PRINCIPAL MEASURES AND SPECTRAL SHIFT MEASURES ASSOCIATED WITH DIXMIER TRACES

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Abstract. We establish, in the context of Dixmier trace, the analogues of two classic trace formulas.

1. Introduction

The purpose of this paper is to develop for Dixmier trace analogues of certain important formulas that are well known in the case of the ordinary trace. We will refer to these formulas for the ordinary trace as the classic case. To motivate the analogues, it is appropriate to first review the classic case.

Suppose that A, B are bounded self-adjoint operators such that the commutator $[A, B]$ is in the trace class. In [15], Helton and Howe showed that there is a compactly-supported, real-valued regular Borel measure dP on $\mathbf{R}^2$ such that

$$\text{tr}([p(A, B), q(A, B)]) = i \int \{p, q\} dP \tag{1.1}$$

for all $p, q \in \mathbf{C}[x, y]$. Here, $\{p, q\}$ is the Poisson bracket for p, q. That is,

$$\{p, q\}(x, y) = \frac{\partial p}{\partial x}(x, y)\frac{\partial q}{\partial y}(x, y) - \frac{\partial p}{\partial y}(x, y)\frac{\partial q}{\partial x}(x, y).$$

Carey and Pincus [21,22,2,4,5] took this trace formula one step further by showing that there is a $g \in L^1(\mathbf{R}^2)$, which is called the *principal function* for the pair A, B, such that

$$\text{tr}([p(A, B), q(A, B)]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) g(y, x) dx dy \tag{1.2}$$

for all $p, q \in \mathbf{C}[x, y]$. In other words, (1.2) tells us that the measure on the right-hand side is absolutely continuous with respect to the two-dimensional Lebesgue measure on $\mathbf{R}^2$. In fact, g is supported on a bounded set in $\mathbf{R}^2$, and, by functional calculus, (1.2) extends to a much larger class of functions than $\mathbf{C}[x, y]$. For an irreducible pair A, B with $\text{rank}([A, B]) = 1$, the principal function g is a complete unitary invariant.

Conversely, every real-valued $g \in L^1(\mathbf{R}^2)$ with a bounded support is the principal function of a pair of self-adjoint operators A, B with trace-class commutator [3].

Keywords: Dixmier trace, principal measure, spectral shift.

The analogue of (1.2) also holds in type-II von Neumann algebras [5]. Once the door of generalization is open, it does not take too much imagination to think about other possibilities.

The motivation for this paper is a very simple question: is there an analogue of (1.2) where the ordinary trace tr is replaced by a Dixmier trace?

The case of Dixmier trace is somewhat different, because Dixmier trace is a singular trace, meaning that one cannot compute Dixmier trace by finite-rank approximation. By contrast, the traces in both type-I and type-II von Neumann algebras are normal. Thus if we want to consider the analogue of (1.2) in the context of Dixmier trace, then new way of thinking is required. On the other hand, one can take advantage of the singular nature of Dixmier trace to simplify certain arguments [6,7]. In any case, the analogue of (1.2) in the context of Dixmier trace is certainly something that is worth exploring.

We are pleased to report the analogue of formulas (1.1-2) for Dixmier trace, as follows. Let A, B be self-adjoint operators such that the commutator $[A, B]$ is in the ideal $\mathcal{C}_1^+$, which is the domain of every Dixmier trace. Suppose that the spectra of A, B are contained in $[a, b]$ and $[c, d]$ respectively. Given a Dixmier trace Tr_ω, there is a real-valued regular Borel measure μ on $R = [a, b] \times [c, d]$ such that

$$(1.3) \qquad \mathrm{Tr}_\omega([p(A, B), q(A, B)]) = \frac{i}{2\pi} \iint_R \{p, q\}(x, y) d\mu(x, y)$$

for all $p, q \in \mathbf{C}[x, y]$. Moreover, the total variation of μ does not exceed $4\pi\|[A, B]\|_1^+$, and the support of μ is contained in the essential spectrum of $A + iB$ under the natural identification $\mathbf{R}^2 \cong \mathbf{C}$. If $i[B, A]$ is non-negative, then so is the measure μ.

Since there are many Dixmier traces, we should be mindful of the possibility that the measure μ in (1.3) may depend on the choice of Tr_ω. Nonetheless, following the terminology in the classic case, we will call the measure μ in (1.3) the *principal measure* for the triple A, B and Tr_ω.

Given (1.3), the first question that comes to mind is, is the principal measure μ absolutely continuous with respect to the two-dimensional Lebesgue measure $dxdy$? The general answer is negative. We think that this is one aspect where the singular nature of Dixmier trace asserts itself. As we mentioned above, in the classic case, every real-valued, boundedly supported $g \in L^1(\mathbf{R}^2)$ is the principal function for some pair with trace-class commutator. In the case of Dixmier trace, we will prove the following. Let μ be any compactly-supported regular Borel measure on $\mathbf{R}^2$. Then there exist self-adjoint operators A, B with $[A, B] \in \mathcal{C}_1^+$ such that for every Dixmier trace Tr_ω, we have

$$(1.4) \qquad \mathrm{Tr}_\omega([p(A, B), q(A, B)]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) d\mu(x, y)$$

for all $p, q \in \mathbf{C}[x, y]$. In other words, every compactly-supported regular Borel measure on $\mathbf{R}^2$ is the principal measure for some pair of self-adjoint operators A, B in the context of Dixmier trace. This is a sharp contrast to the case of ordinary trace.

A closely related topic in the classic case is Krein's trace formula and spectral shift. Suppose that A is a bounded self-adjoint operator and K is a self-adjoint operator of the trace class. Then a famous theorem of Krein tells us that there is a $\xi \in L^1(\mathbf{R})$ with a bounded support such that for every $z \in \mathbf{C}\backslash\mathbf{R}$,

$$\text{tr}((A+K-z)^{-1}-(A-z)^{-1}) = \int \frac{-1}{(t-z)^2}\xi(t)dt. \tag{1.5}$$

If we further assume that K is non-negative, then so is ξ. See [18,19,1,27] for these well-known facts. The function ξ in (1.5) is called the *spectral shift* for the perturbation problem $A \to A+K$. By now, it is a standard exercise to deduce from (1.5) that

$$\text{tr}(\eta(A+K)-\eta(A)) = \int \eta'(t)\xi(t)dt \tag{1.6}$$

for every $\eta \in C_c^\infty(\mathbf{R})$.

Spectral shifts play an extremely important role in the perturbation theory of self-adjoint operators [13,23]. And not coincidentally, analogues of (1.5) and (1.6) hold in type II von Neumann algebras [5]. Thus, again, a natural question presents itself: is there an analogue of (1.5) and (1.6) in the context of Dixmier trace?

We have an answer for this question too. Let A and K be bounded self-adjoint operators with $K \in \mathcal{C}_1^+$. Let any Dixmier trace Tr_ω be given. Then there is a compactly-supported, real-valued regular Borel measure μ on $\mathbf{R}$ such that

$$\text{Tr}_\omega((A+K-z)^{-1}-(A-z)^{-1}) = \int \frac{-1}{(t-z)^2}d\mu(t) \tag{1.7}$$

for every $z \in \mathbf{C}\backslash\mathbf{R}$. Moreover, the total variation of μ does not exceed $\text{Tr}_\omega(|K|)$. We will call the measure μ the spectral shift for the triple A, $A+K$ and Tr_ω.

Again, the existence of spectral shift in the context of Dixmier trace leads to more questions: Is the μ in (1.7) absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$? Or, does every compactly-supported regular Borel measure show up in (1.7) for some triple A, $A+K$ and Tr_ω? And again, the latter turns out to be the case: Let μ be a regular Borel measure on $[a,b]$, where $-\infty < a < b < \infty$. Let A be any bounded self-adjoint operator whose spectrum contains $[a,b]$. We will show that there is a non-negative, self-adjoint operator $K \in \mathcal{C}_1^+$ such that for every Dixmier trace Tr_ω and every $z \in \mathbf{C}\backslash\mathbf{R}$, we have

$$\text{Tr}_\omega((A+K-z)^{-1}-(A-z)^{-1}) = \int \frac{-1}{(t-z)^2}d\mu(t). \tag{1.8}$$

Moreover, this K satisfies the estimate $\|K\|_1^+ \le 2(\log 2)\mu([a,b])$.

The rest of the paper is organized as follows. We prove (1.3), (1.4) and (1.7) in Sections 2, 3 and 4 respectively. We divide the proof of statement (1.8) into two steps, which take up Sections 5 and 6. In Section 7 we examine a commutator property that demonstrates a crucial difference between Dixmier trace and the ordinary trace.

2. Dixmier trace and Poisson brackets

First of all, we cite [9,7] as general references for Dixmier trace. Any discussion of Dixmier trace needs to start from its domain, the operator ideal $\mathcal{C}_1^+$. Given an operator T, write $s_1(T), \ldots, s_k(T), \ldots$ for its s-numbers [14]. The formula

$$\|T\|_1^+ = \sup_{k\geq 1} \frac{s_1(T) + s_2(T) + \cdots + s_k(T)}{1^{-1} + 2^{-1} + \cdots + k^{-1}}$$

defines a symmetric norm for operators. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_1^+ = \{T \in \mathcal{B}(\mathcal{H}) : \|T\|_1^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [14]. In particular, if $T \in \mathcal{C}_1^+$ and X is a bounded operator, then

$$\|XT\|_1^+ \leq \|X\|\|T\|_1^+.$$

This fact will play a prominent role below.

To define Dixmier trace on the ideal $\mathcal{C}_1^+$, one starts with a Banach limit ω on $\ell^\infty(\mathbf{N})$ that has the following "doubling" property:

$$\omega(\{a_k\}_{k\in\mathbf{N}}) = \omega(\{a_1, a_1, a_2, a_2, \ldots, a_k, a_k, \ldots\})$$

for every $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$. Such an ω can be easily constructed. For example, one can start with the doubling operator $D : \ell^\infty(\mathbf{N}) \to \ell^\infty(\mathbf{N})$. That is,

$$D\{a_1, a_2, \ldots, a_k, \ldots\} = \{a_1, a_1, a_2, a_2, \ldots, a_k, a_k, \ldots\}$$

for $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$. Take any Banach limits (cf. [8, Section III.7]) L_1 and L_2, distinct or identical. An elementary exercise shows that the formula

$$\omega(a) = L_2\left(\left\{\frac{1}{k}\sum_{j=1}^{k} L_1(D^j a)\right\}_{k\in\mathbf{N}}\right),$$

$a \in \ell^\infty(\mathbf{N})$, defines a Banach limit that has the doubling property stated above.

Given such an ω, for a *positive* operator $E \in \mathcal{C}_1^+$, its Dixmier trace is defined to be

$$\mathrm{Tr}_\omega(E) = \omega\left(\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} s_j(E)\right\}_{k\in\mathbf{N}}\right).$$

The doubling property of ω ensures the additivity $\mathrm{Tr}_\omega(E + F) = \mathrm{Tr}_\omega(E) + \mathrm{Tr}_\omega(F)$ for positive operators $E, F \in \mathcal{C}_1^+$. Thus Tr_ω naturally extends to a linear functional on $\mathcal{C}_1^+$.

Then the unitary invariance of Tr_ω leads to the identity $\mathrm{Tr}_\omega(XT) = \mathrm{Tr}_\omega(TX)$ for every $T \in \mathcal{C}_1^+$ and every bounded operator X, which is what one expects of a trace.

An important property of the Dixmier trace Tr_ω is that $\mathrm{Tr}_\omega(T) = 0$ whenever T is in the trace class. Because of this property, the computation of Dixmier trace cannot be done through finite-dimensional approximation. Consequently, Tr_ω is often called a *singular trace*. Interestingly, this is also the property on which many previous calculations of Dixmier trace depended. See, e.g., [6,10,11,12,24]. We will also take advantage of this and other properties of Dixmier trace.

In addition to $\mathcal{C}_1^+$, we also need the ideal $\mathcal{C}_1^{+(0)}$, which is the $\|\cdot\|_1^+$-closure in $\mathcal{C}_1^+$ of the collection of finite-rank operators. It is well known that $\mathcal{C}_1^{+(0)}$ is a proper subset of $\mathcal{C}_1^+$ [14]. Indeed we can see that $\mathcal{C}_1^{+(0)} \neq \mathcal{C}_1^+$ from the following obvious fact: For every $T \in \mathcal{C}_1^{+(0)}$ and every Dixmier trace Tr_ω, we have

$$\mathrm{Tr}_\omega(T) = 0.$$

Lemma 2.1. *If $K \in \mathcal{C}_1^+$ and E is a compact operator, then $\mathrm{Tr}_\omega(EK) = 0$ for every Dixmier trace Tr_ω.*

Proof. If $K \in \mathcal{C}_1^+$ and E is a compact operator, then we obviously have $EK \in \mathcal{C}_1^{+(0)}$. □

Lemma 2.2. *Let A, B be self-adjoint operators that essentially commute, i.e., $[A, B]$ is compact. Suppose that the spectra of A, B are contained in finite intervals $[a, b]$ and $[c, d]$ respectively. Let $K \in \mathcal{C}_1^+$, and let Tr_ω be any Dixmier trace. Then for all $f_1, \dots, f_n \in C[a, b]$ and $g_1, \dots, g_n \in C[c, d]$, $n \in \mathbf{N}$, we have*

$$\left|\mathrm{Tr}_\omega\left(\sum_{i=1}^n f_i(A)g_i(B)K\right)\right| \le 2\|K\|_1^+ \sup_{(x,y)\in[a,b]\times[c,d]} \left|\sum_{i=1}^n f_i(x)g_i(y)\right|.$$

Proof. Let $f_1, \dots, f_n \in C[a, b]$ and $g_1, \dots, g_n \in C[c, d]$ be given. Denote

$$C = \max\{\|g_1\|_\infty, \dots, \|g_n\|_\infty\}.$$

Let $\epsilon > 0$ also be given. By the continuity of $f_1, \dots, f_n$, there is a partition

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$$

such that the resulting intervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$, $\cdots$, $I_m = [x_{m-1}, x_m]$ have the property that the inequality

$$\sup_{x,x'\in I_j\cup I_{j+1}} |f_i(x) - f_i(x')| \le \frac{\epsilon}{n(1 + C)} \tag{2.1}$$

holds for every pair of $1 \le j < m$ and $1 \le i \le n$.

Let $\eta_0, \eta_1, \ldots, \eta_m$ be a partition of the unity by continuous functions on $[a,b]$ such that $\eta_0 = 0$ on $[a,b]\backslash I_1$, $\eta_m = 0$ on $[a,b]\backslash I_m$, and $\eta_j = 0$ on $[a,b]\backslash\{I_j \cup I_{j+1}\}$ for every $1 \le j < m$. It follows from (2.1) that for each $1 \le i \le n$, we have

$$\left\| f_i(A) - \sum_{j=0}^{m} f_i(x_j)\eta_j(A) \right\| \le \frac{\epsilon}{n(1+C)}.$$

Obviously, we also have the bound $\|g_i(B)\| \le C$ for every $1 \le i \le n$. Therefore

$$\left\| \sum_{i=1}^{n} f_i(A)g_i(B) - \sum_{i=1}^{n}\sum_{j=0}^{m} f_i(x_j)\eta_j(A)g_i(B) \right\| \le \epsilon. \tag{2.2}$$

Let us denote

$$X_j = \sum_{i=1}^{n} f_i(x_j)g_i(B) \tag{2.3}$$

for each $0 \le j \le m$. Then we can rewrite (2.2) as

$$\left\| \sum_{i=1}^{n} f_i(A)g_i(B) - \sum_{j=0}^{m} \eta_j(A)X_j \right\| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, to prove the lemma, it suffices to show that for every Tr_ω, we have

$$\left| \mathrm{Tr}_\omega\left(\sum_{j=0}^{m} \eta_j(A)X_jK \right) \right| \le 2\|K\|_1^+ \sup_{(x,y)\in[a,b]\times[c,d]} \left| \sum_{i=1}^{n} f_i(x)g_i(y) \right|. \tag{2.4}$$

To prove this, we use the assumption that A, B essentially commute. A consequence of this essential commutativity is that for every $0 \le j \le m$, the commutator $[\eta_j^{1/2}(A), X_j]$ is compact. Hence we can apply Lemma 2.1 to conclude that

$$\mathrm{Tr}_\omega\left(\sum_{j=0}^{m} \eta_j(A)X_jK \right) = \mathrm{Tr}_\omega\left(\sum_{j=0}^{m} \eta_j^{1/2}(A)X_j\eta_j^{1/2}(A)K \right).$$

Thus (2.4) will follow if we can show that

$$\left\| \sum_{j=0}^{m} \eta_j^{1/2}(A)X_j\eta_j^{1/2}(A) \right\| \le 2 \sup_{(x,y)\in[a,b]\times[c,d]} \left| \sum_{i=1}^{n} f_i(x)g_i(y) \right|. \tag{2.5}$$

Let E and F respectively denote the collection of the even numbers and the collection of the odd numbers in $\{0, 1, \ldots, m\}$. Our partition of unity was designed so that for $j \ne k$ in E, we have $\eta_j^{1/2}\eta_k^{1/2} = 0$. Thus by the spectral decomposition of A, we have

$$\left\| \sum_{j\in E} \eta_j^{1/2}(A)X_j\eta_j^{1/2}(A) \right\| = \max_{j\in E} \|\eta_j^{1/2}(A)X_j\eta_j^{1/2}(A)\| \le \max_{j\in E} \|X_j\|.$$

That is, there is a $j^* \in \{0, 1, \ldots, m\}$ such that

$$\Big\| \sum_{j \in E} \eta_j^{1/2}(A) X_j \eta_j^{1/2}(A) \Big\| \le \|X_{j^*}\|.$$

But by (2.3) we have

$$\|X_{j^*}\| \le \sup_{y \in [c,d]} \Big| \sum_{i=1}^n f_i(x_{j^*}) g_i(y) \Big|.$$

These two inequalities obviously imply

$$\Big\| \sum_{j \in E} \eta_j^{1/2}(A) X_j \eta_j^{1/2}(A) \Big\| \le \sup_{(x,y) \in [a,b] \times [c,d]} \Big| \sum_{i=1}^n f_i(x) g_i(y) \Big|. \tag{2.6}$$

Similarly, by design we have $\eta_j^{1/2} \eta_k^{1/2} = 0$ for every pair of $j \ne k$ in F. Thus the same argument shows that

$$\Big\| \sum_{j \in F} \eta_j^{1/2}(A) X_j \eta_j^{1/2}(A) \Big\| \le \sup_{(x,y) \in [a,b] \times [c,d]} \Big| \sum_{i=1}^n f_i(x) g_i(y) \Big|. \tag{2.7}$$

From (2.6) and (2.7) we obtain (2.5). This completes the proof. □

Theorem 2.3. *Let A, B be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $[A, B]$ is compact. Suppose that the spectra of A, B are contained in finite intervals $[a, b]$ and $[c, d]$ respectively. Given any self-adjoint operator $K \in \mathcal{C}_1^+$ and any Dixmier trace Tr_ω, there is a real-valued regular Borel measure μ on $[a, b] \times [c, d]$ such that for all $f \in C[a, b]$ and $g \in C[c, d]$,*

$$\mathrm{Tr}_\omega(f(A)g(B)K) = \frac{1}{2} \int_a^b \int_c^d f(x)g(y)d\mu(x, y). \tag{2.8}$$

Moreover, the following three statements hold true for μ:
(i) *The total variation of μ does not exceed $4\|K\|_1^+$.*
(ii) *If K is non-negative, then so is the measure μ.*
(iii) *Under the natural identification $\mathbf{C} \cong \mathbf{R}^2$, the support of μ is contained in the essential spectrum of $A + iB$.*

Proof. Denote $R = [a, b] \times [c, d]$. Let $\mathcal{E}$ be the collection of functions of the form

$$F(x, y) = \sum_{i=1}^n f_i(x) g_i(y) \tag{2.9}$$

on R, where $n \in \mathbf{N}$, $f_1, \ldots, f_n \in C[a, b]$ and $g_1, \ldots, g_n \in C[c, d]$ are arbitrary. Then $\mathcal{E}$ is a linear subspace of $C(R)$ that is dense with respect to the norm $\| \cdot \|_\infty$. We now define a $\Phi : \mathcal{E} \to \mathbf{C}$ by the formula

$$\Phi(F) = \mathrm{Tr}_\omega\Big(\sum_{i=1}^n f_i(A) g_i(B) K \Big) \tag{2.10}$$

for the F given by (2.9). Then Lemma 2.2 tells us that

$$|\Phi(F)| \leq 2\|K\|_1^+ \|F\|_\infty.$$

Thus Φ is well defined on $\mathcal{E}$. By this bound and the density of $\mathcal{E}$ in $C(R)$, Φ uniquely extends to a bounded linear functional on $C(R)$ with $\|\Phi\| \leq 2\|K\|_1^+$. By the Riesz representation theorem, there is a complex-valued regular Borel measure μ on R such that

$$\Phi(G) = \frac{1}{2}\int_a^b \int_c^d G(x,y)d\mu(x,y)$$

for every $G \in C(R)$. (We put the coefficient 1/2 on the right-hand side by design.) Combining this with (2.10), we have shown that (2.8) holds for this μ. Statement (i) follows from the fact that the total variation of μ equals $2\|\Phi\|$.

Next we show that μ is real valued, and that statement (ii) holds. By (2.8), the first assertion will follow if we can show that for real-valued $f \in C[a,b]$ and $g \in C[c,d]$,

$$\mathrm{Tr}_\omega(f(A)g(B)K) \in \mathbf{R}. \tag{2.11}$$

By the usual combination of continuous function, it suffices to prove (2.11) for non-negative $f \in C[a,b]$ and $g \in C[c,d]$. But for non-negative $f \in C[a,b]$ and $g \in C[c,d]$, by the essential commutativity of A, B and Lemma 2.1, we have

$$\begin{aligned}\mathrm{Tr}_\omega(f(A)g(B)K) &= \mathrm{Tr}_\omega(f^{1/2}(A)g(B)f^{1/2}(A)K)\\ &= \mathrm{Tr}_\omega(\{f^{1/2}(A)g(B)f^{1/2}(A)\}^{1/2}K\{f^{1/2}(A)g(B)f^{1/2}(A)\}^{1/2}).\end{aligned}$$

Since K is assumed to be self-adjoint, the above is a real number. This proves (2.11). If $K \geq 0$, then the above is a non-negative number. By (2.8), this means that μ is non-negative whenever K is non-negative. This proves (ii).

To prove (iii), consider any finite open rectangle

$$E = \{x + iy : \alpha < x < \beta \ \text{ and } \ u < y < v\}$$

that does not intersect the essential spectrum of $A+iB$. It suffices to show that $\mu(E) = 0$. Let ξ, η be arbitrary continuous functions on $\mathbf{R}$ such that $\xi = 0$ on $\mathbf{R}\backslash(\alpha,\beta)$ and $\eta = 0$ on $\mathbf{R}\backslash(u,v)$. The fact $\mu(E) = 0$ will follow if we can show that

$$\iint \xi(x)\eta(y)d\mu(x,y) = 0.$$

By (2.8), this is reduced to the proof of the assertion that

$$\mathrm{Tr}_\omega(\xi(A)\eta(B)K) = 0.$$

By Lemma 2.1, it suffices to show that the operator $\xi(A)\eta(B)$ is compact. To prove this compactness, consider the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Write $\hat{A}$ and $\hat{B}$ for the images of A and B in $\mathcal{Q}$ respectively. Since $[A, B]$ is compact, $\hat{A} + i\hat{B}$ is a normal element in $\mathcal{Q}$. Define the function

$$h(x + iy) = \xi(x)\eta(y).$$

Since $h = 0$ on $\mathbf{C}\backslash E$ and since E does not intersect the spectrum of $\hat{A} + i\hat{B}$, we have $h(\hat{A} + i\hat{B}) = 0$, i.e., $\xi(\hat{A})\eta(\hat{B}) = 0$. Since $\xi(\hat{A})$ and $\eta(\hat{B})$ are, respectively, the images of $\xi(A)$ and $\eta(B)$ in $\mathcal{Q}$, we conclude that $\xi(A)\eta(B)$ is compact. This completes the proof. □

Recall that for $F, G \in C^1(\mathbf{R}^2)$, the Poisson bracket $\{F, G\}$ is defined by the formula

$$\{F, G\}(x, y) = \frac{\partial F}{\partial x}(x, y)\frac{\partial G}{\partial y}(x, y) - \frac{\partial F}{\partial y}(x, y)\frac{\partial G}{\partial x}(x, y).$$

For any $p \in \mathbf{C}[x, y]$ and any two operators A, B, we define

$$p(A, B) = \sum_{(j,k)\in E} c_{j,k}A^j B^k \quad \text{if} \quad p(x, y) = \sum_{(j,k)\in E} c_{j,k}x^j y^k,$$

where E is a finite subset of $\mathbf{Z}_+^2$. For A and B that do not commute, this definition of $p(A, B)$ seems rather contrived and artificial. But for what we are doing here, this does not matter, by the following trivial observation: If A, B, X, Y and Z have the properties that $[A, B] \in \mathcal{C}_1^+$, $[XABY, Z] \in \mathcal{C}_1^+$ and $[XBAY, Z] \in \mathcal{C}_1^+$, then

$$\mathrm{Tr}_\omega([XABY, Z]) = \mathrm{Tr}_\omega([XBAY, Z]).$$

We are now ready to establish (1.3).

Theorem 2.4. *Let A, B be self-adjoint operators such that $[A, B] \in \mathcal{C}_1^+$. Suppose that the spectra of A, B are contained in finite intervals $[a, b]$ and $[c, d]$ respectively. Given a Dixmier trace Tr_ω, there is a real-valued regular Borel measure μ on $[a, b] \times [c, d]$ such that*

$$\mathrm{Tr}_\omega([p(A, B), q(A, B)]) = \frac{i}{2\pi}\int_a^b\int_c^d \{p, q\}(x, y)d\mu(x, y) \tag{2.12}$$

for all $p, q \in \mathbf{C}[x, y]$. Moreover, the following three statements hold true for μ:
(i) *The total variation of μ does not exceed $4\pi\|[A, B]\|_1^+$.*
(ii) *If $i[B, A]$ is non-negative, then so is the measure μ.*
(iii) *Under the natural identification $\mathbf{C} \cong \mathbf{R}^2$, the support of μ is contained in the essential spectrum of $A + iB$.*

Proof. We write $[B, A] = K/\pi i$. Then $K = \pi i[B, A]$ is a self-adjoint operator in $\mathcal{C}_1^+$. We apply Theorem 2.3 to this K and the given Dixmier trace Tr_ω. Thus we have a real-valued regular Borel measure μ on $[a, b] \times [c, d]$ such that for all $f \in C[a, b]$ and $g \in C[c, d]$,

$$\pi i\mathrm{Tr}_\omega(f(A)g(B)[B, A]) = \frac{1}{2}\int_a^b\int_c^d f(x)g(y)d\mu(x, y). \tag{2.13}$$

Moreover, statements (i), (ii) and (iii) follow from the corresponding ones in Theorem 2.3. What remains is to show that (2.12) holds for this μ.

By linearity, to prove (2.12), it suffices to consider monomials $p(x,y) = x^m y^n$ and $q(x,y) = x^j y^k$, where $m,n,j,k \in \mathbf{Z}_+$, $m+j \geq 1$, and $n+k \geq 1$. Obviously,

$$[A^m B^n, A^j B^k] = A^m [B^n, A^j] B^k + A^j [A^m, B^k] B^n. \tag{2.14}$$

By Lemma 2.1, for all bounded operators X, Y, Z we have

$$\mathrm{Tr}_\omega(XABY[A,B]Z) = \mathrm{Tr}_\omega(XBAY[A,B]Z).$$

Because of this, easy algebra now leads to

$$\mathrm{Tr}_\omega(A^m[B^n, A^j]B^k) = nj\mathrm{Tr}_\omega(A^{m+j-1}B^{n+k-1}[B,A]) \quad \text{and} \tag{2.15}$$

$$\mathrm{Tr}_\omega(A^j[A^m, B^k]B^n) = mk\mathrm{Tr}_\omega(A^{m+j-1}B^{n+k-1}[A,B]). \tag{2.16}$$

Combining these two identities with (2.14) and (2.13), we find that

$$\begin{aligned} \mathrm{Tr}_\omega([A^m B^n, A^j B^k]) &= (nj - mk)\mathrm{Tr}_\omega(A^{m+j-1}B^{n+k-1}[B,A]) \\ &= \frac{nj - mk}{2\pi i}\int_a^b \int_c^d x^{m+j-1}y^{n+k-1} d\mu(x,y) \\ &= \frac{i}{2\pi}\int_a^b \int_c^d \{p,q\}(x,y) d\mu(x,y), \end{aligned}$$

where $p(x,y) = x^m y^n$ and $q(x,y) = x^j y^k$. This completes the proof. □

3. Prescribing principal measure

In the classic case for a pair of bounded self-adjoint operators A, B with trace-class commutator [21,2,4,5], the associated principal function g is in $L^1(R)$ for some rectangle R in $\mathbf{R}^2$. In other words, the "principal measure" for A, B associated with the ordinary trace is always absolutely continuous with respect to the two-dimensional Lebesgue measure. In fact, this absolute continuity is a hallmark of the Carey-Pincus theory for pairs with trace-class commutator. What is more, this absolute continuity even holds in type-II von Neumann algebras [5].

Given what we know in the classic case, once Theorem 2.4 is proved, the most pressing question is, is the measure μ in that theorem always absolutely continuous with respect to the two-dimensional Lebesgue measure? Put differently, in the case of Dixmier trace, do we also get a "principal function" rather than just a "principal measure"? The answer is negative. The purpose of this section is to show that, in fact, every compactly-supported regular Borel measure on $\mathbf{R}^2$ is the principal measure for some triple A, B and Tr_ω.

We will use the singular integral operators of the classic case as the building blocks for the construction of our desired pair A, B. The main idea is that we can take orthogonal sums of appropriate operators to produce what we want.

Denote $I = [0,1]$. On the Hilbert space $L^2(I)$, we define

$$(Uf)(x) = xf(x) \quad \text{and} \quad (Vf)(x) = \frac{1}{2}f(x) + \frac{1}{2\pi i}\text{p.v.}\int_I \frac{f(t)}{t-x}dt$$

for every $f \in L^2(I)$. The spectrum of U is, of course, I. We have $V = (1/2)(1+V_0)$, where

$$(V_0 f)(x) = \frac{1}{\pi i}\text{p.v.}\int_I \frac{f(t)}{t-x}dt,$$

$f \in L^2(I)$. If we view $L^2(I)$ as a subspace of $L^2(\mathbf{R})$, then V_0 is the compression of the Hilbert transform to $L^2(I)$. Thus the spectrum of V_0 is contained in $[-1,1]$. Consequently, the spectrum of V is contained in I. Moreover,

$$[V,U] = \frac{1}{2\pi i}\chi_I \otimes \chi_I. \tag{3.1}$$

Here and in what follows, we write χ_I for the constant function 1 on the interval $I = [0,1]$. For the rest of the section, the symbols U, V will always denote the two self-adjoint operators defined above. This pair U, V will be the fundamental building block in our construction.

Lemma 3.1. *Let μ be a regular Borel measure on a rectangle $R = [a,b] \times [c,d]$, where $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. Suppose that μ has the property that*

$$\mu(L) = 0 \tag{3.2}$$

for every vertical line L in R and every horizontal line L in R. Let any $\epsilon > 0$ be given. Then there exist self-adjoint operators A, B satisfying the following four conditions:
(1) *The spectra of A and B are contained in $[a, b+\epsilon]$ and $[c, d+\epsilon]$ respectively.*
(2) $[A,B] \in \mathcal{C}_1^+$ *with* $\|[A,B]\|_1^+ \le 2\pi^{-1}(\log 4)\mu(R)$.
(3) $\sqrt{-1}[B,A] \ge 0$.
(4) *For every Dixmier trace* Tr_ω *and every pair of* $p, q \in \mathbf{C}[x,y]$, *we have*

$$\text{Tr}_\omega([p(A,B), q(A,B)]) = \frac{\sqrt{-1}}{2\pi}\iint_R \{p,q\}(x,y)d\mu(x,y).$$

Proof. Write $M = \mu(R)$, the total mass of μ. We assume that $M > 0$, for otherwise there is nothing to prove. Given an $\epsilon > 0$, there is a $k_0 \in \mathbf{N}$ such that $2^{-k_0}(M\log 4)^{1/2} < \epsilon$.

For each $k \ge k_0$, we decompose R according to the following scheme. First of all, using property (3.2) for vertical lines, there is a partition

$$a = s_{0;k} < s_{1;k} < s_{2;k} < \cdots < s_{2^k-1;k} < s_{2^k;k} = b$$

of the interval $[a,b]$ such that

$$\mu([a, s_{i;k}] \times [c,d]) = \frac{i}{2^k}M \quad \text{for every} \quad 0 \le i \le 2^k. \tag{3.3}$$

By this and (3.2), we have $\mu([s_{i-1;k}, s_{i;k}] \times [c,d]) = 2^{-k}M$ for every $1 \le i \le 2^k$. Thus, using (3.2) for horizontal lines, for every pair of $k \ge k_0$ and $1 \le i \le 2^k$, there is a partition

$$c = t_{i,0;k} < t_{i,1;k} < \cdots < t_{i,2^k-1;k} < t_{i,2^k;k} = d$$

of the interval $[c,d]$ such that

$$\mu([s_{i-1;k}, s_{i;k}] \times [c, t_{i,j;k}]) = \frac{j}{4^k}M \quad \text{for every} \quad 0 \le j \le 2^k. \tag{3.4}$$

With $s_{i;k}$ and $t_{i,j;k}$ chosen as above, we are ready to define our operators.

We begin with the Hilbert space

$$\mathcal{H} = \bigoplus_{k=k_0}^{\infty} \bigoplus_{i=1}^{2^k} \bigoplus_{j=1}^{2^k} \mathcal{H}_{i,j;k}, \tag{3.5}$$

where

$$\mathcal{H}_{i,j;k} = L^2(I)$$

for every triple of $k \ge k_0$ and $i, j \in \{1, 2, \dots, 2^k\}$. Thus the subscripts on the right-hand side of (3.5) are there purely for the purpose of tracking the orthogonal summands. Now, for each triple of $k \ge k_0$ and $i, j \in \{1, 2, \dots, 2^k\}$, we define

$$A_{i,j;k} = s_{i;k} + 2^{-k}(M \log 4)^{1/2}U \quad \text{and} \quad B_{i,j;k} = t_{i,j;k} + 2^{-k}(M \log 4)^{1/2}V \tag{3.6}$$

on $\mathcal{H}_{i,j;k}$. Then on the Hilbert space $\mathcal{H}$ we define

$$A = \bigoplus_{k=k_0}^{\infty} \bigoplus_{i=1}^{2^k} \bigoplus_{j=1}^{2^k} A_{i,j;k} \quad \text{and} \quad B = \bigoplus_{k=k_0}^{\infty} \bigoplus_{i=1}^{2^k} \bigoplus_{j=1}^{2^k} B_{i,j;k}. \tag{3.7}$$

Let us verify that (1)-(4) hold for this pair of self-adjoint operators A, B.

First of all, since $2^{-k_0}(M \log 4)^{1/2} < \epsilon$ and since the spectra of U, V are contained in $[0,1]$, condition (1) obviously holds for A, B. For every triple of $k \ge k_0$ and $i, j \in \{1, 2, \dots, 2^k\}$, let $u_{i,j;k}$ denote the unit vector χ_I in $\mathcal{H}_{i,j;k} = L^2(I)$. By (3.1), we have

$$[B, A] = \frac{M \log 4}{2\pi\sqrt{-1}} \bigoplus_{k=k_0}^{\infty} 4^{-k} \bigoplus_{i=1}^{2^k} \bigoplus_{j=1}^{2^k} u_{i,j;k} \otimes u_{i,j;k}. \tag{3.8}$$

For every $k' \ge k_0$, if $\sum_{k=k_0}^{k'} 4^k < \nu \le \sum_{k=k_0}^{k'+1} 4^k$, then

$$s_\nu([B, A]) = 4^{-k'-1}(2\pi)^{-1}M \log 4 \le (4/\nu)(2\pi)^{-1}M \log 4.$$

This verifies (2). Also, (3) is obvious.

What remains it to verify condition (4), which takes a few steps. For any $(s,t) \in R$, denote $R_{s,t} = [a,s] \times [c,t]$. For $(s,t) \in R$ and $k \geq k_0$, we define

$$E_k(s,t) = \{(i,j) \in \{1,\dots,2^k\} \times \{1,\dots,2^k\} : s_{i;k} \leq s \text{ and } t_{i,j;k} \leq t\}.$$

Then define the operator

$$T_{s,t} = (\log 4)M \bigoplus_{k=k_0}^{\infty} 4^{-k} \bigoplus_{(i,j)\in E_k(s,t)} u_{i,j;k} \otimes u_{i,j;k}.$$

The main step is to show that for every Dixmier trace Tr_ω and every $(s,t) \in R$, we have

$$\mathrm{Tr}_\omega(T_{s,t}) = \mu(R_{s,t}). \tag{3.9}$$

To prove this, we define the rectangle

$$G_{i,j;k} = [s_{i-1;k}, s_{i;k}] \times [t_{i,j-1;k}, t_{i,j;k}]$$

for every triple of $k \geq k_0$ and $i,j \in \{1,\dots,2^k\}$. Note that

$$E_k(s,t) = \{(i,j) \in \{1,\dots,2^k\} \times \{1,\dots,2^k\} : G_{i,j;k} \subset R_{s,t}\}.$$

For each $k \geq k_0$, we also introduce the set

$$C_k(s,t) = \{(i,j) \in \{1,\dots,2^k\} \times \{1,\dots,2^k\} : G_{i,j;k} \cap R_{s,t} \neq \emptyset\}.$$

If $(i,j) \in C_k(s,t)\backslash E_k(s,t)$, then we have either $s_{i-1;k} \leq s < s_{i;k}$ or $t_{i,j-1;k} \leq t < t_{i,j;k}$. Thus

$$\mathrm{card}(C_k(s,t)\backslash E_k(s,t)) \leq 2 \times 2^k. \tag{3.10}$$

By (3.4) and (3.2), we have $\mu(G_{i,j;k}) = 4^{-k}M$ for every triple of $k \geq k_0$ and $i,j \in \{1,\dots,2^k\}$. Using (3.2) again, we have

$$\begin{aligned}\frac{M}{4^k}\mathrm{card}(E_k(s,t)) = \mu\Bigg(\bigcup_{(i,j)\in E_k(s,t)} G_{i,j;k}\Bigg) \leq \mu(R_{s,t}) \leq \mu\Bigg(\bigcup_{(i,j)\in C_k(s,t)} G_{i,j;k}\Bigg) \\ = \frac{M}{4^k}\mathrm{card}(C_k(s,t)).\end{aligned}$$

Combining this with (3.10), we find that for every $k \geq k_0$,

$$\frac{M}{4^k}\mathrm{card}(E_k(s,t)) \leq \mu(R_{s,t}) \leq \frac{M}{4^k}\{\mathrm{card}(E_k(s,t)) + 2^{k+1}\}. \tag{3.11}$$

Consider the following two possibilities.

(i) Suppose that $\mu(R_{s,t}) = 0$. Then (3.11) tells us that, in this case, $E_k(s,t) = \emptyset$ for every $k \geq k_0$. By definition, this means $T_{s,t} = 0$. Therefore (3.9) holds in this case.

(ii) Suppose that $\mu(R_{s,t}) > 0$. Then by (3.11), for each sufficiently large $\nu \in \mathbf{N}$, there is a $k(\nu) \geq k_0$ such that

$$\sum_{k=k_0}^{k(\nu)} \operatorname{card}(E_k(s,t)) \leq \nu < \sum_{k=k_0}^{k(\nu)+1} \operatorname{card}(E_k(s,t)). \tag{3.12}$$

Recalling the definition of $T_{s,t}$, we have

$$\sum_{\ell=1}^{\nu} s_\ell(T_{s,t}) \leq (\log 4)M \sum_{k=k_0}^{k(\nu)+1} 4^{-k}\operatorname{card}(E_k(s,t)) \leq (\log 4)(k(\nu)+1)\mu(R_{s,t}),$$

where the second $\leq$ follows from the lower bound in (3.11). Combining the lower bound in (3.12) with the upper bound in (3.11), we find that

$$\nu \geq \operatorname{card}(E_{k(\nu)}(s,t)) \geq \frac{\mu(R_{s,t})}{M}4^{k(\nu)} - 2^{k(\nu)+1}.$$

Combining these two bounds, we find that

$$\frac{1}{\log(\nu+1)}\sum_{\ell=1}^{\nu} s_\ell(T_{s,t}) \leq \frac{(\log 4)(k(\nu)+1)\mu(R_{s,t})}{k(\nu)\log 4 + \log(\{\mu(R_{s,t})/M\} - 2^{-k(\nu)+1})}. \tag{3.13}$$

By the lower bound in (3.12) and the upper bound in (3.11), we have

$$\begin{aligned}\sum_{\ell=1}^{\nu} s_\ell(T_{s,t}) &\geq (\log 4)M \sum_{k=k_0}^{k(\nu)} 4^{-k}\operatorname{card}(E_k(s,t)) \geq (\log 4)\sum_{k=k_0}^{k(\nu)}\left(\mu(R_{s,t}) - \frac{2M}{2^k}\right)\\ &\geq (\log 4)\{(k(\nu)-k_0)\mu(R_{s,t}) - 2M\}.\end{aligned}$$

Thus

$$\frac{1}{\log(\nu+1)}\sum_{\ell=1}^{\nu} s_\ell(T_{s,t}) \geq \frac{(\log 4)\{(k(\nu)-k_0)\mu(R_{s,t}) - 2M\}}{\log\left(4^{k(\nu)+2}\right)}.$$

This and (3.13) together clearly give us the limit

$$\lim_{\nu\to\infty}\frac{1}{\log(\nu+1)}\sum_{\ell=1}^{\nu} s_\ell(T_{s,t}) = \mu(R_{s,t}).$$

Therefore for every Dixmier trace Tr_ω we have $\operatorname{Tr}_\omega(T_{s,t}) = \mu(R_{s,t})$. This completes the proof of (3.9).

Clearly, a consequence of (3.9) and (3.2) is that

$$\operatorname{Tr}_\omega\left((\log 4)M\bigoplus_{k=k_0}^{\infty}4^{-k}\bigoplus_{(s_{i;k},t_{i,j;k})\in L}u_{i,j;k}\otimes u_{i,j;k}\right)=0 \tag{3.14}$$

if L is either a vertical line or a horizontal line.

We are now ready to verify condition (4) for our pair A, B. By (2.14), (2.15) and (2.16), to verify (4), it suffices to show that for all $m,n\in\mathbf{Z}_+$, we have

$$\operatorname{Tr}_\omega(A^mB^n[B,A])=\frac{1}{2\pi\sqrt{-1}}\iint_R x^my^n d\mu(x,y). \tag{3.15}$$

Recalling (3.7) and (3.8), for each pair of $m,n\in\mathbf{Z}_+$ we have

$$A^mB^n[B,A]=\frac{M\log 4}{2\pi\sqrt{-1}}\bigoplus_{k=k_0}^{\infty}4^{-k}\bigoplus_{i=1}^{2^k}\bigoplus_{j=1}^{2^k}A^m_{i,j;k}B^n_{i,j;k}u_{i,j;k}\otimes u_{i,j;k}.$$

By (3.6), given a pair of $m,n\in\mathbf{Z}_+$, there is a constant $C_{m,n}$ such that

$$\|A^m_{i,j;k}B^n_{i,j;k}u_{i,j;k}\otimes u_{i,j;k}-s^m_{i;k}t^n_{i,j;k}u_{i,j;k}\otimes u_{i,j;k}\|\le C_{m,n}2^{-k}$$

for every triple of $k\ge k_0$ and $i,j\in\{1,\dots,2^k\}$. Therefore if we define

$$Z^{(m,n)}=(\log 4)M\bigoplus_{k=k_0}^{\infty}4^{-k}\bigoplus_{i=1}^{2^k}\bigoplus_{j=1}^{2^k}s^m_{i;k}t^n_{i,j;k}u_{i,j;k}\otimes u_{i,j;k},$$

then $A^mB^n[B,A]-(2\pi\sqrt{-1})^{-1}Z^{(m,n)}$ is in the trace class. Thus the proof of (3.15) is now reduced to that of

$$\operatorname{Tr}_\omega(Z^{(m,n)})=\iint_R x^my^n d\mu(x,y). \tag{3.16}$$

To prove this, pick a large $\nu\in\mathbf{N}$ and divide the intervals $[a,b]$ and $[c,d]$ equally into ν potions. That is, we have

$$a=x_0<x_1<\cdots<x_\nu=b \quad\text{and}\quad c=y_0<y_1<\cdots<y_\nu=d$$

with $x_r=a+(r/\nu)(b-a)$ and $y_r=c+(r/\nu)(d-c)$ for every $0\le r\le\nu$. For every pair of $r,\ell\in\{1,\dots,\nu\}$, we define the rectangle

$$W_{r,\ell}=(x_{r-1},x_r]\times(y_{\ell-1}y_\ell].$$

Accordingly, we define the operators

$$S_{r,\ell} = (\log 4)M \bigoplus_{k=k_0}^{\infty} 4^{-k} \bigoplus_{(s_{i;k},t_{i,j;k})\in W_{r,\ell}} u_{i,j;k} \otimes u_{i,j;k} \quad \text{and}$$

$$Z_{r,\ell}^{(m,n)} = (\log 4)M \bigoplus_{k=k_0}^{\infty} 4^{-k} \bigoplus_{(s_{i;k},t_{i,j;k})\in W_{r,\ell}} s_{i;k}^m t_{i,j;k}^n u_{i,j;k} \otimes u_{i,j;k},$$

$r, \ell \in \{1, \dots, \nu\}$. From (3.9) and (3.14) it is easy to deduce that

$$\mathrm{Tr}_\omega(S_{r,\ell}) = \mu(W_{r,\ell}) \tag{3.17}$$

for all $r, \ell \in \{1, \dots, \nu\}$. If $(s_{i;k}, t_{i,j;k}) \in W_{r,\ell}$, then $|s_{i;k}^m t_{i,j;k}^n - x_r^m y_\ell^n| = O(1/\nu)$. Hence

$$|\mathrm{Tr}_\omega(Z_{r,\ell}^{(m,n)}) - x_r^m y_\ell^n \mathrm{Tr}_\omega(S_{r,\ell})| = O(1/\nu)\mathrm{Tr}_\omega(S_{r,\ell}) \tag{3.18}$$

for all $r, \ell \in \{1, \dots, \nu\}$. Combining (3.17) and (3.18), we now have

$$\begin{aligned}\mathrm{Tr}_\omega(Z^{(m,n)}) = \sum_{r,\ell=1}^{\nu} \mathrm{Tr}_\omega(Z_{r,\ell}^{(m,n)}) &= \sum_{r,\ell=1}^{\nu} (x_r^m y_\ell^n + O(1/\nu))\mathrm{Tr}_\omega(S_{r,\ell}) \\ &= \left(\sum_{r,\ell=1}^{\nu} x_r^m y_\ell^n \mu(W_{r,\ell}) \right) + O(1/\nu)\mu(R).\end{aligned}$$

Since this holds for arbitrarily large $\nu \in \mathbf{N}$, (3.16) follows. This completes the proof. □

Proposition 3.2. *Let μ be a compactly-supported regular Borel measure on $\mathbf{R}^2$. Suppose that μ has no point masses. Then there exist bounded self-adjoint operators A, B satisfying the following three conditions*:
(1) $[A, B] \in \mathcal{C}_1^+$ *with* $\|[A, B]\|_1^+ \le 2\pi^{-1}(\log 4)\mu(\mathbf{R}^2)$.
(2) $i[B, A] \ge 0$.
(3) *For every Dixmier trace* Tr_ω *and every pair of* $p, q \in \mathbf{C}[x, y]$, *we have*

$$\mathrm{Tr}_\omega([p(A, B), q(A, B)]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) d\mu(x, y).$$

Proof. For each $\theta \in [0, \pi)$, let $\mathcal{R}_\theta$ be the counter-clockwise rotation of $\mathbf{R}^2$ of the angle θ. We define the measure μ_θ on $\mathbf{R}^2$ by the formula

$$\mu_\theta(E) = \mu(\mathcal{R}_\theta E)$$

for every Borel set $E \subset \mathbf{R}^2$. Then

$$\iint f(\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y) d\mu_\theta(x, y) = \iint f(x, y) d\mu(x, y) \tag{3.19}$$

for every $f \in C(\mathbf{R}^2)$. Since μ has no point masses, there are at most countably many lines L in $\mathbf{R}^2$ that have the property that $\mu(L) \neq 0$. Since $[0, \pi)$ is uncountable, there is a $\theta_0 \in [0, \pi)$ such that the measure μ_{θ_0} has the property that

$$\mu_{\theta_0}(L) = 0$$

for every vertical line L in $\mathbf{R}^2$ and every horizontal line L in $\mathbf{R}^2$. That is, μ_{θ_0} has property (3.2) required in Lemma 3.1. Thus by Lemma 3.1, there is a pair of bounded self-adjoint operators A_{θ_0}, B_{θ_0} satisfying the following three conditions:
(α) $[A_{\theta_0}, B_{\theta_0}] \in \mathcal{C}_1^+$ with $\|[A_{\theta_0}, B_{\theta_0}]\|_1^+ \leq 2\pi^{-1}(\log 4)\mu_{\theta_0}(\mathbf{R}^2) = 2\pi^{-1}(\log 4)\mu(\mathbf{R}^2)$.
(β) $i[B_{\theta_0}, A_{\theta_0}] \geq 0$.
(γ) For every Dixmier trace Tr_ω and every pair of $p, q \in \mathbf{C}[x, y]$, we have

$$\mathrm{Tr}_\omega([p(A_{\theta_0}, B_{\theta_0}), q(A_{\theta_0}, B_{\theta_0})]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) d\mu_{\theta_0}(x, y).$$

From (2.15) and (γ) we obtain

$$\mathrm{Tr}_\omega(A_{\theta_0}^j B_{\theta_0}^k [B_{\theta_0}, A_{\theta_0}]) = \frac{1}{2\pi i} \iint x^j y^k d\mu_{\theta_0}(x, y) \tag{3.20}$$

for all $j, k \in \mathbf{Z}_+$. We now define the self-adjoint operators

$$A = \cos\theta_0 A_{\theta_0} - \sin\theta_0 B_{\theta_0} \quad \text{and} \quad B = \sin\theta_0 A_{\theta_0} + \cos\theta_0 B_{\theta_0}.$$

Then

$$[A, B] = [A_{\theta_0}, B_{\theta_0}]. \tag{3.21}$$

Thus (1) and (2) follow from (α) and (β) respectively. It follows from (3.21), Lemma 2.1, (3.20) and (3.19) that for every pair of $m, n \in \mathbf{Z}_+$, we have

$$\begin{aligned}
\mathrm{Tr}_\omega(A^m B^n [B, A]) &= \mathrm{Tr}_\omega((\cos\theta_0 A_{\theta_0} - \sin\theta_0 B_{\theta_0})^m (\sin\theta_0 A_{\theta_0} + \cos\theta_0 B_{\theta_0})^n [B_{\theta_0}, A_{\theta_0}]) \\
&= \frac{1}{2\pi i} \iint (\cos\theta_0 x - \sin\theta_0 y)^m (\sin\theta_0 x + \cos\theta_0 y)^n d\mu_{\theta_0}(x, y) \\
&= \frac{1}{2\pi i} \iint x^m y^n d\mu(x, y).
\end{aligned}$$

By (2.14), (2.15) and (2.16), this implies condition (3) for the pair A, B. This completes the proof. □

Proposition 3.3. *Consider the measure $M\delta_\xi$, where $0 < M < \infty$, $\xi = (u, v)$ is a point in $\mathbf{R}^2$, and δ_ξ is the unit point mass at ξ. Let $\epsilon > 0$ be given. Then there exist bounded self-adjoint operators A, B satisfying the following four conditions*:
(1) *The spectra of A, B are contained in the intervals $[u, u + \epsilon]$ and $[v, v + \epsilon]$ respectively.*
(2) *$[A, B] \in \mathcal{C}_1^+$ with $\|[A, B]\|_1^+ \leq (2\pi)^{-1}M$.*

(3) $i[B, A] \geq 0$.
(4) *For every Dixmier trace* Tr_ω *and every pair of* $p, q \in \mathbf{C}[x, y]$, *we have*

$$\mathrm{Tr}_\omega([p(A,B), q(A,B)]) = \frac{iM}{2\pi} \iint \{p, q\}(x, y) d\delta_\xi(x, y) = \frac{iM}{2\pi}\{p, q\}(u, v).$$

Proof. This follows the idea in the proof of Lemma 3.1, but the actual construction is much simpler. Let $k_0 \in \mathbf{N}$ be such that $(M/k_0)^{1/2} < \epsilon$. Define the Hilbert space

$$\mathcal{H} = \bigoplus_{k=k_0}^{\infty} \mathcal{H}_k, \tag{3.22}$$

where

$$\mathcal{H}_k = L^2(I)$$

for every $k \geq k_0$. Thus, again, the subscript on the right-hand side of (3.22) is just for tracking the orthogonal summands. On each $\mathcal{H}_k$, $k \geq k_0$, we define

$$A_k = u + (M/k)^{1/2} U \quad \text{and} \quad B_k = v + (M/k)^{1/2} V.$$

We then define the operators

$$A = \bigoplus_{k=k_0}^{\infty} A_k \quad \text{and} \quad B = \bigoplus_{k=k_0}^{\infty} B_k$$

on $\mathcal{H}$. Obviously, condition (1) is ensured by the choice $(M/k_0)^{1/2} < \epsilon$. For each $k \geq k_0$, let u_k denote the unit vector χ_I in $\mathcal{H}_k$. By (3.1) we have

$$[B, A] = \frac{M}{2\pi i} \bigoplus_{k=k_0}^{\infty} \frac{1}{k} u_k \otimes u_k.$$

Thus conditions (2) and (3) are obviously satisfied.

To verify condition (4), we introduce the operators

$$X_k = (M/k)^{1/2} U \quad \text{and} \quad Y_k = (M/k)^{1/2} V$$

on $\mathcal{H}_k$, $k \geq k_0$. We then define the operators

$$X = \bigoplus_{k=k_0}^{\infty} X_k \quad \text{and} \quad Y = \bigoplus_{k=k_0}^{\infty} Y_k$$

on $\mathcal{H}$. Then $A = u + X$ and $B = v + Y$. Observe that the operators $X[B, A]$ and $Y[B, A]$ are in the trace class. Therefore for any $m, n \in \mathbf{Z}_+$, we have

$$A^m B^n [B, A] = u^m v^n [B, A] + T_{m,n},$$

where $T_{m,n}$ is in the trace class. Thus for every Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(A^m B^n [B, A]) = u^m v^n \mathrm{Tr}_\omega([B, A]) = \frac{M}{2\pi i} u^m v^n = \frac{M}{2\pi i} \iint x^m y^n d\delta_\xi(x, y),$$

$m, n \in \mathbf{Z}_+$. By (2.14), (2.15) and (2.16), this implies condition (4) for the pair A, B. This completes the proof. □

With the above preparation, we can now show that every compactly-supported regular Borel measure on $\mathbf{R}^2$ is the principal measure for some pair of self-adjoint operators.

Theorem 3.4. *Let μ be a compactly-supported regular Borel measure on $\mathbf{R}^2$. Then there exists a pair of bounded self-adjoint operators A, B satisfying the following three conditions*:
(1) $[A, B] \in \mathcal{C}_1^+$ *with* $\|[A, B]\|_1^+ \le 2\pi^{-1}(\log 4)\mu(\mathbf{R}^2)$.
(2) $i[B, A] \ge 0$.
(3) *For every Dixmier trace* Tr_ω *and every pair of* $p, q \in \mathbf{C}[x, y]$, *we have*

$$\mathrm{Tr}_\omega([p(A, B), q(A, B)]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) d\mu(x, y).$$

Proof. There is a (possibly empty) subset E of $\mathbf{N}$ with which μ has the decomposition

$$\mu = \mu_0 + \sum_{k \in E} \mu_k,$$

where μ_0 has no point masses and for each $k \in E$, μ_k is a mass at a single point $\xi_k \in \mathbf{R}^2$. We apply Proposition 3.2 to μ_0 and Proposition 3.3 with $\epsilon = 1$ to μ_k for each $k \in E$. Thus we have a pair of bounded self-adjoint operators A_k, B_k for each $k \in \{0\} \cup E$ such that the following hold true:
(a) $[A_k, B_k] \in \mathcal{C}_1^+$ with $\|[A_k, B_k]\|_1^+ \le 2\pi^{-1}(\log 4)\mu_k(\mathbf{R}^2)$.
(b) $i[B_k, A_k] \ge 0$.
(c) For every Dixmier trace Tr_ω and every pair of $p, q \in \mathbf{C}[x, y]$, we have

$$\mathrm{Tr}_\omega([p(A_k, B_k), q(A_k, B_k)]) = \frac{i}{2\pi} \iint \{p, q\}(x, y) d\mu_k(x, y).$$

Define the operators

$$A = \bigoplus_{k \in \{0\} \cup E} A_k \quad \text{and} \quad B = \bigoplus_{k \in \{0\} \cup E} B_k.$$

One then easily deduces (1), (2) and (3) from (a), (b) and (c). □

4. Spectral shift for Dixmier trace

Having established the analogue of principal function in the case of Dixmier trace, it is logical to take a look at possible analogue of Krein's spectral shift, which plays an extremely important role in the perturbation theory of self-adjoint operators. We begin with a review of the classic case.

Suppose that A is a bounded self-adjoint operator and K is a self-adjoint operator of the trace class. A famous theorem of Krein tells us that there is a $\xi \in L^1(\mathbf{R})$ with a support contained in a finite interval $[a, b]$ such that for every $z \in \mathbf{C}\backslash\mathbf{R}$,

$$\mathrm{tr}((A + K - z)^{-1} - (A - z)^{-1}) = \int \frac{-1}{(t - z)^2}\xi(t)dt.$$

If we further assume that K is non-negative, then so is ξ. See [18,19,1,27] for these well-known facts.

In this section we will establish the analogue of spectral shift for Dixmier trace. We will see that for Dixmier trace, instead of $\xi(t)dt$, we get a real-valued Borel measure $d\mu(t)$ in the formula. We consider this measure $d\mu(t)$ as the analogue of spectral shift in the setting of Dixmier trace. Later we will show that every regular Borel measure $d\mu(t)$ on a finite interval $[a, b]$, $a < b$, is the spectral shift with respect to Dixmier trace for a pair of self-adjoint operators. Moreover, the A in the pair can be *any* self-adjoint operator whose spectrum contains $[a, b]$. At a fundamental level, this phenomenon reflects the fact that Dixmier trace is a singular trace.

Lemma 4.1. *Let K be a non-negative self-adjoint operator in $\mathcal{C}_1^+$. Then for any bounded operator S we have $K^{1/2}SK^{1/2} \in \mathcal{C}_1^+$ and*

$$\mathrm{Tr}_\omega(K^{1/2}SK^{1/2}) = \mathrm{Tr}_\omega(SK)$$

for each Dixmier trace Tr_ω.

Proof. By linearity, it suffices to consider the case where S is a non-negative self-adjoint operator. For such an S, define the operator $T = K^{1/2}S^{1/2}$. We have $T^*T = S^{1/2}KS^{1/2} \in \mathcal{C}_1^+$ by assumption. It is well known that T^*T and TT^* have identical s-numbers. Therefore $K^{1/2}SK^{1/2} = TT^* \in \mathcal{C}_1^+$ and by the properties of Dixmier trace,

$$\mathrm{Tr}_\omega(K^{1/2}SK^{1/2}) = \mathrm{Tr}_\omega(TT^*) = \mathrm{Tr}_\omega(T^*T) = \mathrm{Tr}_\omega(S^{1/2}KS^{1/2}) = \mathrm{Tr}_\omega(SK).$$

This completes the proof. □

Proposition 4.2. *Let A and K be self-adjoint operators on a Hilbert space $\mathcal{H}$. Suppose that the spectrum of A is contained in the interval $[a, b]$ for some $-\infty < a < b < \infty$. Furthermore, suppose that $K \in \mathcal{C}_1^+$ and that K is non-negative. Let a Dixmier trace* Tr_ω *also be given. Then there is a regular Borel measure μ on $[a, b]$ such that*

$$\mathrm{Tr}_\omega((A + K - z)^{-1} - (A - z)^{-1}) = \int \frac{-1}{(t - z)^2}d\mu(t)$$

for every $z \in \mathbf{C}\backslash\mathbf{R}$. Moreover, $\mu([a, b]) = \mathrm{Tr}_\omega(K)$.

Proof. For any $z \in \mathbf{C}\backslash\mathbf{R}$, we have

$$\begin{aligned}(A+K-z)^{-1}-(A-z)^{-1} &= -(A+K-z)^{-1}K(A-z)^{-1}\\ &= -(A-z)^{-1}K(A-z)^{-1}+R,\end{aligned}$$

where

$$R=(A+K-z)^{-1}K(A-z)^{-1}K(A-z)^{-1}.$$

Let a Dixmier trace Tr_ω be given. Then by Lemma 2.1, $\mathrm{Tr}_\omega(R)=0$. Consequently,

$$\begin{aligned}\mathrm{Tr}_\omega((A+K-z)^{-1}-(A-z)^{-1}) &= -\mathrm{Tr}_\omega((A-z)^{-1}K(A-z)^{-1})\\ &= -\mathrm{Tr}_\omega((A-z)^{-2}K).\end{aligned} \tag{4.1}$$

By assumption, we have the spectral decomposition

$$(A-z)^{-2}=\int_a^b \frac{1}{(t-z)^2}dE_t$$

for A, where dE_t is its spectral measure. For each $m \in \mathbf{N}$, we define the intervals

$$I_{1,m}=\left[a, a+\frac{1}{m}(b-a)\right] \quad \text{and} \quad I_{j,m}=\left(a+\frac{j-1}{m}(b-a), a+\frac{j}{m}(b-a)\right]$$

if $1<j\le m$. Pick a $t_{j,m}\in I_{j,m}$ for every pair of $m\in\mathbf{N}$ and $1\le j\le m$. Define

$$T_m=\sum_{j=1}^m \frac{1}{(t_{j,m}-z)^2}E(I_{j,m}), \tag{4.2}$$

$m\in\mathbf{N}$. Then $\|(A-z)^{-2}-T_m\|\to 0$ as $m\to\infty$. Consequently, $\|(A-z)^{-2}K-T_mK\|_1^+\le \|(A-z)^{-2}-T_m\|\|K\|_1^+\to 0$ as $m\to\infty$ and

$$\begin{aligned}\mathrm{Tr}_\omega((A-z)^{-2}K) &= \lim_{m\to\infty}\mathrm{Tr}_\omega(T_mK)=\lim_{m\to\infty}\sum_{j=1}^m\frac{1}{(t_{j,m}-z)^2}\mathrm{Tr}_\omega(E(I_{j,m})K)\\ &= \lim_{m\to\infty}\sum_{j=1}^m\frac{1}{(t_{j,m}-z)^2}\mathrm{Tr}_\omega(K^{1/2}E(I_{j,m})K^{1/2}),\end{aligned} \tag{4.3}$$

where the second = follows from (4.2) and the third = from Lemma 4.1.

Let us consider dE_t as a spectral measure on $\mathbf{R}$ by setting $E(\mathbf{R}\backslash[a,b])=0$. For $s<t$ in $\mathbf{R}$, the operator inequality $E(-\infty,s]\le E(-\infty,t]$ implies the operator inequality

$$K^{1/2}E((-\infty,s])K^{1/2}\le K^{1/2}E((-\infty,t])K^{1/2}.$$

Thus we have a non-decreasing function

$$t\mapsto \mathrm{Tr}_\omega(K^{1/2}E((-\infty,t])K^{1/2}) \tag{4.4}$$

on $\mathbf{R}$, and from (4.3) we obtain

$$\mathrm{Tr}_\omega((A-z)^{-2}K) = \int \frac{1}{(t-z)^2} d\mathrm{Tr}_\omega(K^{1/2}E((-\infty,t])K^{1/2}), \tag{4.5}$$

where the right-hand side is a Riemann-Stieltjes integral. The non-decreasing function in (4.4) is a constant on each of the intervals $(-\infty,a)$ and (b,∞). Thus we can restate the above as

$$\mathrm{Tr}_\omega((A-z)^{-2}K) = \int_a^b \frac{1}{(t-z)^2} d\mu(t), \tag{4.6}$$

where μ is a positive, regular Borel measure on $[a,b]$ with

$$\mu([a,b]) = \mathrm{Tr}_\omega(K^{1/2}E(\mathbf{R})K^{1/2}) = \mathrm{Tr}_\omega(K^{1/2}\cdot 1\cdot K^{1/2}) = \mathrm{Tr}_\omega(K).$$

Combining (4.6) with (4.1), the proof is now complete. □

Proposition 4.3. *Let A and K be self-adjoint operators on a Hilbert space $\mathcal{H}$. Suppose that the spectrum of A is contained in the interval $[a,b]$ for some $-\infty < a < b < \infty$. Furthermore, suppose that $K \in \mathcal{C}_1^+$ and that K is non-negative. Let a Dixmier trace Tr_ω also be given. Let μ be the regular Borel measure provided by Proposition 4.2 for this triple of A, $A+K$* and Tr_ω. *Then*

$$\mathrm{Tr}_\omega(\eta(A+K)-\eta(A)) = \int \eta'(t)d\mu(t)$$

for every $\eta \in C_c^\infty(\mathbf{R})$.

Proof. This follows from Proposition 4.2 by a well-known argument, as follows. First of all, it is obvious that the map

$$z \mapsto -(A+K-z)^{-1}K(A-z)^{-1} = (A+K-z)^{-1} - (A-z)^{-1}$$

from $\mathbf{C}\backslash\mathbf{R}$ into $\mathcal{C}_1^+$ is continuous with respect to the norm $\|\cdot\|_1^+$. Since μ is supported in the finite interval $[a,b]$, combining the identity in Proposition 4.2 with this $\|\cdot\|_1^+$-continuity and with contour integration, we have

$$\mathrm{Tr}_\omega((A+K)^k - A^k) = k\int t^{k-1}d\mu(t) \tag{4.7}$$

for every integer $k \geq 1$. It is easy to see that there is a $0 < C < \infty$ such that

$$\|(A+K)^k - A^k\|_1^+ \leq 2^k C^k$$

for every $k \geq 1$. Using this bound and the power series expansion for the exponential function, from (4.7) we deduce that for every $x \in \mathbf{R}$,

$$\mathrm{Tr}_\omega(e^{ix(A+K)} - e^{ixA}) = ix\int e^{ixt}d\mu(t).$$

By the fundamental theorem of calculus,

$$e^{ix(A+K)} - e^{ixA} = \int_0^1 \frac{d}{ds} e^{isx(A+K)} e^{i(1-s)xA} ds = ix \int_0^1 e^{isx(A+K)} K e^{i(1-s)xA} ds. \tag{4.8}$$

Therefore

$$\|e^{ix(A+K)} - e^{ixA}\|_1^+ \le |x| \|K\|_1^+ \tag{4.9}$$

for every $x \in \mathbf{R}$. Given any $\eta \in C_c^\infty(\mathbf{R})$, let $\hat{\eta}$ denote its Fourier transform. By the Fourier inversion formula and the spectral decompositions of the self-adjoint operators $A + K$ and A, we have

$$\eta(A+K) - \eta(A) = \frac{1}{\sqrt{2\pi}} \int \{e^{ix(A+K)} - e^{ixA}\} \hat{\eta}(x) dx.$$

Using (4.8) and (4.9), it is straightforward to show that the integral on the right-hand side is the limit with respect to the norm $\| \cdot \|_1^+$ of the corresponding Riemann sums. This $\| \cdot \|_1^+$-convergence allows us to switch the order of taking Tr_ω and integration to obtain

$$\begin{aligned} \mathrm{Tr}_\omega(\eta(A+K) - \eta(A)) &= \frac{1}{\sqrt{2\pi}} \int \mathrm{Tr}_\omega(e^{ix(A+K)} - e^{ixA}) \hat{\eta}(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int ix \int e^{ixt} d\mu(t) \hat{\eta}(x) dx \\ &= \int \eta'(t) d\mu(t). \end{aligned}$$

This completes the proof. □

Once Proposition 4.3 is proven, we can drop the condition that K be non-negative, again by a well-known argument:

Theorem 4.4. *Let A and K be self-adjoint operators on a Hilbert space $\mathcal{H}$. Suppose that A is bounded and that $K \in \mathcal{C}_1^+$. Let any Dixmier trace Tr_ω be given. Then there is a real-valued regular Borel measure μ supported on a finite interval in $\mathbf{R}$ such that*

$$\mathrm{Tr}_\omega(\eta(A+K) - \eta(A)) = \int \eta'(t) d\mu(t)$$

for every $\eta \in C_c^\infty(\mathbf{R})$. Moreover, the total variation of μ does not exceed $\mathrm{Tr}_\omega(|K|)$.

Proof. Given any self-adjoint $K \in \mathcal{C}_1^+$, there are non-negative, self-adjoint operators K_1 and K_2 in $\mathcal{C}_1^+$ such that $K_1 - K_2 = K$ and $K_1 + K_2 = |K|$. For any $\eta \in C_c^\infty(\mathbf{R})$, we have

$$\eta(A+K) - \eta(A) = \{\eta(A+K_1) - \eta(A)\} - \{\eta(A+K+K_2) - \eta(A+K)\}.$$

Applying Proposition 4.3 to the two brackets on the right, the theorem follows. □

5. Prescribing spectral shift

From now on, all Hilbert spaces are assumed to be separable and infinite dimensional. We will show that we can prescribe arbitrary spectral shift for Dixmier trace by choosing the right K. We will accomplish this goal in several steps.

First of all, the spirit of Section 3 also applies to the construction of spectral shift. That is, if A_1, A_2 are bounded self-adjoint operators and if K_1 and K_2 are self-adjoint operators in $\mathcal{C}_1^+$, then for every Dixmier trace Tr_ω and every $z \in \mathbf{C}\backslash\mathbf{R}$,

$$\mathrm{Tr}_\omega(((A_1 \oplus A_2)+(K_1 \oplus K_2)-z)^{-1}-((A_1 \oplus A_2)-z)^{-1}) = \sum_{i=1}^{2} \mathrm{Tr}_\omega((A_i+K_i-z)^{-1}-(A_i-z)^{-1}).$$

Thus we can construct spectral shifts additively.

Proposition 5.1. *Consider any $-\infty < a < b < \infty$. Let μ be a regular Borel measure on $[a, b]$ that has no point masses. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ satisfying the following two conditions*:

(1) *A has a pure point spectrum.*

(2) *The spectrum of A contains $[a, b]$.*

Then there is a non-negative, self-adjoint operator $K \in \mathcal{C}_1^+$ such that for every Dixmier trace Tr_ω and every $z \in \mathbf{C}\backslash\mathbf{R}$, we have

$$\mathrm{Tr}_\omega((A + K - z)^{-1} - (A - z)^{-1}) = \int \frac{-1}{(t-z)^2} d\mu(t). \tag{5.1}$$

Moreover, K satisfies the estimate $\|K\|_1^+ \le 2(\log 2)\mu([a, b])$.

Proof. Write $M = \mu([a, b])$, the total mass of μ. We assume that $M > 0$. For each $k \in \mathbf{N}$, since μ has no point masses, there are

$$a \le t_{0,k} < t_{1,k} < t_{2,k} < \cdots < t_{2^k-1,k} < t_{2^k,k} \le b$$

such that

$$\mu([a, t_{j,k}]) = \frac{j}{2^k} M \quad \text{for every} \quad 0 \le j \le 2^k. \tag{5.2}$$

Note that such a $t_{j,k}$ may not be unique, but any $t_{j,k}$ satisfying (5.2) will do. By conditions (1), (2) and the observation preceding the proposition, we only need to consider

$$A = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i,$$

where $\{e_i : i \in \mathbf{N}\}$ is an orthonormal basis for $\mathcal{H}$, $\lambda_i \ne \lambda_{i'}$ if $i \ne i'$, and $\{\lambda_i : i \in \mathbf{N}\}$ is a dense subset of (a, b). Thus $\{\lambda_i : i \in \mathbf{N}\}$ contains pairwise disjoint subsets

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_k \ldots$$

such that for each $k \in \mathbf{N}$,

$$\Lambda_k = \{\lambda_{i(1,k)}, \lambda_{i(2,k)}, \ldots, \lambda_{i(2^k,k)}\},$$

where the elements satisfy the condition

$$t_{j-1,k} < \lambda_{i(j,k)} < t_{j,k} \quad \text{for every } \ 1 \le j \le 2^k. \tag{5.3}$$

The point is that this defines the subscript $i(j,k)$ for every pair of $k \in \mathbf{N}$ and $1 \le j \le 2^k$. With these $i(j,k)$ so chosen, we now define

$$K = \sum_{k=1}^{\infty} \frac{M \log 2}{2^k} \sum_{j=1}^{2^k} e_{i(j,k)} \otimes e_{i(j,k)}. \tag{5.4}$$

It is easy to see that if $2^k \le \nu < 2^{k+1}$ for some $k \in \mathbf{N}$, then

$$s_\nu(K) \le \frac{(\log 2)M}{2^k} < \frac{2(\log 2)M}{\nu}.$$

It follows that $\|K\|_1^+ \le 2(\log 2)M$. What remains it to show that (5.1) holds for this K.

Let E denote the spectral measure for A. Obviously, for each $t \in \mathbf{R}$,

$$E((-\infty, t]) = \sum_{\lambda_i \le t} e_i \otimes e_i.$$

By (4.5) and (4.1), to verify (5.1) for the K defined by (5.4), it suffices to show that

$$\mathrm{Tr}_\omega(K^{1/2} E((-\infty, \tau]) K^{1/2}) = \mu([a, \tau]) \tag{5.5}$$

for every $a \le \tau \le b$. Let such a τ be given.

(i) First, we consider the case where $0 < \mu([a,\tau]) < M$. There is a $k_0 \in \mathbf{N}$ such that if $k \ge k_0$, then $2^{-k}M < \mu([a,\tau])$ and $\mu([a,\tau]) + 2^{-k+1}M < M$. Thus for each $k \ge k_0$, there is a $1 \le J(k) < 2^k - 2$ such that

$$\frac{J(k)}{2^k} M < \mu([a,\tau]) \le \frac{J(k)+1}{2^k} M. \tag{5.6}$$

By (5.2), this means

$$t_{J(k),k} < \tau < t_{J(k)+2,k}. \tag{5.7}$$

By (5.3) and (5.4), we have

$$K^{1/2} E((-\infty, \tau]) K^{1/2} = E((-\infty, \tau]) K \ge G,$$

where we denote

$$G=\sum_{k=k_0}^{\infty}\frac{M\log 2}{2^k}\sum_{j=1}^{J(k)}e_{i(j,k)}\otimes e_{i(j,k)}.$$

Consider any natural number $m\geq 2^{k_0}$. Then there is a $k(m)\geq k_0$ such that

$$\sum_{k=k_0}^{k(m)}J(k)\leq m<\sum_{k=k_0}^{k(m)+1}J(k). \tag{5.8}$$

Thus

$$\frac{1}{\log(m+1)}\sum_{\nu=1}^{m}s_\nu(G)\geq\frac{\log 2}{\log\left(2^{k(m)+2}\right)}\sum_{k=k_0}^{k(m)}\frac{J(k)}{2^k}M. \tag{5.9}$$

By the upper bound in (5.6), we have

$$\sum_{k=k_0}^{k(m)}\frac{J(k)}{2^k}M\geq(k(m)-k_0)\mu([a,\tau])-M.$$

Substituting this in (5.9), for sufficiently large m we have

$$\frac{1}{\log(m+1)}\sum_{\nu=1}^{m}s_\nu(G)\geq\frac{(\log 2)\{(k(m)-k_0)\mu([a,\tau])-M\}}{(\log 2)\{k(m)+2\}}.$$

Thus

$$\begin{aligned}\mathrm{Tr}_\omega(K^{1/2}E((-\infty,\tau])K^{1/2})&\geq\mathrm{Tr}_\omega(G)\\&\geq\liminf_{m\to\infty}\frac{1}{\log(m+1)}\sum_{\nu=1}^{m}s_\nu(G)\geq\mu([a,\tau]).\end{aligned} \tag{5.10}$$

On the other hand, by (5.7), (5.3) and (5.4), we have

$$K^{1/2}E((-\infty,\tau])K^{1/2}=E((-\infty,\tau])K\leq F+H$$

where $\mathrm{rank}(F)<\infty$ and where we denote

$$H=\sum_{k=k_0}^{\infty}\frac{M\log 2}{2^k}\sum_{j=1}^{J(k)+2}e_{i(j,k)}\otimes e_{i(j,k)}.$$

Using (5.8) again, for sufficiently large m we have

$$\frac{1}{\log(m+1)}\sum_{\nu=1}^{m}s_\nu(H)\leq\frac{\log 2}{\log\left(\sum_{k=k_0}^{k(m)}J(k)\right)}\sum_{k=k_0}^{k(m)+1}\frac{J(k)+2}{2^k}M. \tag{5.11}$$

By the lower bound in (5.6), we have

$$\sum_{k=k_0}^{k(m)+1} \frac{J(k)+2}{2^k} M \le \{k(m)+1\}\mu([a,\tau]) + 2M.$$

By the upper bound in (5.6), we have

$$\sum_{k=k_0}^{k(m)} J(k) \ge J(k(m)) \ge \frac{\mu([a,\tau])}{M} 2^{k(m)} - 1.$$

Substituting these in (5.11), we find that for sufficiently large m, we have

$$\frac{1}{\log(m+1)} \sum_{\nu=1}^{m} s_\nu(H) \le \frac{(\log 2)(\{k(m)+1\}\mu([a,\tau]) + 2M)}{\log\big(\{\mu([a,\tau])/M\}2^{k(m)} - 1\big)}.$$

Thus

$$\begin{aligned}\mathrm{Tr}_\omega(K^{1/2}E((-\infty,\tau])K^{1/2}) &\le \mathrm{Tr}_\omega(F+H) = \mathrm{Tr}_\omega(H)\\ &\le \limsup_{m\to\infty} \frac{1}{\log(m+1)} \sum_{\nu=1}^{m} s_\nu(H) \le \mu([a,\tau]).\end{aligned}$$

Combining this with (5.10), (5.5) is proved in the case where $0 < \mu([a,\tau]) < M$.

(ii) Let us consider the case where $\mu([a,\tau]) = 0$. By (5.2) and (5.3), we have $\tau < \lambda_{i(j,k)}$ for every pair of $k \in \mathbf{N}$ and $2 \le j \le 2^k$. Therefore

$$K^{1/2}E((-\infty,\tau])K^{1/2} = E((-\infty,\tau])K \le X,$$

where

$$X = \sum_{k=1}^{\infty} \frac{M \log 2}{2^k} e_{i(1,k)} \otimes e_{i(1,k)}.$$

Obviously, X is a trace-class operator. Thus

$$\mathrm{Tr}_\omega(K^{1/2}E((-\infty,\tau])K^{1/2}) \le \mathrm{Tr}_\omega(X) = 0$$

as promised.

(iii) Finally, we consider the case where $\mu([a,\tau]) = M$. It suffices to show that

$$\mathrm{Tr}_\omega(K^{1/2}E((-\infty,\tau])K^{1/2}) = \mathrm{Tr}_\omega(K) = M. \tag{5.12}$$

By (5.2) and (5.3), we have $\lambda_{i(j,k)} < \tau$ for every pair of $k \in \mathbf{N}$ and $1 \le j \le 2^k - 1$. Thus

$$K^{1/2}E((-\infty,\tau])K^{1/2} = E((-\infty,\tau])K \ge Y,$$

where

$$Y = \sum_{k=1}^{\infty} \frac{M \log 2}{2^k} \sum_{j=1}^{2^k - 1} e_{i(j,k)} \otimes e_{i(j,k)}.$$

Obviously, $K - Y$ is a trace-class operator. Therefore

$$\mathrm{Tr}_\omega(K) \geq \mathrm{Tr}_\omega(K^{1/2} E((-\infty, \tau]) K^{1/2}) \geq \mathrm{Tr}_\omega(Y) = \mathrm{Tr}_\omega(K). \tag{5.13}$$

Thus we need to compute $\mathrm{Tr}_\omega(K)$, which is an easier version of the computation in (i). For each sufficiently large $m \in \mathbf{N}$, there is a $\kappa(m) \in \mathbf{N}$ such that

$$\sum_{k=1}^{\kappa(m)} 2^k \leq m < \sum_{k=1}^{\kappa(m)+1} 2^k.$$

Therefore, by (5.4),

$$\frac{(\log 2) M \kappa(m)}{\log\left(2^{\kappa(m)+2}\right)} \leq \frac{1}{\log(m+1)} \sum_{\nu=1}^{m} s_\nu(K) \leq \frac{(\log 2) M \{\kappa(m) + 1\}}{\log\left(2^{\kappa(m)}\right)}.$$

This obviously implies

$$\lim_{m \to \infty} \frac{1}{\log(m+1)} \sum_{\nu=1}^{m} s_\nu(K) = M.$$

Therefore $\mathrm{Tr}_\omega(K) = M$, and (5.12) follows from this fact and (5.13).

Summarizing (i), (ii) and (iii) above, we have proved (5.5) for every $a \leq \tau \leq b$. This completes the proof of the proposition. □

Lemma 5.2. *Let $-\infty < a < b < \infty$. Suppose that A is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ satisfying the following two conditions*:

(1) *A has a pure point spectrum.*

(2) *The spectrum of A contains $[a, b]$.*

Then A admits an orthogonal decomposition

$$A = \bigoplus_{k=1}^{\infty} A_k$$

such that every A_k satisfies the same two conditions.

This lemma is completely elementary. Therefore its proof will be omitted.

Proposition 5.3. *Consider any $-\infty < a < b < \infty$. Let μ be a regular Borel measure on $[a, b]$ that consists purely of point masses. That is, there is a countable subset C of $[a, b]$ such that $\mu([a, b] \backslash C) = 0$. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ satisfying the following two conditions*:

(1) *A has a pure point spectrum.*

(2) *The spectrum of A contains $[a, b]$.*
Then there is a non-negative, self-adjoint operator $K \in \mathcal{C}_1^+$ such that for every Dixmier trace Tr_ω and every $z \in \mathbf{C} \backslash \mathbf{R}$, we have

$$\mathrm{Tr}_\omega((A + K - z)^{-1} - (A - z)^{-1}) = \int \frac{-1}{(t - z)^2} d\mu(t).$$

Moreover, K satisfies the estimate $\|K\|_1^+ \leq \mu([a, b])$.

Proof. By Lemma 5.2 and by the norm bound $\|K\|_1^+ \leq \mu([a, b])$, it suffices to consider the case where μ is a single point mass. That is, we only need to prove the proposition for $\mu = M\delta_x$, where $0 < M < \infty$, $x \in [a, b]$, and δ_x is the unit point mass at x.

By the two conditions, there is a sequence of mutually distinct eigenvalues $\{\lambda_i\}$ of A in $[a, b]$ such that $|\lambda_i - x| \to 0$ as $i \to \infty$. Passing to a subsequence if necessary, we further require that

$$\sum_{i=1}^{\infty} |\lambda_i - x| < \infty. \tag{5.14}$$

For each i, let e_i be a unit eigenvector of A corresponding to the eigenvalue λ_i. Since the λ_i's are mutually distinct, $\{e_i : i \in \mathbf{N}\}$ is an orthonormal set in $\mathcal{H}$. Let $\mathcal{H}'$ be the closure of $\mathrm{span}\{e_i : i \in \mathbf{N}\}$. Then both $\mathcal{H}'$ and $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$ are invariant subspaces for A. With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$, we have $A = A' \oplus A''$, where

$$A' = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i.$$

Accordingly, we define

$$K' = M \sum_{i=1}^{\infty} \frac{1}{i} e_i \otimes e_i$$

on $\mathcal{H}'$ and $K = K' \oplus 0$ on $\mathcal{H}$. Obviously, we have

$$\|K\|_1^+ = \|K'\|_1^+ = M.$$

Let us also define

$$B' = x \sum_{i=1}^{\infty} e_i \otimes e_i.$$

By (5.14), $A' - B'$ is in the trace class.

For any $z \in \mathbf{C} \backslash \mathbf{R}$ and any Dixmier trace Tr_ω, we have

$$\begin{aligned}\mathrm{Tr}_\omega((A + K - z)^{-1} - (A - z)^{-1}) &= \mathrm{Tr}_\omega((A' + K' - z)^{-1} - (A' - z)^{-1}) \\ &= \mathrm{Tr}_\omega((B' + K' + (A' - B') - z)^{-1} - (B' + (A' - B') - z)^{-1}).\end{aligned}$$

Since $A' - B'$ is in the trace class, the above gives us

$$\begin{aligned}\mathrm{Tr}_\omega((A+K-z)^{-1}-(A-z)^{-1}) &= \mathrm{Tr}_\omega((B'+K'-z)^{-1}-(B'-z)^{-1}) \\ &= \mathrm{Tr}_\omega\left(\sum_{i=1}^{\infty}\left\{\frac{1}{x+(M/i)-z}-\frac{1}{x-z}\right\}e_i\otimes e_i\right) \\ &= -\mathrm{Tr}_\omega\left(\sum_{i=1}^{\infty}\left\{\frac{1}{(x+(M/i)-z)(x-z)}\right\}\frac{M}{i}e_i\otimes e_i\right).\end{aligned}$$

Combining this with the fact that

$$\sum_{i=1}^{\infty}\left|\frac{1}{(x+(M/i)-z)(x-z)}-\frac{1}{(x-z)^2}\right|\frac{M}{i}<\infty,$$

we now have

$$\begin{aligned}&\mathrm{Tr}_\omega((A+K-z)^{-1}-(A-z)^{-1}) \\ &\quad = -\mathrm{Tr}_\omega\left(\frac{1}{(x-z)^2}\sum_{i=1}^{\infty}\frac{M}{i}e_i\otimes e_i\right)=\frac{-M}{(x-z)^2}=M\int\frac{-1}{(t-z)^2}d\delta_x(t).\end{aligned}$$

This completes the proof. □

6. Arbitrary spectrum for A

The condition that A have a pure point spectrum in Section 5, while serving nicely as a stepping stone toward our goal, is not necessary for the end result. In this section we will remove this condition. That is, we will show that A can have any kind of spectrum, so long as its spectrum, as a set, contains $[a, b]$.

Recall that we write $\mathcal{C}_1^{+(0)}$ for the $\|\cdot\|_1^+$-closure in $\mathcal{C}_1^+$ of the collection of finite-rank operators. Because of the property that

$$\mathrm{Tr}_\omega(T) = 0 \tag{6.1}$$

for every $T \in \mathcal{C}_1^{+(0)}$ and every Dixmier trace Tr_ω, the ideal $\mathcal{C}_1^{+(0)}$ played an important role in previous sections. In addition to (6.1), here we need another property of $\mathcal{C}_1^{+(0)}$, namely that every self-adjoint operator can be diagonalized modulo it. More precisely, let A be a bounded self-adjoint operator. Then, because $\|\cdot\|_1^+$ is strictly weaker than the trace norm, by a famous theorem of Kuroda [20,17], there is a self-adjoint operator $X \in \mathcal{C}_1^{+(0)}$ such that $A + X$ is a *diagonal operator*. That is, $A + X$ has a pure point spectrum.

Here is our main result on spectral shift:

Theorem 6.1. *Let μ be any regular Borel measure on $[a, b]$, where $-\infty < a < b < \infty$. Let A be any bounded self-adjoint operator whose spectrum contains $[a, b]$. Then there is* *a non-negative, self-adjoint operator* $K \in \mathcal{C}_1^+$ *such that for every Dixmier trace* Tr_ω *and every* $z \in \mathbf{C}\backslash\mathbf{R}$, *we have*

$$\mathrm{Tr}_\omega((A+K-z)^{-1} - (A-z)^{-1}) = \int \frac{-1}{(t-z)^2} d\mu(t).$$

Moreover, K *satisfies the estimate* $\|K\|_1^+ \leq 2(\log 2)\mu([a,b])$.

Proof. (i) Any μ described above admits a decomposition $\mu = \mu_{\mathrm{c}} + \mu_{\mathrm{p}}$ where μ_{c} has no point masses and μ_{p} either consists purely of point masses or is 0. If A has a pure point spectrum, then by Lemma 5.2 we can apply Proposition 5.1 to μ_{c} and Proposition 5.3 to μ_{p}. Thus the theorem follows from these two propositions if A has a pure point spectrum.

(ii) Suppose that A is an arbitrary bounded self-adjoint operator whose spectrum contains $[a,b]$. By Kuroda's theorem cited above, there is a self-adjoint operator $X \in \mathcal{C}_1^{+(0)}$ such that $A+X$ has a pure point spectrum. Obviously, $[a,b]$ is a part of the essential spectrum of A. Therefore the essential spectrum of $A+X$ also contains $[a,b]$. In particular, the spectrum of $A+X$ contains $[a,b]$. Applying part (i) to $A+X$, we obtain a non-negative self-adjoint operator $K \in \mathcal{C}_1^+$ with $\|K\|_1^+ \leq 2(\log 2)\mu([a,b])$ such that

$$\mathrm{Tr}_\omega((A+X+K-z)^{-1} - (A+X-z)^{-1}) = \int \frac{-1}{(t-z)^2} d\mu(t) \tag{6.2}$$

for every Dixmier trace Tr_ω and every $z \in \mathbf{C}\backslash\mathbf{R}$. Since $X \in \mathcal{C}_1^{+(0)}$, the difference

$$\{(A+K-z)^{-1} - (A-z)^{-1}\} - \{(A+X+K-z)^{-1} - (A+X-z)^{-1}\}$$

is in the ideal $\mathcal{C}_1^{+(0)}$. Hence

$$\mathrm{Tr}_\omega((A+K-z)^{-1} - (A-z)^{-1}) = \mathrm{Tr}_\omega((A+X+K-z)^{-1} - (A+X-z)^{-1}).$$

Combining this identity with (6.2), the theorem follows. □

7. Commutators of compact operators

Having established the analogues of two classic trace formulas in the context of Dixmier trace, we will now examine a commutator property where Dixmier trace and the ordinary trace behave quite differently. We begin with a classic result of Helton and Howe:

Proposition 7.1. [16, Lemma 1.3] *Suppose that* X *is a self-adjoint operator and* C *is a compact operator. If* $[X,C]$ *is in the trace class, then* $\mathrm{tr}([X,C]) = 0$.

This vanishing principle for the ordinary trace is an important tool in operator theory. See [26] for a recent example of its application. If C is a compact self-adjoint operator, then the spectrum of C is discrete, and therefore singular with respect to the Lebesgue measure. In this respect, Proposition 7.1 has a stronger version due to Voiculescu:

Proposition 7.2. [25, Proposition 2.1] *Let X and C be bounded self-adjoint operators such that $[X, C]$ is in the trace class. If the spectral measure of C is purely singular with respect to the Lebesgue measure, then* $\text{tr}([X, C]) = 0$.

From the results in Section 3 we can see that Proposition 7.2 does not generalize to the context of Dixmier trace. In fact, take any bounded self-adjoint operators A, B with $[A, B] \in \mathcal{C}_1^+$ and $\text{Tr}_\omega([A, B]) \neq 0$ (Theorem 3.4 provides plenty of such pairs). By Kuroda's theorem [20,17], there are self-adjoint operators $K, L \in \mathcal{C}_1^{+(0)}$ such that $A + K$ and $B + L$ have pure point spectra. In particular, the spectra of $A + K$ and $B + L$ are purely singular with respect to the Lebesgue measure. On the other hand, since $K, L \in \mathcal{C}_1^{+(0)}$, we have

$$\text{Tr}_\omega([A + K, B + L]) = \text{Tr}_\omega([A, B]) \neq 0.$$

Digging a little deeper, we find that Proposition 7.1, too, does not generalize to the context of Dixmier trace:

Proposition 7.3. *There exist compact self-adjoint operators X, C such that $[X, C] \in \mathcal{C}_1^+$ and $\text{Tr}_\omega([X, C]) \neq 0$ for every Dixmier trace Tr_ω.*

Proof. For each $n \in \mathbf{N}$, let $\mathcal{H}_n = \mathbf{C}^{n+1}$. Let T_n be the $(n + 1) \times (n + 1)$ matrix whose diagonal entries are all 0 and whose other entries are all 1. As usual, we consider T_n as an operator on $\mathcal{H}_n$. We have

$$T_n = S_n - 1,$$

where S_n is the $(n + 1) \times (n + 1)$ matrix whose entries are all equal to 1. Therefore the eigenvalues of T_n are n (with multiplicity 1) and -1 (with multiplicity n).

Define the Hilbert space

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n.$$

Corresponding to this orthogonal sum, we define the operator

$$T = \bigoplus_{n=1}^{\infty} \frac{1}{n^2} T_n.$$

By the discussion in the first paragraph, we have $T = A - B$ with

$$A = \sum_{n=1}^{\infty} \frac{1}{n} u_n \otimes u_n \quad \text{and} \quad B = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{i=1}^{n} v_{n,i} \otimes v_{n,i}, \tag{7.1}$$

where $\{u_n : n \in \mathbf{N}\}$ and $\{v_{n,i} : 1 \leq i \leq n \text{ and } n \in \mathbf{N}\}$ are orthonormal sets. It is obvious that for every Dixmier trace Tr_ω, we have $\text{Tr}_\omega(A) = 1$. Next we compute $\text{Tr}_\omega(B)$.

For a sufficiently large $k \in \mathbf{N}$, there is an $n_k \in \mathbf{N}$ such that

$$\sum_{n=1}^{n_k} n \leq k < \sum_{n=1}^{n_k+1} n. \tag{7.2}$$

Thus
$$\sum_{j=1}^{k} s_j(B) < \sum_{n=1}^{n_k+1} \frac{1}{n} \quad \text{whereas} \quad \log k \geq \log\left(\frac{n_k^2}{2}\right) = 2\log n_k - \log 2.$$

That is,

$$\frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(B) \leq \frac{1}{2\log n_k - \log 2} \sum_{n=1}^{n_k+1} \frac{1}{n}, \tag{7.3}$$

which in particular means that $B \in \mathcal{C}_1^+$. Similarly, from (7.1) and (7.2) we obtain

$$\frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(B) \geq \frac{1}{2\log(n_k+2)} \sum_{n=1}^{n_k} \frac{1}{n}. \tag{7.4}$$

From (7.3) and (7.4) we see that

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(B) = \frac{1}{2}.$$

This implies that $\mathrm{Tr}_\omega(B) = 1/2$ for every Dixmier trace Tr_ω. Consequently, $\mathrm{Tr}_\omega(T) = \mathrm{Tr}_\omega(A) - \mathrm{Tr}_\omega(B) = 1 - (1/2) = 1/2$.

Thus to complete the proof, it suffices to find a pair of compact self-adjoint operators X, C such that $\sqrt{-1}[X, C] = T$. To do this, consider any $n \in \mathbf{N}$. We first write T_n as a commutator. Let X_n be the $(n+1) \times (n+1)$ diagonal matrix whose diagonal entries, from the upper-left corner to the lower-right corner, are $0, 1, \ldots, n$. Let $C_n = [c_{i,j}^{(n)}]$ be the $(n+1) \times (n+1)$ matrix such that $c_{i,i}^{(n)} = 0$ for every $1 \leq i \leq n+1$ and such that $c_{i,j}^{(n)} = (j-i)^{-1}\sqrt{-1}$ for all $i \neq j$ in $\{1, 2, \ldots, n+1\}$. It is easy to verify that

$$\sqrt{-1}[X_n, C_n] = T_n. \tag{7.5}$$

We now define
$$X = \bigoplus_{n=1}^{\infty} \frac{1}{n^{3/2}} X_n \quad \text{and} \quad C = \bigoplus_{n=1}^{\infty} \frac{1}{n^{1/2}} C_n$$
on $\mathcal{H}$. Then from (7.5) we obtain $\sqrt{-1}[X, C] = T$. What remains is to show that the self-adjoint operators X and C are compact.

First of all, we have $\|X_n\| = n$ for every $n \in \mathbf{N}$, from which the compactness of X easily follows. Then note that C_n is the $(n+1) \times (n+1)$ *Toeplitz matrix* with

$$\frac{1}{\sqrt{-1}} \sum_{k=1}^{\infty} \frac{1}{k}\left(e^{k\sqrt{-1}x} - e^{-k\sqrt{-1}x}\right) = \sum_{k=1}^{\infty} \frac{2}{k}\sin(kx) = \pi - x$$

as its “symbol function”, $0 < x < 2\pi$. Therefore $\|C_n\| \le \pi$ for every $n \in \mathbf{N}$. Thus C is also compact. This completes the proof of the proposition. □

References

1. M. Birman and D. Yafaev, The spectral shift function. The papers of M. G. Kren and their further development, St. Petersburg Math. J. **4** (1993), 833-870.
2. R. Carey and J. Pincus, An exponential formula for determining functions, Indiana Univ. Math. J. **23** (1973/74), 1031-1042.
3. R. Carey and J. Pincus, Construction of seminormal operators with prescribed mosaic, Indiana Univ. Math. J. **23** (1973/74), 1155-1165.
4. R. Carey and J. Pincus, An invariant for certain operator algebras, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 1952-1956.
5. R. Carey and J. Pincus, Mosaics, principal functions, and mean motion in von Neumann algebras, Acta Math. **138** (1977), 153-218.
6. A. Connes, The action functional in noncommutative geometry, Comm. Math. Phys. **117** (1988), 673-683.
7. A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.
8. J. Conway, A course in functional analysis, 2nd edition, Graduate Texts in Mathematics **96**, Springer-Verlag, New York, 1990.
9. J. Dixmier, Existence de traces non normales, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1107-A1108.
10. M. Engliš, K. Guo and G. Zhang, Toeplitz and Hankel operators and Dixmier traces on the unit ball of $\mathbf{C}^n$, Proc. Amer. Math. Soc. **137** (2009), 3669-3678.
11. M. Engliš and R. Rochberg, The Dixmier trace of Hankel operators on the Bergman space, J. Funct. Anal. **257** (2009), 1445-1479.
12. M. Engliš and G. Zhang, Hankel operators and the Dixmier trace on the Hardy space, J. Lond. Math. Soc. (2) **94** (2016), 337-356.
13. F. Gesztesy, K. Makarov and S. Naboko, The spectral shift operator, Mathematical results in quantum mechanics (Prague, 1998), 59-90, Oper. Theory Adv. Appl., 108, Birkhuser, Basel, 1999.
14. I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
15. J. Helton and R. Howe, Integral operators: commutators, traces, index and homology. Proceedings of a Conference Operator Theory (Dalhousie Univ., Halifax, N.S., 1973), pp. 141-209. Lecture Notes in Math., **345**, Springer, Berlin, 1973.
16. J. Helton and R. Howe, Traces of commutators of integral operators, Acta Math. **135** (1975), 271-305.
17. T. Kato, Perturbation theory for linear operators, Springer-Verlag, New York, 1976.
18. M. Krein, On the trace formula in perturbation theory (Russian), Mat. Sbornik N.S. **33(75)** (1953), 597-626.
19. M. Krein, On perturbation determinants and a trace formula for unitary and self-adjoint operators (Russian), Dokl. Akad. Nauk SSSR **144** (1962), 268-271.
20. Sh. Kuroda, On a theorem of Weyl-von Neumann, Proc. Japan Acad. **34** (1958), 11-15.

21. J. Pincus, Commutators and systems of singular integral equations. I, Acta Math. **121** (1968), 219-249.
22. J. Pincus, Symmetric singular integral operators, Indiana Univ. Math. J. **23** (1973/74), 537-556.
23. B. Simon, Spectral analysis of rank one perturbations and applications. Mathematical quantum theory. II. Schrdinger operators (Vancouver, BC, 1993), 109149, CRM Proc. Lecture Notes **8**, Amer. Math. Soc., Providence, RI, 1995.
24. H. Upmeier and K. Wang, Dixmier trace for Toeplitz operators on symmetric domains, J. Funct. Anal. **271** (2016), 532-565.
25. D. Voiculescu, On the existence of quasicentral approximate units relative to normed ideals. I, J. Funct. Anal. **91** (1990), 1-36.
26. J. Xia, An integral formula for Schatten norm on the Hardy space: the only high-dimensional case, J. Funct. Anal. **281** (2021), no. 7, 109113.
27. D. Yafaev, Mathematical scattering theory. General theory, Translations of Mathematical Monographs **105**, American Mathematical Society, Providence, RI, 1992.

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