GEOMETRIC ARVESON-DOUGLAS CONJECTURE FOR THE DRURY-ARVESON SPACE: THE CASE OF ONE-DIMENSIONAL VARIETY

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Abstract. We consider a class of analytic subsets $\tilde{M}$ of an open neighborhood of the closed unit ball in $\mathbf{C}^n$. Such an $\tilde{M}$ gives rise to a submodule $\mathcal{R}$ and a quotient module $\mathcal{Q}$ of the Drury-Arveson module H_n^2 in n variables. The geometric Arveson-Douglas conjecture predicts that the quotient module $\mathcal{Q}$ is p-essentially normal for $p > d = \dim_{\mathbf{C}}\tilde{M}$. We prove this conjecture for the case of dimension $d = 1$. In fact, we prove that if $d = 1$, then $\mathcal{Q}$ is 1-essentially normal, which is a stronger result than the original prediction.

1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle} \tag{1.1}$$

as its reproducing kernel [1,16]. Equivalently, H_n^2 can be described as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle h, g\rangle = \sum_{\alpha\in\mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} a_\alpha\zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} b_\alpha\zeta^\alpha.$$

Here and throughout, we follow the standard multi-index notation [26, page 3].

Nowadays, it is common to view H_n^2 as a Hilbert module over the polynomial ring $\mathbf{C}[\zeta_1, \dots, \zeta_n]$ [6,12]. Thus H_n^2 has submodules and quotient modules.

Suppose that $\mathcal{N}$ is either a submodule or a quotient module of the Drury-Arveson module H_n^2. Let $P_{\mathcal{N}} : H_n^2 \to \mathcal{N}$ be the orthogonal projection. Then we have the module operators

$$\mathcal{Z}_{\mathcal{N},j} = P_{\mathcal{N}} M_{\zeta_j}|\mathcal{N}, \quad j = 1, \dots, n,$$

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on $\mathcal{N}$. Recall that $\mathcal{N}$ is said to be p-essentially normal if all commutators $[\mathcal{Z}_{\mathcal{N},i}, \mathcal{Z}^*_{\mathcal{N},j}]$, $1 \leq i, j \leq n$, are in the Schatten class $\mathcal{C}_p$.

The famous Arveson Conjecture [2-4] predicts that every graded submodule of $H^2_n \otimes \mathbf{C}^r$ is p-essentially normal for $p > n$. To date, the best results on the Arveson Conjecture are due to Guo and K. Wang [20].

In addition to graded submodules, quotient modules of the form $H^2_n/[I]$, where I is a homogeneous ideal in $\mathbf{C}[\zeta_1, \ldots, \zeta_n]$, were also studied in [20]. Motived by the results in [20], Douglas observed in [10] that for quotient modules the essential normality should really be $p > d$, where d is the complex dimension of the variety involved. This more refined conjecture is now called Arveson-Douglas Conjecture. See [5,11,13-15,17,19,20-22,27-29] for the tremendous progress that has been made in this direction.

In this paper we consider a very specific class of submodules and the corresponding quotient modules. Our focus is on the quotient modules, because that is where things become really interesting. Let $\tilde{M}$ be an analytic subset [8] of an open neighborhood of $\overline{\mathbf{B}}$ with $1 \leq \dim_{\mathbf{C}} \tilde{M} \leq n - 1$. We will assume that $\tilde{M}$ has no singular points on the sphere $S = \{z \in \mathbf{C}^n : |z| = 1\}$ and that $\tilde{M}$ intersects S transversely. Denote $M = \mathbf{B} \cap \tilde{M}$. Then we have a submodule

$$\mathcal{R} = \{f \in H^2_n : f = 0 \text{ on } M\}$$

of H^2_n. The corresponding quotient module is

$$\mathcal{Q} = H^2_n \ominus \mathcal{R}.$$

In this setting, we have

Geometric Arveson-Douglas Conjecture. The quotient module $\mathcal{Q}$ is p-essentially normal for every $p > d = \dim_{\mathbf{C}} \tilde{M}$.

Since the Drury-Arverson module H^2_n itself is known to be p-essentially normal for $p > n$ [1], by a well-known result of Arveson [2], the geometric Arveson-Douglas conjecture implies that the submodule $\mathcal{R}$ is p-essentially normal for $p > n$.

The analogous problems for the Bergman module $L^2_a(\mathbf{B})$ and the Hardy module $H^2(S)$ were recently solved [15,27,28]. Thus it is logical for us to consider the Arveson-Douglas conjecture for H^2_n. But the case of the Drury-Arverson space H^2_n poses significant challenges. In fact, it is very easy to describe the main difficulty. Note that the power for the denominator on the right-hand side of (1.1) is only 1. By contrast, the reproducing kernels for Bergman space $L^2_a(\mathbf{B})$ and the Hardy space $H^2(S)$ are

$$K^{\text{Berg}}_z(\zeta) = \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1}} \quad \text{and} \quad K^{\text{Har}}_z(\zeta) = \frac{1}{(1 - \langle \zeta, z \rangle)^{n}},$$

which have powers $n+1$ and n for the denominator respectively. Because of the necessary estimates involved, the smaller the power of the reproducing kernel, the harder it is to prove essential normality. Indeed by a comparison of [15,27] with [28], we can already see a significant rise of the level of difficulty when that power is reduced from $n+1$ to n.

The challenge we face in this paper is to reduce the power of the reproducing kernel from n all the way to 1 and still prove essential normality. As of this writing, we have managed to overcome this challenge only in the case $d = 1$. But the case $d = 1$ is also arguably the most interesting one, for in this case we have a stronger result than the prediction of the Arveson-Douglas conjecture:

Theorem 1.1. *Suppose that* $\dim_{\mathbf{C}}\tilde{M} = 1$. *Then the quotient module* $\mathcal{Q}$ *is* 1*-essentially normal, i.e., every commutator* $[\mathcal{Z}_{\mathcal{Q},i}, \mathcal{Z}^*_{\mathcal{Q},j}]$ *is in the trace class* $\mathcal{C}_1$, $i, j \in \{1, \dots, n\}$.

The rest of the paper is devoted to the long proof of this result. We conclude the Introduction with a brief discussion of the main steps in the proof and the organization of the paper, which should give the reader some idea why the proof is as long as it is.

Our proof of Theorem 1.1 begins with the preliminaries in Section 2. Specifically, in Section 2 we first record the precise definitions of $\tilde{M}$, $\mathcal{R}$, $\mathcal{Q}$, etc, and then we collect a number of previously-established results that will be needed in the subsequent sections.

As we have already mentioned, a major difficulty we face is that the power in (1.1) is too small. Our main idea of dealing with this is to increase the power of the denominator by differentiation. Since we only consider the case $d = 1$ in the proof of Theorem 1.1, one order of derivative will increase the power of the denominator to 2, which should be enough based on dimensional considerations. But derivative has to be taken very carefully in the following sense. For $z \in M$, we have $K_z \in \mathcal{Q}$, and differentiation modifies the kernel K_z. We must make sure that the modified kernel still belongs to the quotient module $\mathcal{Q}$. Thus we can only differentiate in the directions tangential to M.

To carry out the idea explained above, starting Section 3 we assume $d = 1$. We consider a smooth part $\mathcal{M}$ of $\tilde{M}$ near S. For each $w \in \mathcal{M}$, let p_w be the orthogonal projection of w on the tangent space T_w. The transversality of $\tilde{M}$ implies that $p_w \neq 0$ if $w \in \tilde{M} \cap S$. Thus if $\mathcal{M}$ is a part of $\tilde{M}$ sufficiently near S, then $p_w \neq 0$ for every $w \in \mathcal{M}$. This gives us a non-vanishing cross section $w \mapsto p_w$ of the complex tangent bundle of $\mathcal{M}$. For $w \in \mathcal{M} \cap \mathbf{B}$, the kernel

$$K_{w,p_w}(\zeta) = \frac{\langle \zeta, p_w \rangle}{(1 - \langle \zeta, w \rangle)^2}$$

reproduces the derivative in the direction of p_w. That is, $\langle f, K_{w,p_w} \rangle = (\partial_{p_w} f)(w)$ for $f \in H^2_n$. Moreover, because $p_w \in T_w$, we have $K_{w,p_w} \in \mathcal{Q}$. Specific to the case $d = 1$, we introduce the measure

$$d\mu(w) = (1 - |w|^2) dv_M(w)$$

on M. This naturally leads to the operator

$$T_1 = \int_{M^{(t_0)}} K_{w,p_w} \otimes K_{w,p_w} d\mu(w)$$

on H^2_n, where $M^{(t_0)}$ is a carefully chosen subset of $\mathcal{M} \cap \mathbf{B}$. A major step in the proof of Theorem 1.1 is Theorem 3.5, which says that the spectrum of the positive operator T_1 does not intersect the interval $(0, c)$ for some $c > 0$. Note that even though the cross section

$w \mapsto p_w$ is non-vanishing on $\mathcal{M}$, in general it is not analytic. But the condition $d = 1$ means that, *locally*, $w \mapsto p_w$ is an analytic cross section multiplied by a scalar function. We will use this fact in the proof of Theorem 3.5.

In addition to T_1, in Section 4 we introduce the more conventional operator

$$T_2 = \int_{M^{(t_0)}} K_w \otimes K_w d\mu(w).$$

For our purpose, the operator that really matters is $T = T_1 + T_2$. Theorem 3.5 allows us to show that there is a $c' > 0$ such that the spectrum of T does not intersect $(0, c')$, and that Q, the orthogonal projection from H_n^2 onto $\mathcal{Q}$, equals the spectral projection of T corresponding to the interval $[c', \infty)$. In other words, we have a practical control of the orthogonal projection $Q : H_n^2 \to \mathcal{Q}$ through the operators T_1 and T_2, which are given by explicit formulas.

We then introduce a particular Hilbert space $\mathcal{L}$ in Section 4, which can be thought of as a collection of functions on $M^{(t_0)}$ with a particular norm $\|\cdot\|_{\#}$. Let $\mathcal{P}$ be the closure of the analytic polynomials $\mathbf{C}[\zeta_1, \dots, \zeta_n]$ in $\mathcal{L}$. The norm $\|\cdot\|_{\#}$ has the property that

$$\langle Tf, f\rangle = \|f\|_{\#}^2$$

for every $f \in H_n^2$. This leads to the operator J, which take each $f \in H_n^2$ to the same function f in $\mathcal{P}$. We think of J as *restricting* each $f \in H_n^2$ to the set $M^{(t_0)}$. The above identity means that

$$J^*J = T.$$

Thus it follows from the properties of T that $J : \mathcal{Q} \to \mathcal{P}$ is invertible. We call $\mathcal{P}$ the *range space* of the restriction operator J. One can think of $\mathcal{P}$ as a representation of the quotient module $\mathcal{Q}$ that is more accessible.

Accordingly, the operators T_1 and T_2 also have their representations $\hat{T}_1$ and $\hat{T}_2$ on $\mathcal{L}$. Individually, the operators $\hat{T}_1$ and $\hat{T}_2$ are not self-adjoint on $\mathcal{L}$. It is, therefore, something of a miracle that the sum $\hat{T} = \hat{T}_1 + \hat{T}_2$ actually is self-adjoint on $\mathcal{L}$. From this self-adjointness it follows that, with respect to the space decomposition $\mathcal{L} = \mathcal{P} \oplus \mathcal{P}^{\perp}$, we have the operator decomposition $\hat{T} = \tilde{T} \oplus 0$, where $\tilde{T} = JJ^*$. This means that the orthogonal projection $P : \mathcal{L} \to \mathcal{P}$ can be expressed in the form $P = \xi(\hat{T})$ for some $\xi \in C_c^{\infty}(\mathbf{R})$. This converts the proof of the 1-essential normality in Theorem 1.1 to a problem in terms of commutators and double commutators on $\mathcal{L}$ that are much more accessible than the ones on the Drury-Arveson space H_n^2. But the actual handling of these commutators and double commutators on $\mathcal{L}$ is quite tedious: it takes the work in Sections 5, 6 and 7 to finally obtain the 1-essential normality promised in Theorem 1.1.

There is a major difference between proving essential normality in the case of H_n^2 and the corresponding task in the case of the Bergman space $L_a^2(\mathbf{B})$ or the Hardy space $H^2(S)$. In the two latter cases, the commuting tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ is jointly subnormal, i.e., it extends to a tuple of multiplication operators $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on an L^2-space. On the L^2-space, we have $M_{\zeta_j}^* = M_{\bar{\zeta}_j}$, $1 \le j \le n$. More to the point, M_{ζ_j} commutes with $M_{\zeta_i}^*$ on the L^2-space, which is a fact heavily involved in [27,28]. In contrast, on the Drury-Arveson space H^2_n, the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ is known not to be jointly subnormal [1]. This creates an additional obstacle to the proof of essential normality on H^2_n.

Because $(M_{\zeta_1}, \dots, M_{\zeta_n})$ is not jointly subnormal on H^2_n, the only way to obtain the desired essential normality for $\mathcal{Q}$ is through the pair of spaces $\mathcal{P} \subset \mathcal{L}$. We have the tuple of multiplication operators $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on $\mathcal{P}$ with the relation

$$M_{\zeta_i} J = J \mathcal{Z}_{\mathcal{Q},i},$$

$1 \leq i \leq n$. The tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on $\mathcal{P}$ naturally extends to the commuting tuple $(\hat{M}_{\zeta_1}, \dots, \hat{M}_{\zeta_n})$ on $\mathcal{L}$. On $\mathcal{L}$, we still have $\hat{M}^*_{\zeta_i} \neq \hat{M}_{\bar{\zeta}_i}$, but the difference $\hat{M}^*_{\zeta_i} - \hat{M}_{\bar{\zeta}_i}$ can be computed explicitly. In fact, the handling of this difference is a significant part of Sections 5, 6 and 7. But what matters is the fact that in the end, this approach does lead to a proof of Theorem 1.1.

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2. Preliminaries

Although the actual work of this paper only concerns the case $\dim_{\mathbf{C}} \tilde{M} = 1$, due to the need to cite a number of existing results, it is necessary to introduce the general technical framework. Much of the material in this section is cited from [28, Section 2].

We begin with the precise definitions of the analytic sets, submodules and quotient modules that we consider in this paper.

Definition 2.1. [8] Let Ω be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of Ω if for each point $a \in \Omega$ there are a neighborhood U of a and functions $f_1, \cdots, f_N$ analytic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a *singular point* of A if it is not regular.

Definition 2.2. Let Y be a manifold and let X, Z be submanifolds of Y. We say that the submanifolds X and Z *intersect transversely* if for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.

Assumption 2.3. Let $\tilde{M}$ be an analytic subset in an open neighborhood of the closed ball $\overline{\mathbf{B}}$. Furthermore, $\tilde{M}$ satisfies the following conditions:

(1) $\tilde{M}$ intersects $\partial\mathbf{B}$ transversely.
(2) $\tilde{M}$ has no singular points on $\partial\mathbf{B}$.
(3) $\tilde{M}$ is of pure dimension d, where $1 \leq d \leq n-1$.

Note that condition (3) implies that $\tilde{M}$ has no isolated singularities in $\mathbf{B}$.

We emphasize that Assumption 2.3 will always be in force for the rest of the paper. Given such an $\tilde{M}$, we fix M, $\mathcal{R}$, $\mathcal{Q}$ and Q as follows.

Notation 2.4. (a) Let $M = \tilde{M} \cap \mathbf{B}$.

(b) Denote $\mathcal{R} = \{f \in H_n^2 : f = 0 \text{ on } M\}$.

(c) Denote $\mathcal{Q} = H_n^2 \ominus \mathcal{R}$.

(d) Let Q be the orthogonal projection from H_n^2 onto $\mathcal{Q}$.

In addition, we will simplify the notation for module operators used in the Introduction. Namely, we will write $Q_{\zeta_j} = \mathcal{Z}_{\mathcal{Q},j}$ for $j = 1, \dots, n$. That is,

$$Q_{\zeta_j} = QM_{\zeta_j}|\mathcal{Q},$$

$j = 1, \dots, n$. Thus the goal of the paper is to show that under the condition $d = 1$, we have $[Q_{\zeta_i}, Q_{\zeta_j}^*] \in \mathcal{C}_1$ for all $i, j \in \{1, \dots, n\}$. But this will take a very long journey.

Denote $S = \{z \in \mathbf{C}^n : |z| = 1\}$, the unit sphere in $\mathbf{C}^n$. For $z \in \mathbf{C}^n$ and $r > 0$, denote

$$B(z, r) = \{w \in \mathbf{C}^n : |z - w| < r\}.$$

By Assumption 2.3, there is an $s \in (0, 1)$ such that

$$\mathcal{M} = \{z \in \tilde{M} : 1 - s < |z| < 1 + s\} \tag{2.1}$$

is a complex manifold of complex dimension d and of finite volume.

For each $z \in \mathcal{M}$, let T_z be the tangent space to $\mathcal{M}$ at the point z, viewed as a natural subspace of $\mathbf{C}^n$. For each $z \in \mathcal{M}$, let p_z be the orthogonal projection of z on T_z. Condition (1) in Assumption 2.3 says that if $z \in \tilde{M} \cap S$, then $p_z \neq 0$. Thus, reducing the value of $s \in (0, 1)$ if necessary, we may assume that there is a $\gamma > 0$ such that

$$|p_z| \geq \gamma \quad \text{for every} \quad z \in \mathcal{M}. \tag{2.2}$$

Denote

$$K = \{z \in \tilde{M} : 1 - (s/2) \leq |z| \leq 1\}. \tag{2.3}$$

Then K is a compact subset of the complex manifold $\mathcal{M}$. By the standard facts known about such a pair of $\mathcal{M}$ and K [23-25], the statements below hold true with constants that are independent of $z \in K$.

There are $a > 0$ and $b > 0$ such that for each $z \in K$, there is a map

$$G_z : T_z \cap B(0, a) \to \mathcal{M} \tag{2.4}$$

that biholomorphically maps $T_z \cap B(0, a)$ onto an open subset of $\mathcal{M}$ with the properties that $G_z(0) = z$ and that

$$\{G_z(w) : w \in T_z \cap B(0, a)\} \supset \mathcal{M} \cap B(z, b). \tag{2.5}$$

Let DG_z be the complex derivative of G_z. For each $w \in T_z \cap B(0, a)$, we have the local Taylor expansion

$$G_z(w+u) = G_z(w) + (DG_z)(w)u + \int_0^1 \{(DG_z)(w+tu) - (DG_z)(w)\}udt,$$

$w + u \in T_z \cap B(0, a)$. In particular, at the point $w = 0$ we have

$$T_z = (DG_z)(0)T_z$$

and

$$G_z(u) = z + (DG_z)(0)u + \int_0^1 \{(DG_z)(tu) - (DG_z)(0)\}udt \quad \text{for} \quad u \in T_z \cap B(0, a).$$

Reducing the values of a and b if necessary, we may assume that there are constants $0 < \alpha \le \beta < \infty$ such that for $w \in T_z \cap B(0, a)$, the linear transformation inequality

$$\alpha \le (DG_z)^*(w)(DG_z)(w) \le \beta$$

holds on T_z.

For each $z \in K$,

$$T_z^\perp = \{u \in T_z : \langle u, p_z \rangle = 0\}$$

is a linear subspace of T_z of dimension $d - 1$. As a subspace of $\mathbf{C}^n$, $T_z^\perp$ is orthogonal to z.

Definition 2.5. (a) For each $z \in K$, we define

$$T_z^{\mathrm{mod}} = T_z^\perp \oplus \{\xi z : \xi \in \mathbf{C}\},$$

which we consider as the *modified* complex tangent space at z.
(b) For each $z \in K$, let P_z be the orthogonal projection from $\mathbf{C}^n$ onto T_z^{mod}.

Lemma 2.6. [28, Lemma 2.7] *There exist $b_0 > 0$ and $c_0 > 0$ such that for every $z \in K$, P_z is a biholomorphic map from $\mathcal{M} \cap B(z, b_0)$ onto an open set in T_z^{mod} that contains $T_z^{\mathrm{mod}} \cap B(z, c_0)$.*

For $z \in K$, let $I_z : T_z^{\mathrm{mod}} \cap B(z, c_0) \to \mathcal{M}$ be the inverse of P_z. For $x \in T_z^{\mathrm{mod}} \cap B(z, c_0)$, the relation $P_z I_z(x) = x$ leads to

$$I_z(x) = x + h_z(x), \quad \text{where} \quad h_z(x) = I_z(x) - P_z I_z(x). \tag{2.6}$$

That is, for each $z \in K$, h_z maps $T_z^{\mathrm{mod}} \cap B(z, c_0)$ into $\mathbf{C}^n \ominus T_z^{\mathrm{mod}}$. We now fix a $c_1 \in (0, c_0)$. By the analysis on page 8 in [28], there are constants $0 < \alpha(c_1) \le \beta(c_1) < \infty$ such that the operator inequality

$$\alpha(c_1) \le (DI_z)^*(x)(DI_z)(x) \le \beta(c_1) \tag{2.7}$$

holds on the linear space $T_z^{\rm mod}$ for all $z \in K$ and $x \in T_z^{\rm mod} \cap B(z, c_1)$. Applying the standard open mapping theorem, there is a $0 < b_1 < b_0$ such that

$$\{I_z(x) : x \in T_z^{\rm mod} \cap B(z, c_1)\} \supset \mathcal{M} \cap B(z, b_1). \tag{2.8}$$

Our analysis also involves the Bergman-metric structure of the ball. As usual, we write β for the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B}.$$

We recall that the Möbius transform φ_z is given by the formula

$$\varphi_z(w) = \frac{1}{1 - \langle w, z\rangle} \left\{ z - \frac{\langle w, z\rangle}{|z|^2} z - (1 - |z|^2)^{1/2} \left(w - \frac{\langle w, z\rangle}{|z|^2} z \right) \right\}$$

when $z \neq 0$, and $\varphi_0(w) = -w$. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z, a) = \{w \in \mathbf{B} : \beta(z, w) < a\}$.

Lemma 2.7. [28, Lemma 2.9] (1) *Let $r > 0$ be given. For each $\epsilon > 0$, there is a $\delta = \delta(r, \epsilon)$ $\in (0, 1)$ such that if $z \in K$ satisfies the condition $1 - \delta \leq |z| < 1$, then the inequality*

$$\beta(w, P_z w) \leq \epsilon$$

holds for every $w \in D(z, r) \cap \mathcal{M}$.
(2) *Let $z \in M \cap K$ and $r > 0$ be such that $D(z, r/2) \subset B(z, c_0)$ and $\beta(w, P_z w) \leq r/3$ for every $w \in D(z, 2r) \cap M$. Then $I_z(D(z, r/2) \cap T_z^{\rm mod}) \subset D(z, r) \cap M$.*

For every $z \in K$, $T_z^{\rm mod}$ is a d-dimensional linear subspace of $\mathbf{C}^n$. For convenience we will write v for the natural volume measure on $T_z^{\rm mod}$, even though for different $z \in K$ this may be a different linear subspace of $\mathbf{C}^n$. But since volume depends only on the Euclidean metric, which $T_z^{\rm mod}$ inherits from $\mathbf{C}^n$, such a simplification of notation is justified.

For each $z \in K$, we have the Jacobian

$$J_z(x) = \det\{(DI_z)^*(x)(DI_z)(x)\}, \tag{2.9}$$

$x \in T_z^{\rm mod} \cap B(z, c_1)$. Let $v_{\mathcal{M}}$ denote the natural volume measure on $\mathcal{M}$. Suppose that $z \in K$ and U is an open set in $\mathcal{M} \cap B(z, b_1)$. By (2.8), we have $P_z U \subset T_z^{\rm mod} \cap B(z, c_1)$. For any positive, continuous function f on U, we have

$$\int_U f(w) dv_{\mathcal{M}}(w) = \int_{P_z U} f(I_z(x)) J_z(x) dv(x). \tag{2.10}$$

Recall that this is in fact how volume is defined on $\mathcal{M}$.

In addition to the volume measure $v_{\mathcal{M}}$ on $\mathcal{M}$, we define the measure v_M on $M = \tilde{M} \cap \mathbf{B}$ by the formula $v_M(E) = v_{\mathcal{M}}(E \cap \mathcal{M})$ for Borel sets $E \subset M$.

Lemma 2.8. [28, Lemma 2.10] *Given any $a > 0$ and $\kappa > -1$, there is a $0 < C_{2.8} < \infty$ such that*

$$\int_M \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le C_{2.8}$$

for every $z \in M$.

Moreover, it is known that if $\kappa > -1$, then

$$\int_M (1-|w|^2)^\kappa dv_M(w) < \infty$$

[28, page 15]. This finiteness is due to the fact that we can use the function $\rho(w) = 1-|w|^2$ as one of the $2d$ real coordinates on M for $w \in M$ near S.

Lemma 2.9. [28, Lemma 2.11] *Given any $a > 0$ and $\kappa > -1$, there are $\delta > 0$ and $0 < C_{2.9}(\delta) < \infty$ such that*

$$\int_{M\backslash D(z,r)} \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le C_{2.9}(\delta)e^{-2\delta r}$$

for all $z \in M$ and $r > 0$.

Following [28], we use a subscript d to indicate a set in $\mathbf{C}^d$. For example, $\mathbf{B}_d = \{w \in \mathbf{C}^d : |w| < 1\}$ and $D_d(z,r) = \{w \in \mathbf{B}_d : \beta(z,w) < r\}$. Similarly, dv_d denotes the volume measure on $\mathbf{C}^d$. In particular, dv_1 is just the area measure on $\mathbf{C}$.

Lemma 2.10. *Let $0 < r < \infty$. If f is a bounded analytic function on $D_1(z,r)$, $z \in \mathbf{B}_1$, then*

$$\int_{D_1(z,r)} f(w)\frac{1-|w|^2}{(1-\langle z,w\rangle)^3} dv_1(w) = \Phi(r)f(z) \tag{2.11}$$

where

$$\Phi(r) = \int_{D_1(0,r)} (1-|\zeta|^2)dv_1(\zeta).$$

Proof. Let $w = \varphi_z(\zeta)$. By the formulas from [26, Theorem 2.2.2], we have

$$1-\langle z,\varphi_z(\zeta)\rangle = \frac{1-|z|^2}{1-\langle z,\zeta\rangle} \quad \text{and} \quad 1-|\varphi_z(\zeta)|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\langle z,\zeta\rangle|^2}.$$

Therefore the left-hand side of (2.11) equals

$$\int_{D_1(0,r)} f(\varphi_z(\zeta))\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\langle z,\zeta\rangle|^2}\left(\frac{1-\langle z,\zeta\rangle}{1-|z|^2}\right)^3 \frac{(1-|z|^2)^2}{|1-\langle z,\zeta\rangle|^4} dv_1(\zeta).$$

After the obvious cancellation, we find that

$$\int_{D_1(z,r)} f(w)\frac{1-|w|^2}{(1-\langle z,w\rangle)^3}dv_1(w) = \int_{D_1(0,r)} \frac{f(\varphi_z(\zeta))}{(1-\langle \zeta,z\rangle)^3}(1-|\zeta|^2)dv_1(\zeta).$$

With respect to the Euclidean metric on $\mathbf{C}$, $D_1(0,r)$ is a disc centered at 0. Hence the above equals $\Phi(r)f(\varphi_z(0))(1-\langle 0,z\rangle)^{-3} = \Phi(r)f(z)$. □

Lemma 2.11. [28, Lemma 3.2] *For each given* $0 < r < \infty$, *we have*

$$\lim_{t\uparrow 1}\sup\left\{\left|1-\frac{1-|x|^2}{1-|I_z(x)|^2}\right| : |z|\geq t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\mathrm{mod}}\right\} = 0$$

and

$$\lim_{t\uparrow 1}\sup\{|J_z(z)-J_z(x)| : |z|\geq t,\ z\in M \text{ and } x\in D(z,r)\cap T_z^{\mathrm{mod}}\} = 0.$$

Lemma 2.12. *Let* $-1 < \tau < \infty$. *Then*

$$d\Omega(w) = (1-|w|^2)^{n-d+\tau}dv_M(w)$$

is a Carleson measure for the weighted Bergman space $L^2_{a,\tau} = L^2_a(\mathbf{B},(1-|z|^2)^\tau dv(z))$.

Proof. For each pair of $\zeta\in S$ and $r>0$, define $Q(\zeta,r) = \{z\in\mathbf{B} : |1-\langle z,\zeta\rangle| < r\}$. By the well-known [7, Theorem 1], to show that Ω is a Carleson measure for $L^2_{a,\tau}$, it suffices to find a C_1 such that

$$\Omega(Q(\zeta,r)) \leq C_1 r^{n+1+\tau} \tag{2.12}$$

for all $\zeta\in S$ and $r>0$. To prove this, note that

$$(1-|w|^2)^{n-d+\tau} = (1-|w|^2)^{1+\tau}\cdot(1-|w|^2)^{n-1-d}$$

If $w\in Q(\zeta,r)\cap M$, then $1-|w|^2\leq 2r$. Since $1+\tau>0$, we have

$$\Omega(Q(\zeta,r)) = \int_{Q(\zeta,r)\cap M}(1-|w|^2)^{n-d+\tau}dv_M(w) \leq (2r)^{1+\tau}\int_{Q(\zeta,r)\cap M}(1-|w|^2)^{n-1-d}dv_M(w).$$

By inequality (2.26) in [28],

$$\int_{Q(\zeta,r)\cap M}(1-|w|^2)^{n-1-d}dv_M(w) \leq Cr^n$$

for all $\zeta\in S$ and $r>0$. Thus (2.12) indeed holds. □

Under the usual identification of $\mathbf{C}$ with $\mathbf{R}^2$, we can also view T_z as a subspace of $\mathbf{R}^{2n}$ of real dimension $2d$, equipped with the real inner product. Thus if $z\in\mathcal{M}$ and h is a real-valued C^1-function on an open neighborhood U of z in $\mathbf{C}^n\cong\mathbf{R}^{2n}$, then we define

$(\nabla_{\mathcal{M}} h)(z)$ to be the orthogonal projection of the real vector $(\nabla h)(z)$ onto the real subspace T_z. If h is complex-valued, we can write $h = h_1 + ih_2$, where h_1 and h_2 are real valued. In this case, we define $(\nabla_{\mathcal{M}} h)(z) = (\nabla_{\mathcal{M}} h_1)(z) + i(\nabla_{\mathcal{M}} h_2)(z)$. This defines the operation $\nabla_{\mathcal{M}}$. We think of $\nabla_{\mathcal{M}}$ as the gradient in the directions tangent to $\mathcal{M}$.

For each $0 < t \leq 1$, we define the sets

$$M^{(t)} = \{z \in M : 1 - |z|^2 < t\} \quad \text{and} \quad N^{(t)} = \{z \in M : 1 - |z|^2 > t\}, \tag{2.13}$$

which will appear frequently in the sequel. For each $-1 < \tau < \infty$, we denote

$$d\nu_\tau(w) = (1 - |w|^2)^\tau dv_M(w).$$

Lemma 2.13. *Given any $-1 < \tau < \infty$ and $0 < t \leq 1$, there are constants $0 < a < b < t$ and $0 < C < \infty$ such that the inequality*

$$\begin{aligned}\int_{M^{(t)}} |f(w)|^2 d\nu_\tau(w) \leq C \int_{M^{(b)}} &|(\nabla_{\mathcal{M}} f)(w)|^2 (1 - |w|^2)^2 d\nu_\tau(w) \\ &+ C \int_{N^{(a)} \cap M^{(t)}} |f(w)|^2 d\nu_\tau(w)\end{aligned}$$

holds for every C^1 function f on any open set containing the closure of $M^{(t)}$.

The proof of this lemma is essentially the same as the proof of [29, Lemma 3.1]. For that reason we leave the proof of Lemma 2.13 to Appendix 1.

As usual, we write $\partial = (\partial_1, \ldots, \partial_n)$, the analytic gradient on $\mathbf{C}^n$. By the multi-index convention that we follow, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, ∂^α denotes $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Lemma 2.14. *Let $-1 < \tau < \infty$ and $k \in \mathbf{N}$ be given. There are constants $0 < a < b < 1$ and $0 < C < \infty$ such that if f is any analytic function on an open set containing the closure of M, then*

$$\begin{aligned}\int_M |f(w)|^2 d\nu_\tau(w) \leq C \sum_{|\alpha|=k} &\int_{M^{(b)}} |(\partial^\alpha f)(w)|^2 (1 - |w|^2)^{2k} d\nu_\tau(w) \\ &+ C \sum_{0 \leq |\beta| \leq k-1} \int_{N^{(a)}} |(\partial^\beta f)(w)|^2 d\nu_\tau(w).\end{aligned}$$

Proof. This follows from Lemma 2.13 by an obvious induction on k. $\square$

Let us recall the family of spaces $\mathcal{H}^{(t)}$ introduced in [19]. For each $-n \leq t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ which has the function

$$K_w^{(t)}(z) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+t}}, \quad z, w \in \mathbf{B},$$

as its reproducing kernel. Equivalently, $\mathcal{H}^{(t)}$ is the completion of $\mathbf{C}[z_1,\dots,z_n]$ with respect to the norm $\|\cdot\|_t$ arising from the inner product $\langle\cdot,\cdot\rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta\rangle_t = 0$ whenever $\alpha\neq\beta$,

$$\langle z^\alpha, z^\alpha\rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n+t+j)} \tag{2.14}$$

if $\alpha\in\mathbf{Z}_+^n\backslash\{0\}$, and $\langle 1,1\rangle_t = 1$. It is well known that $H_n^2 = \mathcal{H}^{(-n)}$ and that for each $-1<t<\infty$, $\mathcal{H}^{(t)}$ is a weighted Bergman space on $\mathbf{B}$.

Recall that the formula

$$R = z_1\partial_1 + \cdots + z_n\partial_n$$

defines the radial derivative on $\mathbf{B}$. Let $m\in\mathbf{N}$ and $-n\le t<\infty$ satisfy the condition $2m+t>-1$. For such a pair of m and t, we define

$$\|f\|_{m,t}^2 = |f(0)|^2 + \int_{\mathbf{B}} |(R^m f)(z)|^2(1-|z|^2)^{2m+t}dv(z)$$

whenever f is an analytic function on $\mathbf{B}$. The following is well known:

Lemma 2.15. *Let $m\in\mathbf{N}$ and $-n\le t<\infty$ satisfy the condition $2m+t>-1$. Then there exist constants $0<c\le C<\infty$ such that*

$$c\|f\|_{m,t}\le\|f\|_t\le C\|f\|_{m,t}$$

for every analytic function f on $\mathbf{B}$.

Proof. When $\alpha\neq\beta$, we have $\langle z^\alpha, z^\beta\rangle_t = 0$ by definition. When $\alpha\neq\beta$, it is easy to see that $\langle z^\alpha, z^\beta\rangle_{m,t} = 0$, where $\langle\cdot,\cdot\rangle_{m,t}$ is the inner product that corresponds to the norm $\|\cdot\|_{m,t}$ defined above. Thus it suffices to find constants $0<c\le C<\infty$ such that

$$c\|z^\alpha\|_{m,t}\le\|z^\alpha\|_t\le C\|z^\alpha\|_{m,t} \tag{2.15}$$

for every $\alpha\in\mathbf{Z}_+^n$. We have $R^m z^\alpha = |\alpha|^m z^\alpha$. Therefore, for any $\alpha\neq 0$,

$$\begin{aligned}\|z^\alpha\|_{m,t}^2 &= |\alpha|^{2m}\int_{\mathbf{B}}|z^\alpha|^2(1-|z|^2)^{2m+t}dv(z)\\ &= \frac{|\alpha|^{2m}(n-1)!\alpha!}{(n-1+|\alpha|)!}2n\int_0^1 r^{2|\alpha|+2n-1}(1-r^2)^{2m+t}dr\\ &= \frac{|\alpha|^{2m}n!\alpha!}{(n-1+|\alpha|)!}\int_0^1 x^{|\alpha|+n-1}(1-x)^{2m+t}dx\\ &= \frac{|\alpha|^{2m}n!\alpha!}{(n-1+|\alpha|)!}\cdot\frac{(|\alpha|+n-1)!}{\prod_{j=1}^{|\alpha|+n}(2m+t+j)} = \frac{|\alpha|^{2m}n!\alpha!}{\prod_{j=1}^{|\alpha|+n}(2m+t+j)}.\end{aligned} \tag{2.16}$$

By Stirling's asymptotic formula (see, e.g., identity (3.3) in [18]), we have

$$\prod_{j=1}^{|\alpha|+n}(2m+t+j) \approx (2m+t+|\alpha|+n)^{2m+t+|\alpha|+n+(1/2)}e^{-|\alpha|} \quad \text{whereas}$$

$$\prod_{j=1}^{|\alpha|}(n+t+j) \approx (n+t+|\alpha|)^{n+t+|\alpha|+(1/2)}e^{-|\alpha|}.$$

Combining these formulas with (2.14) and (2.16), we obtain (2.15). □

Lemma 2.16. *Given $m \in \mathbf{N}$ and $t > -1$, there is a constant $0 < C < \infty$ such that*

$$\int_{\mathbf{B}} |(\partial^\beta f)(z)|^2(1-|z|^2)^t dv(z) \le C\int_{\mathbf{B}} |(R^m f)(z)|^2(1-|z|^2)^t dv(z) \tag{2.17}$$

for every $\beta \in \mathbf{Z}_+^n$ satisfying the condition $|\beta| = m$ and every analytic function f on $\mathbf{B}$.

Proof. Similar to what happened in the previous proof, it suffices to find a $0 < C < \infty$ such that

$$\int_{\mathbf{B}} |\partial^\beta z^\alpha|^2(1-|z|^2)^t dv(z) \le C\int_{\mathbf{B}} |R^m z^\alpha|^2(1-|z|^2)^t dv(z) \tag{2.18}$$

for every $\alpha \in \mathbf{Z}_+^n$. Write $\beta = (\beta_1, \dots, \beta_n)$. Note that for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$, the left-hand side of (2.18) is 0 unless $\alpha_j \ge \beta_j$ for every $j \in \{1, \dots, n\}$. Suppose that this condition is satisfied. Then $\alpha - \beta \in \mathbf{Z}_+^n$ and we have

$$\partial^\beta z^\alpha = z^{\alpha-\beta}\prod_{\beta_\nu>0}\prod_{i_\nu=1}^{\beta_\nu}(\alpha_\nu - \beta_\nu + i_\nu).$$

Therefore

$$\begin{aligned}\int_{\mathbf{B}} |\partial^\beta z^\alpha|^2(1-|z|^2)^t dv(z) &= \prod_{\beta_\nu>0}\prod_{i_\nu=1}^{\beta_\nu}(\alpha_\nu - \beta_\nu + i_\nu)^2\int_{\mathbf{B}} |z^{\alpha-\beta}|^2(1-|z|^2)^t dv(z)\\ &= \prod_{\beta_\nu>0}\prod_{i_\nu=1}^{\beta_\nu}(\alpha_\nu - \beta_\nu + i_\nu)^2 \cdot \frac{n!(\alpha-\beta)!}{\prod_{j=1}^{|\alpha-\beta|+n}(t+j)}\\ &\le \frac{n!\alpha!|\alpha|^m\prod_{j=|\alpha-\beta|+n+1}^{|\alpha|+n}(t+j)}{\prod_{j=1}^{|\alpha|+n}(t+j)}. \qquad (2.19)\end{aligned}$$

On the other hand, since $R^m z^\alpha = |\alpha|^m z^\alpha$, we have

$$\int_{\mathbf{B}} |R^m z^\alpha|^2(1-|z|^2)^t dv(z) = \frac{n!\alpha!|\alpha|^{2m}}{\prod_{j=1}^{|\alpha|+n}(t+j)}. \tag{2.20}$$

Since $|\beta| = m$, we have $|\alpha| = |\alpha-\beta|+m$, and consequently $\prod_{j=|\alpha-\beta|+n+1}^{|\alpha|+n}(t+j) \le C|\alpha|^m$. Combining this inequality with (2.19) and (2.20), we obtain (2.18). □

We end the section with one more notation. For any $u = (u_1, \dots, u_n) \in \mathbf{C}^n$, we write

$$\partial_u = u_1\partial_1 + \cdots + u_n\partial_n \quad \text{and} \quad \overline{\partial_u} = \bar{u}_1\bar{\partial}_1 + \cdots + \bar{u}_n\bar{\partial}_n. \tag{2.21}$$

3. Spectral gap

For the rest of the paper, we assume $d = 1$. Accordingly, we define the measure

$$d\mu(w) = (1 - |w|^2)dv_M(w). \tag{3.1}$$

We fix a $t_0 \in (0, 1)$ such that

$$M^{(2t_0)} \subset K \tag{3.2}$$

(see (2.3) and (2.13)). Recall that for $z \in \mathcal{M}$, p_z is the orthogonal projection of z on T_z.

Definition 3.1. For any C^1 function f on an open set containing M, we define

$$\|f\|_* = \left\{ \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(w) \right\}^{1/2}.$$

Proposition 3.2. *There is a* $0 < C < \infty$ *such that*

$$\int_M |(\partial_j f)(w)|^2 d\mu(w) \le C\|f\|^2 \tag{3.3}$$

for all $f \in H_n^2$ *and* $1 \le j \le n$.

Proof. We pick a natural number $m \ge 2$ such that $2m - n \ge 0$. Given an $f \in H_n^2$, define the function $f_r(w) = f(rw)$ for each $0 < r < 1$. Each f_r is analytic on an open ball containing $\overline{\mathbf{B}}$. By Lemma 2.14, we have

$$\int_M |(\partial_j f_r)(w)|^2 d\mu(w) \le C_1 \sum_{|\alpha|=m-1} \int_{M^{(b)}} |(\partial^\alpha \partial_j f_r)(w)|^2 (1-|w|^2)^{2m-2} d\mu(w)$$
$$+ C_1 \sum_{0 \le |\beta| \le m-2} \int_{N^{(a)}} |(\partial^\beta \partial_j f_r)(w)|^2 d\mu(w).$$

By the definition of $N^{(a)}$, the second term is dominated by $C_2\|f_r\|^2 \le C_2\|f\|^2$. It follows from Lemma 2.12 that for each $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| = m-1$,

$$\int_{M^{(b)}} |(\partial^\alpha \partial_j f_r)(w)|^2 (1-|w|^2)^{2m-2} d\mu(w)$$
$$\le C_3 \int_{\mathbf{B}} |(\partial^\alpha \partial_j f_r)(w)|^2 (1-|w|^2)^{2m-n} dv(w) \le C_4\|f_r\|^2 \le C_4\|f\|^2,$$

where the second $\leq$ follows from Lemmas 2.15 and 2.16. Thus

$$\int_M |(\partial_j f_r)(w)|^2 d\mu(w) \leq C_5 \|f\|^2.$$

Applying Fatou's lemma,

$$\int_M |(\partial_j f)(w)|^2 d\mu(w) \leq \liminf_{r\uparrow 1} \int_M |(\partial_j f_r)(w)|^2 d\mu(w) \leq C_5 \|f\|^2.$$

This completes the proof. $\square$

We will now try to modify the reproducing kernel for H_n^2 given by (1.1). For each pair of $w \in \mathbf{B}$ and $u \in \mathbf{C}^n$, we define

$$K_{w,u}(\zeta) = \frac{\langle \zeta, u\rangle}{(1 - \langle \zeta, w\rangle)^2}. \tag{3.4}$$

For $w \in \mathbf{B}$, $u \in \mathbf{C}^n$ and $f \in H_n^2$, it is easy to see that

$$\langle f, K_{w,u}\rangle = \frac{d}{dt}\langle f, K_{w+tu}\rangle\big|_{t=0} = \frac{d}{dt} f(w+tu)\big|_{t=0} = (\partial_u f)(w). \tag{3.5}$$

That is, $K_{w,u}$ is the reproducing kernel for the directional derivative ∂_u.

Lemma 3.3. *Let $w \in M^{(t_0)}$. If $u \in T_w$, then $K_{w,u} \in \mathcal{Q}$.*

Proof. Since $u \in T_w$, there is a smooth path $\gamma : (-c, c) \to M^{(t_0)}$ such that $\gamma(0) = w$ and $\gamma'(0) = u$. Thus

$$K_{w,u} = \frac{d}{dt} K_{\gamma(t)}\big|_{t=0}.$$

Let $f \in \mathcal{R}$. Since the range of γ is contained in M, we have $\langle f, K_{\gamma(t)}\rangle = 0$ for every $t \in (-c, c)$. Therefore

$$\langle f, K_{w,u}\rangle = \frac{d}{dt}\langle f, K_{\gamma(t)}\rangle\big|_{t=0} = \frac{d}{dt} 0\big|_{t=0} = 0.$$

This shows that $K_{w,u} \perp \mathcal{R}$. That is, $K_{w,u} \in \mathcal{Q}$. $\square$

We now define the operator

$$T_1 = \int_{M^{(t_0)}} K_{w,p_w} \otimes K_{w,p_w} d\mu(w). \tag{3.6}$$

Proposition 3.4. *The operator T_1 defined above is bounded on the Drury-Arveson space H_n^2. Moreover, T_1 maps H_n^2 into the quotient module $\mathcal{Q}$.*

Proof. By (3.5), for $f \in H_n^2$ we have $\langle T_1 f, f\rangle = \|f\|_*^2$. Hence the boundedness of T_1 follows from Proposition 3.2. For $w \in M^{(t_0)}$, since $p_w \in T_w$, Lemma 3.3 tells us that $K_{w,p_w} \in \mathcal{Q}$. Therefore T_1 maps H_n^2 into $\mathcal{Q}$. $\square$

Obviously, T_1 is a positive operator on H_n^2. The main goal of this section is to prove that there is an important gap in the spectrum of T_1:

Theorem 3.5. *There is a $c > 0$ such that the spectrum of T_1 does not intersect $(0, c)$.*

The proof of Theorem 3.5 requires some preparations.

Since we now assume $d = 1$, $B_1(0, r) = \{\xi \in \mathbf{C} : |\xi| < r\}$, $r > 0$. Since $\mathcal{M}$ is a complex manifold with $\dim_{\mathbf{C}}\mathcal{M} = 1$, for each $y \in \mathcal{M}$, there is an open subset V_y of $\mathcal{M}$ containing y and a biholomorphic map

$$\rho_y : B_1(0, 2) \to V_y$$

such that $\rho_y(0) = y$. For each $\xi \in B_1(0, 2)$, $\rho'_y(\xi)$ is obviously a complex tangent vector to $\mathcal{M}$ at the point $\rho_y(\xi)$, and we have $\rho'_y(\xi) \neq 0$ since ρ_y is biholomorphic. Since $\dim_{\mathbf{C}}(T_{\rho_y(\xi)}) = 1$, we have $T_{\rho_y(\xi)} = \mathbf{C}\rho'_y(\xi)$ for every $\xi \in B_1(0, 2)$. Define

$$\eta_y(w) = \rho'_y(\rho_y^{-1}(w)), \quad w \in V_y.$$

Then $T_w = \mathbf{C}\eta_y(w)$ for every $w \in V_y$. The important fact to keep in mind is that η_y is analytic on V_y. Since p_w is the orthogonal projection of w onto T_w, there is a continuous function $s_y : V_y \to \mathbf{C}$ such that

$$p_w = s_y(w)\eta_y(w) \quad \text{for every} \quad w \in V_y. \tag{3.7}$$

This identity embodies one of our main observations: modulo continuous scalar multiples, p_w is locally analytic. This fact will be crucial for the proof of Theorem 3.5.

We now define $U_y = \rho_y B_1(0, 1)$, which is an open subset of $\mathcal{M}$ containing y. Obviously, $\overline{U_y}$ is a compact subset of V_y. Thus s_y is uniformly continuous on $\overline{U_y}$. By (2.2), the infimum of $|s_y|$ on the set $\overline{U_y}$ is greater than 0. Since K is a compact subset of $\mathcal{M}$, there is a finite subset F of K such that

$$\bigcup_{y \in F} U_y \supset K.$$

By general topology, this implies

Lemma 3.6. *There is an $\epsilon > 0$ such that for each $\zeta \in K$, there is a $y = y(\zeta) \in F$ for which the containment $\{w \in \mathcal{M} : |\zeta - w| < \epsilon\} \subset U_y$ holds.*

Lemma 3.7. *For each $y \in M^{(t_0)}$, there is an open neighborhood N_y of y in $M^{(t_0)}$ which has the following property. Let $\{f_k\}$ be a sequence in H_n^2. If the sequence $\{T_1^{1/2} f_k\}$ weakly converges to 0, then*

$$\lim_{k \to \infty} \sup\{|(\partial_{p_w} f_k)(w)| : w \in N_y\} = 0.$$

Proof. For each $y \in M^{(t_0)}$, consider the biholomorphic map $\rho_y : B_1(0, 2) \to V_y$ introduced above. Recall that $\rho_y(0) = y$. Since y is now in $M^{(t_0)}$, there are $\epsilon = \epsilon(y) > 0$ and $r = r(y) \in (0, 1)$ such that $\rho_y(B_1(0, r)) \subset \{w \in M^{(t_0)} : 1 - |w|^2 > \epsilon\}$. We will show that the lemma holds for the open set $N_y = \rho_y(B_1(0, r/2))$.

We begin with the Bergman space $L_a^2(B_1(0,r), dv_1)$. For each $f \in H_n^2$, define

$$(Gf)(\xi) = (\partial_{\rho_y'(\xi)} f)(\rho_y(\xi)), \quad \xi \in B_1(0,r).$$

Using the conditions $r < 1$ and (3.7), we have

$$\begin{aligned}\int_{B_1(0,r)} |(Gf)(\xi)|^2 dv_1(\xi) &\le C_1 \int_{B_1(0,r)} |s_y(\rho_y(\xi))|^2 |(\partial_{\rho_y'(\xi)} f)(\rho_y(\xi))|^2 |\rho_y'(\xi)|^2 dv_1(\xi) \\ &= C_1 \int_{\rho_y(B_1(0,r))} |(\partial_{p_w} f)(w)|^2 dv_M(w) \\ &\le C_1 \epsilon^{-1} \int_{\rho_y(B_1(0,r))} |(\partial_{p_w} f)(w)|^2 (1 - |w|^2) dv_M(w) \\ &= C_1 \epsilon^{-1} \int_{\rho_y(B_1(0,r))} |(\partial_{p_w} f)(w)|^2 d\mu(w) \\ &\le C_1 \epsilon^{-1} \langle T_1 f, f\rangle = C_1 \epsilon^{-1} \|T_1^{1/2} f\|^2. \qquad (3.8)\end{aligned}$$

Thus there is a bounded operator $W : H_n^2 \to L_a^2(B_1(0,r), dv_1)$ such that $G = WT_1^{1/2}$.

Now let $\{f_k\}$ be any sequence in H_n^2 such that $\{T_1^{1/2} f_k\}$ weakly converges to 0. Since $G = WT_1^{1/2}$, the sequence $\{Gf_k\}$ weakly converges to 0 in $L_a^2(B_1(0,r), dv_1)$. Using the reproducing kernel for the Bergman space, we have

$$\lim_{k\to\infty} \sup\{|(\partial_{\rho_y'(\xi)} f_k)(\rho_y(\xi))| : \xi \in B_1(0, r/2)\} = 0.$$

By (3.7) and the boundedness of $s_y \circ \rho_y$ on $B_1(0, r/2)$, the above limit implies

$$\lim_{k\to\infty} \sup\{|(\partial_{p_w} f_k)(w)| : w \in N_y\} = 0$$

as promised. □

Lemma 3.8. *Define the operators B and B_r on $L^2(M, d\mu)$ by the formulas*

$$(Bf)(z) = \int_M \frac{f(w)}{|1 - \langle z, w\rangle|^3} d\mu(w) \quad \text{and}$$
$$(B_r f)(z) = \int_{M \setminus D(z,r)} \frac{f(w)}{|1 - \langle z, w\rangle|^3} d\mu(w)$$

for $f \in L^2(M, d\mu)$, $r > 0$. Then $\|B\| < \infty$ and $\|B_r\| \to 0$ as $r \to \infty$.

Proof. We set $a = 1/2$ and $\kappa = 1/2$. Define $h(w) = (1 - |w|^2)^{-1/2}$, $w \in M$. Then

$$(B_r h)(z) = \int_{M \setminus D(z,r)} \frac{(1 - |w|^2)^\kappa}{|1 - \langle z, w\rangle|^{1+1+a+\kappa}} dv_M(w).$$

By Lemma 2.9, we have $(B_r h)(z) \le C_{2.9}(\delta)e^{-2\delta r}(1-|z|^2)^{-a} = C_{2.9}(\delta)e^{-2\delta r}h(z)$, $z \in M$. Since the kernel function $|1-\langle z,w\rangle|^{-3}$ is symmetric with respect to z and w, we can now apply the Schur test to conclude that $\|B_r\| \le C_{2.9}(\delta)e^{-2\delta r}$. Hence $\|B_r\| \to 0$ as $r \to \infty$.

Similarly, by Lemma 2.8 we have $(Bh)(z) \le C_{2.8}h(z)$, $z \in M$. Thus it follows from the Schur test that $\|B\| \le C_{2.8}$. This completes the proof. □

For each $f \in H_n^2$, define

$$(Xf)(z) = \int_{M^{(t_0)}} \frac{(\partial_{p_w} f)(w)\langle p_z, w\rangle\langle z, p_w\rangle}{(1-\langle z,w\rangle)^3} d\mu(w), \quad z \in \mathbf{B}.$$

Lemma 3.9. *Given any $\delta > 0$, there exist constants $0 < \tau < t_0$ and $0 < C < \infty$ such that*

$$\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C \int_{M^{(t)}} |(Xf)(z)|^2 d\mu(z) + \delta\|f\|_*^2$$

for all $0 < t \le \tau$ and $f \in H_n^2$.

Proof. We begin with a large $1 \le r < \infty$, whose exact value will be determined below. With such an r, there is a $0 < \tau_1 \le t_0$ such that if $0 < t \le \tau_1$, then for $z \in M^{(t)}$ we have $D(z, 2r) \subset B(z, \min\{b_1, c_1\})$ (see (2.8)). By Lemma 2.7(1), there is a $0 < \tau_2 \le \tau_1$ such that if $0 < t \le \tau_2$, then for $z \in M^{(t)}$ and $w \in D(z,r) \cap M$ we have $\beta(w, P_z w) < r$. Thus $P_z w \in D(z,2r) \cap T_z^{\text{mod}}$ and $I_z(P_z w) = w \in D(z,r) \cap M$. That is, if $0 < t \le \tau_2$, then

$$I_z(D(z,2r) \cap T_z^{\text{mod}}) \supset D(z,r) \cap M \quad \text{for every} \quad z \in M^{(t)}. \tag{3.9}$$

By (2.7), there is a constant $1 \le C_1 < \infty$ such that the inequality

$$|I_z(x) - I_z(x')| \le C_1|x - x'| \tag{3.10}$$

holds for every triple of $z \in K$ and $x, x' \in T_z^{\text{mod}} \cap B(z, c_1)$. Therefore there is a $0 < \tau_3 \le \tau_2$ such that

$$I_z(D(z,2r) \cap T_z^{\text{mod}}) \subset M^{(t_0)} \quad \text{if} \quad z \in M^{(\tau_3)}. \tag{3.11}$$

Let us write $U(z) = I_z(D(z,2r) \cap T_z^{\text{mod}})$ for $z \in M^{(\tau_3)}$.

Let $f \in H_n^2$ be given. By (3.11), for $z \in M^{(\tau_3)}$ we have

$$(Xf)(z) = \langle p_z, z\rangle A(z) + B(z) + C(z),$$

where

$$A(z) = \int_{U(z)} (\partial_{p_w} f)(w)\langle z, p_w\rangle \frac{1-|w|^2}{(1-\langle z,w\rangle)^3} dv_M(w),$$

$$B(z) = \int_{U(z)} (\partial_{p_w} f)(w)\langle p_z, w-z\rangle\langle z, p_w\rangle \frac{1-|w|^2}{(1-\langle z,w\rangle)^3} dv_M(w) \quad \text{and}$$

$$C(z) = \int_{M^{(t_0)}\backslash U(z)} (\partial_{p_w} f)(w)\langle p_z, w\rangle\langle z, p_w\rangle \frac{1-|w|^2}{(1-\langle z,w\rangle)^3} dv_M(w).$$

Since $P_zU(z) = D(z,2r) \cap T_z^{\mathrm{mod}}$, $z \in M^{(\tau_3)}$, by (2.10) we have

$$A(z) = \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} (\partial_{p_{I_z(x)}} f)(I_z(x))\langle z, p_{I_z(x)}\rangle \frac{1-|I_z(x)|^2}{(1-\langle z, I_z(x)\rangle)^3} J_z(x)dv_1(x).$$

Recall from (2.6) that $\langle z, I_z(x)\rangle = \langle z, x\rangle$. Writing

$$F(z,x) = 1 - \frac{1-|x|^2}{1-|I_z(x)|^2}\cdot\frac{J_z(z)}{J_z(x)},$$

we have $A(z) = A_1(z) + A_2(z)$, where

$$A_1(z) = J_z(z)\int_{D(z,2r)\cap T_z^{\mathrm{mod}}} (\partial_{p_{I_z(x)}} f)(I_z(x))\langle z, p_{I_z(x)}\rangle \frac{1-|x|^2}{(1-\langle z, x\rangle)^3} dv_1(x) \quad \text{and}$$

$$A_2(z) = \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} (\partial_{p_{I_z(x)}} f)(I_z(x))\langle z, p_{I_z(x)}\rangle \frac{1-|I_z(x)|^2}{(1-\langle z, I_z(x)\rangle)^3} F(z,x)J_z(x)dv_1(x).$$

Let us first consider $A_1(z)$.

There is a $0 < \tau_4 \le \tau_3$ such that if $0 < t \le \tau_4$, then for each $z \in M^{(t)}$, $D(z,3r) \subset B(z, \min\{c_1, C_1^{-1}\epsilon\})$, where ϵ is the constant in Lemma 3.6. Recall that $I_z(z) = z$. Thus by (3.10) and Lemma 3.6, if $0 < t \le \tau_4$ and $z \in M^{(t)}$, then there is a $y(z)$ in the finite set F such that $I_z(D(z,3r)\cap T^{\mathrm{mod}}) \subset U_{y(z)}$. Applying (3.7), we now have

$$p_{I_z(x)} = s_{y(z)}(I_z(x))\eta_{y(z)}(I_z(x)) \tag{3.12}$$

for every $x \in D(z,3r)\cap T_z^{\mathrm{mod}}$. Define

$$\begin{aligned}\lambda_z(x) &= s_{y(z)}(I_z(x))\langle z, p_{I_z(x)}\rangle - s_{y(z)}(z)\langle z, p_z\rangle \\ &= |s_{y(z)}(I_z(x))|^2\langle z, \eta_{y(z)}(I_z(x))\rangle - |s_{y(z)}(z)|^2\langle z, \eta_{y(z)}(z)\rangle,\end{aligned}$$

$x \in D(z,2r)\cap T_z^{\mathrm{mod}}$. By the uniform continuity of η_y and s_y on $\overline{U_y}$, $y \in F$, if we denote

$$\delta(r,t) = \sup_{z\in M^{(t)}}\ \sup_{x\in D(z,2r)\cap T_z^{\mathrm{mod}}} |\lambda_z(x)|,$$

then

$$\lim_{t\downarrow 0}\delta(r,t) = 0 \tag{3.13}$$

for every given $1 \le r < \infty$.

By (3.12) and the definition of $\lambda_z(x)$, we have $A_1(z) = A_{11}(z) + A_{12}(z)$, where

$$A_{11}(z) = J_z(z)s_{y(z)}(z)\langle z, p_z\rangle\int_{D(z,2r)\cap T_z^{\mathrm{mod}}} (\partial_{\eta_{y(z)}(I_z(x))} f)(I_z(x))\frac{1-|x|^2}{(1-\langle z, x\rangle)^3} dv_1(x) \quad \text{and}$$

$$A_{12}(z) = J_z(z)\int_{D(z,2r)\cap T_z^{\mathrm{mod}}} \lambda_z(x)(\partial_{\eta_{y(z)}(I_z(x))} f)(I_z(x))\frac{1-|x|^2}{(1-\langle z, x\rangle)^3} dv_1(x).$$

Being a local inverse of P_z, the map I_z is analytic. Thus the map $x \mapsto \eta_{y(z)}(I_z(x))$ is analytic on $D(z,3r) \cap T_z^{\rm mod}$. Therefore it follows from Lemma 2.10 that

$$\begin{aligned} A_{11}(z) &= \Phi(2r)J_z(z)s_{y(z)}(z)\langle z, p_z\rangle(\partial_{\eta_{y(z)}(I_z(z))}f)(I_z(z)) \\ &= \Phi(2r)J_z(z)s_{y(z)}(z)\langle z, p_z\rangle(\partial_{\eta_{y(z)}(z)}f)(z) \\ &= \Phi(2r)J_z(z)\langle z, p_z\rangle(\partial_{p_z}f)(z), \end{aligned}$$

where the last $=$ follows from (3.12). Recalling (2.2), (2.9) and (2.7), we see that there is a $0 < C_2 < \infty$ such that

$$|(\partial_{p_z}f)(z)| \le C_2|\langle p_z, z\rangle A_{11}(z)| \tag{3.14}$$

for $z \in M^{(t)}$, $0 < t \le \tau_4$.

By Lemma 2.11, there is a $0 < \tau_5 \le \tau_4$ such that if $z \in M^{(\tau_5)}$, then

$$1 - |x|^2 \le 2(1 - |I_z(x)|^2) \quad \text{for every} \quad x \in D(z, 2r) \cap T_z^{\rm mod}.$$

Thus, applying (2.7) and the bounds for $|s_y|$ on $\overline{U_y}$, for $z \in M^{(\tau_5)}$ we have

$$\begin{aligned} |A_{12}(z)| &\le C_3\delta(r,t)\int_{D(z,2r)\cap T_z^{\rm mod}} |(\partial_{\eta_{y(z)}(I_z(x))}f)(I_z(x))|\frac{1-|x|^2}{|1-\langle z,x\rangle|^3}dv_1(x) \\ &\le C_4\delta(r,t)\int_{D(z,2r)\cap T_z^{\rm mod}} |(\partial_{\eta_{y(z)}(I_z(x))}f)(I_z(x))|\frac{1-|I_z(x)|^2}{|1-\langle z,I_z(x)\rangle|^3}J_z(x)dv_1(x) \\ &= C_4\delta(r,t)\int_{U(z)} |(\partial_{\eta_{y(z)}(w)}f)(w)|\frac{1-|w|^2}{|1-\langle z,w\rangle|^3}dv_M(w) \\ &\le C_5\delta(r,t)\int_{M^{(t_0)}} |(\partial_{p_w}f)(w)|\frac{1}{|1-\langle z,w\rangle|^3}d\mu(w). \end{aligned}$$

Using the operator B in Lemma 3.8, for $0 < t \le \tau_5$ we have

$$\begin{aligned} \int_{M^{(t)}} |A_{12}(z)|^2 d\mu(z) &\le \{C_5\delta(r,t)\|B\|\}^2 \int_{M^{(t_0)}} |(\partial_{p_w}f)(w)|^2 d\mu(w) \\ &= \{C_5\delta(r,t)\|B\|\}^2\|f\|_*^2. \end{aligned} \tag{3.15}$$

Denote

$$\epsilon(r,t) = \sup_{z\in M^{(t)}}\left\{\sup_{x\in D(z,2r)\cap T_z^{\rm mod}} |F(z,x)|\right\},$$

$0 < t \le \tau_3$. By the definition of $F(z,x)$, we can rewrite it in the form

$$F(z,x) = \left(1 - \frac{1-|x|^2}{1-|I_z(x)|^2}\right) + \frac{1-|x|^2}{1-|I_z(x)|^2}\cdot\frac{1}{J_z(x)}\cdot(J_z(x) - J_z(z)).$$

From (2.7) and (2.9) we obtain an upper bound for $1/J_z(x)$. Therefore it follows from Lemma 2.11 that

$$\lim_{t\downarrow 0} \epsilon(r,t) = 0 \tag{3.16}$$

for every $1 \le r < \infty$. Applying (2.10) again, we have

$$\begin{aligned} |A_2(z)| &\le \epsilon(r,t) \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} |(\partial_{p_{I_z(x)}} f)(I_z(x))| \frac{1-|I_z(x)|^2}{|1-\langle z, I_z(x)\rangle|^3} J_z(x) dv_1(x) \\ &\le \epsilon(r,t) \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)| \frac{1}{|1-\langle z,w\rangle|^3} d\mu(w) \end{aligned}$$

for $z \in M^{(t)}$, $0 < t \le \tau_3$. Thus it follows from Lemma 3.8 that if $0 < t \le \tau_3$, then

$$\begin{aligned} \int_{M^{(t)}} |A_2(z)|^2 d\mu(z) &\le \{\epsilon(r,t)\|B\|\}^2 \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(w) \\ &= \{\epsilon(r,t)\|B\|\}^2 \|f\|_*^2. \end{aligned} \tag{3.17}$$

For $1 \le r < \infty$ and $0 < t \le \tau_3$, we define

$$\sigma(r,t) = \sup_{z\in M^{(t)}} \sup\{|w-z| : w \in I_z(D(z,2r)\cap T_z^{\mathrm{mod}})\}.$$

Recalling (2.7), we have

$$\lim_{t\downarrow 0} \sigma(r,t) = 0 \tag{3.18}$$

for any given $1 \le r < \infty$. By (3.11), we have

$$|B(z)| \le \sigma(r,t) \int_{U(z)} |(\partial_{p_w} f)(w)| \frac{1-|w|^2}{|1-\langle z,w\rangle|^3} dv_M(w) \le \sigma(r,t) \int_{M^{(t_0)}} \frac{|(\partial_{p_w} f)(w)|}{|1-\langle z,w\rangle|^3} d\mu(w)$$

for $z \in M^{(t)}$, $0 < t \le \tau_3$. Applying Lemma 3.8, for each $0 < t \le \tau_3$ we now have

$$\begin{aligned} \int_{M^{(t)}} |B(z)|^2 d\mu(z) &\le \{\sigma(r,t)\|B\|\}^2 \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(w) \\ &= \{\sigma(r,t)\|B\|\}^2 \|f\|_*^2. \end{aligned} \tag{3.19}$$

Finally, from (3.9) we obtain

$$|C(z)| \le \int_{M^{(t_0)}\setminus D(z,r)} |(\partial_{p_w} f)(w)| \frac{1}{|1-\langle z,w\rangle|^3} d\mu(w),$$

$z \in M^{(\tau_3)}$. Using the operator B_r in Lemma 3.8, for $0 < t \le \tau_3$ we have

$$\int_{M^{(t)}} |C(z)|^2 d\mu(z) \le \|B_r\|^2 \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(w) = \|B_r\|^2 \|f\|_*^2. \tag{3.20}$$

Retracing the above steps, we have

$$\langle p_z, z\rangle A_{11}(z) = (Xf)(z) - \langle p_z, z\rangle(A_{12}(z) + A_2(z)) - B(z) - C(z).$$

Thus, for $0 < t \le \tau_5$, it follows from (3.14), (3.15), (3.17), (3.19) and (3.20) that

$$\begin{aligned} \int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) &\le 5C_2^2 \int_{M^{(t)}} |(Xf)(z)|^2 d\mu(z) \\ &+ 5C_2^2(\{C_5\delta(r,t)\|B\|\}^2 + \{\epsilon(r,t)\|B\|\}^2 + \{\sigma(r,t)\|B\|\}^2 + \|B_r\|^2)\|f\|_*^2. \end{aligned} \tag{3.21}$$

Let a $\delta > 0$ be given. By Lemma 3.8, we can first pick an $r \in [1, \infty)$ such that $5C_2^2\|B_r\|^2 \le \delta/2$. With r so fixed, by (3.13), (3.16) and (3.18), we can pick a $0 < \tau \le \tau_5$ such that

$$5C_2^2(\{C_5\delta(r,t)\|B\|\}^2 + \{\epsilon(r,t)\|B\|\}^2 + \{\sigma(r,t)\|B\|\}^2) \le \delta/2$$

for every $0 < t \le \tau$. Substituting these bounds in (3.21), the lemma is proved. □

For each $f \in H_n^2$, define

$$(Yf)(z) = \int_{M^{(t_0)}} \frac{(\partial_{p_w} f)(w)\langle p_z, p_w\rangle}{(1 - \langle z, w\rangle)^2} d\mu(w), \quad z \in \mathbf{B}.$$

Lemma 3.10. *Given any $\delta > 0$, there is a $0 < \rho < t_0$ such that*

$$\int_{M^{(t)}} |(Yf)(z)|^2 d\mu(z) \le \delta \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(z) \tag{3.22}$$

for all $0 < t \le \rho$ and $f \in H_n^2$.

Proof. On the Hilbert space $L^2(M, d\mu)$, define the operator

$$(L\varphi)(z) = \int_{M^{(t_0)}} \frac{\varphi(w)\langle p_z, p_w\rangle}{(1 - \langle z, w\rangle)^2} d\mu(w)$$

$\varphi \in L^2(M, d\mu)$. It follows from Lemma 2.8 that

$$\iint \frac{1}{|1 - \langle z, w\rangle|^4} d\mu(w) d\mu(z) = \iint \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w\rangle|^4} dv_M(w) dv_M(z) < \infty.$$

Hence L is a Hilbert-Schmidt operator on $L^2(M, d\mu)$. Thus for any given $\delta > 0$, there is a $0 < \rho < t_0$ such that $\|M_{\chi_{M^{(\rho)}}} L\| \le \delta^{1/2}$, where $M_{\chi_{M^{(\rho)}}}$ is the operator of multiplication by the function $\chi_{M^{(\rho)}}$ on $L^2(M, d\mu)$. Obviously, (3.22) holds for this ρ. □

Corollary 3.11. *Given any $\delta > 0$, there exist $0 < t < t_0$ and $0 < C < \infty$ such that*

$$\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C\|T_1 f\|_*^2 + \delta\|f\|_*^2 \tag{3.23}$$

for every $f \in H_n^2$.

Proof. Let $\delta > 0$ be given. Applying Lemma 3.9 to $\delta/2$, we obtain constants $0 < \tau \le t_0$ and $0 < C < \infty$ such that

$$\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C \int_{M^{(t)}} |(Xf)(z)|^2 d\mu(z) + \frac{\delta}{2}\|f\|_*^2 \tag{3.24}$$

for all $0 < t \le \tau$ and $f \in H_n^2$. For each $f \in H_n^2$, we have

$$(T_1 f)(z) = \int_{M^{(t_0)}} \frac{(\partial_{p_w} f)(w)\langle z, p_w\rangle}{(1 - \langle z, w\rangle)^2} d\mu(w), \quad z \in \mathbf{B}.$$

By straightforward differentiation,

$$(\partial_{p_z} T_1 f)(z) = 2(Xf)(z) + (Yf)(z).$$

Thus from (3.24) we obtain the inequality

$$\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C\|T_1 f\|_*^2 + C \int_{M^{(t)}} |(Yf)(z)|^2 d\mu(z) + \frac{\delta}{2}\|f\|_*^2 \tag{3.25}$$

for all $0 < t \le \tau$ and $f \in H_n^2$. By Lemma 3.10, there is a $0 < \rho < t_0$ such that

$$C \int_{M^{(t)}} |(Yf)(z)|^2 d\mu(z) \le \frac{\delta}{2}\|f\|_*^2 \tag{3.26}$$

for all $0 < t \le \rho$ and $f \in H_n^2$. Combining (3.25) and (3.26), we see that (3.23) holds for every $0 < t \le \min\{\tau, \rho\}$. □

Lemma 3.12. *There exist a finite number of open subsets $W_1, \dots, W_m$ of $M^{(t_0)}$ such that*

$$W_1 \cup \dots \cup W_m \supset M^{(t^*)}$$

for some $0 < t^* < t_0$ and such that the following hold true for every $1 \le j \le m$:

(1) $W_j = G_j((0, c_j) \times (-b_j, b_j))$, where $0 < c_j < t_0$, $b_j > 0$, and $G_j : (0, c_j) \times (-b_j, b_j) \to \mathbf{C}^n$ is a one-to-one C^∞ map.

(2) There are $0 < \epsilon_j \le M_j < \infty$ such that DG_j, the derivative of G_j, satisfies the inequality $\epsilon_j \le (DG_j)^*(x, y)(DG_j)(x, y) \le M_j$ for all $(x, y) \in (0, c_j) \times (-b_j, b_j)$.

(3) If $w = G_j(x, y)$ for some $(x, y) \in (0, c_j) \times (-b_j, b_j)$, then $x = 1 - |w|^2$. Equivalently, for each $w \in W_j$, there is a unique $y_w \in (-b_j, b_j)$ such that $w = G_j(1 - |w|^2, y_w)$.

Proof. Consider the function $\rho(w) = 1 - |w|^2$. Since $\mathcal{M}$ intersects $\partial\mathbf{B}$ transversely, the vector $\nabla_{\mathcal{M}}\rho$ does not vanish near $\mathcal{M}\cap\partial\mathbf{B}$. Thus we can use ρ as one of the real coordinates on $\mathcal{M}$ near $\partial\mathbf{B}$. More precisely, if $\zeta \in \mathcal{M}\cap\partial\mathbf{B}$, then ζ has an open neighborhood N_ζ in $\mathcal{M}$ that has the following properties:

(α) $N_\zeta = G((-c,c)\times(-b,b))$, where $0 < c < t_0$, $b > 0$ and $G : (-c,c)\times(-b,b) \to \mathbf{C}^n$ is a one-to-one C^∞ map.

(β) There are $0 < \epsilon \le M < \infty$ such that DG, the derivative of G, satisfies the matrix inequality $\epsilon \le (DG)^*(x,y)(DG)(x,y) \le M$ for all $(x,y) \in (-c,c)\times(-b,b)$.

(γ) If $w = G(x,y)$ for some $(x,y) \in (-c,c)\times(-b,b)$, then $x = 1-|w|^2$. Equivalently, for each $w \in N_\zeta$, there is a unique $y_w \in (-b,b)$ such that $w = G(1-|w|^2, y_w)$.

We then define $W_\zeta = N_\zeta \cap \mathbf{B}$. By ($\gamma$) and ($\alpha$), we have $W_\zeta = G((0,c)\times(-b,b)) \subset M^{(t_0)}$. Since $\mathcal{M}\cap\partial\mathbf{B}$ is compact, there is a finite subset Z of $\mathcal{M}\cap\partial\mathbf{B}$ such that $\cup_{\zeta\in Z}N_\zeta \supset \mathcal{M}\cap\partial\mathbf{B}$. Since $\cup_{\zeta\in Z}N_\zeta$ is an open subset of $\mathcal{M}$, there is a $0 < t^* < t_0$ such that $\cup_{\zeta\in Z}N_\zeta \supset M^{(t^*)}$. Obviously, if we re-enumerate the finite family of sets $\{W_\zeta : \zeta \in Z\}$ as $\{W_1, \dots, W_m\}$, then the lemma holds. $\square$

Proposition 3.13. *The dimension of* $\{f \in \mathcal{Q} : T_1 f = 0\}$ *is finite.*

Proof. Let $g \in H^2_n$. If $T_1 g = 0$, then

$$\int_{M^{(t_0)}} |(\partial_{p_w} g)(w)|^2 d\mu(w) = 0.$$

Let $y \in M^{(t_0)}$. By the argument in the proof of Lemma 3.7 (see (3.8)), the above equality implies that $(g\circ\rho_y)' = 0$ on $B_1(0,r)$ for some $r = r(y) > 0$. That is, $g\circ\rho_y$ is a constant on $B_1(0,r)$. Equivalently, g is a constant on an open subset of $M^{(t_0)}$ containing y.

Let $W_1, \dots, W_m$ be the open subsets of $M^{(t_0)}$ provided by Lemma 3.12. For each $1 \le j \le m$, since W_j is homeomorphic to $(0,c_j)\times(-b_j,b_j)$, it is a connected subset of $M^{(t_0)}$. Thus by the conclusion of the preceding paragraph, if $g \in H^2_n$ and $T_1 g = 0$, then g is a constant on W_j for every $1 \le j \le m$.

Thus we can define a linear map $L : \{f \in \mathcal{Q} : T_1 f = 0\} \to \mathbf{C}^m$ by the formula

$$Lf = (f\big|W_1, \dots, f\big|W_m),$$

where $f\big|W_j$ means the constant value of f on W_j, $1 \le j \le m$. If h is in the kernel of L, then $h = 0$ on $W_1\cup\cdots\cup W_m$. By the fact $W_1\cup\cdots\cup W_m \supset M^{(t^*)}$ and the maximum modulus principle [8, pages 72,73], we have $h = 0$ on M. That is, $h \perp \mathcal{Q}$. Since $h \in \mathcal{Q}$, this means $h = 0$. Hence $\dim\{f \in \mathcal{Q} : T_1 f = 0\} \le m$. $\square$

Proof of Theorem 3.5. Let $t^* \in (0,t_0)$ be the number provided by Lemma 3.12. By Corollary 3.11, there are $0 < t < t^*$ and $0 < C < \infty$ such that

$$\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C\|T_1 f\|_*^2 + \frac{1}{2}\|f\|_*^2$$

for every $f \in H^2_n$. After the obvious cancellation, we obtain the inequality

$$\frac{1}{2}\int_{M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \le C\|T_1 f\|_*^2 + \frac{1}{2}\int_{M^{(t_0)}\backslash M^{(t)}} |(\partial_{p_z} f)(z)|^2 d\mu(z) \tag{3.27}$$

for every $f \in H^2_n$.

Pick a positive number $t_1 > 0$ satisfying the condition $t_1 < \min\{t, c_1, \dots, c_m\}$, where $c_1, \dots, c_m$ are the same as in Lemma 3.12. With this t_1, we define the operator

$$\Phi = \int_{N^{(t_1)}} K_w \otimes K_w d\mu(w)$$

(see (2.13)). Then Φ is obviously a positive operator, and we have

$$\mathrm{tr}(\Phi) = \int_{N^{(t_1)}} \frac{1}{1-|w|^2} d\mu(w) \le \int_M 1 dv_M(w) < \infty.$$

Thus Φ is in the trace class, but here we only need the compactness of Φ. Define

$$S_1 = T_1 + \Phi.$$

We claim that there is an $a > 0$ such that the spectrum of S_1 does not intersect the interval $(0, a)$. Postponing the proof of this claim for a moment, we first show that this claim implies the conclusion of Theorem 3.5.

Of course, both T_1 and Φ map $\mathcal{Q}$ into itself. Since T_1 and Φ are both positive, we have $\{f \in \mathcal{Q} : S_1 f = 0\} \subset \{f \in \mathcal{Q} : T_1 f = 0\}$. Thus it follows from Proposition 3.13 that $\dim\{f \in \mathcal{Q} : S_1 f = 0\} < \infty$. If the spectrum of S_1 does not intersect $(0, a)$ for some $a > 0$, then 0 is not in the essential spectrum of the restricted operator $S_1\big|\mathcal{Q}$. Since Φ is compact, this means that 0 is not in the essential spectrum of the restricted operator $T_1\big|\mathcal{Q}$. Thus there is a $c > 0$ such that the spectrum of $T_1\big|\mathcal{Q}$ does not intersect $(0, c)$. Since $T_1 = 0$ on $\mathcal{Q}^\perp$, it follows that the spectrum of T_1 does not intersect $(0, c)$.

Thus we have reduced the proof of Theorem 3.5 to the proof of the claim that there is an $a > 0$ such that the spectrum of S_1 does not intersect $(0, a)$. To prove this claim, let dE be the spectral measure for the positive operator S_1. That is,

$$S_1 = \int_0^{\|S_1\|} \lambda dE(\lambda).$$

Suppose that $E(0, a) \ne 0$ for every $a > 0$. We will complete the proof by showing that this leads to a contradiction. For each $k \in \mathbf{N}$, since $E(0, 1/k) \ne 0$, we pick an $f_k \in E(0, 1/k)H^2_n$ such that $\langle S_1 f_k, f_k\rangle = 1$. That is,

$$\int_{M^{(t_0)}} |(\partial_{p_w} f_k)(w)|^2 d\mu(w) + \int_{N^{(t_1)}} |f_k(w)|^2 d\mu(w) = 1 \tag{3.28}$$

for every k. Obviously, the sequence $\{S_1^{1/2} f_k\}$ weakly converges to 0 in H_n^2. Since $T_1 \leq S_1$, there is a contraction A such that $T_1^{1/2} = AS_1^{1/2}$. Hence the sequence $\{T_1^{1/2} f_k\}$ also weakly converges to 0.

Let $0 < \epsilon < \min\{t_1, t_0 - t_1\}$. Then the closure of $N^{(t_1-\epsilon)} \cap M^{(t_1+\epsilon)}$ is a compact subset of $M^{(t_0)}$. By Lemma 3.7 and a usual covering argument, the weak convergence to 0 of the sequence $\{T_1^{1/2} f_k\}$ implies

$$\lim_{k\to\infty} \sup\{|(\partial_{p_z} f_k)(z)| : z \in N^{(t_1-\epsilon)} \cap M^{(t_1+\epsilon)}\} = 0. \tag{3.29}$$

Denote $\Delta = \{z \in M : 1 - |z|^2 = t_1 + (\epsilon/2)\}$. By the choices of t_1 and ϵ above, Δ is a compact set in $\{w \in M : 1-|w|^2 > t_1\}$. Thus Δ can be covered by a finite number of open sets $D_1, \ldots, D_\ell$ in $\{w \in M : 1-|w|^2 > t_1\}$ in such a way that each D_i is biholomorphically equivalent to the unit disc $D = \{\xi \in \mathbf{C} : |\xi| < 1\}$. By the Bergman integral formula, there is a constant $0 < C_1 < \infty$ such that

$$\sup_{z\in\Delta} |g(z)|^2 \leq C_1 \int_{N^{(t_1)}} |g(w)|^2 d\mu(w) \tag{3.30}$$

for every $g \in H_n^2$. Combining this with (3.28), we see that

$$\sup_{z\in\Delta} |f_k(z)|^2 \leq C_1 \quad \text{for every} \quad k. \tag{3.31}$$

For $k \in \mathbf{N}$, $1 \leq j \leq m$, $s \in (-b_j, b_j)$ and $u \in [t_1 - (\epsilon/2), t_1 + (\epsilon/2)]$, we can write

$$f_k(G_j(u, s)) = f_k(G_j(t_1 + (\epsilon/2), s)) - \int_u^{t_1+(\epsilon/2)} \frac{d}{dr} f_k(G_j(r, s))dr, \tag{3.32}$$

where G_j and b_j are the same as in Lemma 3.12. Taking the derivative in the integral, combining (3.32) with (3.31), (3.29) and Lemma 3.12, and using the fact $\dim_{\mathbf{C}} T_z = 1$ and lower bound (2.2), we deduce that there is a $0 < C_2 < \infty$ such that

$$\sup\{|f_k(z)| : z \in N^{(t_1-(\epsilon/2))} \cap M^{(t_1+(\epsilon/2))}\} \leq C_2 \quad \text{for every} \quad k.$$

By the maximum modulus principle, this implies that

$$\sup\{|f_k(z)| : z \in N^{(t_1-(\epsilon/2))}\} \leq C_2 \quad \text{for every} \quad k. \tag{3.33}$$

If φ is a bounded analytic function on D, then

$$\varphi'(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_0^1 \varphi(re^{i\theta}) e^{-i\theta} dr d\theta.$$

Using this identity and (3.33), and using the bounds for s_y, $y \in F$, again, we obtain a $0 < C_3 < \infty$ such that

$$\sup\{|(\partial_{p_z} f_k)(z)| : z \in N^{(t_1-(\epsilon/3))} \cap M^{(t_0)}\} \leq C_3 \quad \text{for every} \quad k. \tag{3.34}$$

Since the sequence $\{T_1^{1/2} f_k\}$ weakly converges to 0, Lemma 3.7 tells us that

$$\lim_{k\to\infty} (\partial_{p_w} f_k)(w) = 0 \quad \text{for every} \quad w \in M^{(t_0)}.$$

Combining this pointwise convergence with (3.34) and with the dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_{M^{(t_0)}\setminus M^{(t)}} |(\partial_{p_w} f_k)(w)|^2 d\mu(w) = 0. \tag{3.35}$$

Similar to (3.30), for each $t_1 < u < t_0$ there is a $0 < C(u) < \infty$ such that

$$\sup\{|g(z)|^2 : z \in M \text{ and } 1 - |z|^2 = u\} \le C(u) \int_{N^{(t_1)}} |g(w)|^2 d\mu(w) \quad \text{for every} \quad g \in H_n^2.$$

Again by the maximum modulus principle, the above implies that for each $t_1 < u < t_0$,

$$\sup\{|g(z)|^2 : z \in M \text{ and } 1 - |z|^2 \ge u\} \le C(u) \int_{N^{(t_1)}} |g(w)|^2 d\mu(w) \quad \text{for every} \quad g \in H_n^2.$$

Let $\mathcal{H}$ be the closure of H_n^2 in $L^2(N^{(t_1)}, d\mu)$. The above bound means that for each $z \in N^{(t_1)}$, the map $g \mapsto g(z)$ extends to a bounded linear functional on $\mathcal{H}$. Since $\{S_1^{1/2} f_k\}$ weakly converges to 0 and $\Phi \le S_1$, the sequence $\{\Phi^{1/2} f_k\}$ also weakly converges to 0 in H_n^2. For any $h \in H_n^2$, we have

$$\int_{N^{(t_1)}} f_k(w)\overline{h(w)} d\mu(w) = \langle \Phi^{1/2} f_k, \Phi^{1/2} h\rangle, \quad k \in \mathbf{N}.$$

Thus in the Hilbert space $\mathcal{H}$, the sequence $\{f_k\}$ weakly converges to 0. Hence

$$\lim_{k\to\infty} f_k(z) = 0$$

for every $z \in N^{(t_1)}$. Combining this pointwise convergence with (3.33) and with the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \int_{N^{(t_1)}} |f_k(w)|^2 d\mu(w) = 0.$$

From this limit and (3.28), (3.35) it follows that

$$\lim_{k\to\infty} \int_{M^{(t)}} |(\partial_{p_w} f_k)(w)|^2 d\mu(w) = 1. \tag{3.36}$$

Next we show that

$$\lim_{k\to\infty} \|T_1 f_k\|_*^2 = 0. \tag{3.37}$$

Indeed for each $k \in \mathbf{N}$, we have

$$\|T_1 f_k\|_*^2 = \langle T_1 T_1 f_k, T_1 f_k\rangle = \langle T_1 S_1 f_k, S_1 f_k\rangle - \langle T_1 \Phi f_k, T_1 f_k\rangle - \langle T_1 S_1 f_k, \Phi f_k\rangle. \tag{3.38}$$

Since $T_1 \le S_1$, $f_k \in E(0,1/k)H_n^2$ and $\|S_1^{1/2} f_k\| = 1$, we have

$$\langle T_1 S_1 f_k, S_1 f_k\rangle \le \langle S_1 S_1 f_k, S_1 f_k\rangle = \|S_1^{3/2} f_k\|^2 \le k^{-2}\|S_1^{1/2} f_k\|^2 = k^{-2}. \tag{3.39}$$

Since the sequence $\{\Phi^{1/2} f_k\}$ weakly converges to 0 and $\Phi^{1/2}$ is compact, the sequence $\{\Phi f_k\}$ converges to 0 strongly, i.e.,

$$\lim_{k\to\infty} \|\Phi f_k\| = 0. \tag{3.40}$$

By (3.39), the first term on the right-hand side of (3.38) tends to 0 as $k \to \infty$. Since $\|T_1 f_k\| \le \|T_1^{1/2}\|\|T_1^{1/2} f_k\| \le \|T_1^{1/2}\|$, it follows from (3.40) that the second term on the right-hand side of (3.38) tends to 0 as $k \to \infty$. Since $\|S_1 f_k\| \le \|S_1^{1/2}\|\|S_1^{1/2} f_k\| = \|S_1^{1/2}\|$, (3.40) also implies that the third term on the right-hand side of (3.38) tends to 0 as $k \to \infty$. This proves (3.37).

Now, recalling (3.27), we have

$$\frac{1}{2}\int_{M^{(t)}} |(\partial_{p_z} f_k)(z)|^2 d\mu(z) \le C\|T_1 f_k\|_*^2 + \frac{1}{2}\int_{M^{(t_0)}\setminus M^{(t)}} |(\partial_{p_z} f_k)(z)|^2 d\mu(z) \tag{3.41}$$

for every $k \in \mathbf{N}$. But the combination of (3.41), (3.35), (3.36) and (3.37) gives us the contradiction $1/2 \le 0$. This proves our claim that there is an $a > 0$ such that the spectrum of S_1 does not intersect $(0, a)$, which in turn completes the proof of Theorem 3.5. □

4. The range space

In addition to the operator T_1 given by (3.6), we define the operator

$$T_2 = \int_{M^{(t_0)}} K_w \otimes K_w d\mu(w). \tag{4.1}$$

Again, T_2 is a positive operator on the Drury-Arveson space H_n^2. We have

$$\mathrm{tr}(T_2) = \int_{M^{(t_0)}} \frac{1}{1-|w|^2} d\mu(w) = \int_{M^{(t_0)}} 1 dv_M(w) < \infty.$$

This shows that T_2 belongs to the trace class $\mathcal{C}_1$. Obviously, T_2 maps H_n^2 into $\mathcal{Q}$. We now define the operator

$$T = T_1 + T_2 \tag{4.2}$$

on H_n^2. Then T maps H_n^2 into $\mathcal{Q}$.

Proposition 4.1. *There is a $c' > 0$ such that the spectrum of T does not intersect the interval $(0, c')$. Moreover, Q equals the spectral projection of T corresponding to the interval $[c', \infty)$.*

Proof. It will be convenient to use the following notation for this proof: If A is an operator on H^2_n such that $A\mathcal{Q} \subset \mathcal{Q}$, we write $A\big|\mathcal{Q}$ for the restriction of A to $\mathcal{Q}$.

Theorem 3.5 tells us that the spectrum of T_1 does not intersect $(0, c)$ for some $c > 0$. By Proposition 3.13, $\dim\{f \in \mathcal{Q} : T_1 f = 0\} < \infty$. Thus 0 is not in the essential spectrum of $T_1\big|\mathcal{Q}$. Since $T_2\big|\mathcal{Q}$ is in the trace class, we conclude that 0 is not in the essential spectrum of $T\big|\mathcal{Q}$. Consequently, there is a $c' > 0$ such that the spectrum of $T\big|\mathcal{Q}$ does not intersect the interval $(0, c')$. With respect to the orthogonal decomposition $H^2_n = \mathcal{R} \oplus \mathcal{Q}$, we have $T = 0 \oplus (T\big|\mathcal{Q})$. Therefore the spectrum of T does not intersect $(0, c')$. Once this fact is established, the assertion that Q equals the spectral projection of T corresponding to the interval $[c', \infty)$ is equivalent to the assertion that $\ker(T\big|\mathcal{Q}) = \{0\}$.

To prove this last assertion, let $f \in \mathcal{Q}$ be such that $Tf = 0$. Since both T_1 and T_2 are positive operators, the condition $Tf = 0$ implies $T_2 f = 0$, i.e.,

$$\int_{M^{(t_0)}} |f(w)|^2 d\mu(w) = 0.$$

This means that $f = 0$ on $M^{(t_0)}$. By the maximum modulus principle, we have $f = 0$ on M. Thus $f \perp \mathcal{Q}$. Since $f \in \mathcal{Q}$, f is the zero element. This completes the proof. □

Let f be a C^1 function on an open set containing the closure of M. We define

$$\|f\|_\# = \left\{ \int_{M^{(t_0)}} |(\partial_{p_w} f)(w)|^2 d\mu(w) + \int_{M^{(t_0)}} |(\overline{\partial_{p_w}} f)(w)|^2 d\mu(w) + \int_{M^{(t_0)}} |f(w)|^2 d\mu(w) \right\}^{1/2}$$

(see (2.21)). Let $\mathcal{L}_0$ be the collection of all such f with the property $\|f\|_\# < \infty$. Then $\|\cdot\|_\#$ is a norm on $\mathcal{L}_0$. This norm is designed to have the symmetric property $\|\bar{f}\|_\# = \|f\|_\#$, which will be important later on.

Obviously, the norm $\|\cdot\|_\#$ is induced by the inner product

$$\begin{aligned}\langle f, g\rangle_\# = &\int_{M^{(t_0)}} (\partial_{p_w} f)(w)\overline{(\partial_{p_w} g)(w)} d\mu(w) + \int_{M^{(t_0)}} (\overline{\partial_{p_w}} f)(w)\overline{(\overline{\partial_{p_w}} g)(w)} d\mu(w) \\ &+ \int_{M^{(t_0)}} f(w)\overline{g(w)} d\mu(w),\end{aligned}$$

$f, g \in \mathcal{L}_0$. Let $\mathcal{L}$ denote the completion of $\mathcal{L}_0$ with respect to the norm $\|\cdot\|_\#$. Then $\mathcal{L}$ is a Hilbert space.

Definition 4.2. (a) Let $\mathcal{P}$ be the closure of the analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$ in $\mathcal{L}$.
(b) Let P denote the orthogonal projection from $\mathcal{L}$ onto $\mathcal{P}$.

Obviously, if $f \in H^2_n$, then

$$\|f\|^2_\# = \langle Tf, f\rangle = \|T^{1/2} f\|^2. \tag{4.3}$$

Since $\mathbf{C}[z_1, \ldots, z_n]$ is dense in H^2_n, every $f \in H^2_n$ is naturally an element in $\mathcal{P}$.

Definition 4.3. Let J denote the operator that takes each $f \in H^2_n$ to the same f in $\mathcal{P}$.

Thus we can rewrite (4.3) in the form of the operator identity

$$J^*J = T. \tag{4.4}$$

Intuitively, we think of J as restricting each $f \in H^2_n$ to the set $M^{(t_0)}$. We call $\mathcal{P}$ the *range space* for the restriction operator J. If $f \in \mathcal{R}$, then we obviously have $Jf = 0$. On the other hand, by Proposition 4.1,

$$\|Jf\|_\# = \|T^{1/2}f\| \geq \sqrt{c'}\|f\| \quad \text{for every} \quad f \in \mathcal{Q}.$$

Thus J is an invertible operator that maps $\mathcal{Q}$ onto $\mathcal{P}$.

We define the operators

$$(\hat{T}_1 f)(z) = \int_{M^{(t_0)}} \frac{\langle z, p_w\rangle}{(1 - \langle z, w\rangle)^2}(\partial_{p_w} f)(w)d\mu(w) \quad \text{and}$$
$$(\hat{T}_2 f)(z) = \int_{M^{(t_0)}} \frac{1}{1 - \langle z, w\rangle} f(w)d\mu(w),$$

$f \in \mathcal{L}_0$.

Lemma 4.4. *The operators $\hat{T}_1$ and $\hat{T}_2$ are bounded on $\mathcal{L}_0$. Therefore $\hat{T}_1$ and $\hat{T}_2$ naturally extend to bounded operators on $\mathcal{L}$.*

Proof. By Lemma 2.8 and the Schur test, the kernels

$$\frac{\langle p_z, w\rangle\langle z, p_w\rangle}{(1 - \langle z, w\rangle)^3}, \quad \frac{\langle z, p_w\rangle}{(1 - \langle z, w\rangle)^2}, \quad \frac{\langle p_z, w\rangle}{(1 - \langle z, w\rangle)^2} \quad \text{and} \quad \frac{1}{1 - \langle z, w\rangle}$$

define bounded operators on $L^2(M^{(t_0)}, d\mu)$. Let us verify the details for the case

$$K(z, w) = \frac{\langle p_z, w\rangle\langle z, p_w\rangle}{(1 - \langle z, w\rangle)^3};$$

the other cases are similar.

Define the function $h(z) = (1 - |z|^2)^{-1/2}$ on $M^{(t_0)}$. By (3.1) and Lemma 2.8,

$$\int_{M^{(t_0)}} |K(z, w)|h(w)d\mu(w) \leq \int_{M^{(t_0)}} \frac{(1 - |w|^2)^{1/2}}{|1 - \langle z, w\rangle|^{1+1+(1/2)+(1/2)}} dv_M(w) \leq C_1 h(z).$$

Similarly,

$$\int_{M^{(t_0)}} |K(z, w)|h(z)d\mu(z) \leq C_1 h(w).$$

Thus by the Schur test, the kernel $K(z,w)$ represents a bounded operator on $L^2(M^{(t_0)}, d\mu)$.

Let $f \in \mathcal{L}_0$. Then

$$(\partial_{p_z}\hat{T}_1 f)(z) = \int_{M^{(t_0)}} \frac{2\langle p_z, w\rangle\langle z, p_w\rangle(\partial_{p_w} f)(w)}{(1-\langle z,w\rangle)^3} d\mu(w) + \int_{M^{(t_0)}} \frac{\langle p_z, p_w\rangle(\partial_{p_w} f)(w)}{(1-\langle z,w\rangle)^2} d\mu(w).$$

Note that $\overline{\partial_{p_z}}\hat{T}_1 f = 0$. By the boundedness of the kernels mentioned in the first paragraph, $\hat{T}_1$ is bounded on $\mathcal{L}_0$. Similarly, $\hat{T}_2$ is also bounded on $\mathcal{L}_0$. □

On the Hilbert space $\mathcal{L}$ we now define the operator

$$\hat{T} = \hat{T}_1 + \hat{T}_2.$$

Our next lemma is crucial to the proof of the 1-essential normality of $\mathcal{Q}$. It deals with a rarity in operator theory: a situation where self-adjointness is not so obvious.

Lemma 4.5. *With respect to the inner product $\langle\cdot,\cdot\rangle_\#$, the operator $\hat{T}$ is self-adjoint.*

Proof. For any $f \in \mathcal{L}_0$, straightforward differentiation gives us

$$(\partial_{p_z}\hat{T}_1 f)(z) = \int_{M^{(t_0)}} \frac{2\langle p_z, w\rangle\langle z, p_w\rangle(\partial_{p_w} f)(w)}{(1-\langle z,w\rangle)^3} d\mu(w) + \int_{M^{(t_0)}} \frac{\langle p_z, p_w\rangle(\partial_{p_w} f)(w)}{(1-\langle z,w\rangle)^2} d\mu(w),$$

$$(\partial_{p_z}\hat{T}_2 f)(z) = \int_{M^{(t_0)}} \frac{\langle p_z, w\rangle}{(1-\langle z,w\rangle)^2} f(w) d\mu(w).$$

Also, $(\overline{\partial_{p_z}}\hat{T}_1 f)(z) = 0$ and $(\overline{\partial_{p_z}}\hat{T}_2 f)(z) = 0$. Thus for $f, g \in \mathcal{L}_0$, we have

$$\begin{aligned}\langle \hat{T}f, g\rangle_\# = &\int_{M^{(t_0)}} (\partial_{p_z}\hat{T}_1 f)(z)\overline{(\partial_{p_z} g)(z)} d\mu(z) + \int_{M^{(t_0)}} (\partial_{p_z}\hat{T}_2 f)(z)\overline{(\partial_{p_z} g)(z)} d\mu(z) \\ &+ \int_{M^{(t_0)}} (\hat{T}_1 f)(z)\overline{g(z)} d\mu(z) + \int_{M^{(t_0)}} (\hat{T}_2 f)(z)\overline{g(z)} d\mu(z).\end{aligned} \tag{4.5}$$

By the integral formula for $\partial_{p_z}\hat{T}_1 f$ and a change of order of integration, we have

$$\int_{M^{(t_0)}} (\partial_{p_z}\hat{T}_1 f)(z)\overline{(\partial_{p_z} g)(z)} d\mu(z) = \int_{M^{(t_0)}} (\partial_{p_w} f)(w)\overline{(\partial_{p_w}\hat{T}_1 g)(w)} d\mu(w). \tag{4.6}$$

It is obvious that

$$\int_{M^{(t_0)}} (\hat{T}_2 f)(z)\overline{g(z)} d\mu(z) = \int_{M^{(t_0)}} f(w)\overline{(\hat{T}_2 g)(w)} d\mu(w). \tag{4.7}$$

Finally, by the formula for $\partial_{p_z}\hat{T}_2 f$, we have

(4.8)

$$\int_{M^{(t_0)}} (\partial_{p_z}\hat{T}_2 f)(z)\overline{(\partial_{p_z} g)(z)} d\mu(z) = \int_{M^{(t_0)}} f(w)\overline{(\hat{T}_1 g)(w)} d\mu(w) \quad \text{and}$$

$$\int_{M^{(t_0)}} (\hat{T}_1 f)(z)\overline{g(z)} d\mu(z) = \int_{M^{(t_0)}} (\partial_{p_w} f)(w)\overline{(\partial_{p_w}\hat{T}_2 g)(w)} d\mu(w). \tag{4.9}$$

Combining (4.5)-(4.9) with the fact that $(\overline{\partial_{p_w}\hat{T}g})(w) = 0$, we see that $\langle \hat{T}f, g\rangle_{\#} = \langle f, \hat{T}g\rangle_{\#}$. This completes the proof. □

From the above proof we see that the individual operators $\hat{T}_1$ and $\hat{T}_2$ are not self-adjoint on $\mathcal{L}$. But miraculously, somehow the sum $\hat{T} = \hat{T}_1 + \hat{T}_2$ is self-adjoint.

Proposition 4.6. (a) $\hat{T}$ *maps* $\mathcal{L}$ *into* $\mathcal{P}$.
(b) *Let* $\tilde{T}$ *denote the restriction of* $\hat{T}$ *to the subspace* $\mathcal{P}$. *Then* $\tilde{T} = JJ^*$. *In particular,* $\tilde{T}$ *is invertible on* $\mathcal{P}$.
(c) *With respect to the orthogonal decomposition* $\mathcal{L} = \mathcal{P} \oplus \mathcal{P}^{\perp}$, *we have* $\hat{T} = \tilde{T} \oplus 0$.

Proof. (a) Recall that the kernels listed in the proof of Lemma 4.4 define bounded operators on $L^2(M^{(t_0)}, d\mu)$. Therefore for any $f \in \mathcal{L}_0$, we have

$$\lim_{t\downarrow 0}\left\| \int_{M^{(t)}} (\partial_{p_w} f)(w) JK_{w,p_w} d\mu(w)\right\|_{\#} = 0.$$

Since we already know that $JH_n^2 \subset \mathcal{P}$, we have

$$\int_{M^{(t_0)}\backslash M^{(t)}} (\partial_{p_w} f)(w) JK_{w,p_w} d\mu(w) \in \mathcal{P}$$

for every $0 < t < t_0$. Therefore

$$\hat{T}_1 f = \int_{M^{(t_0)}} (\partial_{p_w} f)(w) JK_{w,p_w} d\mu(w) \in \mathcal{P}.$$

A similar argument shows that $\hat{T}_2 f \in \mathcal{P}$ for $f \in \mathcal{L}_0$. Thus $\hat{T}\mathcal{L}_0 \subset \mathcal{P}$. Since $\mathcal{L}_0$ is dense in $\mathcal{L}$ and since $\hat{T}$ is a bounded operator, it follows that $\hat{T}\mathcal{L} \subset \mathcal{P}$.

(b) For each $f \in \mathcal{Q}$, it is easy to see that $\tilde{T}Jf = JTf$. Combining this with (4.4), we have $\tilde{T}Jf = JTf = JJ^*Jf$. Since $J\mathcal{Q} = \mathcal{P}$, this implies $\tilde{T} = JJ^*$. Since $J : \mathcal{Q} \to \mathcal{P}$ and $J^* : \mathcal{P} \to \mathcal{Q}$ are invertible, so is $\tilde{T}$.

(c) This follows from (a) and the self-adjointness of $\hat{T}$, which is provided by Lemma 4.5. □

In what follows, we write $\zeta_1, \ldots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$.

Definition 4.7. For $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, $\hat{M}_\varphi$ denotes the operator of multiplication by the function φ on $\mathcal{L}$.

Proposition 4.8. *For each* $j \in \{1, \ldots, n\}$, $\mathcal{P}$ *is an invariant subspace for* $\hat{M}_{\zeta_j}$.

Proof. Let $f \in \mathcal{Q}$. Then $Q_{\zeta_j} f = \zeta_j f - g_j$ for some $g_j \in \mathcal{R}$. It follows from (4.4) that $Jg_j = 0$. Therefore

$$JQ_{\zeta_j} f = J\zeta_j f = \zeta_j Jf = \hat{M}_{\zeta_j} Jf. \tag{4.10}$$

That is, for each $f \in \mathcal{Q}$, we have $\hat{M}_{\zeta_j} Jf \in J\mathcal{Q} = \mathcal{P}$, which proves the proposition. □

Proposition 4.8 makes it possible for us to introduce

Definition 4.9. For each $j \in \{1, \dots, n\}$, let M_{ζ_j} denote the restriction of the operator $\hat{M}_{\zeta_j}$ to the invariant subspace $\mathcal{P}$.

Thus we can restate (4.10) in the form

Corollary 4.10. *We have $JQ_{\zeta_j} = M_{\zeta_j} J$ for every $j \in \{1, \dots, n\}$.*

We end the section with two crucial technical results, whose proofs will have to wait until Section 6.

Proposition 4.11. *For every $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \dots, \zeta_n, \bar{\zeta}_n]$, we have $[\hat{M}_\varphi, \hat{T}] \in \mathcal{C}_2$.*

Proposition 4.12. *For analytic polynomials $q, r \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, we have*

$$[\hat{M}_{\bar{r}}, [\hat{M}_q, \hat{T}]]P \in \mathcal{C}_1.$$

5. Operators on $L^2(M, d\mu)$

First, we recall the following:

Proposition 5.1. [29, Proposition 11.1] *Under the assumption $d = 1$, there is a $0 < C < \infty$ such that*

$$|\zeta - w| \le C|1 - \langle \zeta, w \rangle|$$

for all $\zeta, w \in M$.

Lemma 5.2. *If $G(z, w)$ is a bounded Borel function on $M \times M$, then the operator*

$$(A_G\varphi)(z) = \int_M \frac{G(z, w)}{(1 - \langle z, w \rangle)^3} \varphi(w) d\mu(w), \quad \varphi \in L^2(M, \mu), \tag{5.1}$$

is bounded on $L^2(M, \mu)$.

Proof. Consider the function $h(w) = (1 - |w|^2)^{-1/2}$ on M. Recalling (3.1), we have

$$\begin{aligned} \int_M h(w) \left| \frac{G(z, w)}{(1 - \langle z, w \rangle)^3} \right| d\mu(w) &\le \|G\|_\infty \int_M \frac{(1 - |w|^2)^{1/2}}{|1 - \langle z, w \rangle|^{1+1+(1/2)+(1/2)}} dv_M(w) \\ &\le C_1 \|G\|_\infty h(z), \end{aligned}$$

where the last step is an application of Lemma 2.8 in the case $d = 1$. Similarly,

$$\int_M h(z) \left| \frac{G(z, w)}{(1 - \langle z, w \rangle)^3} \right| d\mu(z) \le C_1 \|G\|_\infty h(w).$$

Thus the Schur test gives us $\|A_G\| \le C_1 \|G\|_\infty$. $\square$

On the Hilbert space $L^2(M, d\mu)$, we define the operator

$$(Z\varphi)(z) = \int_M \frac{1}{(1 - \langle z, w \rangle)^3} \varphi(w) d\mu(w), \quad \varphi \in L^2(M, d\mu).$$

Lemma 5.3. *If f is a Lipschitz function on M, then the commutator $[M_f, Z]$ is a Hilbert-Schmidt operator on $L^2(M, d\mu)$.*

Proof. Obviously, the kernel of $[M_f, Z]$ equals $(1-\langle z, w\rangle)^{-3}(f(z)-f(w))$. If f is Lipschitz on M, then it follows from Proposition 5.1 that

$$\iint \left|\frac{f(z)-f(w)}{(1-\langle z, w\rangle)^3}\right|^2 d\mu(w)d\mu(z) \le C_1 \iint \frac{1}{|1-\langle z, w\rangle|^2} dv_M(w)dv_M(z).$$

Note that

$$\iint \frac{1}{|1-\langle z, w\rangle|^2} dv_M(w)dv_M(z) \le \iint \frac{2^{1/2}}{|1-\langle z, w\rangle|^{1+1+(1/2)}} dv_M(w)dv_M(z)$$
$$\le C_2 \int \frac{1}{(1-|z|^2)^{1/2}} dv_M(z) < \infty,$$

where the second $\le$ is an application of Lemma 2.8. Hence $[M_f, Z] \in \mathcal{C}_2$. □

By Lemma 5.2, Z is a bounded operator on $L^2(M, d\mu)$. Obviously, Z is self-adjoint. What is less obvious is the following:

Lemma 5.4. *We have $Z \ge 0$ on $L^2(M, d\mu)$.*

Proof. To prove this we need the Hilbert space $\mathcal{H}^{(2-n)}$ (see, e.g., [19]). Recall that, since $3 = n+1+(2-n)$, $\mathcal{H}^{(2-n)}$ is the Hilbert space of analytic functions on $\mathbf{B}$ which has

$$K_w^{(2-n)}(z) = \frac{1}{(1-\langle z, w\rangle)^3}, \quad z, w \in \mathbf{B}, \tag{5.2}$$

as its reproducing kernel. That is, the space $\mathcal{H}^{(2-n)}$ has "weight" $2-n$ [19].

For each $0 < \alpha < 1$,

$$Y_\alpha = \int_{N^{(\alpha)}} K_w^{(2-n)} \otimes K_w^{(2-n)} d\mu(w)$$

is obviously a positive operator on $\mathcal{H}^{(2-n)}$ (see (2.13)). For each $0 < \alpha < 1$, let I_α be the operator that maps each $f \in \mathcal{H}^{(2-n)}$ to the function $f\big|N^{(\alpha)}$ in $L^2(N^{(\alpha)}, d\mu)$. Then $\|I_\alpha f\|^2 = \langle Y_\alpha f, f\rangle$ for every $f \in \mathcal{H}^{(2-n)}$. Therefore $Y_\alpha = I_\alpha^* I_\alpha$.

Given an $f \in \mathcal{H}^{(2-n)}$, we have $\langle f, K_w^{(2-n)}\rangle_{2-n} = f(w)$, $w \in \mathbf{B}$. Therefore

$$(I_\alpha Y_\alpha f)(z) = \int_{N^{(\alpha)}} \frac{f(w)}{(1-\langle z, w\rangle)^3} d\mu(w) \quad \text{for} \quad z \in N^{(\alpha)}.$$

On the other hand, by the definitions of Z and I_α,

$$(M_{\chi_{N^{(\alpha)}}} Z I_\alpha f)(z) = \int_{N^{(\alpha)}} \frac{f(w)}{(1-\langle z, w\rangle)^3} d\mu(w) \quad \text{for} \quad z \in N^{(\alpha)}.$$

Thus under the natural identification of $L^2(N^{(\alpha)}, d\mu)$ with $\{\varphi \in L^2(M, d\mu) : \varphi = 0$ on $M\backslash N^{(\alpha)}\}$, we have

$$I_\alpha Y_\alpha = M_{\chi_{N(\alpha)}} Z I_\alpha.$$

Therefore for each analytic polynomial $q \in \mathbf{C}[z_1, \ldots, z_n]$,

$$\langle Z I_\alpha q, I_\alpha q\rangle = \langle M_{\chi_{N(\alpha)}} Z I_\alpha q, I_\alpha q\rangle = \langle I_\alpha Y_\alpha q, I_\alpha q\rangle = \langle Y_\alpha q, I_\alpha^* I_\alpha q\rangle = \|Y_\alpha q\|^2 \geq 0.$$

Since this holds for every $0 < \alpha < 1$, we have $\langle Zq, q\rangle \geq 0$ for every $q \in \mathbf{C}[z_1, \ldots, z_n]$. Since the range of Z is obviously contained in the closure of $\mathbf{C}[z_1, \ldots, z_n]$ in $L^2(M, d\mu)$ and since Z is self-adjoint, we conclude that Z is positive on $L^2(M, d\mu)$. □

Definition 5.5. (a) Let $\mathcal{E}$ denote the closure of $\mathbf{C}[z_1, \ldots, z_n]$ in $L^2(M, d\mu)$.
(b) Let E denote the orthogonal projection from $L^2(M, d\mu)$ onto $\mathcal{E}$.

Proposition 5.6. *There is a $\gamma > 0$ such that the spectrum of Z does not intersect the interval $(0, \gamma)$. Moreover, the range of Z equals $\mathcal{E}$.*

The proof of Proposition 5.6 is similar to the work in Section 3, in particular to the proof of Theorem 3.5. For this reason we will present the proof of Proposition 5.6 in Appendix 2.

An immediate consequence of Lemma 5.4 and Proposition 5.6 is that we can write the orthogonal projection E in the form $E = h(Z)$ for some $h \in C_c^\infty(\mathbf{R})$. Applying the standard smooth functional calculus [9], from Lemma 5.3 we obtain

Corollary 5.7. *If f is a Lipschitz function on M, then the commutator $[M_f, E]$ is a Hilbert-Schmidt operator on $L^2(M, d\mu)$.*

We again need the function $\rho(z) = 1 - |z|^2$, $z \in \mathbf{B}$.

Lemma 5.8. *If $r > 1/2$, then $M_{\rho^r} E$ is a Hilbert-Schmidt operator on $L^2(M, d\mu)$.*

Proof. We know that $Z \geq 0$ from Lemma 5.4. Let $\tilde{Z}$ denote the restriction of Z to its range $\mathcal{E}$. Then Proposition 5.6 implies that $\tilde{Z}$ is invertible and that $E = Z(\tilde{Z}^{-1} \oplus 0)$. Therefore it suffices to show that $M_{\rho^r} Z \in \mathcal{C}_2$ for $1/2 < r < 1$.

For each $f \in L^2(M, d\mu)$, we have

$$(M_{\rho^r} Z f)(z) = \int_M \frac{(1 - |z|^2)^r}{(1 - \langle z, w\rangle)^3} f(w) d\mu(w).$$

Moreover,

$$\iint \left| \frac{(1 - |z|^2)^r}{(1 - \langle z, w\rangle)^3} \right|^2 d\mu(w) d\mu(z) \leq \iint \frac{2^{2+2r}}{|1 - \langle z, w\rangle|^{4-2r}} dv_M(w) dv_M(z)$$
$$= \iint \frac{2^{2+2r}}{|1 - \langle z, w\rangle|^{1+1+2(1-r)}} dv_M(w) dv_M(z) \leq \int \frac{C_1}{(1 - |z|^2)^{2(1-r)}} dv_M(z),$$

where the second $\leq$ follows from Lemma 2.8. For $r > 1/2$, we have $2(1-r) < 1$, and the above is finite. Therefore $M_{\rho^r} Z \in \mathcal{C}_2$. $\square$

Lemma 5.9. *For all $f, g \in \mathrm{Lip}(M)$ and $0 < r < 1$, we have $M_{\rho^{-r}}[M_f, [M_g, Z]] \in \mathcal{C}_2$.*

Proof. It suffices to show that

$$\iint \{\rho^{-r}(z)\}^2 \frac{|(f(z)-f(w))(g(z)-g(w))|^2}{|1-\langle z,w\rangle|^6} d\mu(w) d\mu(z) < \infty.$$

By the Lipschitz condition for f, g and Proposition 5.1, the left-hand side does not exceed

$$\begin{aligned} C_1 \iint (1-|z|^2)^{-2r} \frac{|1-\langle z,w\rangle|^4}{|1-\langle z,w\rangle|^6} d\mu(w) d\mu(z) &\leq 2C_1 \iint \frac{(1-|z|^2) dv_M(w) dv_M(z)}{(1-|z|^2)^{2r}|1-\langle z,w\rangle|} \\ &= 2C_1 \iint \frac{1}{(1-|z|^2)^r} \cdot \frac{(1-|z|^2)^{1-r}}{|1-\langle z,w\rangle|} dv_M(w) dv_M(z) \\ &\leq 4C_1 \iint \frac{dv_M(w) dv_M(z)}{(1-|z|^2)^r (1-|w|^2)^r}, \end{aligned}$$

which is finite because $r < 1$. This completes the proof. $\square$

Lemma 5.10. *For $f, g \in \mathrm{Lip}(M)$, the operators $E[M_f, [M_g, Z]]$ and $[M_f, [M_g, Z]]E$ are in the trace class $\mathcal{C}_1$.*

Proof. Take any $1/2 < r < 1$. We have the factorization

$$E[M_f, [M_g, Z]] = EM_{\rho^r} \cdot M_{\rho^{-r}}[M_f, [M_g, Z]].$$

Applying Lemmas 5.8 and 5.9, we obtain the membership $E[M_f, [M_g, Z]] \in \mathcal{C}_1$. Then note that $[M_f, [M_g, Z]]E = \{E[[Z, M_{\bar{g}}], M_{\bar{f}}]\}^* \in \mathcal{C}_1$. $\square$

Proposition 5.11. *For all $f, g \in \mathrm{Lip}(M)$, the double commutator $[M_f, [M_g, Z]]$ is in the trace class.*

Proof. Let us write $A \sim_1 B$ if $A - B \in \mathcal{C}_1$. By Lemma 5.10, we have

$$[M_f, [M_g, Z]] \sim_1 (1-E)[M_f, [M_g, Z]] \sim_1 (1-E)[M_f, [M_g, Z]](1-E).$$

Then note that $(1-E)Z = 0$ and $Z(1-E) = 0$. Therefore

$$\begin{aligned} (1-E)[M_f, [M_g, Z]](1-E) &= -(1-E)M_f Z M_g (1-E) - (1-E) M_g Z M_f (1-E) \\ &= -[1-E, M_f] Z [M_g, 1-E] - [1-E, M_g] Z [M_f, 1-E] \\ &= [M_f, E] Z [M_g, E] + [M_g, E] Z [M_f, E], \end{aligned}$$

which, according to Corollary 5.7, is in the trace class. $\square$

On the Hilbert space $L^2(M, d\mu)$, we also define the operator

$$(\Lambda\varphi)(z) = \int_M \frac{1}{(1-\langle z,w\rangle)^2} \varphi(w) d\mu(w), \quad \varphi \in L^2(M, d\mu).$$

Lemma 5.12. *We have $\Lambda \in \mathcal{C}_2$. Moreover, $E[M_f, \Lambda] \in \mathcal{C}_1$ and $[M_f, \Lambda]E \in \mathcal{C}_1$ for every $f \in \mathrm{Lip}(M)$.*

Proof. We have

$$\iint \frac{1}{|1-\langle z,w\rangle|^4} d\mu(w)d\mu(z) \le \iint \frac{2^{3/2}}{|1-\langle z,w\rangle|^{1+1+(1/2)}} dv_M(w)dv_M(z)$$
$$\le \int \frac{C_1}{(1-|z|^2)^{1/2}} dv_M(z) < \infty,$$

where the second $\le$ is an application of Lemma 2.8. Therefore $\Lambda \in \mathcal{C}_2$. Similar to what we saw in the proof of Lemma 5.9, we have

$$M_{\rho^{-r}}[M_f, \Lambda] \in \mathcal{C}_2$$

for $f \in \mathrm{Lip}(M)$ and $1/2 < r < 1$. Combining this membership with the factorization

$$E[M_f, \Lambda] = EM_{\rho^r} \cdot M_{\rho^{-r}}[M_f, \Lambda]$$

and with Lemma 5.8, we obtain the membership $E[M_f, \Lambda] \in \mathcal{C}_1$. By the relation $[M_f, \Lambda]E = \{E[\Lambda, M_{\bar f}]\}^*$, we also have $[M_f, \Lambda]E \in \mathcal{C}_1$. $\square$

Proposition 5.13. *For every $f \in \mathrm{Lip}(M)$ we have $[M_f, \Lambda] \in \mathcal{C}_1$.*

Proof. Note that $(1-E)\Lambda = 0$ and $\Lambda(1-E) = 0$. Applying Lemma 5.12, we have

$$[M_f, \Lambda] \sim_1 (1-E)[M_f, \Lambda] \sim_1 (1-E)[M_f, \Lambda](1-E) = 0,$$

i.e., $[M_f, \Lambda] \in \mathcal{C}_1$. $\square$

Lemma 5.14. *Write $\|\cdot\|_{2-n}$ for the norm on the reproducing-kernel Hilbert space $\mathcal{H}^{(2-n)}$. There is a $0 < C < \infty$ such that*

$$\int_M |f(w)|^2 d\mu(w) \le C\|f\|_{2-n}^2$$

for every $f \in \mathcal{H}^{(2-n)}$. In other words, $d\mu$ is a Carleson measure for $\mathcal{H}^{(2-n)}$.

Proof. This is an easier version of Proposition 3.2. We pick a natural number $m \ge 2$ such that $2m - n \ge 0$. By Lemma 2.14, we have

$$\int_M |f(w)|^2 d\mu(w) \le C_1 \sum_{|\alpha|=m-1} \int_{M^{(b)}} |(\partial^\alpha f)(w)|^2 (1-|w|^2)^{2m-2} d\mu(w)$$
$$+ C_1 \sum_{0\le|\beta|\le m-2} \int_{N^{(a)}} |(\partial^\beta f)(w)|^2 d\mu(w),$$

$f \in \mathcal{H}^{(2-n)}$. Recalling (2.13), the second term is dominated by $C_2\|f\|_{2-n}^2$. It follows from Lemma 2.12 that for each $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| = m-1$,

$$\int_{M^{(b)}} |(\partial^\alpha f)(w)|^2(1-|w|^2)^{2m-2}d\mu(w)$$
$$\leq C_3 \int_{\mathbf{B}} |(\partial^\alpha f)(w)|^2(1-|w|^2)^{2m-n}dv(w) \leq C_4\|f\|_{2-n}^2,$$

where the second $\leq$ follows from Lemmas 2.15 and 2.16. □

6. Proofs of Propositions 4.11 and 4.12

Let $\mathcal{I} : \mathcal{L} \to L^2(M^{(t_0)}, d\mu)$ be the natural embedding. Obviously, $\|\mathcal{I}\| \leq 1$.

As usual, we identify $L^2(M^{(t_0)}, d\mu)$ with the subspace $\{f \in L^2(M, d\mu) : f = 0$ on $M\backslash M^{(t_0)}\}$ of $L^2(M, d\mu)$.

Lemma 6.1. *If $r < 1$, then the operator $M_{\rho^{-r}}\mathcal{I}$ is bounded.*

Proof. Denote $\tau = 1-2r$. If $r < 1$, then $\tau > -1$, and therefore we can apply Lemma 2.13 to the measure $d\nu_\tau$. By that lemma, there are $0 < a < b < t_0$ and $0 < C < \infty$ such that for each $f \in \mathcal{L}_0$, we have

$$\int_{M^{(t_0)}} |\rho^{-r}(w)f(w)|^2d\mu(w) = \int_{M^{(t_0)}} |f(w)|^2d\nu_\tau(w)$$
$$\leq C\int_{M^{(b)}} |(\nabla_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2d\nu_\tau(w) + C\int_{N^{(a)}\cap M^{(t_0)}} |f(w)|^2d\nu_\tau(w).$$

It follows from (2.2) that if f is real-valued, then $|(\nabla_{\mathcal{M}}f)(w)| \leq (2/\gamma)|(\partial_{p_w}f)(w)|$, $w \in M^{(t_0)}$. Substituting this in the above and noting that $2+\tau > 1$, we have

$$\int_{M^{(t_0)}} |\rho^{-r}(w)f(w)|^2d\mu(w)$$
$$\leq (2/\gamma)^2C\int_{M^{(b)}} |(\partial_{p_w}f)(w)|^2d\mu(w) + a^{-2}C\int_{N^{(a)}\cap M^{(t_0)}} |f(w)|^2d\mu(w)$$

for real-valued $f \in \mathcal{L}_0$. Hence there is a $0 < C_1 < \infty$ such that $\|M_{\rho^{-r}}\mathcal{I}f\| \leq C_1\|f\|_\#$ for real-valued $f \in \mathcal{L}$.

Now we use the fact that the norm $\|\cdot\|_\#$ has the symmetry $\|\bar{g}\|_\# = \|g\|_\#$ for every $g \in \mathcal{L}$. Thus if $g = f_1 + if_2$, where f_1 and f_2 are real-valued, then $\|f_1\|_\# \leq \|g\|_\#$ and $\|f_2\|_\# \leq \|g\|_\#$. Combining this fact with the conclusion of the preceding paragraph, we see that $\|M_{\rho^{-r}}\mathcal{I}g\| \leq 2C_1\|g\|_\#$ for every $g \in \mathcal{L}$. □

Lemma 6.2. *If $\varphi \in L^\infty(M^{(t_0)}, d\mu)$, then $EM_\varphi\mathcal{I}$ is a Hilbert-Schmidt operator.*

Proof. Take any $1/2 < r < 1$ and factor $EM_\varphi\mathcal{I}$ in the form

$$EM_\varphi\mathcal{I} = EM_{\rho^r} \cdot M_\varphi \cdot M_{\rho^{-r}}\mathcal{I}.$$

Since $EM_{\rho^r} = (M_{\rho^r}E)^*$, it follows from Lemmas 5.8 and 6.1 that $EM_\varphi\mathcal{I}$ is a Hilbert-Schmidt operator. □

We will write I for the restriction of $\mathcal{I}$ to the subspace $\mathcal{P}$.

Lemma 6.3. *The embedding $I : \mathcal{P} \to L^2(M^{(t_0)}, d\mu)$ is a Hilbert-Schmidt operator.*

Proof. Recall from Section 4 that the operator $J : \mathcal{Q} \to \mathcal{P}$ is invertible. Thus it suffices to show that $IJ : \mathcal{Q} \to L^2(M^{(t_0)}, d\mu)$ is a Hilbert-Schmidt operator. By Lemma 2.13, there are $0 < a < b < 1$ and $0 < C < \infty$ such that

$$\int_M |f(w)|^2 d\mu(w) \le C \int_{M^{(b)}} |(\nabla_{\mathcal{M}} f)(w)|^2 (1-|w|^2)^2 d\mu(w) + C \int_{N^{(a)}} |f(w)|^2 d\mu(w)$$

for every $f \in H^2_n$. Therefore

$$\int_M |f(w)|^2 d\mu(w)$$

$$\le 2C \sum_{j=1}^n \int_M |(\partial_j f)(w)|^2 (1-|w|^2)^2 d\mu(w) + \frac{C}{a^2} \int_M |f(w)|^2 (1-|w|^2)^2 d\mu(w). \tag{6.1}$$

Write $\|\cdot\|_{2-n}$ for the norm on $\mathcal{H}^{(2-n)}$. Then it is well known that $\|\partial_j g\|_{2-n} \le C_1 \|g\|$ for all $g \in H^2_n$ and $1 \le j \le n$. Moreover, $\|g\|_{2-n} \le \|g\|$ for $g \in H^2_n$. Combining these bounds with Lemma 5.14, we see that

$$Af = (\partial_1 f, \dots, \partial_n f, f), \quad f \in \mathcal{Q},$$

is a bounded operator that maps $\mathcal{Q}$ into $\mathcal{E}^{[n+1]}$, the orthogonal sum of $n+1$ copies of $\mathcal{E}$. Let $\{M_\rho E\}^{[n+1]}$ denote the orthogonal sum of $n+1$ copies of $M_\rho E$. By (6.1), there is a bounded operator $B : \{L^2(M, d\mu)\}^{[n+1]} \to L^2(M^{(t_0)}, d\mu)$ such that

$$IJ = B\{M_\rho E\}^{[n+1]} A.$$

By Lemma 5.8, $\{M_\rho E\}^{[n+1]}$ is a Hilbert-Schmidt operator. Therefore so is $IJ : \mathcal{Q} \to L^2(M^{(t_0)}, d\mu)$. This completes the proof. □

For each $f \in \mathcal{L}_0$, we define

$$(\mathcal{D}f)(z) = (\partial_{p_z} f)(z) \quad \text{and} \quad (\overline{\mathcal{D}}f)(z) = (\overline{\partial_{p_z}} f)(z).$$

We consider $\mathcal{D}$ and $\overline{\mathcal{D}}$ as operators from $\mathcal{L}_0$ into $L^2(M^{(t_0)}, d\mu)$. By the definition of $\|\cdot\|_\#$, $\mathcal{D}$ and $\overline{\mathcal{D}}$ are contractions on $\mathcal{L}_0$. Thus $\mathcal{D}$ and $\overline{\mathcal{D}}$ naturally extend to contractions form $\mathcal{L}$ to $L^2(M^{(t_0)}, d\mu)$.

We will write D for the restriction of $\mathcal{D}$ to the subspace $\mathcal{P}$. Note that the restriction of $\overline{\mathcal{D}}$ to $\mathcal{P}$ is 0.

Notation 6.4. For the rest of the paper, $\psi_i(\zeta)$ denotes the i-the component of the vector p_ζ, $\zeta \in \mathcal{M}$ and $1 \le i \le n$.

Lemma 6.5. *If $\varphi \in L^\infty(M^{(t_0)}, d\mu)$, then $\mathcal{I}^* M_\varphi \mathcal{D}\hat{T}_1 \in \mathcal{C}_2$ and $\mathcal{I}^* M_\varphi \mathcal{D}\hat{T}_2 \in \mathcal{C}_2$.*

Proof. For $f \in \mathcal{L}_0$, straightforward differentiation gives us

$$(\mathcal{D}\hat{T}_1 f)(z) = 2(Af)(z) + (Bf)(z),$$

where

$$\begin{aligned}(Af)(z) &= \int_{M^{(t_0)}} \frac{\langle p_z, w\rangle \langle z, p_w\rangle}{(1-\langle z, w\rangle)^3} (\partial_{p_w} f)(w) d\mu(w) \quad \text{and} \\ (Bf)(z) &= \int_{M^{(t_0)}} \frac{\langle p_z, p_w\rangle}{(1-\langle z, w\rangle)^2} (\partial_{p_w} f)(w) d\mu(w).\end{aligned}$$

Thus

$$A = \sum_{j=1}^n \sum_{i=1}^n M_{\chi_{M^{(t_0)}}} M_{\psi_j} M_{\zeta_i} Z M_{\bar{\zeta}_j} M_{\bar{\psi}_i} \mathcal{D} \quad \text{and} \quad B = \sum_{i=1}^n M_{\chi_{M^{(t_0)}}} M_{\psi_i} \Lambda M_{\bar{\psi}_i} \mathcal{D}.$$

Since $Z = EZ$, for $1 \le i, j \le n$ we have

$$\mathcal{I}^* M_\varphi M_{\chi_{M^{(t_0)}}} M_{\psi_j} M_{\zeta_i} Z = \mathcal{I}^* M_{\varphi \chi_{M^{(t_0)}} \psi_j \zeta_i} EZ = (E M_{\bar{\varphi} \chi_{M^{(t_0)}} \bar{\psi}_j \bar{\zeta}_i} \mathcal{I})^* Z.$$

Thus from Lemma 6.2 we see that $\mathcal{I}^* M_\varphi A$ is a Hilbert-Schmidt operator. On the other hand, it follows from Lemma 5.12 that B is a Hilbert-Schmidt operator. Therefore $\mathcal{I}^* M_\varphi \mathcal{D}\hat{T}_1 = \mathcal{I}^* M_\varphi (2A + B)$ is a Hilbert-Schmidt operator.

Straightforward differentiation gives us

$$(\mathcal{D}\hat{T}_2 f)(z) = \int_{M^{(t_0)}} \frac{\langle p_z, w\rangle}{(1-\langle z, w\rangle)^2} f(w) d\mu(w),$$

$f \in \mathcal{L}_0$. That is,

$$\mathcal{D}\hat{T}_2 = \sum_{i=1}^n M_{\psi_i} \Lambda M_{\bar{\zeta}_i} \mathcal{I}.$$

By Lemma 5.12, $\mathcal{D}\hat{T}_2$ is a Hilbert-Schmidt operator, and so is $\mathcal{I}^* M_\varphi \mathcal{D}\hat{T}_2$. This completes the proof. □

Since $\mathcal{L}$ is not an L^2-space, the adjoint of $\hat{M}_{\zeta_j}$ is not $\hat{M}_{\bar{\zeta}_j}$, $1 \le j \le n$. But using the operators $\mathcal{I}$, $\mathcal{D}$ and $\overline{\mathcal{D}}$, we can give a formula for $\hat{M}_{\zeta_j}^*$:

Proposition 6.6. *For each $1 \le j \le n$, we have $\hat{M}_{\zeta_j}^* = \hat{M}_{\bar{\zeta}_j} + \mathcal{I}^* M_{\bar{\psi}_j} \mathcal{D} - \overline{\mathcal{D}}^* M_{\bar{\psi}_j} \mathcal{I}$.*

Proof. For all $f, g \in \mathcal{L}_0$ and $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\langle f, \hat{M}^*_{\zeta_j} g\rangle_\# = \langle \hat{M}_{\zeta_j} f, g\rangle_\# &= \int_{M^{(t_0)}} (\partial_{p_w} \hat{M}_{\zeta_j} f)(w)\overline{(\partial_{p_w} g)(w)} d\mu(w) \\
&+ \int_{M^{(t_0)}} (\overline{\partial_{p_w}} \hat{M}_{\zeta_j} f)(w)\overline{(\overline{\partial_{p_w}} g)(w)} d\mu(w) + \int_{M^{(t_0)}} w_j f(w)\overline{g(w)} d\mu(w) \\
&= \int_{M^{(t_0)}} w_j (\partial_{p_w} f)(w)\overline{(\partial_{p_w} g)(w)} d\mu(w) + \int_{M^{(t_0)}} w_j (\overline{\partial_{p_w}} f)(w)\overline{(\overline{\partial_{p_w}} g)(w)} d\mu(w) \\
&+ \int_{M^{(t_0)}} \psi_j(w) f(w)\overline{(\partial_{p_w} g)(w)} d\mu(w) + \int_{M^{(t_0)}} w_j f(w)\overline{g(w)} d\mu(w) \\
&= \int_{M^{(t_0)}} (\partial_{p_w} f)(w)\overline{(\partial_{p_w} \hat{M}_{\bar{\zeta}_j} g)(w)} d\mu(w) + \int_{M^{(t_0)}} (\overline{\partial_{p_w}} f)(w)\overline{(\overline{\partial_{p_w}} \hat{M}_{\bar{\zeta}_j} g)(w)} d\mu(w) \\
&+ \int_{M^{(t_0)}} f(w)\overline{(\hat{M}_{\bar{\zeta}_j} g)(w)} d\mu(w) \\
&+ \int_{M^{(t_0)}} \psi_j(w) f(w)\overline{(\partial_{p_w} g)(w)} d\mu(w) - \int_{M^{(t_0)}} \psi_j(w) (\overline{\partial_{p_w}} f)(w)\overline{g(w)} d\mu(w) \\
&= \langle f, \hat{M}_{\bar{\zeta}_j} g\rangle_\# + \langle M_{\psi_j} \mathcal{I} f, \mathcal{D} g\rangle - \langle M_{\psi_j} \overline{\mathcal{D}} f, \mathcal{I} g\rangle \\
&= \langle f, \hat{M}_{\bar{\zeta}_j} g\rangle_\# + \langle f, \mathcal{I}^* M_{\bar{\psi}_j} \mathcal{D} g\rangle_\# - \langle f, \overline{\mathcal{D}}^* M_{\bar{\psi}_j} \mathcal{I} g\rangle_\#.
\end{aligned}$$

This completes the proof. □

Lemma 6.7. *If $\varphi \in L^\infty(M^{(t_0)}, d\mu)$, then $[\mathcal{I}^* M_\varphi \mathcal{D}, \hat{T}] \in \mathcal{C}_2$ and $[\overline{\mathcal{D}}^* M_\varphi \mathcal{I}, \hat{T}] \in \mathcal{C}_2$.*

Proof. We have $\mathcal{I}^* M_\varphi \mathcal{D} \hat{T} \in \mathcal{C}_2$ by Lemma 6.5. On the other hand, by Lemma 6.3, $\hat{T}\mathcal{I}^* = (\mathcal{I}\hat{T})^* = (\mathcal{I}P\hat{T})^* \in \mathcal{C}_2$. Therefore $[\mathcal{I}^* M_\varphi \mathcal{D}, \hat{T}] \in \mathcal{C}_2$.

Similarly, $\mathcal{I}\hat{T} = \mathcal{I}P\hat{T} \in \mathcal{C}_2$ by Lemma 6.3. Then note that $\hat{T}\overline{\mathcal{D}}^* = (\overline{\mathcal{D}}\hat{T})^* = 0$. Hence $[\overline{\mathcal{D}}^* M_\varphi \mathcal{I}, \hat{T}] \in \mathcal{C}_2$. □

Lemma 6.8. *For every $1 \le j \le n$, we have $[\hat{M}_{\bar{\zeta}_j}, \hat{T}](1 - P) \in \mathcal{C}_2$.*

Proof. It follows from Proposition 6.6 that

$$[\hat{M}_{\bar{\zeta}_j}, \hat{T}](1 - P) = [\hat{M}^*_{\zeta_j}, \hat{T}](1 - P) - [\mathcal{I}^* M_{\bar{\psi}_j} \mathcal{D}, \hat{T}](1 - P) + [\overline{\mathcal{D}}^* M_{\bar{\psi}_j} \mathcal{I}, \hat{T}](1 - P).$$

By Lemma 6.7, the last two terms on the right-hand side are in $\mathcal{C}_2$. Then note that

$$[\hat{M}^*_{\zeta_j}, \hat{T}](1 - P) = \{(1 - P)[\hat{T}, \hat{M}_{\zeta_j}]\}^* = 0.$$

This completes the proof. □

Lemma 6.9. *Let $A \in \mathcal{B}(\mathcal{L})$. For any $1 \le p < \infty$, if the operators $\mathcal{D}A, \overline{\mathcal{D}}A, \mathcal{I}A : \mathcal{L} \to L^2(M^{(t_0)}, d\mu)$ are in the Schatten p-class, then A is in the Schatten p-class.*

Proof. By the definition of $\| \cdot \|_\#$, the formula

$$Vf = \mathcal{D}f \oplus \overline{\mathcal{D}}f \oplus \mathcal{I}f,$$

$f \in \mathcal{L}$, defines an isometry that maps $\mathcal{L}$ into $L^2(M^{(t_0)}, d\mu) \oplus L^2(M^{(t_0)}, d\mu) \oplus L^2(M^{(t_0)}, d\mu)$. Thus we can factor any operator A on $\mathcal{L}$ in the form

$$A = V^*(\mathcal{D}A \oplus \overline{\mathcal{D}}A \oplus \mathcal{I}A)W,$$

where $W : \mathcal{L} \to \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}$ is the operator defined by the formula $Wf = f \oplus f \oplus f$, $f \in \mathcal{L}$. Obviously, the desire conclusion follows from this factorization. □

Lemma 6.10. *For each $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, the operator $\hat{T}_2\hat{M}_\varphi P$ is in the trace class.*

Proof. Let $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$. By Lemma 6.9, it suffices to show that the operators $\mathcal{D}\hat{T}_2\hat{M}_\varphi : \mathcal{P} \to L^2(M^{(t_0)}, d\mu)$ and $\mathcal{I}\hat{T}_2\hat{M}_\varphi : \mathcal{P} \to L^2(M^{(t_0)}, d\mu)$ are in the trace class.

For each $f \in \mathcal{P}$, we have

$$(\mathcal{D}\hat{T}_2\hat{M}_\varphi f)(z) = \int_{M^{(t_0)}} \frac{\langle p_z, w\rangle}{(1 - \langle z, w\rangle)^2}\varphi(w)f(w)d\mu(w).$$

This obviously translates to the operator identity

$$\mathcal{D}\hat{T}_2\hat{M}_\varphi P = \sum_{i=1}^n M_{\chi_{M^{(t_0)}}} M_{\psi_i}\Lambda M_{\bar{\zeta}_i} M_\varphi I.$$

Thus it follows from Lemmas 5.12 and 6.3 that $\mathcal{D}\hat{T}_2\hat{M}_\varphi P$ is in the trace class. On the other hand, since $(1 - \langle z, w\rangle)^{-1} = (1 - \langle z, w\rangle)^{-2} \cdot (1 - \langle z, w\rangle)$, we have

$$\mathcal{I}\hat{T}_2\hat{M}_\varphi P = M_{\chi_{M^{(t_0)}}}\Lambda M_\varphi I - \sum_{i=1}^n M_{\chi_{M^{(t_0)}}} M_{\zeta_i}\Lambda M_{\bar{\zeta}_i} M_\varphi I.$$

By Lemmas 5.12 and 6.3, $\mathcal{I}\hat{T}_2\hat{M}_\varphi P$ is also in the trace class. □

Proof of Proposition 4.11. We begin with a sequence of reductions of our task.

By the linearity of commutators, to prove that $[\hat{M}_\varphi, \hat{T}] \in \mathcal{C}_2$ for an arbitrary $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, it suffices to show that $[\hat{M}_{\zeta^\alpha\bar{\zeta}^\beta}, \hat{T}] \in \mathcal{C}_2$ for all $\alpha, \beta \in \mathbf{Z}_+^n$. Since

$$[A_1A_2\cdots A_\nu, \hat{T}] = [A_1, \hat{T}]A_2\cdots A_\nu + A_1[A_2, \hat{T}]A_3\cdots A_\nu + \cdots + A_1\cdots A_{\nu-1}[A_\nu, \hat{T}],$$

our task is reduced to the proof of the fact that $[\hat{M}_{\zeta_k}, \hat{T}] \in \mathcal{C}_2$ and $[\hat{M}_{\bar{\zeta}_k}, \hat{T}] \in \mathcal{C}_2$ for each $1 \le k \le n$. By Proposition 6.6, we have

$$[\hat{T}, \hat{M}_{\zeta_k}]^* = [\hat{M}_{\zeta_k}^*, \hat{T}] = [\hat{M}_{\bar{\zeta}_k}, \hat{T}] + [\mathcal{I}^* M_{\bar{\psi}_k}\mathcal{D}, \hat{T}] - [\overline{\mathcal{D}}^* M_{\bar{\psi}_k}\mathcal{I}, \hat{T}].$$

Lemma 6.7 tells us that the last two terms on the right-hand side are in $\mathcal{C}_2$. Therefore, to prove Proposition 4.11, it suffices to show that $[\hat{M}_{\bar{\zeta}_k}, \hat{T}] \in \mathcal{C}_2$ for every $1 \le k \le n$. Lemma 6.8 further reduces this task to the proof of the membership $[\hat{M}_{\bar{\zeta}_k}, \hat{T}]P \in \mathcal{C}_2$, $1 \le k \le n$.

By Lemma 6.10, we actually have $[\hat{M}_{\bar{\zeta}_k}, \hat{T}_2]P \in \mathcal{C}_1$, $1 \le k \le n$. Since $\hat{T} = \hat{T}_1 + \hat{T}_2$, we only need to show that $[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] \in \mathcal{C}_2$ for every $1 \le k \le n$.

By Lemma 6.9, to prove that $[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] \in \mathcal{C}_2$, it suffices to show that $\mathcal{D}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]$, $\overline{\mathcal{D}}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]$ and $\mathcal{I}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]$ are Hilbert-Schmidt operators. For $f \in \mathcal{L}_0$, we have

$$([\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]f)(z) = \int_{M^{(t_0)}} \frac{\bar{z}_k - \bar{w}_k}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w).$$

By straightforward differentiation,

$$(\mathcal{D}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]f)(z) = (Y_1 f)(z) + (Y_2 f)(z),$$

where

$$(Y_1 f)(z) = 2\int_{M^{(t_0)}} \frac{\bar{z}_k - \bar{w}_k}{(1 - \langle z, w\rangle)^3} \langle p_z, w\rangle\langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w) \quad \text{and}$$
$$(Y_2 f)(z) = \int_{M^{(t_0)}} \frac{\bar{z}_k - \bar{w}_k}{(1 - \langle z, w\rangle)^2} \langle p_z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w).$$

It is easy to see that

$$Y_1 = 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\psi_i} M_{\zeta_j} [M_{\bar{\zeta}_k}, Z] M_{\bar{\zeta}_i} M_{\bar{\psi}_j} \mathcal{D} \quad \text{and}$$
$$Y_2 = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\psi_j} [M_{\bar{\zeta}_k}, \Lambda] M_{\bar{\psi}_j} \mathcal{D}.$$

By Lemma 5.3, Y_1 is a Hilbert-Schmidt operator. By Lemma 5.12, Y_2 is also a Hilbert-Schmidt operator. Hence $\mathcal{D}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] = Y_1 + Y_2$ is a Hilbert-Schmidt operator.

Again by differentiation,

$$(\overline{\mathcal{D}}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]f)(z) = \int_{M^{(t_0)}} \frac{\bar{\psi}_k(z)}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w)$$

for $f \in \mathcal{L}_0$. That is,

$$\overline{\mathcal{D}}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\bar{\psi}_k} M_{\zeta_j} \Lambda M_{\bar{\psi}_j} \mathcal{D}.$$

Thus it follows from Lemma 5.12 that $\overline{\mathcal{D}}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1]$ is a Hilbert-Schmidt operator.

It is easy to see that

$$\mathcal{I}[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] = \sum_{i=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\zeta_i} [M_{\bar{\zeta}_k}, \Lambda] M_{\bar{\psi}_i} \mathcal{D},$$

which is in $\mathcal{C}_2$ according to Lemma 5.12. Combining this with the results of the previous two paragraphs, we obtain the membership $[\hat{M}_{\bar{\zeta}_k}, \hat{T}_1] \in \mathcal{C}_2$. This completes the proof. □

Corollary 6.11. *For every* $\varphi \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \dots, \zeta_n, \bar{\zeta}_n]$, *we have* $[\hat{M}_\varphi, P] \in \mathcal{C}_2$.

Proof. By Proposition 4.6, there is an $h \in C_c^\infty(\mathbf{R})$ such that $P = h(\hat{T})$. Thus the membership $[\hat{M}_\varphi, P] \in \mathcal{C}_2$ follows from Proposition 4.11 and the standard smooth functional calculus. □

The proof of Proposition 4.11 gives us a taste of what is to come. For Proposition 4.12, because it involves double commutators, the proof will be more tedious for an obvious reason: more terms will have to be examined.

Proof of Proposition 4.12. Let $q, r \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$ be given. By the relation $\hat{T} = \hat{T}_1 + \hat{T}_2$ and Lemma 6.10, it suffices to prove that $[\hat{M}_{\bar{r}}, [\hat{M}_q, \hat{T}_1]]P \in \mathcal{C}_1$.

For $f \in \mathcal{L}_0$, we have

$$([\hat{M}_q, \hat{T}_1]f)(z) = (X_1 f)(z) - (X_2 f)(z),$$

where

$$(X_1 f)(z) = \int_{M^{(t_0)}} \frac{q(z) - q(w)}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w) \quad \text{and}$$
$$(X_2 f)(z) = \int_{M^{(t_0)}} \frac{(\partial_{p_w} q)(w)}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle f(w) d\mu(w).$$

Denote $A_1 = [\hat{M}_{\bar{r}}, X_1]P$. For $f \in \mathcal{P}$, we have

$$(A_1 f)(z) = \int_{M^{(t_0)}} \frac{(q(z) - q(w))(\bar{r}(z) - \bar{r}(w))}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w).$$

We again use the operators $\mathcal{D}$, $\overline{\mathcal{D}}$ and $\mathcal{I}$. By differentiation, we have

$$(\mathcal{D}A_1 f)(z) = (Y_{11} f)(z) + (Y_{12} f)(z) + (Y_{13} f)(z),$$

where

$$(Y_{11} f)(z) = 2\int_{M^{(t_0)}} \frac{(q(z) - q(w))(\bar{r}(z) - \bar{r}(w))}{(1 - \langle z, w\rangle)^3} \langle p_z, w\rangle \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w),$$
$$(Y_{12} f)(z) = \int_{M^{(t_0)}} \frac{(\partial_{p_z} q)(z)(\bar{r}(z) - \bar{r}(w))}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w) \quad \text{and}$$
$$(Y_{13} f)(z) = \int_{M^{(t_0)}} \frac{(q(z) - q(w))(\bar{r}(z) - \bar{r}(w))}{(1 - \langle z, w\rangle)^2} \langle p_z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w),$$

$f \in \mathcal{P}$. Denote $\eta(z) = (\partial_{p_z} q)(z)$. We can rewrite the above as the operator identities

$$Y_{11} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\zeta_j} M_{\psi_i}[M_{\bar r}, [M_q, Z]] M_{\bar\zeta_i} M_{\bar\psi_j} D,$$

$$Y_{12} = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_\eta M_{\zeta_j}[M_{\bar r}, \Lambda] M_{\bar\psi_j} D \quad \text{and}$$

$$Y_{13} = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\psi_j}[M_{\bar r}, [M_q, \Lambda]] M_{\bar\psi_j} D.$$

It follows from Proposition 5.11 that $Y_{11} \in \mathcal{C}_1$. By Proposition 5.13, we have $Y_{12} \in \mathcal{C}_1$ and $Y_{13} \in \mathcal{C}_1$. Hence $\mathcal{D}A_1$ is in the trace class.

On the other hand,

$$(\overline{\mathcal{D}}A_1 f)(z) = \int_{M^{(t_0)}} \frac{q(z) - q(w)}{(1 - \langle z, w\rangle)^2} \overline{(\partial_{p_z} r)(z)} \langle z, p_w\rangle (\partial_{p_w} f)(w) d\mu(w)$$

for $f \in \mathcal{P}$. Denote $\xi(z) = (\partial_{p_z} r)(z)$. Then the above translates to the operator identity

$$\overline{\mathcal{D}}A_1 = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\bar\xi} M_{\zeta_j}[M_q, \Lambda] M_{\bar\psi_j} D.$$

Applying Proposition 5.13, we conclude that $\overline{\mathcal{D}}A_1$ is in the trace class.

It is easy to see that

$$\mathcal{I}A_1 = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\zeta_j}[M_{\bar r}, [M_q, \Lambda]] M_{\bar\psi_j} D.$$

Applying Proposition 5.13 again, we see that $\mathcal{I}A_1$ is in the trace class. Lemma 6.9 now allows us to conclude that $[\hat M_{\bar r}, X_1]P = A_1 \in \mathcal{C}_1$.

Let us now consider $A_2 = [\hat M_{\bar r}, X_2]P$. For each $f \in \mathcal{P}$,

$$(A_2 f)(z) = \int_{M^{(t_0)}} \frac{(\bar r(z) - \bar r(w))(\partial_{p_w} q)(w)}{(1 - \langle z, w\rangle)^2} \langle z, p_w\rangle f(w) d\mu(w).$$

Thus

$$(\mathcal{D}A_2 f)(z) = (Y_{21} f)(z) + (Y_{22} f)(z),$$

where

$$(Y_{21} f)(z) = 2\int_{M^{(t_0)}} \frac{(\bar r(z) - \bar r(w))(\partial_{p_w} q)(w)}{(1 - \langle z, w\rangle)^3} \langle p_z, w\rangle \langle z, p_w\rangle f(w) d\mu(w) \quad \text{and}$$

$$(Y_{22} f)(z) = \int_{M^{(t_0)}} \frac{(\bar r(z) - \bar r(w))(\partial_{p_w} q)(w)}{(1 - \langle z, w\rangle)^2} \langle p_z, p_w\rangle f(w) d\mu(w).$$

It is easy to see that

$$Y_{21} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\zeta_j} M_{\psi_i} [M_{\bar{r}}, Z] M_{\bar{\zeta}_i} M_{\bar{\psi}_j} M_\eta I \quad \text{and}$$

$$Y_{22} = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\psi_j} [M_{\bar{r}}, \Lambda] M_{\bar{\psi}_j} M_\eta I.$$

Thus it follows from Lemmas 5.3 and 6.3 that $Y_{21} \in \mathcal{C}_1$. Applying Proposition 5.13 again, we have $Y_{22} \in \mathcal{C}_1$. Hence $\mathcal{D}A_2 \in \mathcal{C}_1$.

On the other hand,

$$(\overline{\mathcal{D}}A_2 f)(z) = \int_{M^{(t_0)}} \frac{1}{(1-\langle z, w\rangle)^2} \overline{(\partial_{p_z} r)(z)} (\partial_{p_w} q)(w) \langle z, p_w\rangle f(w) d\mu(w),$$

$f \in \mathcal{P}$. Thus

$$\overline{\mathcal{D}}A_2 = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\bar{\xi}} M_{\zeta_j} \Lambda M_{\bar{\psi}_j} M_\eta I.$$

Applying Lemmas 5.12 and 6.3, we obtain the membership $\overline{\mathcal{D}}A_2 \in \mathcal{C}_1$.

Finally, it is easy to see that

$$\mathcal{I}A_2 = \sum_{j=1}^{n} M_{\chi_{M^{(t_0)}}} M_{\zeta_j} [M_{\bar{r}}, \Lambda] M_{\bar{\psi}_j} M_\eta I.$$

Thus we have $\mathcal{I}A_2 \in \mathcal{C}_1$ by Proposition 5.13. Having proved the memberships of $\mathcal{D}A_2$, $\overline{\mathcal{D}}A_2$, $\mathcal{I}A_2$ in $\mathcal{C}_1$, Lemma 6.9 allows us to conclude that $[\hat{M}_{\bar{r}}, X_2]P = A_2 \in \mathcal{C}_1$. Since $X_1 - X_2 = [\hat{M}_q, \hat{T}_1]$, we have $[\hat{M}_{\bar{r}}, [\hat{M}_q, \hat{T}_1]]P \in \mathcal{C}_1$ as promised. □

7. Proof of Theorem 1.1

Recall from Section 4 that we write $M_{\zeta_1}, \dots, M_{\zeta_n}$ for the restrictions of $\hat{M}_{\zeta_1}, \dots, \hat{M}_{\zeta_n}$ to $\mathcal{P}$. Thus for each $1 \le j \le n$, $M^*_{\zeta_j}$ means the adjoint of M_{ζ_j} on $\mathcal{P}$.

Lemma 7.1. (a) *For each* $1 \le j \le n$, $M^*_{\zeta_j} = P\hat{M}_{\bar{\zeta}_j}P + I^* M_{\bar{\psi}_j} D$.
(b) *For each* $1 \le j \le n$, *we have* $M^*_{\zeta_j} - P\hat{M}_{\bar{\zeta}_j}P \in \mathcal{C}_2$.

Proof. Since $M^*_{\zeta_j} = P\hat{M}^*_{\zeta_j}P$ and $P\overline{\mathcal{D}}^* = (\overline{\mathcal{D}}P)^* = 0$, (a) follows from Proposition 6.6. Lemma 6.3 tells us that $I \in \mathcal{C}_2$. Thus (b) follows from (a). □

Proposition 7.2. *For every pair of* $i, j \in \{1, \dots, n\}$, *we have* $[M_{\zeta_i}, M^*_{\zeta_j} - P\hat{M}_{\bar{\zeta}_j}P] \in \mathcal{C}_1$.

Proof. By Lemma 7.1, this is equivalent to the assertion that $[M_{\zeta_i}, I^* M_{\bar{\psi}_j} D] \in \mathcal{C}_1$ for all $i, j \in \{1, \dots, n\}$. Note that $DM_{\zeta_i} = M_{\zeta_i} D + M_{\psi_i} I$. Therefore

$$[M_{\zeta_i}, I^* M_{\bar{\psi}_j} D] = (M_{\zeta_i} I^* - I^* M_{\zeta_i}) M_{\bar{\psi}_j} D - I^* M_{\bar{\psi}_j} M_{\psi_i} I.$$

It follows from Lemma 6.3 that $I^* M_{\bar{\psi}_j} M_{\psi_i} I \in \mathcal{C}_1$. Thus it suffices to show that $M_{\zeta_i} I^* - I^* M_{\zeta_i}$ is in the trace class. Equivalently, it suffices to show that $I M_{\zeta_i}^* - M_{\bar{\zeta}_i} I$ is in the trace class. By Lemmas 7.1 and 6.3, $I(M_{\zeta_i}^* - P\hat{M}_{\bar{\zeta}_i} P)$ is in the trace class. Hence our task is reduced to the proof that $IP\hat{M}_{\bar{\zeta}_i} P - M_{\bar{\zeta}_i} I$ is in the trace class.

Then note that $M_{\bar{\zeta}_i} I = M_{\bar{\zeta}_i} \mathcal{I} P = \mathcal{I} \hat{M}_{\bar{\zeta}_i} P$. Hence

$$IP\hat{M}_{\bar{\zeta}_i} P - M_{\bar{\zeta}_i} I = -\mathcal{I}(1-P)\hat{M}_{\bar{\zeta}_i} P = -\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, P]P.$$

To prove that this is in the trace class, we recall Proposition 4.6, which implies that $P = \hat{T}(\tilde{T}^{-1} \oplus 0)$. Thus

$$\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, P]P = \mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}](\tilde{T}^{-1} \oplus 0)P + \mathcal{I}\hat{T}[\hat{M}_{\bar{\zeta}_i}, \tilde{T}^{-1} \oplus 0]P. \tag{7.1}$$

Let us first consider the second term on the right-hand side. Since the range of $\hat{T}$ is contained in $\mathcal{P}$, it follows from Lemma 6.3 that $\mathcal{I}\hat{T}$ is a Hilbert-Schmidt operator. By Proposition 4.6, there is a $\xi \in C_c^\infty(\mathbf{R})$ such that $\tilde{T}^{-1} \oplus 0 = \xi(\hat{T})$. Therefore it follows from Proposition 4.11 and smooth functional calculus that $[\hat{M}_{\bar{\zeta}_i}, \tilde{T}^{-1} \oplus 0] \in \mathcal{C}_2$. Thus the second term on the right-hand side of (7.1) is in the trace class.

The remaining task is to show that the first term on the right-hand side of (7.1) is in the trace class. Since $\hat{T} = \hat{T}_1 + \hat{T}_2$, the proof of the proposition will be complete once we show that both $\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_1]$ and $\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_2]$ are in the trace class.

For this purpose, we define

$$(T_1^\circ \varphi)(z) = \int_{M^{(t_0)}} \frac{\langle z, p_w \rangle}{(1 - \langle z, w \rangle)^2} \varphi(w) d\mu(w) \quad \text{and}$$
$$(T_2^\circ \varphi)(z) = \int_{M^{(t_0)}} \frac{1}{1 - \langle z, w \rangle} \varphi(w) d\mu(w),$$

$\varphi \in L^2(M^{(t_0)}, d\mu)$, which are operators on the space $L^2(M^{(t_0)}, d\mu)$. Note that $\mathcal{I}\hat{T}_1 = T_1^\circ \mathcal{D}$. Since $\bar{\zeta}_i$ is conjugate analytic, we have $\mathcal{D}\hat{M}_{\bar{\zeta}_i} = M_{\bar{\zeta}_i} \mathcal{D}$. Also, $\mathcal{I}\hat{M}_{\bar{\zeta}_i} = M_{\bar{\zeta}_i} \mathcal{I}$. Thus

$$\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_1] = [M_{\bar{\zeta}_i}, T_1^\circ]\mathcal{D}.$$

It is easy to see that

$$[M_{\bar{\zeta}_i}, T_1^\circ] = \sum_{\ell=1}^n M_{\chi_{M^{(t_0)}}} M_{\zeta_\ell} [M_{\bar{\zeta}_i}, \Lambda] M_{\bar{\psi}_\ell} M_{\chi_{M^{(t_0)}}}.$$

Since $[M_{\bar{\zeta}_i}, \Lambda] = [\Lambda, M_{\zeta_i}]^*$, it follows from these identities and Proposition 5.13 that $\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_1]$ is in the trace class. It is also easy to see that

$$\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_2] = [M_{\bar{\zeta}_i}, T_2^\circ]\mathcal{I} = M_{\chi_{M^{(t_0)}}} [M_{\bar{\zeta}_i}, \Lambda]\mathcal{I} - \sum_{\ell=1}^n M_{\chi_{M^{(t_0)}}} M_{\zeta_\ell} [M_{\bar{\zeta}_i}, \Lambda] M_{\bar{\zeta}_\ell} \mathcal{I}.$$

By Proposition 5.13, $\mathcal{I}[\hat{M}_{\bar{\zeta}_i}, \hat{T}_2]$ is also in the trace class. This completes the proof. □

After so much preparation, we are finally able to deal with essential normality.

Proposition 7.3. *On the space $\mathcal{P}$, the commuting tuple $(M_{\zeta_1}, \ldots, M_{\zeta_n})$ is 1-essentially normal. That is, for all $i, j \in \{1, \ldots, n\}$, we have $[M_{\zeta_i}, M^*_{\zeta_j}] \in \mathcal{C}_1$.*

Proof. In view of Proposition 7.2, it suffices to show that $[M_{\zeta_i}, P\hat{M}_{\bar{\zeta}_j}P] \in \mathcal{C}_1$ for $i, j \in \{1, \ldots, n\}$. Since $\hat{M}_{\zeta_i}$ commutes with $\hat{M}_{\bar{\zeta}_j}$, we have

$$\begin{aligned}[M_{\zeta_i}, P\hat{M}_{\bar{\zeta}_j}P] &= [P\hat{M}_{\zeta_i}P, P\hat{M}_{\bar{\zeta}_j}P] = P\hat{M}_{\bar{\zeta}_j}(1-P)\hat{M}_{\zeta_i}P - P\hat{M}_{\zeta_i}(1-P)\hat{M}_{\bar{\zeta}_j}P \\ &= [P, \hat{M}_{\bar{\zeta}_j}](1-P)[\hat{M}_{\zeta_i}, P] - [P, \hat{M}_{\zeta_i}](1-P)[\hat{M}_{\bar{\zeta}_j}, P].\end{aligned}$$

By Corollary 6.11, this is in the trace class. □

Lemma 7.4. *We have $[M_{\zeta_i}, \tilde{T}] \in \mathcal{C}_2$ for every $1 \le i \le n$.*

Proof. Obviously, we have $[M_{\zeta_i}, \tilde{T}] = P[\hat{M}_{\zeta_i}, \hat{T}]P$. Thus the membership $[M_{\zeta_i}, \tilde{T}] \in \mathcal{C}_2$ is a consequence of Proposition 4.11. □

Proposition 7.5. *For all $i, j \in \{1, \ldots, n\}$, we have $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}]] \in \mathcal{C}_1$.*

Proof. It follows from Proposition 7.2 and Lemmas 7.1 and 7.4 that $[M_{\zeta_i}, [M^*_{\zeta_j} - P\hat{M}_{\bar{\zeta}_j}P, \tilde{T}]] \in \mathcal{C}_1$ for all $i, j \in \{1, \ldots, n\}$. Therefore it suffices to show that $[M_{\zeta_i}, [P\hat{M}_{\bar{\zeta}_j}P, \tilde{T}]] \in \mathcal{C}_1$, $i, j \in \{1, \ldots, n\}$. Again, let us write $A \sim_1 B$ when $A - B \in \mathcal{C}_1$. We have

$$[M_{\zeta_i}, [P\hat{M}_{\bar{\zeta}_j}P, \tilde{T}]] = P[\hat{M}_{\zeta_i}, P[\hat{M}_{\bar{\zeta}_j}, \hat{T}]P]P \sim_1 P[\hat{M}_{\zeta_i}, [\hat{M}_{\bar{\zeta}_j}, \hat{T}]]P = P[\hat{M}_{\bar{\zeta}_j}, [\hat{M}_{\zeta_i}, \hat{T}]]P,$$

where the $\sim_1$ follows from Proposition 4.11 and Corollary 6.11. Now an application of Proposition 4.12 completes the proof. □

Corollary 7.6. *For all $i, j \in \{1, \ldots, n\}$, we have $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}]] \in \mathcal{C}_1$.*

Proof. By Proposition 4.6, there are $0 < c < C < \infty$ such that the spectrum of $\tilde{T}$ is contained in the interval $[c, C]$. Consider $H_+ = \{\lambda \in \mathbf{C} : \mathrm{Re}(\lambda) > 0\}$, the right half-plane. Let γ be a simple Jordan curve in $H_+ \backslash [c, C]$ whose winding number about every $x \in [c, C]$ is 1. Taking advantage of the fact that the square-root function $\lambda^{1/2}$ is analytic on H_+, from the Riesz functional calculus we obtain

$$\tilde{T}^{1/2} = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}(\lambda - \tilde{T})^{-1} d\lambda. \tag{7.2}$$

Therefore

$$[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}]] = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}\{A(\lambda) + B(\lambda) + C(\lambda)\} d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= (\lambda - \tilde{T})^{-1}[M_{\zeta_i}, \tilde{T}](\lambda - \tilde{T})^{-1}[M^*_{\zeta_j}, \tilde{T}](\lambda - \tilde{T})^{-1}, \\ B(\lambda) &= (\lambda - \tilde{T})^{-1}[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}]](\lambda - \tilde{T})^{-1} \quad \text{and} \\ C(\lambda) &= (\lambda - \tilde{T})^{-1}[M^*_{\zeta_j}, \tilde{T}](\lambda - \tilde{T})^{-1}[M_{\zeta_i}, \tilde{T}](\lambda - \tilde{T})^{-1}. \end{aligned}$$

Applying Proposition 7.5 to $B(\lambda)$ and Lemma 7.4 to $A(\lambda)$ and $C(\lambda)$, we obtain the membership $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}]] \in \mathcal{C}_1$. □

Corollary 7.7. *We have* $[M_{\zeta_i}, \tilde{T}^{1/2}] \in \mathcal{C}_2$ *for every* $1 \le i \le n$.

Proof. This follows obviously from Lemma 7.4 and identity (7.2). □

Finally, we can accomplish the main goal of the paper:

Proof of Theorem 1.1. Recall from Section 4 that the operator $J : \mathcal{Q} \to \mathcal{P}$ is invertible. Therefore the operator $J^* : \mathcal{P} \to \mathcal{Q}$ is also invertible. By the standard polarization,

$$J^* = U|J^*|,$$

where $U : \mathcal{P} \to \mathcal{Q}$ is a unitary operator and $|J^*| = (JJ^*)^{1/2} = \tilde{T}^{1/2}$ (see Proposition 4.6). Recall from Corollary 4.10 that $JQ_{\zeta_j} = M_{\zeta_j}J$ for every $j \in \{1, \dots, n\}$. Therefore

$$Q_{\zeta_j} = J^{-1}M_{\zeta_j}J = U\tilde{T}^{-1/2}M_{\zeta_j}\tilde{T}^{1/2}U^* = UM_{\zeta_j}U^* + K_j,$$

$j \in \{1, \dots, n\}$, where

$$K_j = U\tilde{T}^{-1/2}[M_{\zeta_j}, \tilde{T}^{1/2}]U^*.$$

Thus for any $i, j \in \{1, \dots, n\}$, we have

$$[Q_{\zeta_i}, Q^*_{\zeta_j}] = U[M_{\zeta_i}, M^*_{\zeta_j}]U^* + [UM_{\zeta_i}U^*, K^*_j] + [K_i, UM^*_{\zeta_j}U^*] + [K_i, K^*_j].$$

By Proposition 7.3 and Corollary 7.7, the first term and the last term on the right-hand side are in the trace class. What remains is to show that the two middle terms on the right-hand side are also in the trace class.

We have

$$\begin{aligned}[UM_{\zeta_i}U^*, K^*_j] &= U[M_{\zeta_i}, \{\tilde{T}^{-1/2}[M_{\zeta_j}, \tilde{T}^{1/2}]\}^*]U^* = U[M_{\zeta_i}, [\tilde{T}^{1/2}, M^*_{\zeta_j}]\tilde{T}^{-1/2}]U^* \\ &= U[M_{\zeta_i}, [\tilde{T}^{1/2}, M^*_{\zeta_j}]]\tilde{T}^{-1/2}U^* + U[\tilde{T}^{1/2}, M^*_{\zeta_j}]\tilde{T}^{-1/2}[\tilde{T}^{1/2}, M_{\zeta_i}]\tilde{T}^{-1/2}U^* \\ &= A + B.\end{aligned}$$

By Corollary 7.6, we have $A \in \mathcal{C}_1$. By Corollary 7.7, we have $B \in \mathcal{C}_1$. Hence $[UM_{\zeta_i}U^*, K^*_j] \in \mathcal{C}_1$. Since $[K_i, UM^*_{\zeta_j}U^*] = [UM_{\zeta_j}U^*, K^*_i]^*$, we also have $[K_i, UM^*_{\zeta_j}U^*] \in \mathcal{C}_1$. This completes the proof of Theorem 1.1. □

Appendix 1

The purpose of this appendix is to give a proof for Lemma 2.13. With minor changes of details, this is essentially the same as the proof of [29, Lemma 3.1].

Let $-1 < \tau < \infty$ be given. Then for any $f \in C^1_c[0, \infty)$, we have

$$\text{(A1.1)} \qquad \int_0^\infty |f(x)|^2 x^\tau \, dx \le \left(\frac{2}{\tau + 1}\right)^2 \int_0^\infty |xf'(x)|^2 x^\tau \, dx.$$

See inequality (3.1) in [29].

Let $0 < b < t \le 1$ be such that $M^{(b)} \subset \mathcal{M}$. Consider the function $\rho(w) = 1 - |w|^2$. Since $\mathcal{M}$ intersects $S = \{\zeta \in \mathbf{C}^n : |\zeta| = 1\}$ transversely, $\nabla_{\mathcal{M}}\rho$ does not vanish near $\mathcal{M} \cap S$. Thus we can use ρ as one of the real coordinates on $\mathcal{M}$ near S. More precisely, if $\zeta \in \mathcal{M} \cap S$, then ζ has an open neighborhood U_ζ in $\mathcal{M}$ that has the following properties:

(1) $U_\zeta = G((-c, c) \times V)$, where $0 < c < b$, V is a bounded open set in $\mathbf{R}^{2d-1}$ and $G : (-c, c) \times V \to \mathbf{C}^n$ is a one-to-one C^∞ map.

(2) there are $0 < \delta < C < \infty$ such that DG, the derivative of G, satisfies the matrix inequality $\delta \le (DG)^*(x, y)(DG)(x, y) \le C$ for all $x \in (-c, c)$ and $y \in V$.

(3) If $w = G(x, y)$ for some $x \in (-c, c)$ and $y \in V$, then $x = 1 - |w|^2$. Equivalently, for each $w \in U_\zeta$, there is a unique $y_w \in V$ such that $w = G(1 - |w|^2, y_w)$.

Obviously, (3) implies $U_\zeta \cap M = G((0, c) \times V) \subset M^{(b)}$.

Once we have this c, by the standard technique of using a smooth cutoff function, we can truncate inequality (A1.1) to the interval $[0, c]$. That is, there are C_1 and C_2 such that

$$\text{(A1.2)} \qquad \int_0^c |h(x)|^2 x^\tau dx \le C_1 \int_0^c |xh'(x)|^2 x^\tau dx + C_2 \int_{c/2}^c |h(x)|^2 x^\tau dx$$

for every $h \in C^1[0, c]$. Let f be any C^1 function on an open set containing the closure of $M^{(t)}$. By the definition of $d\nu_\tau$ and property (3) above,

$$\begin{aligned} \int_{U_\zeta \cap M} |f(w)|^2 d\nu_\tau(w) &= \int_{U_\zeta \cap M} |f(w)|^2 (1 - |w|^2)^\tau dv_M(w) \\ &= \int_V \int_0^c |f(G(x, y))|^2 x^\tau J(x, y) dx dy \le C_3 \int_V \int_0^c |f(G(x, y))|^2 x^\tau dx dy \\ \text{(A1.3)} \qquad &\le C_4 \int_V \int_0^c \left| x \frac{d}{dx} f(G(x, y)) \right|^2 x^\tau dx dy + C_5 \int_V \int_{c/2}^c |f(G(x, y))|^2 x^\tau dx dy, \end{aligned}$$

where the second $\le$ is an application of (A1.2). By the chain rule for differentiation,

$$\frac{d}{dx} f(G(x, y)) = \langle (\nabla f)(G(x, y)), \tau(x, y) \rangle,$$

where $\tau(x, y)$ is a (real) tangent vector to $\mathcal{M}$ at the point $G(x, y)$. Moreover, (2) implies the bound $|\tau(x, y)| \le C^{1/2}$. Hence $|df(G(x, y))/dx| \le C^{1/2} |(\nabla_{\mathcal{M}} f)(G(x, y))|$. Thus

$$\begin{aligned} \int_V \int_0^c \left| x \frac{d}{dx} f(G(x, y)) \right|^2 x^\tau dx dy &\le C \int_V \int_0^c |(\nabla_{\mathcal{M}} f)(G(x, y))|^2 x^{2+\tau} dx dy \\ &\le C_6 \int_V \int_0^c |(\nabla_{\mathcal{M}} f)(G(x, y))|^2 x^{2+\tau} J(x, y) dx dy \\ &= C_6 \int_{U_\zeta \cap M} |(\nabla_{\mathcal{M}} f)(w)|^2 (1 - |w|^2)^{2+\tau} dv_M(w) \\ &= C_6 \int_{U_\zeta \cap M} |(\nabla_{\mathcal{M}} f)(w)|^2 (1 - |w|^2)^2 d\nu_\tau(w), \end{aligned}$$

where the third step uses property (3). Combining this with (A1.3), we find that

$$\int_{U_\zeta \cap M} |f(w)|^2 d\nu_\tau(w) \le C_7 \int_{U_\zeta \cap M} |(\nabla_{\mathcal{M}} f)(w)|^2 (1-|w|^2)^2 d\nu_\tau(w)$$

$$+ C_8 \int_{N^{(c/2)} \cap M^{(t)}} |f(w)|^2 d\nu_\tau(w). \tag{A1.4}$$

Since $\mathcal{M} \cap S$ is compact, there are $\zeta_1, \dots, \zeta_k \in \mathcal{M} \cap S$ such that the corresponding open sets $U_{\zeta_1}, \dots, U_{\zeta_k}$ have the property $U_{\zeta_1} \cup \cdots \cup U_{\zeta_k} \supset \mathcal{M} \cap S = \{w \in \mathcal{M} : 1 - |w|^2 = 0\}$. Thus $U_{\zeta_1} \cup \cdots \cup U_{\zeta_k} \supset M^{(\sigma)}$ for some $0 < \sigma < 1$. Combining this containment with (A1.4), Lemma 2.13 is proved.

Appendix 2

The purpose of this appendix is to give a proof of Proposition 5.6. One can characterize the material in this appendix as an easier version of parts of Sections 3 and 4.

Recall that $\mathcal{H}^{(2-n)}$ is the Hilbert space of analytic functions on $\mathbf{B}$ which has (5.2) as its reproducing kernel. We begin with an easier version of Lemma 3.9:

Lemma A2.1. *Given any $\delta > 0$, there exist $0 < \tau < t_0$ and $0 < C < \infty$ such that*

$$\int_{M^{(t)}} |f(z)|^2 d\mu(z) \le C \int_{M^{(t)}} |(Zf)(z)|^2 d\mu(z) + \delta \int_{M^{(t_0)}} |f(z)|^2 d\mu(z)$$

for all $0 < t \le \tau$ and $f \in \mathcal{H}^{(2-n)}$.

Proof. As in the proof of Lemma 3.9, we begin with a large $1 \le r < \infty$, whose exact value will be determined later. For such an r, let $0 < \tau_3 \le \tau_2 \le \tau_1 \le t_0$ be the same as in the proof of Lemma 3.9. Thus if $0 < t \le \tau_2$, then

$$I_z(D(z, 2r) \cap T_z^{\mathrm{mod}}) \supset D(z, r) \cap M \quad \text{for every} \quad z \in M^{(t)}. \tag{A2.1}$$

And if $0 < t \le \tau_3$, then

$$I_z(D(z, 2r) \cap T_z^{\mathrm{mod}}) \subset M^{(t_0)} \quad \text{for every} \quad z \in M^{(t)}. \tag{A2.2}$$

Write $U(z) = I_z(D(z, 2r) \cap T_z^{\mathrm{mod}})$ as before, $z \in M^{(\tau_3)}$. Let $f \in \mathcal{H}^{(2-n)}$ be given. Then

$$(Zf)(z) = A(z) + B(z),$$

where

$$A(z) = \int_{U(z)} f(w) \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^3} dv_M(w) \quad \text{and}$$

$$B(z) = \int_{M^{(t_0)} \setminus U(z)} f(w) \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^3} dv_M(w),$$

$z \in M^{(\tau_3)}$. Since $P_z U(z) = D(z, 2r) \cap T_z^{\mathrm{mod}}$, by (2.10) we have

$$A(z) = \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{1 - |I_z(x)|^2}{(1 - \langle z, I_z(x)\rangle)^3} J_z(x) dv_1(x).$$

Recall from (2.6) that $\langle z, I_z(x)\rangle = \langle z, x\rangle$. Writing

$$F(z, x) = 1 - \frac{1 - |x|^2}{1 - |I_z(x)|^2} \cdot \frac{J_z(z)}{J_z(x)},$$

we have $A(z) = A_1(z) + A_2(z)$, where

$$A_1(z) = J_z(z) \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{1 - |x|^2}{(1 - \langle z, x\rangle)^3} dv_1(x) \quad \text{and}$$
$$A_2(z) = \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} f(I_z(x)) \frac{1 - |I_z(x)|^2}{(1 - \langle z, I_z(x)\rangle)^3} F(z, x) J_z(x) dv_1(x).$$

Let us analyze $A_1(z)$, and $A_2(z)$ and $B(z)$.

Being a local inverse of P_z, the map I_z is analytic. Thus the function $x \mapsto f(I_z(x))$ is analytic on $D(z, 3r) \cap T_z^{\mathrm{mod}}$. Therefore it follows from Lemma 2.10 that

$$A_1(z) = \Phi(2r) J_z(z) f(I_z(z)) = \Phi(2r) J_z(z) f(z).$$

Recalling (2.9) and (2.7), we see that there is a $0 < C_1 < \infty$ such that

(A2.3) $$|f(z)| \le C_1 |A_1(z)| \quad \text{for} \quad z \in M^{(t)}, \quad 0 < t \le \tau_3.$$

Denote

$$\epsilon(r, t) = \sup_{z \in M^{(t)}} \left\{ \sup_{x \in D(z,2r)\cap T_z^{\mathrm{mod}}} |F(z, x)| \right\}.$$

As we explained in the proof of Lemma 3.9,

(A2.4) $$\lim_{t \downarrow 0} \epsilon(r, t) = 0$$

for every $1 \le r < \infty$. Applying (2.10) again, we have

$$|A_2(z)| \le \epsilon(r, t) \int_{D(z,2r)\cap T_z^{\mathrm{mod}}} |f(I_z(x))| \frac{1 - |I_z(x)|^2}{|1 - \langle z, I_z(x)\rangle|^3} J_z(x) dv(x)$$
$$\le \epsilon(r, t) \int_{M^{(t_0)}} |f(w)| \frac{1}{|1 - \langle z, w\rangle|^3} d\mu(w).$$

Thus it follows from Lemma 3.8 that

(A2.5) $$\int_{M^{(t)}} |A_2(z)|^2 d\mu(z) \le \{\epsilon(r, t)\}^2 \|B\|^2 \int_{M^{(t_0)}} |f(w)|^2 d\mu(w).$$

Finally, from (A2.1) we obtain

$$|B(z)| \le \int_{M^{(t_0)}\setminus D(z,r)} |f(w)| \frac{1}{|1-\langle z,w\rangle|^3} d\mu(w)$$

for $z \in M^{(t)}$, $0 < t \le \tau_3$. Using the operator B_r in Lemma 3.8, we have

$$\int_{M^{(t)}} |B(z)|^2 d\mu(z) \le \|B_r\|^2 \int_{M^{(t_0)}} |f(w)|^2 d\mu(w). \tag{A2.6}$$

Retracing the above steps, we have

$$A_1(z) = (Zf)(z) - (A_2(z) + B(z)).$$

Thus it follows from (A2.3), (A2.5) and (A2.6) that

$$\begin{aligned} \int_{M^{(t)}} |f(z)|^2 d\mu(z) &\le 3C_1^2 \int_{M^{(t)}} |(Zf)(z)|^2 d\mu(z) \\ &\quad + 3C_1^2(\{\epsilon(r,t)\}^2\|B\|^2 + \|B_r\|^2) \int_{M^{(t_0)}} |f(z)|^2 d\mu(z). \end{aligned} \tag{A2.7}$$

Let any $\delta > 0$ be given. By Lemma 3.8, we can first pick an $r \in [1,\infty)$ such that $3C_1^2\|B_r\|^2 \le \delta/2$. With r so fixed, by (A2.4), we can pick a $0 < \tau \le \tau_3$ such that

$$3C_1^2\{\epsilon(r,t)\}^2\|B\|^2 \le \delta/2$$

for every $0 < t \le \tau$. Substitution these bounds in (A2.7), the lemma is proved. □

On the space $\mathcal{H}^{(2-n)}$, we now define the operator

$$T^\dagger = \int_M K_w^{(2-n)} \otimes K_w^{(2-n)} d\mu(w).$$

By (5.2) and the reproducing property of $K_w^{(2-n)}$, we have

$$(T^\dagger f)(z) = (Zf)(z) \quad \text{for} \quad z \in M, \tag{A2.8}$$

$f \in \mathcal{H}^{(2-n)}$. Write $\langle \cdot, \cdot \rangle_{2-n}$ for the inner product on $\mathcal{H}^{(2-n)}$. Then

$$\langle T^\dagger f, f\rangle_{2-n} = \int_M |f(w)|^2 d\mu(w), \tag{A2.9}$$

$f \in \mathcal{H}^{(2-n)}$. Thus it follows from Lemma 5.14 that

$$\langle T^\dagger f, f\rangle_{2-n} \le C\|f\|_{2-n}^2 \quad \text{for every} \quad f \in \mathcal{H}^{(2-n)}.$$

In other words, $T^\dagger$ is a bounded operator on $\mathcal{H}^{(2-n)}$. Obviously, $T^\dagger \geq 0$.

Lemma A2.2. *For each* $y \in \mathcal{M} \cap M = \mathcal{M} \cap \mathbf{B}$, *there is an open neighborhood* N_y *of* y *in* $\mathcal{M} \cap M$ *which has the following property. Let* $\{f_k\}$ *be a sequence in* $\mathcal{H}^{(2-n)}$. *If the sequence* $\{(T^\dagger)^{1/2} f_k\}$ *weakly converges to* 0, *then*

$$\lim_{k\to\infty} \sup\{|f_k(w)| : w \in N_y\} = 0.$$

Proof. This is very similar to the proof of Lemma 3.7. Indeed we again use the biholomorphic map $\rho_y : B_1(0,2) \to V_y$ introduced in Section 3, $y \in \mathcal{M}$. Recall that $\rho_y(0) = y$. For each $y \in \mathcal{M} \cap M$, there are $\epsilon = \epsilon(y) > 0$ and $r = r(y) \in (0,2)$ such that $\rho_y(B_1(0,r)) \subset \{w \in \mathcal{M} \cap M : 1 - |w|^2 > \epsilon\}$. We will show that the lemma holds for the open set $N_y = \rho_y(B_1(0,r/2))$.

Again, consider the Bergman space $L^2_a(B_1(0,r), dv_1)$. This time, we define

$$(Gf)(\xi) = f(\rho_y(\xi)), \quad \xi \in B_1(0,r),$$

$f \in \mathcal{H}^{(2-n)}$. By the condition $r < 2$, we have

$$\begin{aligned}
\int_{B_1(0,r)} |(Gf)(\xi)|^2 dv_1(\xi) &\leq C_1 \int_{B_1(0,r)} |f(\rho_y(\xi))|^2 |\rho_y'(\xi)|^2 dv_1(\xi) \\
&= C_1 \int_{\rho_y(B_1(0,r))} |f(w)|^2 dv_M(w) \\
&\leq C_1 \epsilon^{-1} \int_{\rho_y(B_1(0,r))} |f(w)|^2 (1-|w|^2) dv_M(w) \\
&= C_1 \epsilon^{-1} \int_{\rho_y(B_1(0,r))} |f(w)|^2 d\mu(w) \\
&\leq C_1 \epsilon^{-1} \langle T^\dagger f, f\rangle_{2-n} = C_1 \epsilon^{-1} \|(T^\dagger)^{1/2} f\|^2_{2-n}.
\end{aligned}$$

Thus $G = W(T^\dagger)^{1/2}$, where $W : \mathcal{H}^{(2-n)} \to L^2_a(B_1(0,r), dv_1)$ is a bounded operator.

Now let $\{f_k\}$ be any sequence in $\mathcal{H}^{(2-n)}$ such that $\{(T^\dagger)^{1/2} f_k\}$ weakly converges to 0. Since $G = W(T^\dagger)^{1/2}$, the sequence $\{Gf_k\}$ weakly converges to 0 in $L^2_a(B_1(0,r), dv_1)$. Using the reproducing kernel for the Bergman space, we have

$$\lim_{k\to\infty} \sup\{|f_k(\rho_y(\xi))| : \xi \in B_1(0,r/2)\} = 0.$$

Since $N_y = \rho_y(B_1(0,r/2))$, the proof is complete. □

Proposition A2.3. *There is a* $\gamma > 0$ *such that the spectrum of* $T^\dagger$ *does not intersect the interval* $(0,\gamma)$.

Proof. By Lemma A2.1, there are $0 < t < t_0$ and $0 < C < \infty$ such that

$$\int_{M^{(t)}} |f(z)|^2 d\mu(z) \leq C \int_{M^{(t)}} |(Zf)(z)|^2 d\mu(z) + \frac{1}{2} \int_{M^{(t_0)}} |f(z)|^2 d\mu(z)$$

for every $f \in \mathcal{H}^{(2-n)}$. After the obvious cancellation, we have

$$\frac{1}{2}\int_{M^{(t)}} |f(z)|^2 d\mu(z) \le C \int_{M^{(t)}} |(Zf)(z)|^2 d\mu(z) + \frac{1}{2}\int_{M^{(t_0)}\backslash M^{(t)}} |f(z)|^2 d\mu(z).$$

Combining this with (A2.8) and (A2.9), we find that

$$\text{(A2.10)} \qquad \frac{1}{2}\int_{M^{(t)}} |f(z)|^2 d\mu(z) \le C\|(T^\dagger)^{3/2} f\|_{2-n}^2 + \frac{1}{2}\int_{M^{(t_0)}\backslash M^{(t)}} |f(z)|^2 d\mu(z)$$

for every $f \in \mathcal{H}^{(2-n)}$.

Let dE be the spectral measure for the positive operator $T^\dagger$. Suppose that $E(0,\gamma) \neq 0$ for every $\gamma > 0$. We will complete the proof by showing that this leads to a contradiction. For each $k \in \mathbf{N}$, since $E(0,1/k) \neq 0$, we pick an $f_k \in E(0,1/k)\mathcal{H}^{(2-n)}$ such that $\langle T^\dagger f_k, f_k\rangle_{2-n} = 1$. By (A2.9). this means

$$\text{(A2.11)} \qquad \int_M |f_k(w)|^2 d\mu(w) = 1$$

for every k. Obviously, the sequence $\{(T^\dagger)^{1/2} f_k\}$ weakly converges to 0 in $\mathcal{H}^{(2-n)}$.

Let H be the closure of $M^{(t_0)}\backslash M^{(t)}$. Then H is a compact subset of $\mathcal{M} \cap M$. By Lemma A2.2 and a usual covering argument, the weak convergence to 0 of the sequence $\{(T^\dagger)^{1/2} f_k\}$ implies

$$\lim_{k\to\infty} \sup\{|f_k(z)| : z \in H\} = 0.$$

By the maximum modulus principle, this implies

$$\lim_{k\to\infty} \sup\{|f_k(z)| : z \in M\backslash M^{(t)}\} = 0.$$

Therefore

$$\text{(A2.12)} \qquad \lim_{k\to\infty} \int_{M\backslash M^{(t)}} |f_k(z)|^2 d\mu(z) = 0.$$

Combining this with (A2.11), we find that

$$\text{(A2.13)} \qquad \lim_{k\to\infty} \int_{M^{(t)}} |f_k(z)|^2 d\mu(z) = 1.$$

Since $f_k \in E(0,1/k)\mathcal{H}^{(2-n)}$, we have

$$\text{(A2.14)} \qquad \|(T^\dagger)^{3/2} f\|_{2-n}^2 \le k^{-2}\langle T^\dagger f_k, f_k\rangle = k^{-2},$$

$k \in \mathbf{N}$. On the other hand, specializing (A2.10) to each f_k, we see that

$$\text{(A2.15)} \qquad \frac{1}{2}\int_{M^{(t)}} |f_k(z)|^2 d\mu(z) \le C\|(T^\dagger)^{3/2} f_k\|_{2-n}^2 + \frac{1}{2}\int_{M^{(t_0)}\backslash M^{(t)}} |f_k(z)|^2 d\mu(z),$$

$k \in \mathbf{N}$. Obviously, the combination of (A2.15) with (A2.12), (A2.13) and (A2.14) leads to the contradiction $1/2 \leq 0$. This completes the proof. $\square$

Define

$$\mathcal{R}^{(2-n)} = \{f \in \mathcal{H}^{(2-n)} : f = 0 \ \text{ on } \ M\}.$$

Then $\mathcal{R}^{(2-n)}$ is a submodule of the Hilbert module $\mathcal{H}^{(2-n)}$, just as $\mathcal{R}$ is a submodule of the Drury-Arveson module H_n^2.

Proof of Proposition 5.6. Define

$$\mathcal{Q}^{(2-n)} = \mathcal{H}^{(2-n)} \ominus \mathcal{R}^{(2-n)},$$

which is the quotient module of $\mathcal{H}^{(2-n)}$ corresponding to the submodule $\mathcal{R}^{(2-n)}$. Let $J^{(2-n)}$ denote the operator that takes each $f \in \mathcal{H}^{(2-n)}$ to the same f in $L^2(M, d\mu)$. That is, $J^{(2-n)}f$ is the restriction of $f \in \mathcal{H}^{(2-n)}$ to the subset M of $\mathbf{B}$. Then (A2.9) translates to the operator identity

$$(J^{(2-n)})^* J^{(2-n)} = T^\dagger. \tag{A2.16}$$

If $f \in \mathcal{R}^{(2-n)}$, then we obviously have $J^{(2-n)}f = 0$. By (A2.9) and the maximum modulus principle, we have $\ker(T^\dagger) = \mathcal{R}^{(2-n)}$. Therefore it follows from Proposition A2.3 that

$$\int_M |(J^{(2-n)}f)(w)|^2 d\mu(w) = \langle T^\dagger f, f\rangle_{2-n} \geq \gamma \|f\|^2_{n-2} \quad \text{for every} \quad f \in \mathcal{Q}^{(2-n)}.$$

Thus $J^{(2-n)}$ is an invertible operator that maps $\mathcal{Q}^{(2-n)}$ onto $\mathcal{E}$.

Obviously, $\mathcal{E}$ contains the range of the self-adjoint operator Z. Therefore $Z = 0$ on $\mathcal{E}^\perp = L^2(M, d\mu) \ominus \mathcal{E}$. Let $\tilde{Z}$ denote the restriction of Z to the invariant subspace $\mathcal{E}$. Then we have the operator decomposition

$$Z = \tilde{Z} \oplus 0 \tag{A2.17}$$

corresponding to the space decomposition $L^2(M, d\mu) = \mathcal{E} \oplus \mathcal{E}^\perp$. For each $f \in \mathcal{Q}^{(2-n)}$, it is obvious that $\tilde{Z}J^{(2-n)}f = J^{(2-n)}T^\dagger f$. Combining this with (A2.16), we have

$$\tilde{Z}J^{(2-n)}f = J^{(2-n)}T^\dagger f = J^{(2-n)}(J^{(2-n)})^* J^{(2-n)} f.$$

Since $J^{(2-n)}\mathcal{Q}^{(2-n)} = \mathcal{E}$, this implies $\tilde{Z} = J^{(2-n)}(J^{(2-n)})^*$. Since $J^{(2-n)} : \mathcal{Q}^{(2-n)} \to \mathcal{E}$ and $(J^{(2-n)})^* : \mathcal{E} \to \mathcal{Q}^{(2-n)}$ are invertible, $\tilde{Z}$ is invertible on $\mathcal{E}$. Combining this invertibility with (A2.17), the proof of Proposition 5.6 is complete. $\square$

Data availability

No data was used for the research described in the article.

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