ESSENTIAL NORMALITY OF POLYNOMIAL-GENERATED SUBMODULES: HARDY SPACE AND BEYOND

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Abstract. Recently, Douglas and Wang proved that for each polynomial q, the submodule $[q]$ of the Bergman module generated by q is essentially normal [9]. Using improved techniques, we show that the Hardy-space analogue of this result holds, and more.

1. Introduction

Let $\mathbf{B}$ be the unit ball in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ with $(1 - \langle \zeta, z\rangle)^{-1}$ as its reproducing kernel. The space H_n^2 is naturally considered as a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$. In [3-6], Arveson raised the question of whether graded submodules $\mathcal{M}$ of H_n^2 are essentially normal. That is, for the restricted operators

$$Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \quad 1 \leq j \leq n,$$

on $\mathcal{M}$, do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,7,10,13,14]. In particular, Guo and Wang showed that the answer to the above question is affirmative if $\mathcal{M}$ is generated by a homogeneous polynomial [14]. In [8], Douglas proposed analogous essential normality problems for submodules of the Bergman module $L_a^2(\mathbf{B}, dv)$.

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [9] for *arbitrary* polynomials. In that paper, Douglas and Wang showed that for any polynomial $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule $[q]$ of the Bergman module generated by q is p-essentially normal for $p > n$. What is especially remarkable is that [9] contains many novel ideas.

The present paper grew out of a remark in [9]. Toward the end of [9], Douglas and Wang commented

> "*It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from* [9,8] *may be needed to complete the proofs.*"

While the Drury-Arveson space case is out of reach at the moment, in this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

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The key realization is that Bergman space, Hardy space and Drury-Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on $\mathbf{B}$ parametrized by a real-valued parameter $-n \leq t < \infty$. In fact, the spaces corresponding to the values $t \in \mathbf{Z}_+$ were used in an essential way in the proofs in [9]. Our main observation is that if one considers other values of t, then one will see how to extend the techniques in [9] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [9] for spaces with parameter $-2 < t < \infty$. Before stating the result, let us first introduce these spaces.

For each real number $-n \leq t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \dots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Here and throughout the paper, we use the conventional multi-index notation [15,page 3].

Obviously, $\mathcal{H}^{(0)}$ is the Bergman space $L_a^2(\mathbf{B}, dv)$. One can view the Bergman space $\mathcal{H}^{(0)} = L_a^2(\mathbf{B}, dv)$ as a benchmark, against which the other spaces in the family should be compared. Note that for each $-1 < t < \infty$, $\mathcal{H}^{(t)}$ is a weighted Bergman space.

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Let σ be the positive, regular Borel measure on S that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ that fix 0. We take the usual normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$.

Obviously, $\mathcal{H}^{(-1)}$ is just the Hardy space $H^2(S)$. Moreover, $\mathcal{H}^{(-n)}$ is none other than the Drury-Arveson space H_n^2.

It is well known that for each $-n \leq t < -1$, the tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$ is not jointly subnormal on $\mathcal{H}^{(t)}$ [1,Theorem 3.9]. In other words, if $-n \leq t < -1$, then $\mathcal{H}^{(t)}$ is more like the Drury-Arveson space than the Hardy space. The practical consequence of this is that it is difficult to do estimates on $\mathcal{H}^{(t)}$ if $-n \leq t < -1$.

Let $q \in \mathbf{C}[z_1, \dots, z_n]$. For each $-n \leq t < \infty$, let $[q]^{(t)}$ denote the closure of

$$\{qf : f \in \mathbf{C}[z_1, \dots, z_n]\}$$

in $\mathcal{H}^{(t)}$. Since $\mathcal{H}^{(t)}$ is a Hilbert module over $\mathbf{C}[z_1, \dots, z_n]$, $[q]^{(t)}$ is a submodule. For each $j \in \{1, \dots, n\}$, define submodule operator

$$Z_{q,j}^{(t)} = M_{z_j} | [q]^{(t)}.$$

Recall that the submodule $[q]^{(t)}$ is said to be p-essentially normal if the commutators $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}]$, $i, j \in \{1, \dots, n\}$, all belong to the Schatten class $\mathcal{C}_p$. With the foregoing preparation, we are now ready to state our result.

Theorem 1.1. *Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \dots, z_n]$. Then for each real number $-2 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > n$.*

Clearly, the Hardy-space case mentioned in [9] is settled by applying Theorem 1.1 to the special case $t = -1$.

On the other hand, it is a real pity that the requirement $t > -2$ in Theorem 1.1 does not allow us to capture any Drury-Arveson space in dimensions $n \geq 2$. But as a consolation, Theorem 1.1 does cover spaces $\mathcal{H}^{(t)}$ for $-2 < t < -1$, which, as we mentioned, are more Drury-Arveson-like than Hardy-like.

On the technical side, this paper does offer some improvement over [9]. As the authors of [9] stated, the key step in the proof of their result rests on weighted norm estimates given in Section 3 in that paper. At the core of their weighted estimates is an argument using a covering lemma. This is where we offer the most significant improvement. In this paper, the covering-lemma argument of [9] is done away with entirely. In its place, we use a much simpler argument based on Fubini's theorem.

In fact, using Fubini's theorem-based argument in place of covering-lemma argument is a situation with which we are quite familiar. See, for example, the proofs of Proposition 2.6 and Lemma 5.2 in [11].

There are many technical contributions made in [9]. Perhaps, the most important among these is Lemma 3.2 in that paper. This lemma will again be the basis for analysis here. The reader will see that with the combination of [9,Lemma 3.2] and our Fubini's theorem-based argument, the analysis part of the proof is actually easy.

As it was the case in [9], an essential role in the proof is played by the number operator N introduced by Arveson in [2]. Recall that, for a polynomial $f(z) = \sum_\alpha c_\alpha z^\alpha$,

$$(Nf)(z) = \sum_\alpha c_\alpha |\alpha| z^\alpha.$$

Here as well as in [9], the proof boils down to the estimate of an operator series where the k-th term has the operator

$$(N + 1 + n + t)^{-k-1}$$

as a factor, $k \geq 0$. Douglas and Wang's idea is to factor the above in the form

$$(N + 1 + n + t)^{-k-1} = (N + 1 + n + t)^{-1/2} \cdot (N + 1 + n + t)^{-k-(1/2)},$$

"reserve" the factor $(N + 1 + n + t)^{-1/2}$ for establishing the requisite Schatten-class membership, and use the other factor, $(N+1+n+t)^{-k-(1/2)}$, to boost the weight of the space. This is another place where [9] and the present paper differ. Instead of factoring, we will apply the whole of $(N + 1 + n + t)^{-k-1}$ to boost weight. Proposition 4.2 below allows us to recover an equivalent of $(N+1+n+t)^{-1/2}$ at the end of the estimate. This is why we are able to push t below -1.

The rest of the paper is organized as follows. Since the analysis part of the proof is now easy, we will take care of that first, in Sections 2 and 3. Section 4 contains a brief discussion of the relation between the natural embedding $\mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ and norm ideals. Section 5, which mirrors Section 2 in [9], contains the proof of our result.

2. Derivative on the Disc

Write D for the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. Let dA be the area measure on D with the normalization $A(D) = 1$. The unit circle $\{\tau \in \mathbf{C} : |\tau| = 1\}$ will be denote by $\mathbf{T}$. Furthermore, let dm be the Lebesgue measure on $\mathbf{T}$ with the normalization $m(\mathbf{T}) = 1$. For convenience, we write ∂ for the one-variable differentiation d/dz on $\mathbf{C}$.

Our first lemma is basically a restatement of Lemma 3.2 in [9].

Lemma 2.1. *Suppose that g is a one-variable polynomial of degree $K \geq 1$, and that f is analytic on D. Then for each $k \in \mathbf{N}$ we have*

$$|(\partial^k g)(0)f(0)|^2 \leq 2^{2k+2}(K!)^2 \int |gf|^2 dA.$$

Proof. For each $0 \leq r < 1$, let $g_r(z) = g(rz)$ and $f_r(z) = f(rz)$. We only need to consider the case $1 \leq k \leq K$. For such a k, Lemma 3.2 in [9] tells us that $|(\partial^k g_r)(0)f_r(0)| \leq K! \int_{\mathbf{T}} |g_r f_r| dm$. Since $(\partial^k g_r)(0) = r^k(\partial^k g)(0)$ and $f_r(0) = f(0)$, we have

$$\begin{aligned} |(\partial^k g)(0)f(0)| &= 2\int_{1/2}^{1} r^{-k}|(\partial^k g_r)(0)f_r(0)|dr \leq 2K! \int_{1/2}^{1} r^{-k} \int_{\mathbf{T}} |g_r(\tau)f_r(\tau)|dm(\tau)dr \\ &\leq 2^{k+1}K! \int_{1/2}^{1} 2r \int_{\mathbf{T}} |g(r\tau)f(r\tau)|dm(\tau)dr \leq 2^{k+1}K! \int |gf|dA. \end{aligned}$$

Squaring both sides and applying the Cauchy-Schwarz inequality, the lemma follows. □

For each $z \in D$, define the disc $D(z) = \{w \in D : |w - z| < (1/2)(1 - |z|)\}$.

Lemma 2.2. *For all $w \in D$ and $x \in (-1, \infty)$, we have*

$$\int \frac{(1-|z|^2)^x}{A(D(z))} \chi_{D(z)}(w) dA(z) \leq 2^{2\max\{x,0\}+5}(1-|w|^2)^x.$$

Proof. Let $w \in D$, and let $z \in D$ be such that $w \in D(z)$. Then we have $1 - |w| \leq 1 - |z| + |z - w| < (3/2)(1 - |z|)$. Also, $1 - |z| \leq 1 - |w| + |w - z| \leq 1 - |w| + (1/2)(1 - |z|)$. After cancellation, we find $(1/2)(1 - |z|) \leq 1 - |w|$. Thus

$$(2.1) \qquad (2/3)(1 - |w|) \leq 1 - |z| \leq 2(1 - |w|) \quad \text{whenever} \quad w \in D(z).$$

From this we obtain that for $w \in D(z)$ and $x \in (-1, \infty)$,

$$(1-|z|^2)^x \le \begin{cases} 2^{2x}(1-|w|^2)^x & \text{if} \quad 0 \le x < \infty \\ \\ 3(1-|w|^2)^x & \text{if} \quad -1 < x < 0 \end{cases} .$$

Thus, to complete the proof, it suffices to show that

$$\int \frac{\chi_{D(z)}(w)}{A(D(z))} dA(z) \le 9 \tag{2.2}$$

for every $w \in D$. For each $w \in D$, let $G(w) = \{z \in D : w \in D(z)\}$. If $z \in G(w)$, then $|z-w| \le (1/2)(1-|z|) \le 1-|w|$ by (2.1). Hence $A(G(w)) \le (1-|w|)^2$. On the other hand, if $z \in G(w)$, then $A(D(z)) = (1/4)(1-|z|)^2 \ge (1/3)^2(1-|w|)^2$, also by (2.1). Clearly, (2.2) follows from these two inequalities. □

Proposition 2.3. *Suppose that g is a one-variable polynomial of degree $K \ge 1$, and that f is analytic on D. Then for all $k \in \mathbf{N}$ and $t \in (0, \infty)$ satisfying the condition $t - 2k > -1$,*

$$\int |(\partial^k g)(z) f(z)|^2 (1-|z|^2)^t dA(z)$$
$$\le 2^{6k+2\max\{t-2k,0\}+7}(K!)^2 \int |g(w)f(w)|^2(1-|w|^2)^{t-2k} dA(w).$$

Proof. Define $g_z(u) = g(z + (1/2)(1-|z|)u)$ and $f_z(u) = f(z + (1/2)(1-|z|)u)$ for each $z \in D$. Then $2^{-k}(1-|z|)^k(\partial^k g)(z) = (\partial^k g_z)(0)$ and $f(z) = f_z(0)$. By Lemma 2.1,

$$\begin{aligned} |(\partial^k g)(z) f(z)|^2 &= \frac{2^{2k}|(\partial^k g_z)(0) f_z(0)|^2}{(1-|z|)^{2k}} \le \frac{2^{4k+2}(K!)^2}{(1-|z|)^{2k}} \int |g_z(u) f_z(u)|^2 dA(u) \\ &= \frac{2^{4k+2}(K!)^2}{(1-|z|)^{2k}} \cdot \frac{1}{A(D(z))} \int_{D(z)} |g(w)f(w)|^2 dA(w). \end{aligned}$$

Therefore, if $t - 2k > -1$, then

$$\begin{aligned} &\int |(\partial^k g)(z) f(z)|^2 (1-|z|^2)^t dA(z) \\ &\le 2^{6k+2}(K!)^2 \int \frac{(1-|z|^2)^{t-2k}}{A(D(z))} \left(\int_{D(z)} |g(w)f(w)|^2 dA(w) \right) dA(z) \\ &= 2^{6k+2}(K!)^2 \int \left\{ \int \frac{(1-|z|^2)^{t-2k}}{A(D(z))} \chi_{D(z)}(w) dA(z) \right\} |g(w)f(w)|^2 dA(w) \\ &\le 2^{6k+2\max\{t-2k,0\}+7}(K!)^2 \int (1-|w|^2)^{t-2k} |g(w)f(w)|^2 dA(w), \end{aligned}$$

where the last step is an application of Lemma 2.2. This completes the proof. □

3. Derivatives on the Ball

Recall that there is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n}r^n \le \sigma(\{\xi \in S : |1 - \langle u, \xi\rangle| < r\}) \le A_0 r^n \tag{3.1}$$

for all $u \in S$ and $0 < r \le 2$ [15,Proposition 5.1.4]. For each $z \in \mathbf{B}$, define the subset

$$T(z) = \{w \in \mathbf{B} : |1 - \langle w, z\rangle| < 2(1 - |z|^2),\ 1 - |w|^2 > (1/2)(1 - |z|^2)\}$$

of the unit ball. We begin our estimates with the properties of the set $T(z)$.

Let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$.

Lemma 3.1. *There is a constant* $0 < C_{3.1} < \infty$ *such that for all* $\zeta \in \mathbf{B}$ *and* $x \in (-1, \infty)$,

$$\int (1 - |z|^2)^{x-n-1} \chi_{T(z)}(\zeta) dv(z) \le C_{3.1} 2^{\max\{x,0\}} (1 - |\zeta|^2)^x.$$

Proof. Let $\zeta, z \in \mathbf{B}$ be such that $\zeta \in T(z)$. Then we have $1-|\zeta|^2 \le 2(1-|\zeta|) \le 2|1-\langle \zeta, z\rangle|$ $< 4(1 - |z|^2)$. Combining this with the condition $1 - |\zeta|^2 > (1/2)(1 - |z|^2)$, we have

$$(1/4)(1 - |\zeta|^2) \le 1 - |z|^2 \le 2(1 - |\zeta|^2). \tag{3.2}$$

Therefore, for $x \in (-1, \infty)$ we have

$$(1 - |z|^2)^x \le \begin{cases} 2^x (1 - |\zeta|^2)^x & \text{if} \quad 0 \le x < \infty \\ 4(1 - |\zeta|^2)^x & \text{if} \quad -1 < x < 0 \end{cases}.$$

Thus, to complete the proof, it suffices to show that there is a $0 < C < \infty$ such that

$$\int \frac{\chi_{T(z)}(\zeta)}{(1 - |z|^2)^{n+1}} dv(z) \le C \tag{3.3}$$

for every $\zeta \in \mathbf{B}$. Given a $\zeta \in \mathbf{B}$, consider the set $\Omega(\zeta) = \{z \in \mathbf{B} : \zeta \in T(z)\}$. Write $\zeta = |\zeta|\eta$ with $\eta \in S$. If $z = |z|\xi \in \Omega(\zeta)$, where $\xi \in S$, then $|1 - \langle \eta, \xi\rangle| \le 2|1 - \langle \zeta, z\rangle| < 4(1 - |z|^2)$ $< 8(1 - |\zeta|^2)$. Also, $1 - |z| \le |1 - \langle \zeta, z\rangle| < 4(1 - |\zeta|^2)$ if $z \in \Omega(\zeta)$. Hence

$$\Omega(\zeta) \subset \{r\xi : 0 < 1 - r < 4(1 - |\zeta|^2);\ \xi \in S,\ |1 - \langle \eta, \xi\rangle| < 8(1 - |\zeta|^2)\}.$$

By (3.1) and the decomposition $dv = 2nr^{2n-1} dr d\sigma$, there is a $0 < C_1 < \infty$ such that $v(\Omega(\zeta)) \le C_1 (1 - |\zeta|^2)^{n+1}$ for every $\zeta \in \mathbf{B}$. By (3.2), $(1 - |z|^2)^{-n-1} \le 4^{n+1}(1 - |\zeta|^2)^{-n-1}$ when $z \in \Omega(\zeta)$. Clearly, (3.3) follows from these two inequalities. □

Lemma 3.2. *There is a constant* $0 < \epsilon < 1$ *such that for each* $0 \le a < 1$, *the set* $T((a, 0, \dots, 0))$ *contains the polydisc*

$$P_a = \{(a + u, \zeta_2, \dots, \zeta_n) : |u| < \epsilon(1 - a^2),\ |\zeta_j| < \epsilon\sqrt{1 - a^2},\ 2 \le j \le n\}. \tag{3.4}$$

Proof. Given an $a \in [0,1)$, write $\alpha = (a, 0, \ldots, 0)$. Let $0 < \epsilon < 1$, and suppose that u and $\zeta_2, \ldots, \zeta_n$ satisfy the conditions $|u| < \epsilon(1-a^2)$ and $|\zeta_j| < \epsilon\sqrt{1-a^2}$, $2 \le j \le n$. Then consider the vector $w = (a+u, \zeta_2, \ldots, \zeta_n)$. We have $|1 - \langle w, \alpha\rangle| = |1-a^2-au| < (1+\epsilon)(1-a^2)$. Moreover, $1-|w|^2 = 1-|a+u|^2-(|\zeta_2|^2+\cdots+|\zeta_n|^2) \ge 1-|a+u|^2-(n-1)\epsilon^2(1-a^2)$. On the other hand, $1-|a+u|^2 = 1-(a^2+2\mathrm{Re}(au)+|u|^2) \ge 1-a^2-3|u| \ge (1-3\epsilon)(1-a^2)$. Hence $1-|w|^2 \ge (1-(n+2)\epsilon)(1-a^2)$. Thus $\epsilon = \{3(n+2)\}^{-1}$ suffices for our purpose. $\square$

As usual, write $\partial_1, \ldots, \partial_n$ for the differentiations with respect to the complex variables $z_1, \ldots, z_n$. For each vector $b = (b_1, \ldots, b_n) \in \mathbf{C}^n$, define the directional derivative

$$\partial_b = b_1\partial_1 + \cdots + b_n\partial_n.$$

Lemma 3.3. *There is a constant* $0 < C_{3.3} < \infty$ *such that the following estimate holds: Suppose that* $q \in \mathbf{C}[z_1, \ldots, z_n]$ *and that* $\deg(q) = K \ge 1$. *Let* $f \in \mathbf{C}[z_1, \ldots, z_n]$. *If* z *and* b *are vectors in* $\mathbf{B}\backslash\{0\}$ *satisfying the relation* $\langle b, z\rangle = 0$, *then*

$$|(\partial_b q)(z)f(z)|^2 \le \frac{C_{3.3}(K!)^2}{(1-|z|^2)^{n+2}} \int_{T(z)} |qf|^2 dv.$$

Proof. Consider the special case where $z = \alpha = (a, 0, \ldots, 0)$ for some $0 < a < 1$. Let ϵ be the constant provided by Lemma 3.2. Define the polydisc

$$Y = \{(a+u, 0, \zeta_3, \ldots, \zeta_n) : |u| < \epsilon(1-a^2),\ |\zeta_j| < \epsilon\sqrt{1-a^2},\ 3 \le j \le n\}.$$

For each $y \in Y$, we define the one-varible polynomial $q_y(w) = q(y + \epsilon\sqrt{1-a^2}we_2)$, where $e_2 = (0, 1, 0, \ldots, 0)$. Similarly, define $f_y(w) = f(y + \epsilon\sqrt{1-a^2}we_2)$ on D. Since $(\partial q_y)(0) = \epsilon\sqrt{1-a^2}(\partial_2 q)(y)$ and $f_y(0) = f(y)$, we apply Lemma 2.1 to obtain

$$|(\partial_2 q)(y)f(y)|^2 = \frac{|(\partial q_y)(0)f_y(0)|^2}{\epsilon^2(1-a^2)} \le \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)|^2 dA(w).$$

Making the substitution $\zeta_2 = \epsilon\sqrt{1-a^2}w$, we find that

$$|(\partial_2 q)(y)f(y)|^2 \le \frac{16(K!)^2}{\epsilon^4(1-a^2)^2} \int_{|\zeta_2|<\epsilon\sqrt{1-a^2}} |q(y+\zeta_2 e_2)f(y+\zeta_2 e_2)|^2 dA(\zeta_2).$$

Now, integrating both sides over Y, we see that

$$\epsilon^{2n-2}(1-a^2)^n|(\partial_2 q)(\alpha)f(\alpha)|^2 \le \int_Y |(\partial_2 q)(y)f(y)|^2 dy \le \frac{16C(K!)^2}{\epsilon^4(1-a^2)^2} \int_{P_a} |qf|^2 dv,$$

where P_a is given by (3.4) and C accounts for the normalization constants for the measures involved. Since Lemma 3.2 tells us that $P_a \subset T(\alpha)$, we have

$$|(\partial_2 q)(\alpha)f(\alpha)|^2 \le \frac{16\epsilon^{-(2n+2)}C(K!)^2}{(1-a^2)^{n+2}} \int_{T(\alpha)} |qf|^2 dv.$$

Obviously, the above inequality also holds if we replace ∂_2 by ∂_j for any $2 \le j \le n$. Applying these and the Cauchy-Schwarz inequality, we see that

$$|(\partial_b q)(\alpha)f(\alpha)|^2 \le (n-1)\frac{16\epsilon^{-(2n+2)}C(K!)^2}{(1-a^2)^{n+2}}\int_{T(\alpha)}|qf|^2 dv \quad \text{if } \langle b,\alpha\rangle = 0,\ b \in \mathbf{B}.$$

This proves the lemma in the special case where $z = \alpha = (a, 0, \dots, 0)$, $0 < a < 1$. The general case follows from this special case and the following easily-verified relations: If U is any unitary transformation on $\mathbf{C}^n$ and $w, b \in \mathbf{B}$, then $UT(w) = T(Uw)$ and $(\partial_b(q \circ U))(w) = (\partial_{Ub}q)(Uw)$. □

Following [9], for each pair of $i \ne j$ in $\{1, \dots, n\}$ we define $L_{i,j} = \bar{z}_j\partial_i - \bar{z}_i\partial_j$.

Proposition 3.4. *There is a constant $1 \le C_{3.4} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \dots, z_n]$ and that $\deg(q) = K \ge 1$. Let $f \in \mathbf{C}[z_1, \dots, z_n]$. Then for every positive number $t > 0$ and all integers $i \ne j$ in $\{1, \dots, n\}$, we have*

$$\int |(L_{i,j}q)(z)f(z)|^2(1-|z|^2)^t dv(z) \le C_{3.4}2^t(K!)^2 \int |q(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{t-1}dv(\zeta).$$

Proof. It follows from Lemma 3.3 that

$$|(L_{i,j}q)(z)f(z)|^2 \le \frac{C_{3.3}(K!)^2}{(1-|z|^2)^{n+2}}\int_{T(z)}|q(\zeta)f(\zeta)|^2 dv(\zeta),$$

$z \in \mathbf{B}$. Multiplying both sides by $(1-|z|^2)^t$ and integrating, we find that

$$\begin{aligned}
&\int |(L_{i,j}q)(z)f(z)|^2(1-|z|^2)^t dv(z) \\
&\qquad \le C_{3.3}(K!)^2 \int \left((1-|z|^2)^{t-n-2}\int_{T(z)}|q(\zeta)f(\zeta)|^2 dv(\zeta) \right) dv(z) \\
&\qquad = C_{3.3}(K!)^2 \int \left\{ \int (1-|z|^2)^{t-n-2}\chi_{T(z)}(\zeta)dv(z) \right\} |q(\zeta)f(\zeta)|^2 dv(\zeta).
\end{aligned}$$

Applying Lemma 3.1 with $x = t-1$ to the $\{\cdots\}$ above, the proposition follows. □

Write $R = z_1\partial_1 + \cdots + z_n\partial_n$, the radial derivative in n variables. We will denote the one-variable radial derivative by $\mathcal{R}$. For each polynomial h and each $\xi \in S$, define the "slice" function $h_\xi(z) = h(z\xi)$, $z \in D$. If q is a polynomial in n variables, then for every $\xi \in S$ we have the relation $(\mathcal{R}q_\xi)(z) = (Rq)_\xi(z)$.

Proposition 3.5. *There is a constant $1 \le C_{3.5} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \dots, z_n]$ and that $\deg(q) = K \ge 1$. Let $f \in \mathbf{C}[z_1, \dots, z_n]$. Then for each pair of $k \in \mathbf{N}$ and $t \in (0, \infty)$ satisfying the condition $t - 2k > -1$,*

$$\int |(R^k q)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta) \le C_{3.5}^{K(k+t)}(K!)^2 \int |q(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{t-2k}dv(\zeta).$$

Proof. As in [9], we need the following relation between dv, $d\sigma$ and dA: Since $dv = 2nr^{2n-1}drd\sigma$, $dA = 2rdrdm$, and $d\sigma$ is invariant under rotation, we have

$$\int g dv = n \int \left(\int g(z\xi)|z|^{2n-2} dA(z) \right) d\sigma(\xi). \tag{3.5}$$

By Lemma 3.6 in [9], for each $k \in \mathbf{N}$,

$$\mathcal{R}^k = \sum_{j=1}^{k} a_j^{(k)} z^j \partial^j \quad \text{with} \quad |a_j^{(k)}| < (j+1)^k. \tag{3.6}$$

Since the degree of q equals K, for each $\xi \in S$ we have

$$(R^k q)_\xi(z) = (\mathcal{R}^k q_\xi)(z) = \sum_{j=1}^{\min\{k,K\}} a_j^{(k)} z^j (\partial^j q_\xi)(z).$$

Given f, for each $\xi \in S$ we define the "rigged" slice function $f^{(\xi)}(z) = z^{n-1} f(z\xi)$, $z \in D$. Applying first (3.6) and then Proposition 2.3, when $t - 2k > -1$, we have

$$\begin{aligned}
&\int |(\mathcal{R}^k q_\xi)(z) f^{(\xi)}(z)|^2 (1-|z|^2)^t dA(z) \\
&\le K(K+1)^{2k} \sum_{j=1}^{\min\{k,K\}} \int |(\partial^j q_\xi)(z) f^{(\xi)}(z)|^2 (1-|z|^2)^t dA(z) \\
&\le K(K+1)^{2k} \sum_{j=1}^{\min\{k,K\}} 2^{6j+2\max\{t-2j,0\}+7} (K!)^2 \int |q_\xi(z) f^{(\xi)}(z)|^2 (1-|z|^2)^{t-2j} dA(z) \\
&\le K^2 (K+1)^{2k} 2^{6k+2t+7} (K!)^2 \int |q_\xi(z) f^{(\xi)}(z)|^2 (1-|z|^2)^{t-2k} dA(z) \\
&\le C_{3.5}^{K(k+t)} (K!)^2 \int |q_\xi(z) f^{(\xi)}(z)|^2 (1-|z|^2)^{t-2k} dA(z).
\end{aligned}$$

By the relations $(\mathcal{R}^k q_\xi)(z) = (R^k q)_\xi(z)$, $f^{(\xi)}(z) = z^{n-1} f(z\xi)$ and $|z| = |z\xi|$, we now have

$$\begin{aligned}
&\int |(R^k q)(z\xi) f(z\xi)|^2 (1-|z\xi|^2)^t |z|^{2n-2} dA(z) \\
&\qquad \le C_{3.5}^{K(k+t)} (K!)^2 \int |q(z\xi) f(z\xi)|^2 (1-|z\xi|^2)^{t-2k} |z|^{2n-2} dA(z).
\end{aligned}$$

Integrating both sides with respect to the measure $d\sigma$ on S and applying (3.5), the proposition follows. □

Proposition 3.6. *There is a constant* $1 \le C_{3.6} < \infty$ *such that the following estimate holds: Suppose that* $q \in \mathbf{C}[z_1, \dots, z_n]$ *and that* $\deg(q) = K \ge 1$. *Let* $f \in \mathbf{C}[z_1, \dots, z_n]$. *Then for each* $t \in (1, \infty)$ *and each* $j \in \{1, \dots, n\}$, *we have*

$$\int |(\partial_j q)(\zeta) f(\zeta)|^2 (1-|\zeta|^2)^t dv(\zeta) \le C_{3.6}^{Kt} (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1-|\zeta|^2)^{t-2} dv(\zeta). \tag{3.7}$$

Proof. There is a C such that for every analytic function h on $\mathbf{B}$ and every $t > 0$, we have

$$\int_{|\zeta|<1/2} |h(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta) \le C\left(\frac{16}{7}\right)^t \int_{1/2\le|\zeta|<3/4} |h(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta). \tag{3.8}$$

Now apply Proposition 3.4 and the case $k = 1$ in Proposition 3.5: by the identity $|z|^2\partial_j = \bar{z}_j R + \sum_{i\neq j} z_i L_{j,i}$, (3.7) obviously holds if $(\partial_j q)(\zeta)$ is replaced by $|\zeta|^2(\partial_j q)(\zeta)$ on the left-hand side. The extra factor $|\zeta|^2$ is then removed by using (3.8). □

4. Embedding and Norm Ideals

For a bounded operator A, we write its s-numbers as $s_1(A), \dots, s_k(A), \dots$ as usual. Recall that, for each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k\ge 1} \frac{s_1(A)+s_2(A)+\cdots+s_k(A)}{1^{-1/p}+2^{-1/p}+\cdots+k^{-1/p}} \tag{4.1}$$

defines a symmetric norm for operators [12,Section III.14]. On any Hilbert space $\mathcal{H}$, the set $\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$ is a norm ideal [12,Section III.2]. It is well known that if $p < p'$, then $\mathcal{C}_p^+$ is contained in the Schatten class $\mathcal{C}_{p'}$.

For a non-increasing sequence of non-negative numbers $\{a_1, \dots, a_k, \dots\}$, if $a_1 + \cdots + a_k \le C(1^{-1/p} + \cdots + k^{-1/p})$, then $ka_k \le C(1^{-1/p} + \cdots + k^{-1/p})$. It follows that if $p > 1$ and if $T \in \mathcal{C}_p^+$, then there is a $0 < C(T) < \infty$ such that $s_k(T) \le C(T)k^{-1/p}$ for every $k \in \mathbf{N}$. Thus if $p > 1$ and if B is a bounded operator such that $B^*B \in \mathcal{C}_p^+$, then $B \in \mathcal{C}_{2p}^+$.

Proposition 4.1. *For each $t \ge -n$, let $I^{(t)} : \mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ be the natural embedding. Then $I^{(t)*}I^{(t)} \in \mathcal{C}_n^+$.*

Proof. Expanding the reproducing kernel $(1 - \langle\zeta, z\rangle)^{-(n+1+t)}$, we see that the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\{e_\alpha^{(t)} : \alpha \in \mathbf{Z}_+^n\}$, where

$$e_\alpha^{(t)}(\zeta) = \left(\frac{1}{\alpha!}\prod_{j=1}^{|\alpha|}(n+t+j)\right)^{1/2} \zeta^\alpha, \quad \alpha \neq 0, \tag{4.2}$$

and $e_0^{(t)}(\zeta) = 1$. Given these orthonormal bases, it is straightforward to verify that

$$I^{(t)*}I^{(t)}e_\alpha^{(t)} = \frac{n+1+t}{n+1+|\alpha|+t}e_\alpha^{(t)}, \quad \alpha \in \mathbf{Z}_+^n.$$

This formula gives us all the s-numbers for $I^{(t)*}I^{(t)}$. By (4.1), $I^{(t)*}I^{(t)} \in \mathcal{C}_n^+$. □

Proposition 4.2. *Suppose that E is a linear subspace of $\mathbf{C}[z_1, \dots, z_n]$ and that $t \ge -n$. Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in \mathcal{B}(\mathcal{H}^{(t)})$, and suppose that there is a C such that*

$$\|Ag\|_t \le C\|g\|_{t+1} \tag{4.3}$$

for every $g \in E$. Then $A\mathcal{E}^{(t)} \in \mathcal{C}^+_{2n}$.

Proof. By (4.3), for each $g \in E$ we have

$\langle A^*Ag, g\rangle_t = \|Ag\|^2_t \le C^2\|g\|^2_{t+1} = C^2\|I^{(t)}g\|^2_{t+1} = C^2\langle I^{(t)}g, I^{(t)}g\rangle_{t+1} = C^2\langle I^{(t)*}I^{(t)}g, g\rangle_t.$

That is, the operator inequality $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \le C^2\mathcal{E}^{(t)}I^{(t)*}I^{(t)}\mathcal{E}^{(t)}$ holds on $\mathcal{H}^{(t)}$. Thus $s_j((A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)}) \le s_j(C^2\mathcal{E}^{(t)}I^{(t)*}I^{(t)}\mathcal{E}^{(t)})$ for each $j \in \mathbf{N}$ [12,Lemma II.1.1]. By Proposition 4.1, $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \in \mathcal{C}^+_n$. Since $n \ge 2$, this implies $A\mathcal{E}^{(t)} \in \mathcal{C}^+_{2n}$. □

5. Proof of Theorem 1.1

For each $t \ge -n$ and each polynomial q, we write $M^{(t)}_q$ for the operator of multiplication by q on the space $\mathcal{H}^{(t)}$. Keep in mind that the notation "$*$" is t-specific: $M^{(t)*}_q$ means the adjoint of $M^{(t)}_q$ with respect to the inner product $\langle \cdot, \cdot\rangle_t$.

Proposition 5.1. *Let $q \in \mathbf{C}[z_1, \dots, z_n]$, $1 \le j \le n$ and $t \ge -n$. For $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $f(0) = 0$, we have*

$$M^{(t)*}_{z_j}M^{(t)}_q f - M^{(t)}_q M^{(t)*}_{z_j} f = \sum_{k=0}^{\infty}(N + 1 + n + t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j}M^{(t)}_{R^{k+1}q})f.$$

Proof. The main idea is that both sides are linear with respect to both q and f. Therefore the proof is a matter of straightforward verification in the special case of $q = z^\alpha$ and $f = z^\beta$, $\beta \ne 0$, using (4.2). The details of the verification are similar to the Bergman space case (see the proof of Proposition 2.1 in [9]). □

Proposition 5.2. *Let $t \ge -n$* and *$\ell \in \mathbf{N}$.* (1) *For each $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell$ and each non-negative integer k, we have*

$$\|(N + 1 + n + t)^{-k-1}f\|^2_t \le \frac{(n + 2k + 2 + t + \ell)^\ell}{(\ell + 1 + n + t)^{2k+2}}\|f\|^2_{2k+2+t}.$$

(2) *For each $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell + 1$, each non-negative integer k and each $1 \le j \le n$, we have*

$$\|(N+1+n+t)^{-k-1}(M^{(t)*}_{z_j} - M^{(t+2k+2)*}_{z_j})f\|^2_t \le (2k+4)^4\frac{(n + 2k + 4 + t + \ell)^{2\ell}}{(\ell + 1 + n + t)^{2k+2}}\|f\|^2_{2k+4+t}.$$

Proof. For (1), it suffices to consider the case where f is a homogeneous polynomial of degree $m \ge \ell$, as it was the case for the corresponding part in [9]. For such an f,

$$\begin{aligned}\|(N + 1 + n + t)^{-k-1}f\|^2_t &= \frac{\|f\|^2_t}{(m + 1 + n + t)^{2k+2}} \\ &= \frac{\|f\|^2_{2k+2+t}}{(m + 1 + n + t)^{2k+2}}\prod_{j=1}^{m}\frac{n + 2k + 2 + t + j}{n + t + j} \quad \text{(see (4.2))} \\ &= \frac{\|f\|^2_{2k+2+t}}{(m + 1 + n + t)^{2k+2}}\prod_{j=1}^{2k+2}\frac{n + m + t + j}{n + t + j},\end{aligned}$$

where the last $=$ is obtained by considering $\prod_{j=1}^{2k+2+m}(n+t+j)$. Since $m \geq \ell$, for each $j \geq 1$ we have $(n+m+t+j)/(m+1+n+t) \leq (n+\ell+t+j)/(\ell+1+n+t)$. Hence

$$\|(N+1+n+t)^{-k-1}f\|_t^2 \leq \|f\|_{2k+2+t}^2 \prod_{j=1}^{2k+2} \frac{n+\ell+t+j}{(\ell+1+n+t)(n+t+j)}$$

$$= \frac{\|f\|_{2k+2+t}^2}{(\ell+1+n+t)^{2k+2}} \prod_{j=1}^{\ell} \frac{n+2k+2+t+j}{n+t+j} \leq \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \|f\|_{2k+2+t}^2.$$

This proves (1).

Let e_j be the element in $\mathbf{Z}_+^n$ whose j-th component is 1 and whose other components are 0. To prove (2), first note that (4.2) gives us

$$M_{z_j}^{(t)*} z^\alpha = \frac{\alpha_j}{n+t+|\alpha|} z^{\alpha-e_j}$$

whenever the j-th component α_j of α is greater than 0. Hence

$$(M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*}) z^\alpha = \frac{\alpha_j(2k+2)}{(n+t+|\alpha|)(n+2k+2+t+|\alpha|)} z^{\alpha-e_j}$$

(5.1)
$$= \frac{2k+2}{n+t+|\alpha|} M_{z_j}^{(2k+2+t)*} z^\alpha.$$

For $f \in \mathbf{C}[z_1, \ldots, z_n]$ with $f(0) = 0$, $(N+n+t)^{-1}f$ is well defined. Thus we can define

$$f_{t,k} = (N+1+n+2k+2+t)(N+n+t)^{-1}f.$$

We have $\|f_{t,k}\|_\tau \leq (2k+4)\|f\|_\tau$ for every $\tau \geq -n$. Obviously, (5.1) implies

$$(M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*})f = (2k+2)M_{z_j}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1} f_{t,k}$$

Now suppose that $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell + 1$. Applying (1) twice, we have

$$\begin{aligned}
&\|(N+1+n+t)^{-k-1}(M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*})f\|_t^2 \\
&\quad = (2k+2)^2 \|(N+1+n+t)^{-k-1} M_{z_j}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1} f_{t,k}\|_t^2 \\
&\quad \leq (2k+2)^2 \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \|M_{z_j}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1} f_{t,k}\|_{2k+2+t}^2 \\
&\quad \leq (2k+2)^2 \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \|(N+1+n+2k+2+t)^{-1} f_{t,k}\|_{2k+2+t}^2 \\
&\quad \leq (2k+2)^2 \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \cdot \frac{(n+2k+4+t+\ell)^\ell}{(\ell+1+n+2k+2+t)^2} \|f_{t,k}\|_{2k+4+t}^2 \\
&\quad \leq (2k+2)^2 \frac{(n+2k+4+t+\ell)^{2\ell}}{(\ell+1+n+t)^{2k+2}} (2k+4)^2 \|f\|_{2k+4+t}^2.
\end{aligned}$$

This completes the proof of (2). $\square$

For each real number $t > -1$, define

$$a_{n,t} = \frac{1}{n!}\prod_{j=1}^{n}(t+j).$$

Using (4.2) and [15,Proposition 1.4.9], it is straightforward to verify that

$$\langle f, g\rangle_t = a_{n,t}\int f(\zeta)\overline{g(\zeta)}(1-|\zeta|^2)^t dv(\zeta) \tag{5.2}$$

for $f, g \in \mathcal{H}^{(t)}$, $t > -1$. In other words, if $t > -1$, then $\mathcal{H}^{(t)}$ is the weighted Bergman space $L_a^2(\mathbf{B}, a_{n,t}(1-|\zeta|^2)^t dv(\zeta))$.

Our next step requires the assumption that $t > -2$.

Proposition 5.3. *Let real number $t > -2$ and integer $K \geq 1$ be given. Then there is a constant $C_{5.3} = C_{5.3}(n, K, t)$ such that the following estimate holds: Let $q \in \mathbf{C}[z_1, \dots, z_n]$ be such that $\deg(q) = K$. Suppose that $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfies the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| \leq \ell + 1$, where $\ell \in \mathbf{N}$. Then for every integer $k \geq 0$ and every $j \in \{1, \dots, n\}$,*

$$\begin{aligned}\|(N+1+n+t)^{-k-1}&(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j}M^{(t)}_{R^{k+1}q})f\|_t \\ &\leq \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}}C^{k+1}_{5.3}\|qf\|_{t+1}.\end{aligned}$$

Proof. Since

$$M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j}M^{(t)}_{R^{k+1}q} = (M^{(t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j}M^{(t)}_{R^{k+1}q}) - (M^{(t)*}_{z_j} - M^{(2k+2+t)*}_{z_j})M^{(t)}_{R^{k+1}q},$$

we have

$$\|(N+1+n+t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j}M^{(t)}_{R^{k+1}q})f\|_t \leq A + B, \tag{5.3}$$

where

$$\begin{aligned} A &= \|(N+1+n+t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j}M^{(t)}_{R^{k+1}q})f\|_t \quad \text{and} \\ B &= \|(N+1+n+t)^{-k-1}(M^{(t)*}_{z_j} - M^{(2k+2+t)*}_{z_j})M^{(t)}_{R^{k+1}q}f\|_t. \end{aligned}$$

We estimate A and B separately. For A, we apply Proposition 5.2(1), which gives us

$$\begin{aligned} A &\leq \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{k+1}}\|(M^{(t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j}M^{(t)}_{R^{k+1}q})f\|_{2k+2+t} \\ &= \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{k+1}}\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j}M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t}. \end{aligned} \tag{5.4}$$

Since $t > -2$, we have $2k + 2 + t > 0$ for each $k \geq 0$. Hence $\mathcal{H}^{(2k+2+t)}$ is a weighted Bergman space. By (5.2), we have

$$\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \\ \leq a^{1/2}_{n,2k+2+t} \left(\int |\{(\partial_j R^k q)(z) - \bar{z}_j (R^{k+1}q)(z)\} f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.$$

The identity $\partial_j - \bar{z}_j R = (1 - |z|^2)\partial_j + \sum_{i \neq j} z_i L_{j,i}$ then leads to

$$\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \\ \leq a^{1/2}_{n,2k+2+t} \left(\int |(\partial_j R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+4+t} dv(z) \right)^{1/2} \\ + a^{1/2}_{n,2k+2+t} \sum_{i \neq j} \left(\int |(L_{j,i} R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}. \tag{5.5}$$

Applying Propositions 3.6 and 3.5, we have

$$\int |(\partial_j R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+4+t} dv(z) \\ \leq C^{K(2k+4+t)}_{3.6} (K!)^2 \int |(R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \\ \leq (C_{3.6} C_{3.5})^{K(3k+4+t)} (K!)^4 \int |q(z) f(z)|^2 (1 - |z|^2)^{2+t} dv(z). \tag{5.6}$$

Since $1 + t > -1$, we can apply Propositions 3.4 and 3.5 to obtain

$$\int |(L_{j,i} R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \\ \leq C_{3.4} 2^{2k+2+t} (K!)^2 \int |(R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+1+t} dv(z) \\ \leq C_{3.4} (2C_{3.5})^{K(3k+2+t)} (K!)^4 \int |q(z) f(z)|^2 (1 - |z|^2)^{1+t} dv(z). \tag{5.7}$$

By the assumption $t > -2$, we have $a_{n,1+t} \geq (n!)^{-1}(2 + t)^n$. Also note that $a_{n,2k+2+t} \leq (n!)^{-1}(n + 2k + 2 + t)^n$. Combining (5.5), (5.6), (5.7) and (5.2), we see that there is a C_1 that depends only on n, K and t (> -2) such that

$$\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \leq C^{k+1}_1 \|qf\|_{t+1}.$$

Recalling (5.4), this gives us

$$A \leq \frac{(n + 2k + 2 + t + \ell)^\ell}{(\ell + 1 + n + t)^{k+1}} C^{k+1}_1 \|qf\|_{t+1}. \tag{5.8}$$

It follows from Proposition 5.2(2) that

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} \|M^{(t)}_{R^{k+1}q} f\|_{2k+4+t}.$$

Applying (5.2) and Proposition 3.5, we obtain

$$\begin{aligned}\|M^{(t)}_{R^{k+1}q} f\|^2_{2k+4+t} &= a_{n,2k+4+t} \int |(R^{k+1}q)(z)f(z)|^2(1-|z|^2)^{2k+4+t}dv(z) \\ &\le a_{n,2k+4+t} C_{3.5}^{K(3k+5+t)}(K!)^2 \int |q(z)f(z)|^2(1-|z|^2)^{2+t}dv(z).\end{aligned}$$

Thus there is a C_2 that depends only on n, K and t (> -2) such that $\|M^{(t)}_{R^{k+1}q} f\|_{2k+4+t} \le C_2^{k+1}\|qf\|_{t+1}$. Consequently,

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_2^{k+1}\|qf\|_{t+1}.$$

Combining this with (5.8) and (5.3), the proof of the proposition is complete. □

Proof of Theorem 1.1. Let $q \in \mathbf{C}[z_1, \dots, z_n]$ be such that $\deg(q) = K$, $K \ge 1$. Let $t > -2$ also be given. For this pair of K and t, let $C_{5.3} = C_{5.3}(n, K, t)$ be the constant provided by Proposition 5.3. Let $\ell \in \mathbf{N}$ satisfy the condition

$$\ell + 1 + n + t > 2C_{5.3}. \tag{5.9}$$

With this ℓ, we now define

$$E = \{qf : f \in \mathbf{C}[z_1, \dots, z_n],\ (\partial^\alpha f)(0) = 0 \text{ for } |\alpha| \le \ell + 1\}.$$

For the given q, let $Q^{(t)}$ denote the orthogonal projection from $\mathcal{H}^{(t)}$ onto $\mathcal{H}^{(t)} \ominus [q]^{(t)}$. Let $j \in \{1, \dots, n\}$, and let $f \in \mathbf{C}[z_1, \dots, z_n]$ be such that $(\partial^\alpha f)(0) = 0$ for $|\alpha| \le \ell + 1$. Then

$$Q^{(t)}M^{(t)*}_{z_j} qf = Q^{(t)}M^{(t)*}_{z_j} M^{(t)}_q f = Q^{(t)}(M^{(t)*}_{z_j} M^{(t)}_q - M^{(t)}_q M^{(t)*}_{z_j})f.$$

Applying Propositions 5.1 and 5.3, we have

$$\begin{aligned}\|Q^{(t)}M^{(t)*}_{z_j} qf\|_t &\le \sum_{k=0}^\infty \|(N+1+n+t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j} M^{(t)}_{R^{k+1}q})f\|_t \\ &\le \sum_{k=0}^\infty \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}\|qf\|_{t+1}. \end{aligned} \tag{5.10}$$

Set

$$C = \sum_{k=0}^\infty \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}.$$

Then (5.9) ensures that $C < \infty$. Thus (5.10) can be restated as

$$\|Q^{(t)} M_{z_j}^{(t)*} g\|_t \leq C\|g\|_{t+1} \quad \text{for every} \quad g \in E.$$

Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)} : \mathcal{H}^{(t)} \to E^{(t)}$ be the orthogonal projection. By Proposition 4.2, the above implies that

$$Q^{(t)} M_{z_j}^{(t)*} \mathcal{E}^{(t)} \in \mathcal{C}_{2n}^{+}.$$

Obviously, $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension. That is, if $P^{(t)}$ denotes the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$, then $\operatorname{rank}(P^{(t)} - \mathcal{E}^{(t)}) < \infty$. Therefore

$$Q^{(t)} M_{z_j}^{(t)*} P^{(t)} \in \mathcal{C}_{2n}^{+}.$$

Combining this with the well-known fact that $[M_{z_j}^{(t)*}, M_{z_i}^{(t)}] \in \mathcal{C}_n^{+}$, it follows from a routine argument that $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}] \in \mathcal{C}_n^{+}$, $i, j \in \{1, \ldots, n\}$. This completes the proof. □

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