mathbf{C}^d$. The rest of Section 2 contains local analysis on M, which includes the Forelli-Rudin estimates on M and more. Basically, the use of T_z^{mod} allows us to convert the local analysis on M to analysis on $\mathbf{C}^d$.

Section 3 is devoted to the proof of Theorem 1.2, where the reader will see the precise definition of the measure μ. One can consider Section 4 as an operator version of the atomic decomposition for the quotient module $\mathcal{Q}$. More specifically, in Section 4 we introduce two classes of operators on $\mathcal{Q}$, $\mathcal{D}_0$ and $\mathcal{D}$, both of which consist of discrete sums constructed from normalized reproducing kernel over lattices in M, but $\mathcal{D}_0 \subset \mathcal{D}$. In Proposition 4.3 we show that T_μ can be approximated in operator norm by operators in $\text{span}(\mathcal{D}_0)$, which is the atomic decomposition for $\mathcal{Q}$. As consequences of Proposition 4.3, we obtain a compactness test on $\mathcal{Q}$ and a membership test for $C^*(\mathcal{D})$, the C^*-algebra generated by $\mathcal{D}$. Both of these tests will be needed in the proof of Theorem 1.3.

The main result in Section 5 is Lemma 5.3, which says in a very precise way that the operators in $C^*(\mathcal{D})$ are localized. With Lemma 5.3 and a lot more work, in Section 6 we show that for $A \in C^*(\mathcal{D})$, if

$$\lim_{\substack{z\in M\\|z|\to 1}} \langle Ak_z, k_z\rangle = 0,$$

then A is a compact operator. Then in Section 7, we complete the proof of Theorem 1.3 by showing that $\mathcal{T}\mathcal{Q} \subset C^*(\mathcal{D})$.

Finally, Section 8 contains the proof of Theorem 1.1, where Proposition 4.3 also plays an essential role.

2. Local estimates

We begin with the Bergman-metric structure of the ball. As usual, we write β for the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B}.$$

We recall that the Möbius transform φ_z is given by the formula

$$\varphi_z(w) = \frac{1}{1 - \langle w, z\rangle}\left\{z - \frac{\langle w, z\rangle}{|z|^2}z - (1 - |z|^2)^{1/2}\left(w - \frac{\langle w, z\rangle}{|z|^2}z\right)\right\} \tag{2.1}$$

when $z \neq 0$, and $\varphi_0(w) = -w$. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z, a) = \{w \in \mathbf{B} : \beta(z, w) < a\}$.

Definition 2.1. (i) Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z, a) \cap D(w, a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) A subset Γ of $\mathbf{B}$ is simply said to be separated if it is a-separated for some $a > 0$.

Next let us give the precise definitions of the analytic sets, submodules and quotient modules that we consider in this paper.

Definition 2.2. [9] Let Ω be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of Ω if for each point $a \in \Omega$ there are a neighborhood U of a and functions $f_1, \cdots, f_N$ analytic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a *singular point* of A if it is not regular.

Definition 2.3. Let Y be a manifold and let X, Z be submanifolds of Y. We say that the submanifolds X and Z *intersect transversely* if for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.

Assumption 2.4. Let $\tilde{M}$ be an analytic subset in an open neighborhood of the closed ball $\overline{\mathbf{B}}$. Furthermore, $\tilde{M}$ satisfies the following conditions:

(1) $\tilde{M}$ intersects $\partial\mathbf{B}$ transversely.

(2) $\tilde{M}$ has no singular points on $\partial\mathbf{B}$.

(3) $\tilde{M}$ is of pure dimension d, where $1 \leq d \leq n-1$.

Note that condition (3) implies that $\tilde{M}$ has no isolated singularities in $\mathbf{B}$. The reader will see that our work actually allows a condition that is slightly broader than condition (3). In fact, we could allow $\tilde{M}$ to be the union of components $\tilde{C}_1, \dots, \tilde{C}_m$, where $\dim_{\mathbf{C}}\tilde{C}_i = d_i$ for each $1 \leq i \leq m$, with each d_i satisfying $1 \leq d_i \leq n-1$. But for simplicity, we have decided to go with a single complex dimension d, as stated in (3).

Thus we emphasize that Assumption 2.4 will always be in force for the rest of the paper. Given such an $\tilde{M}$, we fix M, $\mathcal{R}$, $\mathcal{Q}$ and Q as follows.

Notation 2.5. (a) Let $M = \tilde{M} \cap \mathbf{B}$.

(b) Denote $\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$.

(c) Denote $\mathcal{Q} = H^2(S) \ominus \mathcal{R}$.

(d) Let Q be the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{Q}$.

For $z \in \mathbf{C}^n$ and $r > 0$, denote

$$B(z, r) = \{w \in \mathbf{C}^n : |z - w| < r\}.$$

By Assumption 2.4, there is an $s \in (0, 1)$ such that

$$\mathcal{M} = \{z \in \tilde{M} : 1 - s < |z| < 1 + s\}$$

is a complex manifold of complex dimension d and of finite volume. Thus

$$K = \{z \in \tilde{M} : 1 - (s/2) \leq |z| \leq 1\}$$

is a compact subset of the complex manifold $\mathcal{M}$. By the standard facts known about such a pair of $\mathcal{M}$ and K, for which we cite [23,24,25] as general references, the statements we make below hold true with constants that are independent of $z \in K$.

For each $z \in K$, let T_z be the tangent space to $\mathcal{M}$ at the point z, viewed as a natural subspace of $\mathbf{C}^n$. Then there are $a > 0$ and $b > 0$ such that for each $z \in K$, there is a map

$$G_z : T_z \cap B(0, a) \to \mathcal{M}$$

that biholomorphically maps $T_z \cap B(0, a)$ onto an open subset of $\mathcal{M}$ with the properties that $G_z(0) = z$ and that

$$\{G_z(w) : w \in T_z \cap B(0, a)\} \supset \mathcal{M} \cap B(z, b). \tag{2.2}$$

Let DG_z be the complex derivative of G_z. For each $w \in T_z \cap B(0, a)$, we have the local Taylor expansion

$$G_z(w + u) = G_z(w) + (DG_z)(w)u + \int_0^1 \{(DG_z)(w + tu) - (DG_z)(w)\}u\,dt, \tag{2.3}$$

$w + u \in T_z \cap B(0,a)$. In particular, at the point $w = 0$ we have

$$T_z = (DG_z)(0)T_z$$

and

$$(2.4)\quad G_z(u) = z + (DG_z)(0)u + \int_0^1 \{(DG_z)(tu) - (DG_z)(0)\}udt \quad \text{for} \quad u \in T_z \cap B(0,a).$$

Reducing the values of a and b if necessary, we may assume that there are constants $0 < \alpha \leq \beta < \infty$ such that for $w \in T_z \cap B(0,a)$, the linear transformation inequality

$$(2.5)\qquad \alpha \leq (DG_z)^*(w)(DG_z)(w) \leq \beta$$

holds on T_z.

For each $z \in K$, let p_z be the orthogonal projection of z on T_z. Condition (1) in Assumption 2.4 says that if $z \in \tilde{M} \cap S$, then $p_z \neq 0$. Thus, reducing the value of $s \in (0,1)$ if necessary, we may assume that $p_z \neq 0$ for every $z \in K$. Thus for each $z \in K$,

$$T_z^{\perp} = \{u \in T_z : \langle u, p_z \rangle = 0\}$$

is a linear subspace of T_z of dimension $d-1$. As a subspace of $\mathbf{C}^n$, $T_z^{\perp}$ is orthogonal to z.

Definition 2.6. (a) For each $z \in K$, we define

$$T_z^{\mathrm{mod}} = T_z^{\perp} \oplus \{\xi z : \xi \in \mathbf{C}\},$$

which we consider as the *modified* complex tangent space at z.
(b) For each $z \in K$, let P_z be the orthogonal projection from $\mathbf{C}^n$ onto T_z^{mod}.

Lemma 2.7. *There exist $b_0 > 0$ and $c_0 > 0$ such that for every $z \in K$, P_z is a biholomorphic map from $\mathcal{M} \cap B(z, b_0)$ onto an open set in T_z^{mod} that contains $T_z^{\mathrm{mod}} \cap B(z, c_0)$.*

Proof. By (2.4), for $z \in K$ we can write

$$G_z(w) = z + (DG_z)(0)w + H_z(w),$$

$w \in T_z \cap B(0,a)$. We now make a change of variable on T_z. That is, we define

$$(2.6)\qquad \tilde{G}_z(w) = z + w + \tilde{H}_z(w), \quad \text{where} \quad \tilde{H}_z(w) = H_z((DG_z)^{-1}(0)w),$$

for $w \in (DG_z)(0)\{T_z \cap B(0,a)\}$. We have $\tilde{G}_z(0) = z$. By (2.4), (2.5), the mapping properties of G_z, and the compactness of K, there is an $a_1 > 0$ such that $\tilde{G}_z$ biholomorphically maps $T_z \cap B(0, a_1)$ onto an open subset of $\mathcal{M}$. For each $z \in K$, define

$$F_z(w) = P_z \tilde{G}_z(w)$$

for $w \in T_z \cap B(0, a_1)$. Obviously, $F_z(0) = P_z\tilde{G}_z(0) = P_z z = z$. We claim that there is an $a_0 \in (0, a_1)$ such that for each $z \in K$, F_z is a biholomorphic map between $T_z \cap B(0, a_0)$ and an open set in T_z^{mod}.

To find such an a_0, we define $v_z = p_z/|p_z|$. Then every $w \in T_z$ has the orthogonal decomposition $w = \xi v_z + u$, where $\xi \in \mathbf{C}$ and $u \in T_z^\perp$. For a pair of $\xi \in \mathbf{C}$ and $u \in T_z^\perp$, if $|\xi|^2 + |u|^2 < a_1^2$, then

$$F_z(\xi v_z + u) = z + (|p_z|/|z|)\xi e_z + u + P_z\tilde{H}_z(\xi v_z + u), \quad \text{where} \;\; e_z = z/|z|.$$

From (2.6) and (2.4) we see that $(DP_z\tilde{H}_z)(w) = O(|w|)$. Using Taylor expansion again, we see that are $a_0 \in (0, a_1)$ and $\delta > 0$ such that

$$|F_z(w) - F_z(w')| \geq \delta|w - w'| \quad \text{for} \;\; w, w' \in T_z \cap B(0, a_0).$$

By the standard inverse mapping theorem, F_z is biholomorphic on $T_z \cap B(0, a_0)$. Since $\tilde{G}_z$ is biholomorphic on $T_z \cap B(0, a_1)$, by the standard open mapping theorem and the compactness of K, there is a $b_0 > 0$ such that

$$\{\tilde{G}_z(w) : w \in T_z \cap B(0, a_0)\} \supset \mathcal{M} \cap B(z, b_0) \tag{2.7}$$

for every $z \in K$. Hence P_z is biholomorphic on $\mathcal{M} \cap B(z, b_0)$. The existence of $c_0 > 0$ is obtained by applying the open mapping theorem to the map P_z on $\mathcal{M} \cap B(z, b_0)$. □

For $z \in K$, let $I_z : T_z^{\mathrm{mod}} \cap B(z, c_0) \to \mathcal{M}$ be the inverse of P_z. For $x \in T_z^{\mathrm{mod}} \cap B(z, c_0)$, the relation $P_z I_z(x) = x$ leads to

$$I_z(x) = x + h_z(x), \quad \text{where} \;\; h_z(x) = I_z(x) - P_z I_z(x). \tag{2.8}$$

That is, for each $z \in K$, h_z maps $T_z^{\mathrm{mod}} \cap B(z, c_0)$ into $\mathbf{C}^n \ominus T_z^{\mathrm{mod}}$. We now fix a $0 < c_1 < c_0$ By the analysis in the proof of Lemma 2.7, there are constants $0 < \alpha(c_1) \leq \beta(c_1) < \infty$ such that the operator inequality

$$\alpha(c_1) \leq (DI_z)^*(x)(DI_z)(x) \leq \beta(c_1) \tag{2.9}$$

holds on the linear space T_z^{mod} for all $z \in K$ and $x \in T_z^{\mathrm{mod}} \cap B(z, c_1)$. Applying the standard open mapping theorem, there is a $0 < b_1 < b_0$ such that

$$\{I_z(x) : x \in T_z^{\mathrm{mod}} \cap B(z, c_1)\} \supset \mathcal{M} \cap B(z, b_1). \tag{2.10}$$

Lemma 2.8. *There is a constant* $0 < C_{2.8} < \infty$ *such that for every* $z \in K$, *if* $u \in T_z^\perp \cap B(0, c_1)$ *(cf. Definition* 2.6), *then* $|h_z(z + u)| \leq C_{2.8}|u|^2$.

Proof. Let such a pair of z and u be given. By (2.6) and (2.7), there is a $w \in T_z \cap B(0, a_0)$ such that $I_z(z + u) = \tilde{G}_z(w)$. Thus

$$z + u = P_z I_z(z + u) = P_z\tilde{G}_z(w) = z + P_z w + P_z\tilde{H}_z(w).$$

We can write w in the form $w = \xi v_z + \eta$ for some $\xi \in \mathbf{C}$ and $\eta \in T_z^\perp$. Hence $P_z w = \xi\langle v_z, e_z\rangle e_z + \eta$. Substituting this in the above, we find that

$$u = \xi\langle v_z, e_z\rangle e_z + \eta + P_z\tilde{H}_z(\xi v_z + \eta).$$

Taking the inner product with v_z on both sides and solving for ξ, we obtain

$$\xi = -\langle P_z\tilde{H}_z(\xi v_z + \eta), v_z\rangle/|\langle v_z, e_z\rangle|^2. \tag{2.11}$$

By (2.4) we have $\tilde{H}_z(x) = O(|x|^2)$. Thus when $|\xi|$ and $|\eta|$ are small enough, in order for (2.11) to hold, we have to have $|\xi| \le |\eta|$ at the very least. Consequently, $\xi = O(|\eta|^2)$ and $u - \eta = O(|\eta|^2)$. Thus $|\eta| = O(|u|)$ and $\xi = O(|u|^2)$. We have

$$\begin{aligned} z + u + h_z(z+u) = I_z(z+u) = \tilde{G}_z(w) &= z + \xi v_z + \eta + \tilde{H}_z(w) \\ &= z + \xi(v_z - \langle v_z, e_z\rangle e_z) + u + \tilde{H}_z(w) - P_z\tilde{H}_z(w). \end{aligned}$$

That is,

$$h_z(z+u) = \xi(v_z - \langle v_z, e_z\rangle e_z) + \tilde{H}_z(w) - P_z\tilde{H}_z(w).$$

Since $|\xi| \le |\eta|$, we have $\tilde{H}_z(w) = O(|w|^2) = O(|\xi v_z + \eta|^2) = O(|\eta|^2) = O(|u|^2)$. This completes the proof. $\square$

Lemma 2.9. (1) *Let $r > 0$ be given. For each $\epsilon > 0$, there is a $\delta = \delta(r, \epsilon) \in (0, 1)$ such that if $z \in K$ satisfies the condition $1 - \delta \le |z| < 1$, then the inequality*

$$\beta(w, P_z w) \le \epsilon$$

holds for every $w \in D(z, r) \cap \mathcal{M}$.
(2) *Let $z \in M \cap K$ and $r > 0$ be such that $D(z, r/2) \subset B(z, c_0)$ and $\beta(w, P_z w) \le r/3$ for every $w \in D(z, 2r) \cap M$. Then $I_z(D(z, r/2) \cap T_z^{\mathrm{mod}}) \subset D(z, r) \cap M$.*

Proof. (1) We know that for a fixed $r > 0$, the Euclidean diameter of $D(z, r)$ tends to 0 as $|z| \uparrow 1$. By (2.10), for $z \in \mathbf{B} \cap \mathcal{M}$ that is sufficiently close to S, once a $w \in D(z, r) \cap \mathcal{M}$ is given, we can write it in the form $w = I_z(x)$ for some $x \in T_z^{\mathrm{mod}} \cap B(z, c_1)$. We have $x = P_z I_z(x) = P_z w$. That is, $w = I_z(P_z w) = P_z w + h_z(P_z w)$.

Now (2.1) gives us

$$\varphi_{P_z w}(w) = -(1 - |P_z w|^2)^{-1/2}(w - P_z w) = -(1 - |P_z w|^2)^{-1/2} h_z(P_z w).$$

We have $P_z w = \langle w, e_z\rangle e_z + u$, where $e_z = z/|z|$ and $u \in T_z^\perp$. If we set $\zeta = z + u$, then

$$|\varphi_{P_z w}(w)| \le (1 - |w|^2)^{-1/2}\{|h_z(P_z w) - h_z(\zeta)| + |h_z(\zeta)|\}.$$

Since $\zeta = z + u$ with $u \in T_z^\perp$, Lemma 2.8 tells us that

$$|h_z(\zeta)| \le C_{2.8}|u|^2 = C_{2.8}|P_z w - \langle w, e_z\rangle e_z|^2 \le C_{2.8}|w - \langle w, e_z\rangle e_z|^2.$$

On the other hand, we obviously have

$$|h_z(P_zw) - h_z(\zeta)| \le C_1|P_zw - \zeta| = C_1|\langle w, e_z\rangle e_z - z|.$$

Therefore

$$|\varphi_{P_zw}(w)| \le C_2(1-|w|^2)^{-1/2}\{|z - \langle w, e_z\rangle e_z| + |w - \langle w, e_z\rangle e_z|^2\}. \tag{2.12}$$

Using (2.1) again, we have

$$\frac{|z - \langle w, e_z\rangle e_z|}{|1 - \langle w, z\rangle|} \le |\varphi_z(w)| \le 1.$$

Combining this with the well-known identity

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1 - \langle w, z\rangle|^2}$$

[26, Theorem 2.2.2], we obtain

$$\frac{|z - \langle w, e_z\rangle e_z|}{(1-|w|^2)^{1/2}} \le \frac{(1-|z|^2)^{1/2}}{(1-|\varphi_z(w)|^2)^{1/2}}. \tag{2.13}$$

Similarly, from (2.1) we obtain

$$\frac{1-|z|^2}{|1-\langle w, z\rangle|^2}|w - \langle w, e_z\rangle e_z|^2 \le |\varphi_z(w)|^2 \le 1.$$

Consequently

$$\frac{|w - \langle w, e_z\rangle e_z|^2}{(1-|w|^2)^{1/2}} \le \frac{(1-|w|^2)^{1/2}}{1-|\varphi_z(w)|^2} \le C(r)\frac{(1-|z|^2)^{1/2}}{1-|\varphi_z(w)|^2},$$

where the second $\le$ follows from the fact that $\beta(z, w) < r$. Combining this with (2.13) and (2.12), we obtain the inequality

$$|\varphi_{P_zw}(w)| \le C_3(r)\frac{(1-|z|^2)^{1/2}}{1-|\varphi_z(w)|^2}. \tag{2.14}$$

The condition $\beta(z, w) < r$ obviously means that $1 - |\varphi_z(w)|^2 \ge c(r)$ for some $c(r) > 0$ that depends only on r. Substituting this lower bound in (2.14), (1) is proved.

(2) Suppose that there were some $x^* \in D(z, r/2) \cap T_z^{\rm mod}$ such that $\beta(z, I_z(x^*)) \ge r$. We will show that this leads to a contradiction. Since $x^* \in D(z, r/2) \cap T_z^{\rm mod}$, there is a geodesic $\gamma : [0, 1] \to D(z, r/2) \cap T_z^{\rm mod}$ with respect to the Bergman metric on $T_z^{\rm mod}$ such that $\gamma(0) = z$ and $\gamma(1) = x^*$. Since $\beta(z, I_z(\gamma(1))) = \beta(z, I_z(x^*)) \ge r$, there is a $t_0 \in [0, 1]$ such that $\beta(z, I_z(\gamma(t_0))) = r$. By the assumption on z and r, we have

$$\beta(I_z(\gamma(t_0)), \gamma(t_0)) = \beta(I_z(\gamma(t_0)), P_zI_z(\gamma(t_0))) \le r/3.$$

Therefore $\beta(z,\gamma(t_0)) \geq \beta(z,I_z(\gamma(t_0))) - \beta(I_z(\gamma(t_0)),\gamma(t_0)) \geq r-(r/3) = 2r/3$, which contradicts the fact that $\gamma(t_0) \in D(z,r/2)$. □

For every $z \in K$, T_z^{mod} is a d-dimensional linear subspace of $\mathbf{C}^n$. For convenience we will write v for the natural volume measure on T_z^{mod}, even though for different $z \in K$ this may be a different linear subspace of $\mathbf{C}^n$. But since volume depends only on the Euclidean metric, which T_z^{mod} inherits from $\mathbf{C}^n$, such a simplification of notation is justified.

For each $z \in K$, we have the Jacobian

$$J_z(x) = \det\{(DI_z)^*(x)(DI_z)(x)\}, \tag{2.15}$$

$x \in T_z^{\text{mod}} \cap B(z,c_1)$. Let $v_{\mathcal{M}}$ denote the natural volume measure on $\mathcal{M}$. Suppose that $z \in K$ and U is an open set in $\mathcal{M} \cap B(z,b_1)$. By (2.10), we have $P_zU \subset T_z^{\text{mod}} \cap B(z,c_1)$. For any positive, continuous function f on U, we have

$$\int_U f(w)dv_{\mathcal{M}}(w) = \int_{P_zU} f(I_z(x))J_z(x)dv(x). \tag{2.16}$$

As we recall, this is in fact how volume is *defined* on $\mathcal{M}$.

In addition to the volume measure $v_{\mathcal{M}}$ on $\mathcal{M}$, we define the measure v_M on $M = \tilde{M} \cap \mathbf{B}$ by the formula $v_M(E) = v_{\mathcal{M}}(E \cap \mathcal{M})$ for Borel sets $E \subset M$.

Lemma 2.10. *Given any $a > 0$ and $\kappa > -1$, there is a $0 < C_{2.10} < \infty$ such that*

$$\int_M \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}}dv_M(w) \leq C_{2.10}$$

for every $z \in M$.

Proof. (1) First we suppose that $z \in M \cap K$. Recalling (2.10), let $0 < b_2 < b_1$ be a number whose exact value will be determined below. With this b_2 we have

$$\int_M \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}}dv_M(w) = A(z) + B(z),$$

where

$$A(z) = \int_{M\cap\mathcal{M}\cap B(z,b_2)} \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}}dv_{\mathcal{M}}(w) \quad \text{and}$$
$$B(z) = \int_{M\backslash\{\mathcal{M}\cap B(z,b_2)\}} \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}}dv_M(w).$$

We estimate $A(z)$ and $B(z)$ separately.

For $A(z)$, note that every $x \in T_z^{\text{mod}}$ has the representation $x = (\xi_1 + i\xi_2)z + u$, where $\xi_1, \xi_2 \in \mathbf{R}$ and $u \in T_z^\perp$. We will identify the vector u with its real version. Then

$$(|z|\xi_1, |z|\xi_2, u)$$

is a set of $2d$-dimensional real coordinates for $x = (\xi_1 + i\xi_2)z + u \in T_z^{\mathrm{mod}} \cap B(z, c_1)$. Let $0 < c_2 < c_1$ be a number whose exact value will be determined below. Define

$$U = \{(|z|\xi_1, |z|\xi_2, u) : (\xi_1 + i\xi_2)z + u \in T_z^{\mathrm{mod}} \cap B(z, c_2)\},$$

and let L be the $2d$-dimensional real linear space that is the linear span of U. We now define the map

$$F : U \to L$$

by the formula

$$F(|z|\xi_1, |z|\xi_2, u) = (1 - |I_z((\xi_1 + i\xi_2)z + u)|^2, |z|\xi_2, u). \tag{2.17}$$

We claim that if c_2 is small enough, then there are $0 < \alpha \le \beta < \infty$ such that

$$\alpha \le \left| \frac{\partial}{\partial \xi_1} |I_z((\xi_1 + i\xi_2)z + u)|^2 \right| \le \beta \tag{2.18}$$

for $(|z|\xi_1, |z|\xi_2, u) \in U$. To prove this, we use (2.8), which tells us that $h_z(x) \perp x$. Hence

$$|I_z((\xi_1 + i\xi_2)z + u)|^2 = (\xi_1^2 + \xi_2^2)|z|^2 + |u|^2 + |h_z((\xi_1 + i\xi_2)z + u)|^2.$$

Consequently,

$$\frac{\partial}{\partial \xi_1} |I_z((\xi_1 + i\xi_2)z + u)|^2 = 2\xi_1|z|^2 + \frac{\partial}{\partial \xi_1} |h_z((\xi_1 + i\xi_2)z + u)|^2.$$

Since $P_z z = z$, we have $I_z(z) = z$, i.e., $h_z(z) = 0$. Thus the second term on the right-hand side is of the form $O(|(\xi_1 - 1 + i\xi_2)z + u|)$. For the first term on the right-hand side, recall that for this part we assume $z \in M \cap K$. Hence (2.18) holds if c_2 is small enough.

We now apply the inverse mapping theorem to F. Reducing the value of c_2 if necessary, we may assume that FU is open and that the map $F : U \to FU$ is invertible. Furthermore, from (2.18) we deduce that there is a $0 < C_1 < \infty$ such that

$$|\det\{(DF^{-1})(y)\}| \le C_1 \quad \text{for every} \quad y \in FU, \tag{2.19}$$

where $F^{-1} : FU \to U$ is the inverse of F.

With c_2 determined in the above, the open mapping theorem provides a $0 < b_2 < b_1$ such that

$$\{I_z(x) : x \in T_z^{\mathrm{mod}} \cap B(z, c_2)\} \supset \mathcal{M} \cap B(z, b_2). \tag{2.20}$$

We emphasize that these constants are determined by the property of the manifold $\mathcal{M}$ and are independent of the $z \in K$ that we are considering.

Having found the desired b_2, we will now estimate $A(z)$. By (2.20), there is an open set $V(z) \subset T_z^{\text{mod}} \cap B(z, c_2)$ such that $I_z V(z) = M \cap \mathcal{M} \cap B(z, b_2)$. By (2.16), we have

(2.21)
$$A(z) = \int_{I_z V(z)} \Phi(w) dv_{\mathcal{M}}(w) = \int_{V(z)} \Phi(I_z(x)) J_z(x) dv(x) \le C_2 \int_{V(z)} \Phi(I_z(x)) dv(x),$$

where
$$\Phi(w) = \frac{(1-|z|^2)^a (1-|w|^2)^\kappa}{|1-\langle w, z\rangle|^{d+1+a+\kappa}}.$$

Let $x = (\xi_1 + i\xi_2)z + u \in T_z^{\text{mod}} \cap B(z, c_2)$, where $\xi_1, \xi_2 \in \mathbf{R}$ and $u \in T_z^\perp$. By (2.8), we have
$$|z - I_z((\xi_1 + i\xi_2)z + u)|^2 = |(1 - \xi_1 - i\xi_2)z|^2 + |u|^2 + |h_z((\xi_1 + i\xi_2)z + u)|^2$$

and $\langle I_z(x), z\rangle = \langle x, z\rangle$. Thus from the identity
$$4|1 - \langle w, z\rangle|^2 = (1 - |z|^2 + 1 - |w|^2 + |z - w|^2)^2 + 4(\text{Im}\langle w, z\rangle)^2$$

we deduce

(2.22) $$8|1-\langle I_z((\xi_1+i\xi_2)z+u), z\rangle| \ge 1-|z|^2+1-|I_z((\xi_1+i\xi_2)z+u)|^2+|u|^2+2|\xi_2||z|^2.$$

On the linear space L we define the function
$$G(t, |z|\xi_2, u) = \frac{(1-|z|^2)^a t^\kappa}{(1-|z|^2+t+|z||\xi_2|+|u|^2)^{d+1+a+\kappa}}.$$

From (2.17) and (2.22) we obtain
$$\Phi(I_z((\xi_1 + i\xi_2)z + u)) \le C_4 G(F(|z|\xi_1, |z|\xi_2, u)).$$

Write $\tilde{V}(z) = \{(|z|\xi_1, |z|\xi_2, u) : (\xi_1 + i\xi_2)z + u \in V(z)\}$. Continuing with (2.21), we have
$$A(z) \le C_2 C_4 \int_{\tilde{V}(z)} G(F(|z|\xi_1, |z|\xi_2, u)) dv(|z|\xi_1, |z|\xi_2, u)$$

(2.23) $$= C_2 C_4 \int_{F\tilde{V}(z)} G(y) dv(F^{-1}(y)) \le C_2 C_4 C_1 \int_{F\tilde{V}(z)} G(y) dv(y),$$

where the second $\le$ follows from (2.19). Obviously,
$$\int_{F\tilde{V}(z)} G(y) dv(y) \le \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{2d-2}} \frac{2(1-|z|^2)^a t^\kappa}{(1-|z|^2+t+\xi_2+|u|^2)^{d+1+a+\kappa}} dm_{2d-2}(u) d\xi_2 dt,$$

where dm_{2d-2} denotes the Lebesgue measure on $\mathbf{R}^{2d-2}$, and where we assume $d > 1$. Using the radial-spherical coordinates on $\mathbf{R}^{2d-2}$, we have
$$\begin{aligned}\int_{F\tilde{V}(z)} G(y) dv(y) &\le C_5 \int_0^\infty \int_0^\infty \int_0^\infty \frac{(1-|z|^2)^a t^\kappa \rho^{2d-3}}{(1-|z|^2+t+\xi_2+\rho^2)^{d+1+a+\kappa}} d\rho d\xi_2 dt \\ &= C_6 \int_0^\infty \int_0^\infty \frac{(1-|z|^2)^a t^\kappa}{(1-|z|^2+t+\xi_2)^{2+a+\kappa}} d\xi_2 dt \\ &= C_7 \int_0^\infty \frac{(1-|z|^2)^a t^\kappa}{(1-|z|^2+t)^{1+a+\kappa}} dt = C_7 \int_0^\infty \frac{s^\kappa}{(1+s)^{1+a+\kappa}} ds,\end{aligned}$$

where the last step is the substitution $s = t/(1-|z|^2)$. Since $a > 0$ and $\kappa > -1$, the s-integral above is finite. Combining this with (2.23), we find that $A(z)$ is bounded in the case $d > 1$. In the case $d = 1$, we omit the integral on $\mathbf{R}^{2d-2}$ and the rest of the argument is still valid. Hence $A(z)$ is bounded on $M \cap K$ in all cases of $1 \le d \le n-1$.

As for $B(z)$, observe that once b_2 is fixed, we have

$$B(z) \le C_8 \int_M (1-|w|^2)^\kappa dv_M(w).$$

By (2.18), the function $1-|w|^2$ serves as one of the $2d$ real coordinates for $w \in M$ near S. Hence the above integral is finite. This proves the desired bound on $B(z)$. Thus the lemma is proved for $z \in M \cap K$.

(2) Suppose that $z \in M\backslash K$. For such a z we obviously have

$$\int_M \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le C_9 \int_M (1-|w|^2)^\kappa dv_M(w).$$

As we have already explained, the right-hand side is finite. This completes the proof of the lemma. □

Lemma 2.11. *Given any $a > 0$ and $\kappa > -1$, there are $\delta > 0$ and $0 < C_{2.11}(\delta) < \infty$ such that*

$$\int_{M\backslash D(z,r)} \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le C_{2.11}(\delta)e^{-2\delta r} \tag{2.24}$$

for all $z \in M$ and $r > 0$.

Proof. Given any $a > 0$ and $\kappa > -1$, we pick a $\delta > 0$ such that the quantities $a' = a - \delta$ and $\kappa' = \kappa - \delta$ also satisfy the conditions $a' > 0$ and $\kappa' > -1$. We have

$$\frac{(1-|z|^2)^\delta(1-|w|^2)^\delta}{|1-\langle w,z\rangle|^{2\delta}} = (1-|\varphi_z(w)|^2)^\delta \le 4^\delta e^{-2\delta\beta(z,w)}.$$

Thus from the factorization

$$\frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} = \frac{(1-|z|^2)^\delta(1-|w|^2)^\delta}{|1-\langle w,z\rangle|^{2\delta}} \cdot \frac{(1-|z|^2)^{a'}(1-|w|^2)^{\kappa'}}{|1-\langle w,z\rangle|^{d+1+a'+\kappa'}}$$

we obtain

$$\int_{M\backslash D(z,r)} \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le 4^\delta e^{-2\delta r} \int_M \frac{(1-|z|^2)^{a'}(1-|w|^2)^{\kappa'}}{|1-\langle w,z\rangle|^{d+1+a'+\kappa'}} dv_M(w).$$

Applying Lemma 2.10 with the values $a' > 0$ and $\kappa' > -1$, (2.24) is proved. □

Definition 2.12. We define the measure μ on M by the formula

$$d\mu(w) = (1 - |w|^2)^{n-1-d} dv_M(w). \tag{2.25}$$

We further extend μ to a measure on $\mathbf{B}$ by setting $\mu(\mathbf{B}\backslash M) = 0$.

Proposition 2.13. *The μ defined above is a Carleson measure for the Hardy space $H^2(S)$.*

Proof. For each pair of $z \in \overline{\mathbf{B}}$ and $r > 0$, define

$$Q(z, r) = \{w \in \mathbf{B} : |1 - \langle w, z\rangle| < r\}.$$

To show that μ is a Carleson measure for $H^2(S)$, it suffices to find a C such that

$$\mu(Q(\zeta, r)) \le Cr^n \tag{2.26}$$

for all $\zeta \in S$ and $r > 0$. See [8,20]. Here, because the power $n - 1 - d$ in (2.25) is non-negative, we do not need to use $1 - |w|^2$ as a coordinate, which saves a lot of trouble.

Let $\zeta \in S$ and $r > 0$ be given. If $Q(\zeta, r)\cap M = \emptyset$, then $\mu(Q(\zeta, r)) = 0$. If $Q(\zeta, r)\cap M \ne \emptyset$, pick a $z \in Q(\zeta, r) \cap M$. Recall that the quantity $d(u, v) = |1 - \langle u, v\rangle|^{1/2}$ satisfies the triangle inequality on the closed ball $\overline{\mathbf{B}}$ [26]. Hence $Q(\zeta, r) \subset Q(z, 4r)$ and, consequently,

$$\mu(Q(\zeta, r)) \le \int_{M\cap Q(z,4r)} (1 - |w|^2)^{n-1-d} dv_M(w).$$

It suffices to prove (2.26) for $r > 0$ that is sufficiently small. Obviously, there is a $\rho > 0$ such that if $0 < r \le \rho$, then $Q(\zeta, r) \cap M \subset K$ and $Q(z, 4r) \cap M \subset \mathcal{M} \cap B(z, b_1)$. Suppose that r satisfies the condition $0 < r \le \rho$. Then we can apply (2.10) and (2.16) to obtain

$$\mu(Q(\zeta, r)) \le \int_{P_z\{Q(z,4r)\cap M\}} (1 - |I_z(x)|^2)^{n-1-d} J_z(x) dv(x).$$

As we recall, $I_z(x) = x + h_z(x)$ and $h_z(x) \perp x$. Hence $1 - |I_z(x)|^2 \le 1 - |x|^2$. Recalling (2.9) and using the fact that $n - 1 - d \ge 0$, we now have

$$\mu(Q(\zeta, r)) \le C_2 \int_{P_z\{Q(z,4r)\cap M\}} (1 - |x|^2)^{n-1-d} dv(x).$$

Since $\langle w, z\rangle = \langle P_z w, z\rangle$, we have $P_z\{Q(z, 4r) \cap M\} \subset Q_z(z, 4r)$, where $Q_z(z, 4r) = \{x \in T_z^{\text{mod}} : |1 - \langle x, z\rangle| < 4r$ and $|x| < 1\}$. Therefore

$$\mu(Q(\zeta, r)) \le C_2 \int_{Q_z(z,4r)} (1 - |x|^2)^{n-1-d} dv(x).$$

Note that the condition $z \in Q(\zeta, r) \cap M$ implies $1 - |z| < r$. Since T_z^{mod} is a copy of $\mathbf{C}^d$, by a standard exercise, the integral on the right-hand side is dominated by $C_3 r^n$. $\square$

Also by a standard exercise, for each $r > 0$, there are $0 < c(r) \leq C(r) < \infty$ such that

$$c(r)(1 - |z|^2)^{d+1} \leq v(D(z, r) \cap T_z^{\text{mod}}) \leq C(r)(1 - |z|^2)^{d+1} \tag{2.27}$$

for every $z \in M \cap K$.

Proposition 2.14. (a) *For each $r \geq 1$, there exist $0 < c_{2.14}(r) \leq C_{2.14}(r) < \infty$ such that for every $z \in M \cap K$, we have*

$$c_{2.14}(r)(1 - |z|^2)^{d+1} \leq v_M(D(z, r)) \leq C_{2.14}(r)(1 - |z|^2)^{d+1}. \tag{2.28}$$

(b) *For each $r \geq 1$, there exist $0 < c'_{2.14}(r) \leq C'_{2.14}(r) < \infty$ such that for every $z \in M \cap K$, we have*

$$c'_{2.14}(r)(1 - |z|^2)^n \leq \mu(D(z, r)) \leq C'_{2.14}(r)(1 - |z|^2)^n.$$

Proof. (a) Let $r \geq 1$ be given. It suffices to find a $0 < \rho(r) < 1$ and $0 < c_{2.14}(r) \leq C_{2.14}(r) < \infty$ such that (2.28) holds for $z \in M$ satisfying the condition $|z| \geq \rho(r)$.

By definition, we have $K \supset \{z \in M : |z| \geq \rho_1\}$ for some $\rho_1 < 1$. By Lemma 2.9(1), there is a $\rho_2 < 1$ such that if $z \in M$ and $|z| \geq \rho_2$, then

$$\beta(w, P_z w) \leq r/5 \quad \text{for every} \quad w \in D(z, 2r) \cap M. \tag{2.29}$$

There is a $\rho_3 < 1$ such that if $\rho_3 \leq |z| < 1$, then $D(z, 2r) \subset B(z, \min\{b_1, c_1\})$ (cf. (2.10)). Set $\rho(r) = \max\{\rho_1, \rho_2, \rho_3\}$. Let $z \in M$ be such that $|z| \geq \rho(r)$. By (2.16), we have

$$v_M(D(z, r)) = \int_{P_z\{D(z,r) \cap M\}} J_z(x) dv(x). \tag{2.30}$$

We have (2.9) to bound $J_z(x)$, and (2.29) tells us that $P_z D(z, r) \subset D(z, 2r)$. Hence

$$v_M(D(z, r)) \leq C_1 v(D(z, 2r) \cap T_z^{\text{mod}}) \leq C_2(r)(1 - |z|^2)^{d+1},$$

proving the upper bound in (2.28).

To prove the lower bound in (2.28), we recall Lemma 2.9(2), which says

$$I_z(D(z, r/2) \cap T_z^{\text{mod}}) \subset D(z, r) \cap M.$$

That is, $P_z\{D(z, r) \cap M\} \supset D(z, r/2) \cap T_z^{\text{mod}}$. Continuing with (2.30) and recalling (2.9), we find that

$$v_M(D(z, r)) \geq \int_{D(z,r/2) \cap T_z^{\text{mod}}} J_z(x) dv(x) \geq c_1 v(D(z, r/2) \cap T_z^{\text{mod}}) \geq c_2(r)(1 - |z|^2)^{d+1},$$

which proves the lower bound in (2.28) and completes the proof of (a).

(b) Given any $r > 0$, there are $0 < c(r) \leq C(r) < \infty$ such that

$$c(r)(1 - |z|^2) \leq 1 - |w|^2 \leq C(r)(1 - |z|^2)$$

for every pair of $z \in \mathbf{B}$ and $w \in D(z, r)$. By this inequality, (b) follows from (a). □

3. Measure μ and the corresponding Toeplitz operator

With the measure μ in Definition 2.12, we define the Toeplitz operator T_μ on the Hardy space $H^2(S)$ by the formula

$$(T_\mu f)(z) = \int \frac{f(w)}{(1 - \langle z, w \rangle)^n} d\mu(w),$$

$f \in H^2(S)$. It is straightforward to verify that we can also write T_μ as

$$T_\mu = \int K_w \otimes K_w d\mu(w), \tag{3.1}$$

where $K_w(z) = (1 - \langle z, w \rangle)^{-n}$, the reproducing kernel for $H^2(S)$. Thus T_μ is a positive operator with

$$\langle T_\mu f, f \rangle = \int |f(w)|^2 d\mu(w)$$

for each $f \in H^2(S)$. By Proposition 2.13, the Toeplitz operator T_μ is bounded. If we consider each K_w as a vector in $L^2(S, d\sigma)$, then (3.1) automatically extends T_μ to an operator on $L^2(S, d\sigma)$.

In our next lemma, a subscript d indicates a set in $\mathbf{C}^d$. For example, $\mathbf{B}_d = \{w \in \mathbf{C}^d : |w| < 1\}$ and $D_d(z, r) = \{w \in \mathbf{B}_d : \beta(z, w) < r\}$. Let dv be the volume measure on $\mathbf{C}^d$.

Lemma 3.1. *If f is an analytic function on $\mathbf{B}_d$, then*

$$\int_{D_d(z,r)} f(w) \frac{(1 - |w|^2)^{n-1-d}}{(1 - \langle z, w \rangle)^n} dv(w) = C(d, r) f(z) \tag{3.2}$$

for every $z \in \mathbf{B}_d$ and every $r > 0$, where

$$C(d, r) = \int_{D_d(0,r)} (1 - |\zeta|^2)^{n-1-d} dv(\zeta).$$

Proof. Let $w = \varphi_z(\zeta)$. By the formulas from [26, Theorem 2.2.2], we have

$$1 - \langle z, \varphi_z(\zeta) \rangle = \frac{1 - |z|^2}{1 - \langle z, \zeta \rangle} \quad \text{and} \quad 1 - |\varphi_z(\zeta)|^2 = \frac{(1 - |z|^2)(1 - |\zeta|^2)}{|1 - \langle z, \zeta \rangle|^2}.$$

Therefore the left-hand side of (3.2) equals

$$\int_{D_d(0,r)} f(\varphi_z(\zeta))\left(\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\langle z,\zeta\rangle|^2}\right)^{n-1-d}\left(\frac{1-\langle z,\zeta\rangle}{1-|z|^2}\right)^n \frac{(1-|z|^2)^{d+1}}{|1-\langle z,\zeta\rangle|^{2d+2}}dv(\zeta).$$

After the obvious cancellation, we find that

$$\int_{D_d(z,r)} f(w)\frac{(1-|w|^2)^{n-1-d}}{(1-\langle z,w\rangle)^n}dv(w) = \int_{D_d(0,r)} \frac{f(\varphi_z(\zeta))}{(1-\langle \zeta,z\rangle)^n}(1-|\zeta|^2)^{n-1-d}dv(\zeta).$$

With respect to the Euclidean metric, $D_d(0,r)$ is also a ball centered at 0. Hence the above equals $C(d,r)f(\varphi_z(0))(1-\langle 0,z\rangle)^{-n} = C(d,r)f(z)$. □

Lemma 3.2. *For each given* $0 < r < \infty$, *we have*

$$\lim_{t\uparrow 1}\sup\left\{\left|1-\frac{1-|x|^2}{1-|I_z(x)|^2}\right| : |z|\ge t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\rm mod}\right\} = 0 \tag{3.3}$$

and

$$\lim_{t\uparrow 1}\sup\{|J_z(z)-J_z(x)| : |z|\ge t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\rm mod}\} = 0. \tag{3.4}$$

Proof. By Lemma 2.9, if $|z|$ is sufficiently close to 1, then $I_z(x)\in D(z,2r)\cap M$ for every $x\in D(z,r)\cap T_z^{\rm mod}$. Since $P_zI_z(x) = x$, it now follows from Lemma 2.9 that

$$\lim_{t\uparrow 1}\sup\{\beta(I_z(x),x) : |z|\ge t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\rm mod}\} = 0. \tag{3.5}$$

On the other hand, for any pair of $a,b\in\mathbf{B}$, if we write $c=\varphi_a(b)$, then $b=\varphi_a(c)$ and

$$\frac{1-|a|^2}{1-|b|^2} = \frac{|1-\langle a,c\rangle|^2}{1-|c|^2} = 1+O(|c|)$$

when $|c|$ is small. Since $\beta(0,c)=\beta(a,b)$, we see that (3.5) implies (3.3).

With the $a,b\in\mathbf{B}$ and $c=\varphi_a(b)$, we also have

$$|a-b|^2 \le 2|1-\langle a,b\rangle| = 2\frac{1-|a|^2}{|1-\langle a,c\rangle|}.$$

Hence for any given $0<r<\infty$,

$$\lim_{t\uparrow 1}\sup\{|z-x| : |z|\ge t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\rm mod}\} = 0. \tag{3.6}$$

Recall that $J_z(x) = \det\{(DI_z)^*(x)(DI_z)(x)\}$. By the construction in Section 2, the continuity of the map $x\mapsto DI_z(x)$ is uniform as z varies over K. Obviously, (3.4) follows from this uniform continuity and (3.6). □

Lemma 3.3. *Define the operators B and B_r on $L^2(M, d\mu)$ by the formulas*

$$(Bf)(z) = \int_M \frac{f(w)}{|1-\langle z,w\rangle|^n} d\mu(w) \quad \textit{and}$$
$$(B_r f)(z) = \int_{M\setminus D(z,r)} \frac{f(w)}{|1-\langle z,w\rangle|^n} d\mu(w)$$

for $f \in L^2(M, d\mu)$, $r > 0$. Then $\|B\| < \infty$ and $\|B_r\| \to 0$ as $r \to \infty$.

Proof. We set $a = 1/2$ and $\kappa = n - 1 - d - (1/2)$. Then $\kappa \geq -1/2$ and we have $n = d + 1 + a + \kappa$. Define $h(w) = (1-|w|^2)^{-1/2}$, $w \in M$. Then

$$(B_r h)(z) = \int_{M\setminus D(z,r)} \frac{(1-|w|^2)^\kappa}{|1-\langle z,w\rangle|^{d+1+a+\kappa}} dv_M(w).$$

By Lemma 2.11, we have $(B_r h)(z) \leq C_{2.11}(\delta)e^{-2\delta r}(1-|z|^2)^{-a} = C_{2.11}(\delta)e^{-2\delta r}h(z)$, $z \in M$. Since the kernel function $|1-\langle z,w\rangle|^{-n}$ is symmetric with respect to z and w, we can now apply the Schur test to conclude that $\|B_r\| \leq C_{2.11}(\delta)e^{-2\delta r}$. Hence $\|B_r\| \to 0$ as $r \to \infty$.

Similarly, by Lemma 2.10 we have $(Bh)(z) \leq C_{2.10}h(z)$, $z \in M$. Thus it follows from the Schur test that $\|B\| \leq C_{2.10}$. This completes the proof. □

Proposition 3.4. *There is a $c_{3.4} > 0$ such that the operator inequality*

$$T_\mu^2 \geq c_{3.4} T_\mu \tag{3.7}$$

holds on $L^2(S, d\sigma)$.

Proof. For each $0 < t < 1$ we define

$$M^{(t)} = \{z \in M : 1 - |z|^2 < t\}.$$

There is a $\tau_0 > 0$ such that if $0 < t \leq \tau_0$, then $M^{(t)} \subset K$. We will show that there is a small enough $t > 0$ such that the inequality

$$\int_{M^{(t)}} |(T_\mu f)(z)|^2 d\mu(z) + \frac{\delta}{2}\int_M |f(w)|^2 d\mu(w) \geq \delta \int_{M^{(t)}} |f(z)|^2 d\mu(z) \tag{3.8}$$

holds for a constant $\delta > 0$ and for all $f \in H^2(S)$.

We begin with the choice of δ. By (2.9), there is an $a > 0$ such that

$$J_z(z) \geq a \tag{3.9}$$

for every $z \in K$. We set

$$C(d) = \int_{\mathbf{B}_d} (1-|\zeta|^2)^{n-1-d} dv(\zeta) \quad \text{and} \quad \delta = \frac{\{aC(d)/2\}^2}{3}.$$

There is an $R > 0$ such that if $r \geq R$, then $C(d, 2r) \geq C(d)/2$ (cf. Lemma 3.1). That is, if $r \geq R$, then

$$\{aC(d, 2r)\}^2/3 \geq \delta. \tag{3.10}$$

Lemma 3.3 allows us to pick an $r \geq R$ such that

$$\|B_r\|^2 \leq \delta/4. \tag{3.11}$$

With r so fixed, there is a $0 < \tau_1 \leq \tau_0$ such that if $0 < t \leq \tau_1$, then for $z \in M^{(t)}$ we have $D(z, 2r) \subset B(z, \min\{b_1, c_1\})$ (cf. (2.10)). By Lemma 2.9(1), there is a $0 < \tau_2 \leq \tau_1$ such that if $0 < t \leq \tau_2$, then for $z \in M^{(t)}$ and $w \in D(z, r) \cap M$ we have $\beta(w, P_z w) < r$. Thus $P_z w \in D(z, 2r) \cap T_z^{\mathrm{mod}}$ and $I_z(P_z w) = w \in D(z, r) \cap M$. That is, if $0 < t \leq \tau_2$, then

$$I_z(D(z, 2r) \cap T_z^{\mathrm{mod}}) \supset D(z, r) \cap M \quad \text{for every} \quad z \in M^{(t)}. \tag{3.12}$$

We write $U(z) = I_z(D(z, 2r) \cap T_z^{\mathrm{mod}})$ for $z \in M^{(t)}$. Let $f \in H^2(S)$ be given. Then

$$(T_\mu f)(z) = A(z) + B(z),$$

where

$$A(z) = \int_{U(z)} f(w) \frac{(1 - |w|^2)^{n-1-d}}{(1 - \langle z, w \rangle)^n} dv_M(w) \quad \text{and}$$
$$B(z) = \int_{M \setminus U(z)} f(w) \frac{(1 - |w|^2)^{n-1-d}}{(1 - \langle z, w \rangle)^n} dv_M(w),$$

$z \in M^{(t)}$. Since $P_z U(z) = D(z, 2r) \cap T_z^{\mathrm{mod}}$, by (2.16) we have

$$A(z) = \int_{D(z,2r) \cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{(1 - |I_z(x)|^2)^{n-1-d}}{(1 - \langle z, I_z(x) \rangle)^n} J_z(x) dv(x).$$

Recall from (2.8) that $\langle z, I_z(x) \rangle = \langle z, x \rangle$. Writing

$$F(z, x) = 1 - \left(\frac{1 - |x|^2}{1 - |I_z(x)|^2} \right)^{n-1-d} \cdot \frac{J_z(z)}{J_z(x)},$$

we have

$$A(z) = A_1(z) + A_2(z),$$

where

$$A_1(z) = J_z(z) \int_{D(z,2r) \cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{(1 - |x|^2)^{n-1-d}}{(1 - \langle z, x \rangle)^n} dv(x) \quad \text{and}$$
$$A_2(z) = \int_{D(z,2r) \cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{(1 - |I_z(x)|^2)^{n-1-d}}{(1 - \langle z, I_z(x) \rangle)^n} F(z, x) J_z(x) dv(x).$$

Being a local inverse of P_z, the map I_z is analytic. Therefore Lemma 3.1 tells us that

$$A_1(z) = C(d,2r)J_z(z)f(I_z(z)) = C(d,2r)J_z(z)f(z). \tag{3.13}$$

Define

$$\epsilon(r,t) = \sup_{z\in M^{(t)}} \Big\{ \sup_{x\in D(z,2r)\cap T_z^{\mathrm{mod}}} |F(z,x)| \Big\}.$$

Applying (2.16) again, we have

$$\begin{aligned} |A_2(z)| &\le \epsilon(r,t)\int_{D(z,2r)\cap T_z^{\mathrm{mod}}} |f(I_z(x))| \frac{(1-|I_z(x)|^2)^{n-1-d}}{|1-\langle z, I_z(x)\rangle|^n} J_z(x)dv(x) \\ &\le \epsilon(r,t)\int_M \frac{|f(w)|}{|1-\langle z,w\rangle|^n}(1-|w|^2)^{n-1-d}dv_M(w). \end{aligned}$$

Thus it follows from Lemma 3.3 that

$$\int_{M^{(t)}} |A_2(z)|^2 d\mu(z) \le \{\epsilon(r,t)\}^2\|B\|^2 \int_M |f(w)|^2 d\mu(w). \tag{3.14}$$

Finally, from (3.12) we obtain

$$|B(z)| \le \int_{M\setminus D(z,r)} \frac{|f(w)|}{|1-\langle z,w\rangle|^n} d\mu(w)$$

for $z \in M^{(t)}$. Using the operator B_r in Lemma 3.3, we have

$$\int_{M^{(t)}} |B(z)|^2 d\mu(z) \le \|B_r\|^2 \int_M |f(w)|^2 d\mu(w). \tag{3.15}$$

Recalling (3.13), for $z \in M^{(t)}$ we have

$$C(d,2r)J_z(z)f(z) = A_1(z) = (T_\mu f)(z) - A_2(z) - B(z).$$

Combining this with (3.9), (3.14) and (3.15), we see that

$$\begin{aligned} \frac{\{aC(d,2r)\}^2}{3}\int_{M^{(t)}} |f(z)|^2 d\mu(z) &\le \int_{M^{(t)}} |(T_\mu f)(z)|^2 d\mu(z) \\ &\quad + (\{\epsilon(r,t)\}^2\|B\|^2 + \|B_r\|^2)\int_M |f(w)|^2 d\mu(w). \end{aligned} \tag{3.16}$$

Since r is fixed, by (3.9), Lemma 3.2, and Lemma 3.3, we can pick a $0 < t \le \tau_2$ such that $\{\epsilon(r,t)\}^2\|B\|^2 \le \delta/4$. With this t, (3.8) follows from (3.16), (3.10) and (3.11).

Recall that v_M is concentrated on $\mathcal{M} \cap M = \mathcal{M} \cap \mathbf{B}$. If Δ is a compact set in $\mathcal{M} \cap M$, then Δ can be covered by open sets $U_1, \ldots, U_m$ in $\mathcal{M} \cap M$ such that each U_j is biholomorphically equivalent to $\mathbf{B}_d$. By the Bergman integral formula, there is a constant $0 < C(\Delta) < \infty$ such that

$$|g(z)|^2 \leq C(\Delta) \int_M |g(w)|^2 d\mu(w) \tag{3.17}$$

for all $g \in H^2(S)$ and $z \in \Delta$. Let $\mathcal{P}$ denote the closure of $H^2(S)$ in $L^2(M, d\mu)$.

By our choice of t, $\{w \in M : |w| = t\}$ is a compact subset of $\mathcal{M} \cap M$. As we mentioned before, Assumption 2.4(3) implies that $\tilde{M}$ has no isolated singularities in $\mathbf{B}$. Thus it follows from the maximum principle and (3.17) that there is a $0 < C_1 < \infty$ such that

$$\sup_{z \in M \backslash M^{(t)}} |g(z)|^2 \leq C_1 \int_M |g(w)|^2 d\mu(w) \tag{3.18}$$

for every $g \in H^2(S)$. Hence for each $z \in M \backslash M^{(t)}$, the map $g \mapsto g(z)$ extends to a linear functional on $\mathcal{P}$ whose norm is at most $C_1^{1/2}$. Thus if $\{u_k\}$ is a sequence in $\mathcal{P}$ that converges to 0 weakly, then

$$\lim_{k \to \infty} |u_k(z)| = 0 \tag{3.19}$$

for every $z \in M \backslash M^{(t)}$.

Let dE be the spectral measure for the positive operator T_μ. That is,

$$T_\mu = \int_0^{\|T_\mu\|} \lambda dE(\lambda).$$

Obviously, (3.7) is equivalent to the statement that there is a $c > 0$ such that $E(0, c) = 0$. Suppose that such a c did not exist. We will show that this leads to a contradiction. In fact, the supposed non-existence of such a c allows us to pick, for each $k \in \mathbf{N}$, a vector $f_k \in E(0, 1/k)H^2(S)$ such that $\langle T_\mu f_k, f_k \rangle = 1$. That is,

$$\int_M |f_k(w)|^2 d\mu(w) = 1. \tag{3.20}$$

Obviously, the sequence $\{T_\mu^{1/2} f_k\}$ weakly converges to 0 in $H^2(S)$. Let $R : H^2(S) \to L^2(M, d\mu)$ be the restriction operator. Then $R^* R = T_\mu$, and consequently $R = V T_\mu^{1/2}$ for some partial isometry V. Hence the sequence $\{Rf_k\}$ weakly converges to 0 in the space $\mathcal{P}$ introduced above. By (3.19), (3.18) and the dominated convergence theorem, we have

$$\lim_{k \to \infty} \int_{M \backslash M^{(t)}} |f_k(w)|^2 d\mu(w) = 0. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\lim_{k \to \infty} \int_{M^{(t)}} |f_k(w)|^2 d\mu(w) = 1. \tag{3.22}$$

Since $f_k \in E(0,1/k)H^2(S)$, we have $\langle T_\mu T_\mu f_k, T_\mu f_k\rangle \le k^{-2}\langle T_\mu f_k, f_k\rangle = k^{-2}$. Thus

$$\lim_{k\to\infty}\int_{M^{(t)}} |(T_\mu f_k)(z)|^2 d\mu(z) \le \lim_{k\to\infty}\langle T_\mu T_\mu f_k, T_\mu f_k\rangle = 0. \tag{3.23}$$

Substituting (3.20), (3.22) and (3.23) in (3.8), we see the contradiction that $\delta/2 \ge \delta$. Hence there is a $c > 0$ such that $E(0,c) = 0$, which proves the proposition. □

Theorem 3.5. *There are scalars $0 < c \le C < \infty$ such that the operator inequality*

$$cQ \le T_\mu \le CQ$$

holds on $L^2(S, d\sigma)$.

Proof. We already know from Proposition 2.13 that T_μ is bounded. Thus the upper bound $T_\mu \le CQ$ simply reflects the fact that range$(T_\mu) \subset \mathcal{Q}$, which is obviously true.

To prove the lower bound, we again consider the spectral decomposition

$$T_\mu = \int_0^{\|T_\mu\|} \lambda dE(\lambda)$$

of T_μ on $L^2(S, d\sigma)$. By Proposition 3.4 we have $T_\mu^2 \ge c_{3.4}T_\mu$, which implies $E(0, c_{3.4}) = 0$. Therefore

$$T_\mu \ge c_{3.4}E[c_{3.4}, \infty) = c_{3.4}E(0,\infty).$$

Thus the desired lower bound will follow if we can show that $E(0,\infty) = Q$, i.e., if we can show that range(T_μ) is dense in $\mathcal{Q}$. Equivalently, it suffices to show that $\{h \in \mathcal{Q} : T_\mu h = 0\} = \{0\}$. Let $h \in \mathcal{Q}$ be such that $T_\mu h = 0$. Using the $M^{(t)}$ in (3.18), the condition $\langle T_\mu h, h\rangle = 0$ implies that h vanishes on both $M^{(t)}$ and $M\backslash M^{(t)}$. That is, $h(w) = 0$ for every $w \in M$. This means that $h \perp \mathcal{Q}$. Since $h \in \mathcal{Q}$, h is the zero element. This proves the density of range(T_μ) in $\mathcal{Q}$ and completes the proof. □

4. Discrete sums

We will approximate the Toeplitz operator T_μ by discrete sums constructed from the reproducing kernel for $H^2(S)$.

Lemma 4.1. *There are constants $t_{4.1} > 0$ and $0 < C_{4.1} < \infty$ such that for every $z \in M$ satisfying the condition $1 - |z|^2 < t_{4.1}$ and every $f \in H^2(S)$, we have*

$$|f(z)| \le \frac{C_{4.1}}{(1-|z|^2)^{d+1}}\int_{D(z,1)\cap M} |f(u)|dv_M(u) \tag{4.1}$$

and

$$|f(z) - f(w)| \le C_{4.1}\frac{\beta(z,w)}{(1-|z|^2)^{d+1}}\int_{D(z,1)\cap M} |f(u)|dv_M(u) \tag{4.2}$$

if $w \in D(z, 1/4) \cap M$.

Proof. We pick a $t_{4.1} > 0$ satisfying the following four requirements:

(1) $M^{(t_{4.1})} = \{z \in M : 1 - |z|^2 < t_{4.1}\} \subset K$.

(2) If $z \in M^{(t_{4.1})}$ and $w \in D(z, 1/4)$, then $P_z w \in D(z, 1/3)$.

(3) For each $z \in M^{(t_{4.1})}$, we have $I_z(D(z, 1/2) \cap T_z^{\mathrm{mod}}) \subset D(z, 1) \cap M$.

(4) For each $z \in M^{(t_{4.1})}$, $D(z, 1) \subset B(z, \min\{b_2, c_2\})$.

Note that requirements (2) and (3) are justified by Lemma 2.9.

Let $f \in H^2(S)$ be given. Given a $z \in M^{(t_{4.1})}$, we define the analytic function $g(x) = f(I_z(x))$ on $T_z^{\mathrm{mod}} \cap D(z, 1)$ (cf. (4) above and (2.20)). We have

$$\begin{aligned} |f(z)| = |g(z)| = |g(\varphi_z(0))| &\leq C_1 \int_{D(0,1/2) \cap T_z^{\mathrm{mod}}} |g(\varphi_z(\zeta))| dv(\zeta) \\ &= C_1 \int_{D(z,1/2) \cap T_z^{\mathrm{mod}}} |g(x)| \frac{(1 - |z|^2)^{d+1}}{|1 - \langle x, z\rangle|^{2d+2}} dv(x) \\ &\leq \frac{C_1 C_2}{(1 - |z|^2)^{d+1}} \int_{D(z,1/2) \cap T_z^{\mathrm{mod}}} |f(I_z(x))| J_z(x) dv(x), \end{aligned}$$

where for the last step we use (2.9) and the fact that $1 - |z|^2 \leq 2|1 - \langle x, z\rangle|$. Applying (3) above and (2.16), we obtain

$$|f(z)| \leq \frac{C_1 C_2}{(1 - |z|^2)^{d+1}} \int_{D(z,1) \cap M} |f(u)| dv_M(u),$$

which proves (4.1).

To prove (4.2), consider any $z \in M^{(t_{4.1})}$ and $w \in D(z, 1/4) \cap M$. By (2), there is a $\xi \in D(z, 1/3) \cap T_z^{\mathrm{mod}}$ such that $w = I_z(\xi)$. Furthermore, there is an $\eta \in D(0, 1/3) \cap T_z^{\mathrm{mod}}$ such that $\xi = \varphi_z(\eta)$. Using the function $g(x) = f(I_z(x))$ again, we have

$$|f(z) - f(w)| = |g(\varphi_z(0)) - g(\varphi_z(\eta))| \leq C_3 \beta(0, \eta) \int_{D(0,1/2) \cap T_z^{\mathrm{mod}}} |g(\varphi_z(\zeta))| dv(\zeta),$$

where the $\leq$ follows from the fact that $|y| \approx \beta(0, y)$ for $y \in D(0, 1/3)$. Note that $\beta(0, \eta) = \beta(z, \xi) = \beta(z, P_z w)$. Since $\varphi_z(P_z w) = P_z \varphi_z(w)$, we have $\beta(z, P_z w) \leq \beta(z, w)$. Thus

$$|f(z) - f(w)| \leq C_3 \beta(z, w) \int_{D(0,1/2) \cap T_z^{\mathrm{mod}}} |g(\varphi_z(\zeta))| dv(\zeta). \tag{4.3}$$

In the proof for (4.1) above, we showed that

$$\int_{D(0,1/2) \cap T_z^{\mathrm{mod}}} |g(\varphi_z(\zeta))| dv(\zeta) \leq \frac{C_2}{(1 - |z|^2)^{d+1}} \int_{D(z,1) \cap M} |f(u)| dv_M(u).$$

Combining this with (4.3), (4.2) is proved. □

Lemma 4.2. *There is a constant* $0 < C_{4.2} < \infty$ *such that if* Γ *is a* 1*-separated set contained in* M *and if* $\{c_z : z \in \Gamma\}$ *is a bounded set of coefficients, then*

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes e_z \right\| \le C_{4.2} \sup_{z\in\Gamma} |c_z|,$$

where $\{e_z : z \in \Gamma\}$ *is any orthonormal set.*

Proof. There is an $\ell \in \mathbf{N}$ such that if Γ is a 1-separated set contained in M, then $\mathrm{card}(\Gamma \cap \{M \backslash M^{(t_{4.1})}\}) \le \ell$. Hence it suffices to consider a 1-separated set Γ contained in $M^{(t_{4.1})}$. Let such a Γ be given and denote

$$A = \sum_{z\in\Gamma} c_z k_z \otimes e_z.$$

For any $f \in H^2(S)$, we have

$$\|A^* f\|^2 = \sum_{z\in\Gamma} |c_z|^2 (1 - |z|^2)^n |f(z)|^2.$$

Applying Lemma 4.1, the Cauchy-Schwarz inequality and Proposition 2.14, we have

$$\begin{aligned}
\|A^* f\|^2 &\le C_1 \sum_{z\in\Gamma} |c_z|^2 (1 - |z|^2)^{n-1-d} \int_{D(z,1)\cap M} |f(u)|^2 dv_M(u) \\
&\le C_2 \sup_{z\in\Gamma} |c_z|^2 \sum_{z\in\Gamma} \int_{D(z,1)\cap M} |f(u)|^2 (1 - |u|^2)^{n-1-d} dv_M(u) \\
&\le C_2 \sup_{z\in\Gamma} |c_z|^2 \int_M |f(u)|^2 (1 - |u|^2)^{n-1-d} dv_M(u) \\
&= C_2 \sup_{z\in\Gamma} |c_z|^2 \langle T_\mu f, f\rangle \le C_2 \sup_{z\in\Gamma} |c_z|^2 \|T_\mu\| \|f\|^2.
\end{aligned}$$

Recalling Proposition 2.13, the conclusion of the lemma follows from this. □

We define the measure $d\lambda$ on M by the formula

$$d\lambda(w) = \frac{dv_M(w)}{(1 - |w|^2)^{d+1}}.$$

Obviously, this $d\lambda$ tries to mimic the Möbius invariant measure on the ball. But keep in mind that there are no Möbius transforms on M. Nonetheless, this $d\lambda$ has all the right properties for our analysis on M. In particular, we have the representation

$$T_\mu = \int_M k_w \otimes k_w d\lambda(w). \tag{4.4}$$

Proposition 4.3. *For each $0 < \epsilon < 1$, let Γ_ϵ be a subset of M that is maximal with respect to the property of being ϵ-separated. By a standard construction, there is a partition*

$$M = \bigcup_{w\in\Gamma_\epsilon} E_w \tag{4.5}$$

such that $D(z,\epsilon) \cap M \subset E_w \subset D(w, 2\epsilon) \cap M$ for every $w \in \Gamma_\epsilon$. Define the operator

$$T_\epsilon = \sum_{w\in\Gamma_\epsilon} \lambda(E_w) k_w \otimes k_w.$$

Then we have

$$\lim_{\epsilon\downarrow 0} \|T_\mu - T_\epsilon\| = 0.$$

Proof. Given (4.5), we partition the set Γ_ϵ in the form $\Gamma_\epsilon = G_\epsilon \cup H_\epsilon$, where

$$G_\epsilon = \{w \in \Gamma_\epsilon : E_w \cap \{M\backslash M^{(t_{4.1})}\} = \emptyset\} \quad \text{and}$$
$$H_\epsilon = \{w \in \Gamma_\epsilon : E_w \cap \{M\backslash M^{(t_{4.1})}\} \neq \emptyset\}.$$

Accordingly, we have the decomposition $T_\epsilon = V_\epsilon + W_\epsilon$, where

$$V_\epsilon = \sum_{w\in G_\epsilon} \lambda(E_w) k_w \otimes k_w \quad \text{and} \quad W_\epsilon = \sum_{w\in H_\epsilon} \lambda(E_w) k_w \otimes k_w.$$

Define the sets

$$A_\epsilon = \bigcup_{w\in G_\epsilon} E_w \quad \text{and} \quad B_\epsilon = \bigcup_{w\in H_\epsilon} E_w.$$

By (4.4), we have $T_\mu = X_\epsilon + Y_\epsilon$, where

$$X_\epsilon = \int_{A_\epsilon} k_\zeta \otimes k_\zeta d\lambda(\zeta) \quad \text{and} \quad Y_\epsilon = \int_{B_\epsilon} k_\zeta \otimes k_\zeta d\lambda(\zeta).$$

Since the whole of B_ϵ is within 4ϵ of $M\backslash M^{(t_{4.1})}$ in terms of the Bergman distance, it is elementary that $\|Y_\epsilon - W_\epsilon\|$ tends to 0 as ϵ descends to 0. Thus it suffices to show that

$$\lim_{\epsilon\downarrow 0} \|X_\epsilon - V_\epsilon\| = 0. \tag{4.6}$$

To prove (4.6), consider any $f \in H^2(S)$. Then

$$\begin{aligned}(X_\epsilon f)(z) - (V_\epsilon f)(z) &= \sum_{w\in G_\epsilon} \int_{E_w} (f(\zeta)K_\zeta(z)(1-|\zeta|^2)^n - f(w)K_w(z)(1-|w|^2)^n) d\lambda(\zeta) \\ &= p_\epsilon(z) + q_\epsilon(z),\end{aligned}$$

where

$$p_\epsilon(z) = \sum_{w\in G_\epsilon} \int_{E_w} (f(\zeta) - f(w))K_w(z)(1-|w|^2)^n d\lambda(\zeta) \quad \text{and}$$
$$q_\epsilon(z) = \sum_{w\in G_\epsilon} \int_{E_w} f(\zeta)(K_\zeta(z)(1-|\zeta|^2)^n - K_w(z)(1-|w|^2)^n) d\lambda(\zeta).$$

By Lemma 4.1, when $2\epsilon < 1/4$, we have

$$|f(\zeta) - f(w)| \le C_{4.1}\frac{2\epsilon}{(1-|\zeta|^2)^{d+1}} \int_{D(\zeta,1)\cap M} |f(u)|dv_M(u)$$

for $\zeta \in E_w$, $w \in G_\epsilon$. Also, $|K_w(z)| \le C_1|K_\zeta(z)|$ and $1-|w|^2 \le C_2(1-|\zeta|^2)$. Therefore

$$\begin{aligned}|p_\epsilon(z)| &\le C_3\epsilon \sum_{w\in G_\epsilon} \int_{E_w}\int_{D(\zeta,1)\cap M} |f(u)|dv_M(u)|K_\zeta(z)|(1-|\zeta|^2)^{n-1-d} d\lambda(\zeta)\\ &\le C_3\epsilon \int_M |f(u)| \int_{D(u,1)\cap M} |K_\zeta(z)|(1-|\zeta|^2)^{n-1-d} d\lambda(\zeta) dv_M(u).\end{aligned}$$

It follows from Proposition 2.14 that $\lambda(D(u,1)\cap M) \le C_4$. Hence

$$|p_\epsilon(z)| \le C_5\epsilon \int_M |f(u)|\frac{(1-|u|^2)^{n-1-d}}{|1-\langle z,u\rangle|^n} dv_M(u) = C_5\epsilon\int_M \frac{|f(u)|}{|1-\langle z,u\rangle|^n} d\mu(u). \tag{4.7}$$

To estimate $|q_\epsilon(z)|$, note that

$$K_\zeta(z)(1-|\zeta|^2)^n - K_w(z)(1-|w|^2)^n = \frac{(1-|\zeta|^2)^n}{(1-\langle z,\zeta\rangle)^n}\left\{1 - \left(\frac{1-|w|^2}{1-|\zeta|^2}\right)^n\left(\frac{1-\langle z,\zeta\rangle}{1-\langle z,w\rangle}\right)^n\right\}.$$

If $\zeta \in E_w$, then $\zeta = \varphi_w(\xi)$ for some $\xi \in D(0, 2\epsilon)$. Thus by a standard exercise, we have

$$|K_\zeta(z)(1-|\zeta|^2)^n - K_w(z)(1-|w|^2)^n| \le C_6\epsilon\frac{(1-|\zeta|^2)^n}{|1-\langle z,\zeta\rangle|^n}$$

for $\zeta \in E_w$, $w \in G_\epsilon$. Therefore

$$|q_\epsilon(z)| \le C_6\epsilon\int_M |f(\zeta)|\frac{(1-|\zeta|^2)^n}{|1-\langle z,\zeta\rangle|^n} d\lambda(\zeta) = C_6\epsilon\int_M \frac{|f(\zeta)|}{|1-\langle z,\zeta\rangle|^n} d\mu(\zeta).$$

Combining this with (4.7), if we write $C_7 = C_5 + C_6$, then

$$|(X_\epsilon f)(z) - (V_\epsilon f)(z)| \le C_7\epsilon\int_M \frac{|f(u)|}{|1-\langle z,u\rangle|^n} d\mu(u).$$

Applying Lemma 3.3, we have

$$(4.8)\qquad \int_M |(X_\epsilon f)(z) - (V_\epsilon f)(z)|^2 d\mu(z) \le (C_7\epsilon\|B\|)^2 \langle T_\mu f, f\rangle \le (C_7\epsilon\|B\|)^2\|T_\mu\|\|f\|^2.$$

Theorem 3.5 tells us that $\|h\|^2 \le (1/c)\langle T_\mu h, h\rangle$ for every $h \in \mathcal{Q}$. Clearly, $X_\epsilon f - V_\epsilon f \in \mathcal{Q}$. Continuing with (4.8), we have

$$\begin{aligned}\|X_\epsilon f - V_\epsilon f\|^2 &\le (1/c)\langle T_\mu(X_\epsilon f - V_\epsilon f), X_\epsilon f - V_\epsilon f\rangle \\ &= (1/c)\int_M |(X_\epsilon f)(z) - (V_\epsilon f)(z)|^2 d\mu(z) \le (1/c)(C_7\epsilon\|B\|)^2\|T_\mu\|\|f\|^2.\end{aligned}$$

Since $f \in H^2(S)$ is arbitrary, we conclude that $\|X_\epsilon - V_\epsilon\|^2 \le (1/c)(C_7\epsilon\|B\|)^2\|T_\mu\|$. This proves (4.6) and completes the proof of the proposition. □

Definition 4.4. (a) The class $\mathcal{D}_0$ consists of operators of the form

$$\sum_{z\in\Gamma} c_z k_z \otimes k_z,$$

where $\Gamma \subset M$ and Γ is a-separated for some $a > 0$, and where $\{c_z : z \in \Gamma\}$ is any bounded set of complex coefficients.
(b) The class $\mathcal{D}$ consists of operators of the form

$$\sum_{z\in\Gamma} c_z k_z \otimes k_{\gamma(z)},$$

where $\Gamma \subset M$ and Γ is a-separated for some $a > 0$, where $\{c_z : z \in \Gamma\}$ is any bounded set of complex coefficients, and where $\gamma : \Gamma \to M$ is a map for which there is a $0 < C < \infty$ such that

$$\beta(z, \gamma(z)) \le C$$

for every $z \in \Gamma$.
(c) Let $C^*(\mathcal{D})$ be the C^*-algebra generated by $\mathcal{D}$.

Proposition 4.5. *$\mathcal{D}_0$ contains an operator that is invertible on $\mathcal{Q}$.*

Proof. Let T_ϵ be the operator defined in the statement of Proposition 4.3, $0 < \epsilon < 1$. Then $T_\epsilon \in \mathcal{D}_0$ by definition. Theorem 3.5 tells us that T_μ is invertible on $\mathcal{Q}$. It follows from the invertibility of T_μ on $\mathcal{Q}$ and Proposition 4.3 that if ϵ is small enough, then T_ϵ is invertible on $\mathcal{Q}$. □

This immediately leads to a compactness test and a membership test, both of which will play an essential role later in the paper.

Corollary 4.6. *Let A be a bounded operator on $\mathcal{Q}$.*
(a) *If XAY is compact for all $X, Y \in \mathcal{D}_0$, then A is a compact operator.*
(b) *If $XAY \in C^*(\mathcal{D})$ for all $X, Y \in \mathcal{D}_0$, then $A \in C^*(\mathcal{D})$.*

Proof. (a) follows immediately from Proposition 4.5. (b) follows from Proposition 4.5 and the fact that $C^*(\mathcal{D})$ is a C^*-algebra. Specifically, it uses the property that if $T \in C^*(\mathcal{D})$ and if T is invertible on $\mathcal{Q}$, then $T^{-1} \in C^*(\mathcal{D})$. □

We end the section with the obvious:

Proposition 4.7. *The norm closure of* $\mathrm{span}(\mathcal{D})$ *contains every compact operator on* $\mathcal{Q}$.

Proof. By definition, we have $k_z \otimes k_w \in \mathcal{D}$ for all $z, w \in M$. Since $\mathcal{Q}$ is the closure of $\mathrm{span}\{k_z : z \in M\}$, for any $f, g \in \mathcal{Q}$, $f \otimes g$ is in the closure of $\mathrm{span}(\mathcal{D})$ with respect to the operator norm. Once this is clear, the proposition follows. □

5. The C^*-algebra $C^*(\mathcal{D})$

This section is devoted to estimates related to the C^*-algebra $C^*(\mathcal{D})$.

Lemma 5.1. *Let $0 \le \eta \le 1/4$ be given. For any $\epsilon > 0$, there is an $r = r(\eta, \epsilon) > 1$ such that the following holds true: Suppose that Γ and G are 1-separated sets contained in $M \cap K$, and that E is a subset of $\Gamma \times G$ satisfying the condition*

$$\beta(z, w) \ge r \quad \text{for every} \quad (z, w) \in E.$$

Let $\{a_{z,w} : (z, w) \in E\}$ be a set of complex coefficients such that

$$|a_{z,w}| \le \frac{(1 - |z|^2)^{(n/2)-\eta}(1 - |w|^2)^{(n/2)-\eta}}{|1 - \langle z, w\rangle|^{n-2\eta}} \quad \text{for every} \quad (z, w) \in E.$$

Then for any orthonormal sets $\{e_z : z \in \Gamma\}$ and $\{u_w : w \in G\}$, we have

$$\left\| \sum_{(z,w)\in E} a_{z,w} e_z \otimes u_w \right\| \le \epsilon.$$

Proof. We will bring the Schur test to bear. Define $h(w) = (1 - |w|^2)^{(n-1)/2}$ for $w \in G$. For $w \in G$ and $\zeta \in D(w, 1)$, we have $1 - |\zeta|^2 \le C_1(1 - |w|^2)$ and $|1 - \langle z, \zeta\rangle| \le C_2|1 - \langle z, w\rangle|$ Thus for each $z \in \Gamma$,

$$\sum_{w\in G\setminus D(z,r)} \frac{(1 - |z|^2)^{(n/2)-\eta}(1 - |w|^2)^{(n/2)-\eta}}{|1 - \langle z, w\rangle|^{n-2\eta}} h(w)$$
$$\le C_3 \sum_{w\in G\setminus D(z,r)} \frac{(1 - |w|^2)^n}{\mu(D(w, 1) \cap M)} \int_{D(w,1)\cap M} \frac{(1 - |z|^2)^{(n/2)-\eta}(1 - |\zeta|^2)^{-(1/2)-\eta}}{|1 - \langle z, \zeta\rangle|^{n-2\eta}} d\mu(\zeta).$$

Since G is 1-separated, from this inequality and Proposition 2.14 we obtain

$$\sum_{w\in G\setminus D(z,r)} \frac{(1 - |z|^2)^{(n/2)-\eta}(1 - |w|^2)^{(n/2)-\eta}}{|1 - \langle z, w\rangle|^{n-2\eta}} h(w)$$
$$\le C_4 \int_{M\setminus D(z,r-1)} \frac{(1 - |z|^2)^{(n/2)-\eta}(1 - |\zeta|^2)^{-(1/2)-\eta}}{|1 - \langle z, \zeta\rangle|^{n-2\eta}} d\mu(\zeta)$$
$$= C_4\tilde{h}(z) \int_{M\setminus D(z,r-1)} \frac{(1 - |z|^2)^{a}(1 - |\zeta|^2)^{\kappa}}{|1 - \langle z, \zeta\rangle|^{d+1+a+\kappa}} dv_M(\zeta),$$

where $a = (1/2) - \eta$, $\kappa = n - 1 - d - (1/2) - \eta$ and $\tilde{h}(z) = (1 - |z|^2)^{(n-1)/2}$ for $z \in \Gamma$. We have $a > 0$ and $\kappa > -1$. Applying Lemma 2.11, we obtain

$$\sum_{w \in G \setminus D(z,r)} \frac{(1-|z|^2)^{(n/2)-\eta}(1-|w|^2)^{(n/2)-\eta}}{|1-\langle z,w\rangle|^{n-2\eta}} h(w) \le C_4 C_{2.11}(\delta) e^{-2\delta(r-1)} \tilde{h}(z)$$

for every $z \in \Gamma$. Similarly, for each $w \in G$ we have

$$\sum_{z \in \Gamma \setminus D(w,r)} \frac{(1-|z|^2)^{(n/2)-\eta}(1-|w|^2)^{(n/2)-\eta}}{|1-\langle z,w\rangle|^{n-2\eta}} \tilde{h}(z) \le C_4 C_{2.11}(\delta) e^{-2\delta(r-1)} h(w).$$

From these two inequalities and the Schur test it now follows that

$$\left\| \sum_{(z,w) \in E} a_{z,w} e_z \otimes u_w \right\| \le C_4 C_{2.11}(\delta) e^{-2\delta(r-1)}.$$

This completes the proof. □

Proposition 5.2. *The C^*-algebra $C^*(\mathcal{D})$ is the closure with respect to the operator norm of the linear span of $\mathcal{D}$.*

Proof. Suppose that Γ is a separated set in $\mathbf{B}$ and that $\gamma : \Gamma \to \mathbf{B}$ is a map for which there is a $0 < C < \infty$ such that $\beta(z, \gamma(z)) \le C$ for every $z \in \Gamma$. Then there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for each $1 \le j \le m$, we have $D(\gamma(z), 1) \cap D(\gamma(z'), 1) = \emptyset$ for all $z \ne z'$ in Γ_j. This implies that if A is in the linear span of $\mathcal{D}$, so is A^*. Therefore the proof will be complete if we can show that for all $A, B \in \mathcal{D}$, the product AB is in the closure with respect to the operator norm of the linear span of $\mathcal{D}$.

Recalling Proposition 4.7, it suffices to consider $A, B \in \mathcal{D}$ with representations

$$A = \sum_{z \in \Gamma} a_z k_z \otimes k_{\gamma(z)} \quad \text{and} \quad B = \sum_{w \in G} b_w k_w \otimes k_{g(w)},$$

where Γ and G are 1-separated sets in $M \cap K$, $\{a_z : z \in \Gamma\}$ and $\{b_w : w \in G\}$ are bounded sets of coefficients, and $\gamma : \Gamma \to M$ and $g : G \to M$ are maps for which there is a C such that $\beta(z, \gamma(z)) \le C$ for every $z \in \Gamma$ and $\beta(w, g(w)) \le C$ for every $w \in G$. Moreover, partitioning G by a finite number of subsets if necessary, we may further assume that $D(g(w), 1) \cap D(g(w'), 1) = \emptyset$ for all $w \ne w'$ in G.

For each $r > 0$, we have the partition $\Gamma \times G = E_r \cup F_r$, where

$$E_r = \{(z,w) \in \Gamma \times G : \beta(z, g(w)) \ge r\} \quad \text{and} \quad F_r = \{(z,w) \in \Gamma \times G : \beta(z, g(w)) < r\}.$$

Accordingly, $AB = S_r + T_r$, where

$$S_r = \sum_{(z,w) \in E_r} a_z b_w \langle k_w, k_{\gamma(z)} \rangle k_z \otimes k_{g(w)} \quad \text{and} \quad T_r = \sum_{(z,w) \in F_r} a_z b_w \langle k_w, k_{\gamma(z)} \rangle k_z \otimes k_{g(w)}.$$

By definition, if $(z,w) \in F_r$, then $\beta(z, g(w)) < r$. Also, if $(z,w) \in F_r$, then

$$\beta(z, w) \le \beta(z, g(w)) + \beta(g(w), w) \le r + C.$$

Since G is 1-separated, there is a $C_1(r)$ such that for every $z \in \Gamma$ we have $\operatorname{card}\{w \in G : (z,w) \in F_r\} \le C_1(r)$. Therefore T_r is in the linear span of $\mathcal{D}$.

To complete the proof, we will show that $\|S_r\|$ is small when r is large. To do that we pick orthonormal sets $\{e_z : z \in \Gamma\}$ and $\{u_w : w \in G\}$. We then define

$$X = \sum_{z\in\Gamma} a_z k_z \otimes e_z \quad \text{and} \quad Y = \sum_{w\in G} b_w u_w \otimes k_{g(w)}.$$

Then $S_r = X S_r' Y$, where

$$S_r' = \sum_{(z,w)\in E_r} \langle k_w, k_{\gamma(z)}\rangle e_z \otimes u_w.$$

By Lemma 4.2, we have $\|X\| \le C_{4.2} a$ and $\|Y\| \le C_{4.2} b$, where $a = \sup_{z\in\Gamma} |a_z|$ and $b = \sup_{w\in G} |b_w|$. Thus it suffices to show that $\|S_r'\|$ is small when r is large.

To estimate $\|S_r'\|$, note that

$$|\langle k_w, k_{\gamma(z)}\rangle| = \frac{(1 - |\gamma(z)|^2)^{n/2}(1 - |w|^2)^{n/2}}{|1 - \langle \gamma(z), w\rangle|^n} \le C_2 \frac{(1 - |z|^2)^{n/2}(1 - |w|^2)^{n/2}}{|1 - \langle z, w\rangle|^n}$$

for $(z,w) \in E_r$, where the $\le$ follows from the condition $\beta(z, \gamma(z)) \le C$. Also,

$$\beta(z, w) \ge \beta(z, g(w)) - \beta(g(w), w) \ge r - C$$

for $(z,w) \in E_r$. That is, $E_r \subset \{(z,w) \in \Gamma \times G : \beta(z,w) \ge r - C\}$. Thus it follows from Lemma 5.1 that $\|S_r'\| \to 0$ as $r \to \infty$. This completes the proof. $\square$

Lemma 5.3. *Let $A \in C^*(\mathcal{D})$ be given. Then for every $\epsilon > 0$, there is an $r > 1$ such that the following holds true: Suppose that Γ and G are 1-separated sets contained in $M \cap K$, and that $\{e_z : z \in \Gamma\}$ and $\{u_w : w \in G\}$ are orthonormal sets. Denote*

$$X = \sum_{z\in\Gamma} e_z \otimes k_z \quad \textit{and} \quad Y = \sum_{w\in G} k_w \otimes u_w.$$

If Γ and G satisfy the condition $\beta(z,w) \ge r$ for every $(z,w) \in \Gamma \times G$, then $\|XAY\| \le \epsilon$.

Proof. First of all, Lemma 4.2 provides the bounds $\|X\| \le C_{4.2}$ and $\|Y\| \le C_{4.2}$. Because of these bounds, by the approximation in Proposition 5.2 we only need to consider $A \in \mathcal{D}$. More specifically, we assume

$$A = \sum_{\xi\in E} c_\xi k_\xi \otimes k_{\gamma(\xi)}$$

where E is a 1-separated set in M, $\gamma : E \to M$ is a map for which there is a C such that $\beta(\xi, \gamma(\xi)) \le C$ for every $\xi \in E$, and $\sup_{\xi \in E} |c_\xi| < \infty$.

Multiplying out the product, we have

$$XAY = \sum_{z \in \Gamma} \sum_{w \in G} a_{z,w} e_z \otimes u_w,$$

where

$$a_{z,w} = \sum_{\xi \in E} c_\xi \langle k_\xi, k_z \rangle \langle k_w, k_{\gamma(\xi)} \rangle$$

for $z \in \Gamma$ and $w \in G$. We have the partition $E = E_1 \cup E_2$, where $E_1 = E \cap K$ and $E_2 = E \cap \{M \backslash K\}$. Accordingly, $a_{z,w} = a^{(1)}_{z,w} + a^{(2)}_{z,w}$, where

$$a^{(i)}_{z,w} = \sum_{\xi \in E_i} c_\xi \langle k_\xi, k_z \rangle \langle k_w, k_{\gamma(\xi)} \rangle$$

for $i = 1, 2$ and $(z, w) \in \Gamma \times G$.

Writing $c = \sup_{\xi \in E} |c_\xi|$, we have

$$\begin{aligned} |a^{(1)}_{z,w}| &\le c \sum_{\xi \in E_1} \frac{\{(1-|\xi|^2)(1-|z|^2)(1-|w|^2)(1-|\gamma(\xi)|^2)\}^{n/2}}{|1-\langle z, \xi \rangle|^n |1 - \langle \gamma(\xi), w \rangle|^n} \\ &\le C_1 c \sum_{\xi \in E_1} \frac{(1-|\xi|^2)^n (1-|z|^2)^{n/2} (1-|w|^2)^{n/2}}{|1-\langle z, \xi \rangle|^n |1 - \langle \xi, w \rangle|^n}, \end{aligned}$$

where for the second $\le$ we use the fact that $\beta(\xi, \gamma(\xi)) \le C$ for every $\xi \in E$. Thus

$$\begin{aligned} |a^{(1)}_{z,w}| &\le C_2 c \sum_{\xi \in E_1} \frac{(1-|\xi|^2)^n}{\mu(D(\xi,1) \cap M)} \int_{D(\xi,1)\cap M} \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle z, \zeta \rangle|^n |1-\langle \zeta, w \rangle|^n} d\mu(\zeta) \\ &\le C_3 c \int_M \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle z, \zeta \rangle|^n |1-\langle \zeta, w \rangle|^n} d\mu(\zeta), \end{aligned} \tag{5.1}$$

where the second $\le$ follows from Proposition 2.14 and the fact that E is 1-separated. The fact that E is 1-separated also ensures $\mathrm{card}(E_2) \le C_4$. Therefore it is trivial that

$$|a^{(2)}_{z,w}| \le C_5 c \int_M \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle z, \zeta \rangle|^n |1-\langle \zeta, w \rangle|^n} d\mu(\zeta).$$

Combining this with (5.1), we see that

$$|a_{z,w}| \le C_6 c \int_M \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle z, \zeta \rangle|^n |1-\langle \zeta, w \rangle|^n} d\mu(\zeta) \tag{5.2}$$

for all $z \in \Gamma$ and $w \in G$.

Recall that we have the triangle inequality

$$|1 - \langle z, w\rangle|^{1/2} \le |1 - \langle z, \zeta\rangle|^{1/2} + |1 - \langle \zeta, w\rangle|^{1/2} \tag{5.3}$$

[26, Proposition 5.1.2]. Thus if we define

$$U_{z,w} = \{\zeta \in M : |1 - \langle z, \zeta\rangle| \ge (1/4)|1 - \langle z, w\rangle|\} \quad \text{and}$$
$$V_{z,w} = \{\zeta \in M : |1 - \langle \zeta, w\rangle| \ge (1/4)|1 - \langle z, w\rangle|\},$$

then $U_{z,w} \cup V_{z,w} = M$. Using this decomposition of M in (5.2), we obtain

$$|a_{z,w}| \le C_7 c \frac{(1 - |z|^2)^{n/2}(1 - |w|^2)^{n/2}}{|1 - \langle z, w\rangle|^{n-(1/9)}} \times$$
$$\biggl(\int_M \frac{1}{(1 - |\zeta|^2)^{1/9}|1 - \langle \zeta, w\rangle|^n} d\mu(\zeta) + \int_M \frac{1}{|1 - \langle z, \zeta\rangle|^n (1 - |\zeta|^2)^{1/9}} d\mu(\zeta)\biggr).$$

By Lemma 2.10, we have

$$\int_M \frac{1}{(1 - |\zeta|^2)^{1/9}|1 - \langle \zeta, w\rangle|^n} d\mu(\zeta) = \int_M \frac{(1 - |\zeta|^2)^{n-1-d-(1/9)}}{|1 - \langle \zeta, w\rangle|^n} dv_M(\zeta)$$
$$\le 2^{n-1-d} \int_M \frac{(1 - |\zeta|^2)^{-1/9}}{|1 - \langle \zeta, w\rangle|^{d+1}} dv_M(\zeta) \le C_8(1 - |w|^2)^{-1/9}.$$

Similarly,

$$\int_M \frac{1}{|1 - \langle z, \zeta\rangle|^n (1 - |\zeta|^2)^{1/9}} d\mu(\zeta) \le C_8(1 - |z|^2)^{-1/9}.$$

Therefore

$$|a_{z,w}| \le C_9 c \frac{(1 - |z|^2)^{n/2}(1 - |w|^2)^{n/2}}{|1 - \langle z, w\rangle|^{n-(1/9)}}((1 - |w|^2)^{-1/9} + (1 - |z|^2)^{-1/9})$$
$$\le C_{10} c \frac{(1 - |z|^2)^{(n/2)-(1/9)}(1 - |w|^2)^{(n/2)-(1/9)}}{|1 - \langle z, w\rangle|^{n-(2/9)}}.$$

Recall that we assume that $\beta(z, w) \ge r$ for every $(z, w) \in \Gamma \times G$. Thus, applying Lemma 5.1 with $\eta = 1/9$, we see that $\|XAY\|$ is small when r is large. □

6. Compactness criterion for operators in $C^*(\mathcal{D})$

In this section, our goal is to prove

Theorem 6.1. *Let $A \in C^*(\mathcal{D})$. If*

$$\lim_{\substack{z \in M \\ |z| \to 1}} \langle Ak_z, k_z\rangle = 0,$$

then A is a compact operator.

In addition to the material from the previous section, the proof of this theorem requires more preparations, not the least of which is the radial-spherical decomposition of the unit ball from [30, Section 4]. We begin the proof with a review of this decomposition.

In the spherical direction, the decomposition begins with the metric

$$d(u,\xi) = |1 - \langle u, \xi\rangle|^{1/2}, \qquad u, \xi \in S,$$

defined on S [26, page 66]. For any pair of $u \in S$ and $r > 0$, we write

$$S(u,r) = \{\xi \in S : d(u,\xi) < r\}.$$

There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \le \sigma(S(u,r)) \le A_0 r^{2n} \tag{6.1}$$

for all $u \in S$ and $0 < r \le \sqrt{2}$ [26, Proposition 5.1.4].

In the radial direction of the ball, we set

$$\rho_k = 1 - 2^{-2k}$$

for every $k \in \mathbf{Z}_+$. For each pair of natural numbers $m \ge 6$ and $j \in \mathbf{N}$, let us denote

$$\alpha_{m,j} = m(1 - \rho_{jm}^2)^{1/2} = m \cdot 2^{-jm} \cdot (2 - 2^{-2jm})^{1/2}. \tag{6.2}$$

Note that $8\alpha_{m,j} \le \sqrt{2}$ for all $m \ge 6$ and $j \in \mathbf{N}$. For each pair of $m \ge 6$ and $j \in \mathbf{N}$, let $E_{m,j}$ be a subset of S that is *maximal* with respect to the property

$$S(u, \alpha_{m,j}/2) \cap S(v, \alpha_{m,j}/2) = \emptyset \quad \text{for all} \quad u \ne v \quad \text{in} \quad E_{m,j}. \tag{6.3}$$

It follows from the maximality of $E_{m,j}$ that

$$\bigcup_{u \in E_{m,j}} S(u, \alpha_{m,j}) = S. \tag{6.4}$$

For each triple of $m \ge 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we define

$$\begin{aligned} A_{m,j,u} &= \{r\xi : \xi \in S(u,\alpha_{m,j}), r \in [\rho_{(j+2)m}, \rho_{(j+3)m}]\} \quad \text{and} \\ B_{m,j,u} &= \{r\xi : \xi \in S(u,3\alpha_{m,j}), r \in [\rho_{jm}, \rho_{(j+5)m}]\}. \end{aligned} \tag{6.5}$$

Then it follows from (6.4) that

$$\bigcup_{j=1}^{\infty} \bigcup_{u \in E_{m,j}} A_{m,j,u} = \{z \in \mathbf{B} : \rho_{3m} \le |z| < 1\}. \tag{6.6}$$

Lemma 6.2. [30, Lemma 4.3] *For each triple of* $m \geq 6$, $j \in \mathbf{N}$ *and* $u \in E_{m,j}$, *define*

$$z_{m,j,u} = \rho_{jm} u. \tag{6.7}$$

Then we have $B_{m,j,u} \subset D(z_{m,j,u}, R_m)$, *where* $R_m = 2 + 5m + \log\left(1 + 2^{10m} \times 18m^2\right)$.

By (6.1) and (6.3), there is a natural number N_0 such that for every triple of $m \geq 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we have

$$\operatorname{card}\{v \in E_{m,j} : d(u,v) < 7\alpha_{m,j}\} \leq N_0. \tag{6.8}$$

By a standard maximality argument, each $E_{m,j}$ admits a partition

$$E_{m,j} = E^{(1)}_{m,j} \cup \cdots \cup E^{(N_0)}_{m,j}$$

such that for every $\nu \in \{1, \dots, N_0\}$, we have $d(u,v) \geq 7\alpha_{m,j}$ for all $u \neq v$ in $E^{(\nu)}_{m,j}$. This number N_0 and the above partition will be fixed for the rest of the section.

Lemma 6.3. [30, Lemma 4.2] (a) *Let* $m \geq 6$, $j \in \mathbf{N}$ *and* $\nu \in \{1, \dots, N_0\}$. *If* $u, v \in E^{(\nu)}_{m,j}$ *and* $u \neq v$, *then we have* $\beta(z,w) > 2$ *for all* $z \in B_{m,j,u}$ *and* $w \in B_{m,j,v}$.
(b) *Let* $m \geq 6$. *If* $u \in E_{m,j}$, $v \in E_{m,k}$ *and* $k \geq j + 6$, *then we have* $\beta(z,w) > 3$ *for all* $z \in B_{m,j,u}$ *and* $w \in B_{m,k,v}$.
(c) *Let* $m \geq 6$, $j \in \mathbf{N}$ *and* $u \in E_{m,j}$. *Then* $\beta(z,w) \geq 2\log m$ *for all* $z \in \mathbf{B}\backslash B_{m,j,u}$ *and* $w \in A_{m,j,u}$.

Definition 6.4. Let $m \geq 6$ be given. (a) For each pair of $\kappa \in \{1,2,3,4,5,6\}$ and $\nu \in \{1, \dots, N_0\}$, where N_0 is the integer that appears in (6.8), let $I^{(\nu,\kappa)}_m$ denote the collection of all triples $m, 6i+\kappa, u$ satisfying the conditions $i \in \mathbf{Z}_+$ and $u \in E^{(\nu)}_{m,6i+\kappa}$.
(b) For $\kappa \in \{1,2,3,4,5,6\}$, $\nu \in \{1, \dots, N_0\}$ and $J \in \mathbf{N}$, let $I^{(\nu,\kappa)}_{m,J}$ denote the collection of all triples $m, 6i+\kappa, u$ satisfying the conditions $0 \leq i \leq J$ and $u \in E^{(\nu)}_{m,6i+\kappa}$.
(c) Denote $I_m = \cup_{\kappa=1}^{6} \cup_{\nu=1}^{N_0} I^{(\nu,\kappa)}_m$.

As in [30], we will try to avoid triple subscripts when possible. That is, we use ω to represent $(m,j,u) \in I_m$ and write A_ω and B_ω for $A_{m,j,u}$ and $B_{m,j,u}$ respectively.

From Definition 6.4(a) and Lemma 6.3(a), (b) we immediately obtain

Corollary 6.5. *Given any* $\kappa \in \{1,2,3,4,5,6\}$ *and* $\nu \in \{1, \dots, N_0\}$, *if* $\omega, \omega' \in I^{(\nu,\kappa)}_m$ *and* $\omega \neq \omega'$, *then for every pair of* $z \in B_\omega$ *and* $w \in B_{\omega'}$ *we have* $\beta(z,w) > 2$.

Lemma 6.6. *Let* $U_1, \dots, U_\ell$ *be subsets of* $\mathbf{B}$ *such that* $U_j \cap U_k = \emptyset$ for all $j \neq k$. *For each* $1 \leq k \leq \ell$, *let* E_k *and* F_k *be finite subsets of* U_k. *Denote* $E = \cup_{k=1}^{\ell} E_k$ *and* $F = \cup_{k=1}^{\ell} F_k$. *Suppose that* $\{e_z : z \in E\}$ *and* $\{\epsilon_w : w \in F\}$ *are orthonormal sets in Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *respectively. Define*

$$X_k = \sum_{z \in E_k} e_z \otimes k_z \quad \textit{and} \quad Y_k = \sum_{w \in F_k} k_w \otimes \epsilon_w$$

for each $1 \le k \le \ell$. *Let* A *be any bounded operator on the Hardy space* $H^2(S)$. *Then there exists a subset* L of $\{1,\dots,\ell\}$ *such that if we define*

$$X = \sum_{k\in L} X_k, \quad Y = \sum_{k\in L} Y_k, \quad X' = \sum_{k\in\{1,\dots,\ell\}\setminus L} X_k \quad \textit{and} \quad Y' = \sum_{k\in\{1,\dots,\ell\}\setminus L} Y_k,$$

then

$$\left\| \sum_{j\neq k} X_j A Y_k \right\| \le 4\{\|XAY'\| + \|X'AY\|\}.$$

Proof. We may assume that $\mathcal{H}_1 = \ell^2(E)$, $\mathcal{H}_2 = \ell^2(F)$, and that $\{e_z : z \in E\}$ and $\{\epsilon_w : w \in F\}$ are the standard orthonormal bases for $\ell^2(E)$ and $\ell^2(F)$ respectively. For a function f defined on $\mathbf{B}$, we define the multiplication operator M_f on $\ell^2(E)$ and $\ell^2(F)$ by the formulas

$$M_f \sum_{z\in E} a_z e_z = \sum_{z\in E} f(z) a_z e_z \quad \text{and} \quad M_f \sum_{w\in F} b_w \epsilon_w = \sum_{w\in F} f(w) b_w \epsilon_w$$

respectively. The rest of the proof is an adaptation of the proof of [30, Lemma 5.1].

It suffices to consider the case $\ell \ge 2$. Write

$$Z = \sum_{j\neq k} X_j A Y_k \quad \text{and} \quad Z_\theta = \sum_{j\neq k} e^{i(j-k)\theta} X_j A Y_k, \quad \theta \in \mathbf{R}.$$

Then obviously we have

$$Z = \frac{1}{2\pi}\int_0^{2\pi} (Z - Z_\theta)\, d\theta.$$

This shows that there is a $\theta^* \in [0, 2\pi]$ such that $\|Z\| \le \|Z - Z_{\theta^*}\|$.

Write $\gamma_k = e^{ik\theta^*}$ for every $k \in \{1,\dots,\ell\}$. Define the operators

$$B = \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} X_j A Y_k \quad \text{and} \quad B' = \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} \gamma_j \bar{\gamma}_k X_j A Y_k$$

from $\ell^2(F)$ to $\ell^2(E)$. Also, define the function

$$\psi = \sum_{k=1}^{\ell} \gamma_k \chi_{U_k}$$

on $\mathbf{B}$. Since $E_k \subset U_k$, $F_k \subset U_k$ and $U_j \cap U_k = \emptyset$ for $j \neq k$, we have

$$B - B' = B - M_\psi B M_{\bar\psi} = M_\psi (M_{\bar\psi} B - B M_{\bar\psi}).$$

For each $k \in \{1, \dots, \ell\}$, let us write $\gamma_k = c_k + id_k$, where $c_k, d_k \in [-1, 1]$. Define

$$p = \sum_{k=1}^{\ell} c_k \chi_{U_k} \quad \text{and} \quad q = \sum_{k=1}^{\ell} d_k \chi_{U_k}.$$

Then the above gives us $B - B' = M_\psi V - iM_\psi W$, where

$$V = M_p B - BM_p \quad \text{and} \quad W = M_q B - BM_q.$$

Since $\gamma_k \bar{\gamma}_k = 1$ for every $k \in \{1, \dots, \ell\}$, we have $Z - Z_{\theta^*} = B - B'$. Consequently, we have either $\|Z\| \le \|Z - Z_{\theta^*}\| \le 2\|V\|$ or $\|Z\| \le \|Z - Z_{\theta^*}\| \le 2\|W\|$.

In the case $\|Z\| \le 2\|V\|$, consider $c_1, \dots, c_\ell$, which are real numbers in $[-1, 1]$. There is a permutation $\tau(1), \dots, \tau(\ell)$ of the integers $1, \dots, \ell$ such that

$$c_{\tau(j)} \ge c_{\tau(j-1)} \quad \text{for every} \quad j \in \{2, \dots, \ell\}.$$

For each $j \in \{1, \dots, \ell\}$, define the subset $L_j = \{\tau(k) : j \le k \le \ell\}$ of $\{1, \dots, \ell\}$. Then

$$p = \sum_{k=1}^{\ell} c_{\tau(k)} \chi_{U_{\tau(k)}} = c_{\tau(1)} \sum_{\alpha \in L_1} \chi_{U_\alpha} + \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) \sum_{\alpha \in L_j} \chi_{U_\alpha}.$$

Obviously, $M_{\chi_{U_j}} X_k = 0$ when $j \ne k$ and $M_{\chi_{U_k}} X_k = X_k$. Thus

$$\sum_{k=1}^{\ell} c_k X_k = M_p \sum_{k=1}^{\ell} X_k = c_{\tau(1)} S_1 + \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) S_j, \quad \text{where} \quad S_j = \sum_{\alpha \in L_j} X_\alpha$$

for every $1 \le j \le \ell$. Similarly,

$$\sum_{k=1}^{\ell} c_k Y_k = \sum_{k=1}^{\ell} Y_k M_p = c_{\tau(1)} T_1 + \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) T_j, \quad \text{where} \quad T_j = \sum_{\alpha \in L_j} Y_\alpha$$

for every $1 \le j \le \ell$. Note that $L_1 = \{1, \dots, \ell\}$. Therefore

$$\begin{aligned} V = M_p B - BM_p &= \sum_{j=1}^{\ell} c_j X_j A T_1 - S_1 A \sum_{j=1}^{\ell} c_j Y_j \\ &= \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)})(S_j A T_1 - S_1 A T_j) = \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)})(S_j A T_j' - S_j' A T_j), \end{aligned}$$

where

$$S_j' = S_1 - S_j = \sum_{\alpha \in \{1, \dots, \ell\} \setminus L_j} X_\alpha \quad \text{and} \quad T_j' = T_1 - T_j = \sum_{\alpha \in \{1, \dots, \ell\} \setminus L_j} Y_\alpha,$$

$1 \le j \le \ell$. Since $(c_{\tau(2)} - c_{\tau(1)}) + \cdots + (c_{\tau(\ell)} - c_{\tau(\ell-1)}) = c_{\tau(\ell)} - c_{\tau(1)} \le 2$, we have

$$\|V\| \le \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) \|S_j A T'_j - S'_j A T_j\| \le 2 \max_{2 \le j \le \ell} (\|S_j A T'_j\| + \|S'_j A T_j\|).$$

Thus there is a $j_0 \in \{2, \dots, \ell\}$ such that

$$\|V\| \le 2(\|S_{j_0} A T'_{j_0}\| + \|S'_{j_0} A T_{j_0}\|).$$

If we simply let $L = L_{j_0}$, then $X = S_{j_0}$, $Y = T_{j_0}$, $X' = S'_{j_0}$ and $Y' = T'_{j_0}$. This proves the lemma in the case where $\|Z\| \le 2\|V\|$.

In the case $\|Z\| \le 2\|W\|$, we just apply the argument in the preceding paragraph with $d_1, \dots, d_\ell$ in place of $c_1, \dots, c_\ell$. This completes the proof of the lemma. □

Proposition 6.7. *Let A be a bounded operator on $\mathcal{Q}$. If*

$$\lim_{\substack{z \in M \\ |z| \to 1}} \langle A k_z, k_z \rangle = 0, \tag{6.9}$$

then for every $0 < r < \infty$ we have

$$\lim_{\substack{z \in M \\ |z| \to 1}} \sup\{|\langle A k_z, k_w \rangle| : w \in D(z, r) \cap M\} = 0.$$

Proof. Assuming the contrary, we would have an $r > 0$ and sequences $\{z_j\}$ and $\{w_j\}$ in M satisfying the following three conditions:

(1) $\lim_{j \to \infty} |z_j| = 1$;

(2) $\beta(z_j, w_j) < r$ for every j;

(3) $\lim_{j \to \infty} \langle A k_{z_j}, k_{w_j} \rangle = a \ne 0$.

We will show that this leads to a contradiction.

Combining (1) above with Lemma 2.9, discarding a finite number of j's if necessary, we may further assume that $D(z_j, 3r) \cap T_{z_j}^{\text{mod}} \subset B(z_j, c_0) \cap T_{z_j}^{\text{mod}}$ (cf. (2.8)),

$$I_{z_j}(D(z_j, 2r) \cap T_{z_j}^{\text{mod}}) \supset D(z_j, r) \cap M \quad \text{and} \quad I_{z_j}(D(z_j, 3r) \cap T_{z_j}^{\text{mod}}) \subset D(z_j, 6r) \cap M$$

for every j. There are $0 < s < t < 1$ such that

$$\varphi_{z_j}(B(0, s) \cap T_{z_j}^{\text{mod}}) = D(z_j, 2r) \cap T_{z_j}^{\text{mod}} \quad \text{and} \quad \varphi_{z_j}(B(0, t) \cap T_{z_j}^{\text{mod}}) = D(z_j, 3r) \cap T_{z_j}^{\text{mod}}$$

for every j. For each j, let $V_j : \mathbf{C}^d \to \mathbf{C}^n$ be an isometry such that

$$V_j \mathbf{C}^d = T_{z_j}^{\text{mod}}.$$

Recall that we write $\mathbf{B}_d$ for the unit ball in $\mathbf{C}^d$. For each j, define the map $\alpha_j : \mathbf{B}_d \to M$ by the formula

$$\alpha_j(\xi) = I_{z_j}(\varphi_{z_j}(tV_j\xi)),$$

$\xi \in \mathbf{B}_d$. Obviously, each α_j is analytic, and we have $\alpha_j(0) = z_j$. By (2), for each j there is a $\xi_j \in B_d(0, s/t) = \{\zeta \in \mathbf{C}^d : |\zeta| < s/t\}$ such that $\alpha_j(\xi_j) = w_j$.

For each j, we now define the analytic function F_j on $\mathbf{B}_d \times \mathbf{B}_d$ by the formula

$$F_j(\xi, \eta) = (1 - |z_j|^2)^n \langle AK_{\alpha_j(\bar{\xi})}, K_{\alpha_j(\eta)}\rangle, \quad (\xi, \eta) \in \mathbf{B}_d \times \mathbf{B}_d.$$

A review of the above finds that $\alpha_j(\xi) \in D(z_j, 6r) \cap M$ for all j and $\xi \in \mathbf{B}_d$. Therefore there are $0 < c_1 \le C_1 < \infty$ such that

$$c_1(1 - |\alpha_j(\xi)|^2) \le 1 - |z_j|^2 \le C_1(1 - |\alpha_j(\xi)|^2) \quad \text{for all } j \text{ and } \xi \in \mathbf{B}_d.$$

Thus $|F_j(\xi, \eta)| \le C_2$ for all j, ξ and η. Hence there exist a subsequence $\{F_{j_\nu}\}$ of $\{F_j\}$ and an analytic function F on $\mathbf{B}_d \times \mathbf{B}_d$ such that $\{F_{j_\nu}\}$ uniformly converges to F on every compact subset of $\mathbf{B}_d \times \mathbf{B}_d$. For each $\xi \in \mathbf{B}_d$, since $\beta(\alpha_j(\xi), z_j) < 6r$, it follows from (1) that $\lim_{j\to\infty} |\alpha_j(\xi)| = 1$. By (6.9), we have

$$F(\bar{\xi}, \xi) = \lim_{\nu\to\infty} F_{j_\nu}(\bar{\xi}, \xi) = \lim_{\nu\to\infty} (1 - |z_{j_\nu}|^2)^n \langle AK_{\alpha_{j_\nu}(\xi)}, K_{\alpha_{j_\nu}(\xi)}\rangle = 0.$$

Since this holds for every $\xi \in \mathbf{B}_d$, it is well known that it implies that F is identically 0 on $\mathbf{B}_d \times \mathbf{B}_d$. Therefore $\{F_{j_\nu}\}$ uniformly converges to 0 on every compact subset of $\mathbf{B}_d \times \mathbf{B}_d$. Since $\xi_{j_\nu} \in B_d(0, s/t)$ for every ν, in particular we have

$$\lim_{\nu\to\infty} F_{j_\nu}(0, \xi_{j_\nu}) = 0. \tag{6.10}$$

On the other hand, since $\alpha_{j_\nu}(0) = z_{j_\nu}$ and $\alpha_{j_\nu}(\xi_{j_\nu}) = w_{j_\nu}$, we have

$$F_{j_\nu}(0, \xi_{j_\nu}) = (1 - |z_{j_\nu}|^2)^n \langle AK_{z_{j_\nu}}, K_{w_{j_\nu}}\rangle = \left(\frac{1 - |z_{j_\nu}|^2}{1 - |w_{j_\nu}|^2}\right)^{n/2} \langle Ak_{z_{j_\nu}}, k_{w_{j_\nu}}\rangle.$$

Since $1 - |z_{j_\nu}|^2 \ge c_1(1 - |w_{j_\nu}|^2)$, (6.10) contradicts (3). This completes the proof. □

Lemma 6.8. *Let Γ be a separated set contained in M, and let $\gamma : \Gamma \to M$ be a map for which there is a $0 < C < \infty$ such that $\beta(z, \gamma(z)) \le C$ for every $z \in \Gamma$. Suppose that A is a bounded operator on $\mathcal{Q}$ which has the property*

$$\lim_{\substack{z\in M\\ |z|\to 1}} \langle Ak_z, k_z\rangle = 0. \tag{6.11}$$

Then for every bounded set of coefficients $\{c_z : z \in \Gamma\}$, the operator

$$\sum_{z\in\Gamma} c_z \langle Ak_{\gamma(z)}, k_z\rangle k_z \otimes k_{\gamma(z)}$$

is compact.

Proof. Let $\{e_z : z \in \Gamma\}$ be an orthonormal set. We have the factorization

$$\sum_{z\in\Gamma} c_z\langle Ak_{\gamma(z)}, k_z\rangle k_z \otimes k_{\gamma(z)} = XTY,$$

where

$$X = \sum_{z\in\Gamma} c_z k_z \otimes e_z, \quad T = \sum_{z\in\Gamma}\langle Ak_{\gamma(z)}, k_z\rangle e_z \otimes e_z \quad \text{and} \quad Y = \sum_{z\in\Gamma} e_z \otimes k_{\gamma(z)}.$$

By Lemma 4.2, X and Y are bounded operators. Since γ has the property that $\beta(z, \gamma(z)) \le C$ for every $z \in \Gamma$, Proposition 6.7 tells us that (6.11) implies

$$\lim_{\substack{z\in\Gamma \\ |z|\to 1}} \langle Ak_{\gamma(z)}, k_z\rangle = 0.$$

Hence T is a compact operator. This completes the proof. $\square$

Proof of Theorem 6.1. By Corollary 4.6, it suffices to show that for any given $X, Y \in \mathcal{D}_0$, the operator XAY is compact. Furthermore, it suffices to assume that

$$X = \sum_{z\in\Gamma} a_z k_z \otimes k_z \quad \text{and} \quad Y = \sum_{w\in G} b_w k_w \otimes k_w,$$

where Γ and G are 1-separated sets in $M \cap K$ and the sets of coefficients $\{a_z : z \in \Gamma\}$ and $\{b_w : w \in G\}$ are bounded. We will decompose X and Y using the sets in Definition 6.4.

Let a large $m \ge 6$ be given. Define

$$F_m = \{z \in \Gamma : |z| < \rho_{3m}\} \quad \text{and} \quad \Gamma_m = \{z \in \Gamma : |z| \ge \rho_{3m}\}.$$

Then $X = T_m + X_m$, where

$$T_m = \sum_{z\in F_m} a_z k_z \otimes k_z \quad \text{and} \quad X_m = \sum_{z\in\Gamma_m} a_z k_z \otimes k_z.$$

Obviously, $\text{rank}(T_m) < \infty$. We need to further decompose X_m. By (6.6) and Definition 6.4, we have $\cup_{\omega\in I_m} A_\omega \supset \Gamma_m$. Therefore there is a partition

$$\Gamma_m = \bigcup_{\omega\in I_m} \Gamma_\omega \quad \text{such that } \Gamma_\omega \subset A_\omega \text{ for every } \omega \in I_m. \tag{6.12}$$

Accordingly, for each $\omega \in I_m$ we define

$$X_\omega = \sum_{z\in\Gamma_\omega} a_z k_z \otimes k_z. \tag{6.13}$$

Also, for each pair of $\kappa \in \{1, 2, 3, 4, 5, 6\}$ and $\nu \in \{1, \ldots, N_0\}$ we define

$$X_m^{(\nu,\kappa)} = \sum_{\omega \in I_m^{(\nu,\kappa)}} X_\omega. \tag{6.14}$$

Thus

$$X = T_m + \sum_{\kappa=1}^{6} \sum_{\nu=1}^{N_0} X_m^{(\nu,\kappa)}.$$

Because N_0 is a constant (see (6.8)), and because $\mathrm{rank}(T_m) < \infty$, to complete the proof, it suffices to show that for each pair of $\kappa \in \{1, 2, 3, 4, 5, 6\}$ and $\nu \in \{1, \ldots, N_0\}$, $X_m^{(\nu,\kappa)} AY$ is the sum of a compact operator and an operator of small norm when m is large.

To do that, let a pair of $\kappa \in \{1, 2, 3, 4, 5, 6\}$ and $\nu \in \{1, \ldots, N_0\}$ be given. We will decompose Y accordingly. Define

$$B_m^{(\nu,\kappa)} = \bigcup_{\omega \in I_m^{(\nu,\kappa)}} B_\omega. \tag{6.15}$$

Then

$$Y = S_m^{(\nu,\kappa)} + Y_m^{(\nu,\kappa)},$$

where

$$S_m^{(\nu,\kappa)} = \sum_{w \in G \backslash B_m^{(\nu,\kappa)}} b_w k_w \otimes k_w \quad \text{and} \quad Y_m^{(\nu,\kappa)} = \sum_{w \in G \cap B_m^{(\nu,\kappa)}} b_w k_w \otimes k_w.$$

Let us first show that $\|X_m^{(\nu,\kappa)} A S_m^{(\nu,\kappa)}\|$ is small when m is large. By (6.12) and (6.15), if $z \in \Gamma_\omega$ for some $\omega \in I_m^{(\nu,\kappa)}$ and if $w \in G \backslash B_m^{(\nu,\kappa)}$, then $w \notin B_\omega$. By Lemma 6.3(c), we have

$$\beta(z, w) \geq 2 \log m. \tag{6.16}$$

In other words, if we define

$$\Gamma_m^{(\nu,\kappa)} = \bigcup_{\omega \in I_m^{(\nu,\kappa)}} \Gamma_\omega,$$

then (6.16) holds for every pair of $z \in \Gamma_m^{(\nu,\kappa)}$ and $w \in G \backslash B_m^{(\nu,\kappa)}$. Since the union in (6.12) is a partition, i.e., $\Gamma_\omega \cap \Gamma_{\omega'} = \emptyset$ if $\omega \neq \omega'$, from (6.13) and (6.14) we see that

$$X_m^{(\nu,\kappa)} = \sum_{z \in \Gamma_m^{(\nu,\kappa)}} a_z k_z \otimes k_z.$$

Recall that we assume $A \in C^*(\mathcal{D})$. Hence it follows from (6.16) and Lemmas 5.3 and 4.2 that $\|X_m^{(\nu,\kappa)} A S_m^{(\nu,\kappa)}\|$ is small when m is large.

Thus what remains is to show that $X_m^{(\nu,\kappa)}AY_m^{(\nu,\kappa)}$ is the sum of a compact operator and an operator of small norm when m is large. To accomplish that goal, we partition the set $G \cap B_m^{(\nu,\kappa)}$ in the form

$$G \cap B_m^{(\nu,\kappa)} = \bigcup_{\omega \in I_m^{(\nu,\kappa)}} G_\omega, \quad \text{where } G_\omega \subset B_\omega \text{ for each } \omega \in I_m^{(\nu,\kappa)}.$$

Accordingly, we have

$$Y_m^{(\nu,\kappa)} = \sum_{\omega \in I_m^{(\nu,\kappa)}} Y_\omega, \quad \text{where } Y_\omega = \sum_{w \in G_\omega} b_w k_w \otimes k_w \text{ for each } \omega \in I_m^{(\nu,\kappa)}.$$

Recalling (6.14), we now have $X_m^{(\nu,\kappa)}AY_m^{(\nu,\kappa)} = D + W$, where

$$D = \sum_{\omega \in I_m^{(\nu,\kappa)}} X_\omega A Y_\omega \quad \text{and} \quad W = \sum_{\substack{\omega,\omega' \in I_m^{(\nu,\kappa)} \\ \omega \neq \omega'}} X_\omega A Y_{\omega'}.$$

Obviously,

$$D = \sum_{\omega \in I_m^{(\nu,\kappa)}} \sum_{(z,w) \in \Gamma_\omega \times G_\omega} a_z b_w \langle A k_w, k_z \rangle k_z \otimes k_w.$$

Recall from Lemma 6.2 that $B_\omega \subset D(z_\omega, R_m)$ for every $\omega \in I_m^{(\nu,\kappa)}$. Since $\Gamma_\omega \subset A_\omega$ and $G_\omega \subset B_\omega$, we have $\beta(z,w) < 2R_m$ for every $(z,w) \in \Gamma_\omega \times G_\omega$, $\omega \in I_m^{(\nu,\kappa)}$. Since G is 1-separated, there is a constant C_m such that $\text{card}(G_\omega) \le C_m$ for every $\omega \in I_m^{(\nu,\kappa)}$. Therefore it follows from Lemma 6.8 that D is a compact operator.

As the last step of the proof, we need to show that $\|W\|$ is small when m is large. To that end, we pick orthonormal sets $\{e_z : z \in \Gamma_m^{(\nu,\kappa)}\}$ and $\{u_w : w \in G \cap B_m^{(\nu,\kappa)}\}$. Define

$$K_\omega = \sum_{z \in \Gamma_\omega} e_z \otimes k_z \quad \text{and} \quad L_\omega = \sum_{w \in G_\omega} k_w \otimes u_w \tag{6.17}$$

for each $\omega \in I_m^{(\nu,\kappa)}$. We also define

$$U = \sum_{z \in \Gamma_m^{(\nu,\kappa)}} a_z k_z \otimes e_z \quad \text{and} \quad V = \sum_{w \in G \cap B_m^{(\nu,\kappa)}} b_w u_w \otimes k_w.$$

Then we can factor W in the form $W = UHV$, where

$$H = \sum_{\substack{\omega,\omega' \in I_m^{(\nu,\kappa)} \\ \omega \neq \omega'}} K_\omega A L_{\omega'}.$$

By Lemma 4.2 we have $\|U\| \le C_{4.2}a$ and $\|V\| \le C_{4.2}b$, where $a = \sup_{z\in\Gamma}|a_z|$ and $b = \sup_{w\in G}|b_w|$. Hence the proof will be complete if we can show that $\|H\|$ is small when m is large. To estimate $\|H\|$, for each $J \in \mathbf{N}$ we define

$$H_J = \sum_{\substack{\omega,\omega'\in I^{(\nu,\kappa)}_{m,J}\\ \omega\neq\omega'}} K_\omega A L_{\omega'}$$

(cf. Definition 6.4(b)). We have the strong convergence $H_J \to H$ as $J \to \infty$. Therefore there is a $J^* \in \mathbf{N}$ such that $\|H\| \le 2\|H_{J^*}\|$. Since $I^{(\nu,\kappa)}_{m,J^*}$ is a finite set, and since Corollary 6.5 tells us that $B_\omega \cap B_{\omega'} = \emptyset$ for $\omega \neq \omega'$ in $I^{(\nu,\kappa)}_{m,J^*}$, by Lemma 6.6, there is a subset F of $I^{(\nu,\kappa)}_{m,J^*}$ such that if we define

$$\Sigma = \sum_{\omega\in F} K_\omega, \quad \Lambda = \sum_{\omega\in F} L_\omega, \quad \Sigma' = \sum_{\omega\in I^{(\nu,\kappa)}_{m,J^*}\backslash F} K_\omega \quad \text{and} \quad \Lambda' = \sum_{\omega\in I^{(\nu,\kappa)}_{m,J^*}\backslash F} L_\omega,$$

then

$$\|H\| \le 2\|H_{J^*}\| \le 8\{\|\Sigma A\Lambda'\| + \|\Sigma' A\Lambda\|\}.$$

By (6.17), we have

$$\Sigma = \sum_{\omega\in F}\sum_{z\in\Gamma_\omega} e_z \otimes k_z \quad \text{and} \quad \Lambda' = \sum_{\omega\in I^{(\nu,\kappa)}_{m,J^*}\backslash F}\sum_{w\in G_\omega} k_w \otimes u_w.$$

Recall that $\Gamma_\omega \subset A_\omega$ and $G_{\omega'} \subset B_{\omega'}$. Again, for any pair of $\omega \in F$ and $\omega' \in I^{(\nu,\kappa)}_{m,J^*}\backslash F$, we have $B_\omega \cap B_{\omega'} = \emptyset$ by Corollary 6.5. Thus by Lemma 6.3(c), for such a pair of ω and ω', if $z \in \Gamma_\omega$ and $w \in G_{\omega'}$, then $\beta(z,w) \ge 2\log m$. Since $A \in C^*(\mathcal{D})$, we can apply Lemma 5.3 to conclude that $\|\Sigma A\Lambda'\|$ is small when m is large. Similarly, $\|\Sigma' A\Lambda\|$ is small when m is large. Therefore $\|H\|$ is small when m is large. This completes the proof. □

7. Compactness criterion in the Toeplitz algebra $\mathcal{TQ}$

Recall that for any $f \in L^\infty(S, d\sigma)$, we define the "Toeplitz operator"

$$Q_f h = Q(fh), \quad h \in \mathcal{Q},$$

on the quotient module $\mathcal{Q}$. We write $\mathcal{TQ}$ for the C^*-algebra generated by $\{Q_f : f \in L^\infty(S, d\sigma)\}$. We think of $\mathcal{TQ}$ as the "Toeplitz algebra" on the quotient module.

Lemma 7.1. *Given any $0 < \eta < 1$, there is a constant $0 < C_{7.1.} < \infty$ such that*

$$\int_S |k_z(u)||k_w(u)|d\sigma(u) \le C_{7.1}\frac{(1-|z|^2)^{(n/2)-\eta}(1-|w|^2)^{(n/2)-\eta}}{|1-\langle z,w\rangle|^{n-2\eta}} \tag{7.1}$$

for all $z, w \in \mathbf{B}$.

Proof. Given any $z, w \in \mathbf{B}$, let us write $x = \varphi_w(z)$. For $u \in S$, we have

$$1 - \langle \varphi_w(u), w\rangle = \frac{1-|w|^2}{1-\langle u, w\rangle} \quad \text{and} \quad 1 - \langle \varphi_w(u), z\rangle = \frac{(1-|w|^2)(1-\langle u, x\rangle)}{(1-\langle u, w\rangle)(1-\langle w, x\rangle)}.$$

Therefore

$$\frac{1-\langle \varphi_w(u), w\rangle}{1-\langle \varphi_w(u), z\rangle} = \frac{1-\langle w, x\rangle}{1-\langle u, x\rangle}.$$

Let $0 < \eta < 1$ be given. Starting with the unnormalized K_z and K_w, we have

$$\begin{aligned}
\int |K_z(u)K_w(u)|d\sigma(u) &= \frac{1}{(1-|w|^2)^n}\int |K_z(u)K_w^{-1}(u)||k_w(u)|^2 d\sigma(u) \\
&= \frac{1}{(1-|w|^2)^n}\int |K_z(\varphi_w(u))K_w^{-1}(\varphi_w(u))|d\sigma(u) \\
&= \frac{|1-\langle w, x\rangle|^n}{(1-|w|^2)^n}\int \frac{1}{|1-\langle u, x\rangle|^n} d\sigma(u) \le \frac{|1-\langle w, x\rangle|^n}{(1-|w|^2)^n}\cdot\frac{C_1}{(1-|x|^2)^\eta},
\end{aligned} \tag{7.2}$$

where for the $\le$ we cite [26, Proposition 1.4.10]. Since $x = \varphi_w(z)$, we have

$$\frac{|1-\langle w, x\rangle|^n}{(1-|w|^2)^n} = \frac{1}{|1-\langle w, z\rangle|^n} \quad \text{and} \quad \frac{1}{(1-|x|^2)^\eta} = \frac{|1-\langle w, z\rangle|^{2\eta}}{(1-|w|^2)^\eta(1-|z|^2)^\eta}.$$

Substituting these identities in (7.2), (7.1) follows. □

Proposition 7.2. *We have* $\mathcal{TQ} \subset C^*(\mathcal{D})$.

Proof. It suffices to show that $Q_f \in C^*(\mathcal{D})$ for every $f \in L^\infty(S, d\sigma)$. By Corollary 4.6(b), we only need to show that $XQ_fY \in C^*(\mathcal{D})$ for every pair of $X, Y \in \mathcal{D}_0$. As in the proof of Theorem 6.1, we can be more specific about X and Y; we assume that

$$X = \sum_{z\in\Gamma} a_z k_z \otimes k_z \quad \text{and} \quad Y = \sum_{w\in G} b_w k_w \otimes k_w,$$

where Γ and G are 1-separated sets in $M \cap K$ and the sets of coefficients $\{a_z : z \in \Gamma\}$ and $\{b_w : w \in G\}$ are bounded. Denote $a = \sup_{z\in\Gamma}|a_z|$ and $b = \sup_{w\in G}|b_w|$.

We can regard Q_f, X, Y as operators on $L^2(S, d\sigma)$. Thus

$$XQ_fY = XM_fY = \sum_{(z,w)\in\Gamma\times G} a_z b_w c_{z,w} k_z \otimes k_w,$$

where

$$c_{z,w} = \langle M_f k_w, k_z\rangle.$$

For any $r > 0$, we have the partition $\Gamma \times G = E_r \cup F_r$, where

$$E_r = \{(z, w) \in \Gamma \times G : \beta(z, w) \le r\} \quad \text{and} \quad F_r = \{(z, w) \in \Gamma \times G : \beta(z, w) > r\}.$$

Accordingly, $XQ_fY = D_r + W_r$, where

$$D_r = \sum_{(z,w)\in E_r} a_z b_w c_{z,w} k_z \otimes k_w \quad \text{and} \quad W_r = \sum_{(z,w)\in F_r} a_z b_w c_{z,w} k_z \otimes k_w.$$

Obviously, the set $\{a_z b_w c_{z,w} : (z,w) \in \Gamma \times G\}$ is bounded. There is a $C(r)$ such that for every $z \in \Gamma$, $\text{card}\{w \in G : \beta(z,w) \le r\} \le C(r)$. Hence D_r is in the linear span of $\mathcal{D}$. Thus the proof will be complete if we can show that $\|W_r\|$ is small when r is large.

To that end, we pick orthonormal sets $\{e_z : z \in \Gamma\}$, $\{u_w : w \in G\}$ and factor W_r in the form $W_r = UH_rV$, where

$$U = \sum_{z\in\Gamma} a_z k_z \otimes e_z, \quad H_r = \sum_{(z,w)\in F_r} c_{z,w} e_z \otimes u_w \quad \text{and} \quad V = \sum_{w\in G} b_w u_w \otimes k_w.$$

By Lemma 4.2, we have $\|U\| \le C_{4.2}a$ and $\|V\| \le C_{4.2}b$. Let $0 < \eta \le 1/4$ be chosen. Then from Lemma 7.1 we obtain

$$|c_{z,w}| \le \|f\|_\infty \langle |k_z|, |k_w| \rangle \le C_{7.1}\|f\|_\infty \frac{(1-|z|^2)^{(n/2)-\eta}(1-|w|^2)^{(n/2)-\eta}}{|1-\langle z,w\rangle|^{n-2\eta}}$$

for all $(z,w) \in \Gamma \times G$. Recalling the definition of F_r, from Lemma 5.1 we see that $\|H_r\|$ is small when r is large. Thus $\|W_r\|$ is small when r is large. This completes the proof. □

Below is the most significant application of Theorem 6.1:

Theorem 7.3. *Let $A \in \mathcal{TQ}$. If*

$$\lim_{\substack{z\in M \\ |z|\to 1}} \langle Ak_z, k_z \rangle = 0,$$

then A is a compact operator.

Proof. This is an immediate consequence of Proposition 7.2 and Theorem 6.1. □

8. Essential normality

We will now show that the quotient module $\mathcal{Q}$ is p-essentially normal for $p > d$. For this purpose, just as in [28], it will be convenient to get certain Lorentz-like ideals involved.

For each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k\ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}}$$

defines a symmetric norm for operators. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [17]. It is well known that $\mathcal{C}_p^+ \subset \mathcal{C}_{p'}$ for all $1 \le p < p' < \infty$.

The reason why the $\mathcal{C}_p^+$'s are the preferred ideals in the study of the Arveson-Douglas conjecture is that norm estimates in these ideals are particularly easy:

Lemma 8.1. [28, Lemma 2.9] *Given any positive numbers $0 < a \le b < \infty$, there is a constant $0 < B(a,b) < \infty$ such that the following holds true: Let $\mathcal{H}$ be a Hilbert space, and suppose that $F_0, F_1, \dots, F_k, \dots$ are operators on $\mathcal{H}$ such that the following two conditions are satisfied for every k:*
(1) $\|F_k\| \le 2^{-ak}$,
(2) $\mathrm{rank}(F_k) \le 2^{bk}$.
Then the operator $F = \sum_{k=0}^{\infty} F_k$ satisfies the estimate $\|F\|_{b/a}^+ \le B(a,b)$. In particular, $F \in \mathcal{C}_{b/a}^+$.

Lemma 8.2. *Given any $\epsilon > 0$, there is a constant $0 < C_{8.2} = C_{8.2}(\epsilon) < \infty$ such that the following holds true: Let Γ be a 1-separated set in $M \cap K$ and let $\{e_z : z \in \Gamma\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Then the operator*

$$T = \sum_{z,w\in\Gamma} \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|w|^2)^{(d+\epsilon)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} e_z \otimes e_w$$

satisfies the estimate $\|T\| \le C_{8.2}$.

Proof. Recall from Proposition 2.14 that $(1-|w|^2)^{d+1} \le C_1 v_M(D(w,1)\cap M)$ for $w \in M\cap K$. Also, if $\xi \in D(w,1)\cap M$, then

$$\frac{(1-|w|^2)^{-1+(\epsilon/2)}}{|1-\langle z,w\rangle|^{d+\epsilon}} \le C_2 \frac{(1-|\xi|^2)^{-1+(\epsilon/2)}}{|1-\langle z,\xi\rangle|^{d+\epsilon}}.$$

Define $h(w) = (1-|w|^2)^{d/2}$ for $w \in \Gamma$. For each $z \in \Gamma$ we have

$$\begin{aligned}
&\sum_{w\in\Gamma} \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|w|^2)^{(d+\epsilon)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} h(w) \\
&\quad\le C_3 \sum_{w\in\Gamma} \int_{D(w,1)\cap M} \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|\xi|^2)^{-1+(\epsilon/2)}}{|1-\langle z,\xi\rangle|^{d+\epsilon}} dv_M(\xi) \\
&\quad\le C_3(1-|z|^2)^{d/2} \int_M \frac{(1-|z|^2)^{\epsilon/2}(1-|\xi|^2)^{-1+(\epsilon/2)}}{|1-\langle z,\xi\rangle|^{d+1+(\epsilon/2)-1+(\epsilon/2)}} dv_M(\xi) \\
&\quad\le C_4(1-|z|^2)^{d/2} = C_4 h(z),
\end{aligned}$$

where the third $\le$ follows from Lemma 2.10. By the Schur test, we have $\|T\| \le C_4$. □

Proposition 8.3. *Let $X \in \mathcal{D}_0$, which we also consider as an operator on $L^2(S, d\sigma)$. If f is a Lipschitz function on S, then $[M_f, X]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

Proof. As before, we can be more specific about X. That is, we only need to consider

$$X = \sum_{z\in\Gamma} c_z k_z \otimes k_z,$$

where Γ is a 1-separated set in $M \cap K$ and the set $\{c_z : z \in \Gamma\}$ is bounded. Let $p > 2d$ be given. Then pick an $0 < \epsilon < 1/2$ such that

$$2d/(1-\epsilon) < p. \tag{8.1}$$

Given an $f \in \mathrm{Lip}(S)$, we have $[M_f, X] = F - G$, where

$$F = \sum_{z\in\Gamma} c_z\{(f - f(z/|z|))k_z\} \otimes k_z \quad \text{and} \quad G = \sum_{z\in\Gamma} c_z k_z \otimes \{\overline{(f - f(z/|z|))}k_z\}.$$

Since G^* is just another F, it suffices to deal with F.

For each $k \geq 0$, define

$$M_k = \{z \in M : 1 - 2^{-2k} \leq |z| < 1 - 2^{-2(k+1)}\}$$

and $\Gamma_k = \Gamma \cap M_k$. For each $k \geq 0$, we further define

$$F_k = \sum_{z\in\Gamma_k} c_z\{(f - f(z/|z|))k_z\} \otimes k_z.$$

Since $F = \sum_{k=0}^{\infty} F_k$, our goal is to apply Lemma 8.1. For this purpose, we need to estimate $\|F_k\|$ and $\mathrm{rank}(F_k)$. But since the estimate for $\mathrm{rank}(F_k)$ only involves $\mathrm{card}(\Gamma_k)$, it is the same as that in the proof of [28, Proposition 3.5]. In fact, by (3.5) in [28], we have

$$\mathrm{rank}(F_k) \leq C2^{2dk} \tag{8.2}$$

for every $k \geq 0$. (See [28, page 1080] for the proof.) But the estimate for $\|F_k\|$ is different, because we are now working on the Hardy space, not the Bergman space in [28].

Let $\{e_z : z \in \Gamma\}$ be an orthonormal set. Then we have $F_k = A_k H$, where

$$A_k = \sum_{z\in\Gamma_k} \{(f - f(z/|z|))k_z\} \otimes e_z \quad \text{and} \quad H = \sum_{z\in\Gamma} c_z e_z \otimes k_z.$$

By Lemma 4.2, $\|H\| \leq C_{4.2}c$, where $c = \sup_{z\in\Gamma} |c_z|$. For each $k \geq 0$, we have

$$A_k^* A_k = \sum_{z,w\in\Gamma_k} a_{z,w} e_z \otimes e_w,$$

where

$$a_{z,w} = \langle (f - f(w/|w|))k_w, (f - f(z/|z|))k_z \rangle$$

for $z, w \in \Gamma$. For $z \in \Gamma$ and $u \in S$, we have

$$\begin{aligned} |f(u) - f(z/|z|)| &\le L(f)|u - (z/|z|)| \\ &\le \sqrt{2}L(f)|1 - \langle u, z/|z|\rangle|^{1/2} \le 2L(f)|1 - \langle u, z\rangle|^{1/2}, \end{aligned}$$

where $L(f)$ is the Lipschitz constant for f. Thus for every pair of $z, w \in \Gamma$,

$$|a_{z,w}| \le C_1 \int_S \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle u,z\rangle|^{n-(1/2)}|1-\langle u,w\rangle|^{n-(1/2)}} d\sigma(u).$$

Note that $n - (1/2) = \{n - 1 + \epsilon\} + \{(1/2) - \epsilon\}$. Using triangle inequality (5.3) again, by the argument following it in the proof of Lemma 5.3, this time we have

$$|a_{z,w}| \le C_2 \frac{(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}}{|1-\langle z,w\rangle|^{n-1+\epsilon}},$$

$z, w \in \Gamma$. Since $d \le n - 1$, this means

$$\begin{aligned} |a_{z,w}| &\le C_3 \frac{(1-|z|^2)^{(d+1)/2}(1-|w|^2)^{(d+1)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} \\ &= C_3 \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|w|^2)^{(d+\epsilon)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} (1-|z|^2)^{(1-\epsilon)/2}(1-|w|^2)^{(1-\epsilon)/2}. \end{aligned}$$

But for $z, w \in \Gamma_k$ specifically, this means

$$|a_{z,w}| \le C_3 \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|w|^2)^{(d+\epsilon)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} (2^{-2k+1})^{1-\epsilon}.$$

Combining this with Lemma 8.2, we find that $\|A_k^* A_k\| \le C_3 C_{8.2} (2^{-2k+1})^{1-\epsilon}$. Thus

$$\|F_k\| \le \|A_k\|\|H\| \le C_4 2^{-(1-\epsilon)k}$$

for every $k \ge 0$. Recalling (8.2), we can now apply Lemma 8.1 to conclude that $F \in \mathcal{C}^+_{2d/(1-\epsilon)}$. By (8.1), this means $F \in \mathcal{C}_p$ as promised. This completes the proof. □

Proposition 8.4. *For any Lipschitz function f on S, the commutator $[M_f, Q]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

Proof. Again, consider the operator T_ϵ defined in the statement of Proposition 4.3, $0 < \epsilon < 1$. As we explained in the proof of Proposition 4.5, if ϵ is small enough, then it follows from Theorem 3.5 and Proposition 4.3 that T_ϵ is invertible on $\mathcal{Q}$. This means that on $L^2(S, d\sigma)$, the spectrum of the positive operator T_ϵ is contained in $\{0\} \cup [c, C]$ for some $0 < c < C < \infty$, and that the spectral measure of T_ϵ corresponding to the interval $[c, C]$ equals Q. Therefore there is an $h \in C_c^\infty(\mathbf{R})$ such that $Q = h(T_\epsilon)$.

We have $T_\epsilon \in \mathcal{D}_0$ by definition. Therefore, by Proposition 8.3, if $f \in \mathrm{Lip}(S)$, then $[M_f, T_\epsilon] \in \mathcal{C}_p$ for every $p > 2d$. By the well-known facts about smooth functional calculus, we have $[M_f, h(T_\epsilon)] \in \mathcal{C}_p$ for every $p > 2d$. Since $h(T_\epsilon) = Q$, this completes the proof. □

We end the paper with

Theorem 8.5. *The quotient module $\mathcal{Q}$ is p-essentially normal for every $p > d$.*

Proof. Recalling (1.1), for $i, j \in \{1, \dots, n\}$ we have

$$\begin{aligned}[\mathcal{Z}_{\mathcal{Q},i}^*, \mathcal{Z}_{\mathcal{Q},j}] &= QM_{\bar{z}_i}QM_{z_j}Q - QM_{z_j}QM_{\bar{z}_i}Q \\ &= [Q, M_{z_j}](1-Q)[M_{\bar{z}_i}, Q] - [Q, M_{\bar{z}_i}](1-Q)[M_{z_j}, Q].\end{aligned}$$

Proposition 8.4 tells us that $[Q, M_{\bar{z}_i}]$ and $[M_{z_j}, Q]$ are in the Schatten class $\mathcal{C}_t$ for every $t > 2d$. Consequently, $[\mathcal{Z}_{\mathcal{Q},i}^*, \mathcal{Z}_{\mathcal{Q},j}]$ is in the Schatten class $\mathcal{C}_p$ for every $p > d$. □

References

1. W. Arveson, p-Summable commutators in dimension d, J. Operator Theory **54** (2005), 101-117.
2. W. Arveson, Myhill Lectures, SUNY Buffalo, April 2006.
3. W. Arveson, Quotients of standard Hilbert modules, Trans. Amer. Math. Soc. **359** (2007), 6027-6055.
4. S. Axler and D. Zheng, Compact operators via the Berezin transform, Indiana Univ. Math. J. **47** (1998), 387-400.
5. W. Bauer and J. Isralowitz, Compactness characterization of operators in the Toeplitz algebra of the Fock space F_α^p, J. Funct. Anal. **263** (2012), 1323-1355.
6. S. Biswas and O. Shalit, Stable division and essential normality: the non-homogeneous and quasi-homogeneous cases, Indiana Univ. Math. J. **67** (2018), 169-185.
7. X. Chen and K. Guo, Analytic Hilbert modules. Chapman & Hall/CRC Research Notes in Mathematics, **433**. Chapman & Hall/CRC, Boca Raton, 2003.
8. J. Cima and W. Wogen, A Carleson measure theorem for the Bergman space on the ball, J. Operator Theory **7** (1982), 157-165.
9. E. Chirka, Complex Analytic Sets. Translated from the Russian by R. A. M. Hoksbergen. Mathematics and its Applications (Soviet Series), **46** Kluwer Academic Publishers Group, Dordrecht, 1989.
10. R. Douglas, A new kind of index theorem, Analysis, geometry and topology of elliptic operators, 369-382, World Sci. Publ., Hackensack, NJ, 2006.
11. R. Douglas and V. Paulsen, Hilbert modules over function algebras. Pitman Research Notes in Mathematics Series, **217**. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.
12. R. Douglas and K. Wang, Essential normality of cyclic submodule generated by any polynomial, J. Funct. Anal. **261** (2011), 3155-3180.
13. R. Douglas, X. Tang and G. Yu, An analytic Grothendieck Riemann Roch theorem, Adv. Math. **294** (2016), 307-331.
14. R. Douglas and Yi Wang, Geometric Arveson Douglas conjecture and holomorphic extension, Indiana Univ. Math. J. **66** (2017), 1499-1535.

15. M. Engliš and J. Eschmeier, Geometric Arveson-Douglas conjecture, Adv. Math. **274** (2015), 606-630.
16. Q. Fang, Y. Wang and J. Xia, The Helton-Howe trace formula for submodules, J. Funct. Anal. **281** (2021), no. 1, 108997.
17. I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
18. K. Guo and K. Wang, Essentially normal Hilbert modules and K-homology, Math. Ann. **340** (2008), 907-934.
19. K. Guo and D. Zheng, The distribution function inequality for a finite sum of finite products of Toeplitz operators, J. Funct. Anal. **218** (2005), 1-53.
20. L. Hörmander, L^p Estimates for (pluri-) harmonic functions, Math. Scand. **20** (1967), 65-78.
21. J. Isralowitz, M. Mitkovski and B. Wick, Localization and compactness in Bergman and Fock spaces, Indiana Univ. Math. J. **64** (2015), 1553-1573.
22. M. Kennedy and O. Shalit, Essential normality and the decomposability of algebraic varieties, New York J. Math. **18** (2012), 877-890.
23. S. Krantz, Function theory of several complex variables. Pure and Applied Mathematics. A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1982.
24. J. Munkres, Analysis on manifolds, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1991.
25. R. Range, Holomorphic functions and integral representations in several complex variables. Graduate Texts in Mathematics **108**, Springer-Verlag, New York, 1986.
26. W. Rudin, Function theory in the unit ball of $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
27. D. Suárez, The essential norm of operators in the Toeplitz algebra on Ap(Bn), Indiana Univ. Math. J. **56** (2007), 2185-2232.
28. Y. Wang and J. Xia, Essential normality for quotient modules and complex dimensions, J. Funct. Anal. **276** (2019), 1061-1096.
29. Y. Wang and J. Xia, Essential commutants on strongly pseudo-convex domains, J. Funct. Anal. **280** (2021), no. 1, 108775.
30. J. Xia, A double commutant relation in the Calkin algebra on the Bergman space, J. Funct. Anal. **274** (2018), 1631-1656.
31. J. Xia and D. Zheng, Localization and Berezin transform on the Fock space, J. Funct. Anal. **264** (2013), 97-117.

Yi Wang
Department of Mathematics, Vanderbilt University, Nashville, TN 37240
E-mail: yiwangfdu@gmail.com

Jingbo Xia
Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260
E-mail: jxia@acsu.buffalo.edu