Abstract. Recently, Douglas and Wang proved that for each polynomial $q$, the submodule $[q]$ of the Bergman module generated by $q$ is essentially normal [9]. Using improved techniques, we show that the Hardy-space analogue of this result holds, and more.

1. Introduction

Let $B$ be the unit ball in $\mathbb{C}^n$. Throughout the paper, the complex dimension $n$ is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space $H^2_n$ is the Hilbert space of analytic functions on $B$ with $(1 - \langle \zeta, \overline{z} \rangle)^{-1}$ as its reproducing kernel. The space $H^2_n$ is naturally considered as a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$. In [3-6], Arveson raised the question of whether graded submodules $\mathcal{M}$ of $H^2_n$ are essentially normal. That is, for the restricted operators

$$Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \quad 1 \leq j \leq n,$$

on $\mathcal{M}$, do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,7,10,13,14]. In particular, Guo and Wang showed that the answer to the above question is affirmative if $\mathcal{M}$ is generated by a homogeneous polynomial [14]. In [8], Douglas proposed analogous essential normality problems for submodules of the Bergman module $L^2_a(B, dv)$.

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [9] for arbitrary polynomials. In that paper, Douglas and Wang showed that for any polynomial $q \in \mathbb{C}[z_1, \ldots, z_n]$, the submodule $[q]$ of the Bergman module generated by $q$ is $p$-essentially normal for $p > n$. What is especially remarkable is that [9] contains many novel ideas.

The present paper grew out of a remark in [9]. Toward the end of [9], Douglas and Wang commented

"It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from [9,8] may be needed to complete the proofs."

While the Drury-Arveson space case is out of reach at the moment, in this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

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The key realization is that Bergman space, Hardy space and Drury-Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on \( B \) parametrized by a real-valued parameter \(-n \leq t < \infty\). In fact, the spaces corresponding to the values \( t \in \mathbb{Z}^+ \) were used in an essential way in the proofs in [9]. Our main observation is that if one considers other values of \( t \), then one will see how to extend the techniques in [9] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [9] for spaces with parameter \(-2 < t < \infty\). Before stating the result, let us first introduce these spaces.

For each real number \(-n \leq t < \infty\), let \( H(t) \) be the Hilbert space of analytic functions on \( B \) with the reproducing kernel

\[
\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.
\]

Alternately, one can describe \( H(t) \) as the completion of \( C[z_1, \ldots, z_n] \) with respect to the norm \( \| \cdot \|_t \) arising from the inner product \( \langle \cdot, \cdot \rangle_t \) defined according to the following rules:

\[
\langle z^\alpha, z^\beta \rangle_t = 0 \quad \text{whenever } \alpha \neq \beta,
\]

\[
\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{\alpha} (n + t + j)}
\]

if \( \alpha \in \mathbb{Z}_n^+ \setminus \{0\} \), and \( \langle 1, 1 \rangle_t = 1 \). Here and throughout the paper, we use the conventional multi-index notation [15, page 3].

Obviously, \( H(0) \) is the Bergman space \( L^2_a(B, dv) \). One can view the Bergman space \( H(0) = L^2_a(B, dv) \) as a benchmark, against which the other spaces in the family should be compared. Note that for each \(-1 < t < \infty\), \( H(t) \) is a weighted Bergman space.

Let \( S \) denote the unit sphere \( \{ z \in \mathbb{C}^n : |z| = 1 \} \) in \( \mathbb{C}^n \). Let \( \sigma \) be the positive, regular Borel measure on \( S \) that is invariant under the orthogonal group \( O(2n) \), i.e., the group of isometries on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) that fix 0. We take the usual normalization \( \sigma(S) = 1 \). Recall that the Hardy space \( H^2(S) \) is the closure of \( C[z_1, \ldots, z_n] \) in \( L^2(S, d\sigma) \).

Obviously, \( H(-1) \) is just the Hardy space \( H^2(S) \). Moreover, \( H(-n) \) is none other than the Drury-Arveson space \( H_n^2 \).

It is well known that for each \(-n \leq t < -1\), the tuple of multiplication operators \((M_{z_1}, \ldots, M_{z_n})\) is not jointly subnormal on \( H(t) \) [1, Theorem 3.9]. In other words, if \(-n \leq t < -1\), then \( H(t) \) is more like the Drury-Arveson space than the Hardy space. The practical consequence of this is that it is difficult to do estimates on \( H(t) \) if \(-n \leq t < -1\).

Let \( q \in C[z_1, \ldots, z_n] \). For each \(-n \leq t < \infty\), let \([q]^{(t)}\) denote the closure of

\[
\{ qf : f \in C[z_1, \ldots, z_n] \}
\]

in \( H(t) \). Since \( H(t) \) is a Hilbert module over \( C[z_1, \ldots, z_n] \), \([q]^{(t)}\) is a submodule. For each \( j \in \{1, \ldots, n\} \), define submodule operator

\[
Z_{q,j}^{(t)} = M_{z_j} [q]^{(t)}.
\]
Recall that the submodule \([q](t)\) is said to be \(p\)-essentially normal if the commutators 
\([Z_{q,j}^{(t)}, Z_{q,i}^{(t)}], i, j \in \{1, \ldots, n\}\), all belong to the Schatten class \(C_p\). With the foregoing preparation, we are now ready to state our result.

**Theorem 1.1.** Let \(q\) be an arbitrary polynomial in \(C[z_1, \ldots, z_n]\). Then for each real number \(-2 < t < \infty\), the submodule \([q](t)\) of \(H^{(t)}\) is \(p\)-essentially normal for every \(p > n\).

Clearly, the Hardy-space case mentioned in [9] is settled by applying Theorem 1.1 to the special case \(t = -1\).

On the other hand, it is a real pity that the requirement \(t > -2\) in Theorem 1.1 does not allow us to capture any Drury-Arveson space in dimensions \(n \geq 2\). But as a consolation, Theorem 1.1 does cover spaces \(H^{(t)}\) for \(-2 < t < -1\), which, as we mentioned, are more Drury-Arveson-like than Hardy-like.

On the technical side, this paper does offer some improvement over [9]. As the authors of [9] stated, the key step in the proof of their result rests on weighted norm estimates given in Section 3 in that paper. At the core of their weighted estimates is an argument using a covering lemma. This is where we offer the most significant improvement. In this paper, the covering-lemma argument of [9] is done away with entirely. In its place, we use a much simpler argument based on Fubini’s theorem.

In fact, using Fubini’s theorem-based argument in place of covering-lemma argument is a situation with which we are quite familiar. See, for example, the proofs of Proposition 2.6 and Lemma 5.2 in [11].

There are many technical contributions made in [9]. Perhaps, the most important among these is Lemma 3.2 in that paper. This lemma will again be the basis for analysis here. The reader will see that with the combination of [9, Lemma 3.2] and our Fubini’s theorem-based argument, the analysis part of the proof is actually easy.

As it was the case in [9], an essential role in the proof is played by the number operator \(N\) introduced by Arveson in [2]. Recall that, for a polynomial \(f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}\),

\[(Nf)(z) = \sum_{\alpha} c_{\alpha} |\alpha| z^{\alpha}.\]

Here as well as in [9], the proof boils down to the estimate of an operator series where the \(k\)-th term has the operator

\[(N + 1 + n + t)^{-k-1}\]

as a factor, \(k \geq 0\). Douglas and Wang’s idea is to factor the above in the form

\[(N + 1 + n + t)^{-k-1} = (N + 1 + n + t)^{-1/2} \cdot (N + 1 + n + t)^{-k-(1/2)},\]

“reserve” the factor \((N + 1 + n + t)^{-1/2}\) for establishing the requisite Schatten-class membership, and use the other factor, \((N + 1 + n + t)^{-k-(1/2)}\), to boost the weight of the space. This is another place where [9] and the present paper differ. Instead of factoring, we will apply the whole of \((N + 1 + n + t)^{-k-1}\) to boost weight. Proposition 4.2 below allows us
to recover an equivalent of \((N + 1 + n + t)^{-1/2}\) at the end of the estimate. This is why we are able to push \(t\) below \(-1\).

The rest of the paper is organized as follows. Since the analysis part of the proof is now easy, we will take care of that first, in Sections 2 and 3. Section 4 contains a brief discussion of the relation between the natural embedding \(\mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}\) and norm ideals. Section 5, which mirrors Section 2 in [9], contains the proof of our result.

2. Derivative on the Disc

Write \(D\) for the open unit disc \(\{z \in \mathbb{C} : |z| < 1\}\) in the complex plane. Let \(dA\) be the area measure on \(D\) with the normalization \(A(D) = 1\). The unit circle \(\{\tau \in \mathbb{C} : |\tau| = 1\}\) will be denote by \(T\). Furthermore, let \(dm\) be the Lebesgue measure on \(T\) with the normalization \(m(T) = 1\). For convenience, we write \(\partial\) for the one-variable differentiation \(d/dz\) on \(\mathbb{C}\).

Our first lemma is basically a restatement of Lemma 3.2 in [9].

**Lemma 2.1.** Suppose that \(g\) is a one-variable polynomial of degree \(K \geq 1\), and that \(f\) is analytic on \(D\). Then for each \(k \in \mathbb{N}\) we have

\[
| (\partial^k g)(0) f(0) |^2 \leq 2^{2k+2} (K!)^2 \int |gf|^2 dA.
\]

**Proof.** For each \(0 \leq r < 1\), let \(g_r(z) = g(rz)\) and \(f_r(z) = f(rz)\). We only need to consider the case \(1 \leq k \leq K\). For such a \(k\), Lemma 3.2 in [9] tells us that

\[
| (\partial^k g_r)(0) f_r(0) | \leq K! \int_T |g_r f_r| dm.
\]

Since \((\partial^k g_r)(0) = r^k (\partial^k g)(0)\) and \(f_r(0) = f(0)\), we have

\[
| (\partial^k g)(0) f(0) | = 2 \int_{1/2}^1 r^{-k} |(\partial^k g_r)(0) f_r(0)| dr \leq 2K! \int_{1/2}^1 r^{-k} \int_T |g_r(\tau) f_r(\tau)| dm(\tau) dr
\]

\[
\leq 2^{k+1} K! \int_{1/2}^1 2r \int_T |g(r\tau) f(r\tau)| dm(\tau) dr \leq 2^{k+1} K! \int |gf| dA.
\]

Squaring both sides and applying the Cauchy-Schwarz inequality, the lemma follows. \(\square\)

For each \(z \in D\), define the disc \(D(z) = \{w \in D : |w - z| < (1/2)(1 - |z|)\}\).

**Lemma 2.2.** For all \(w \in D\) and \(x \in (-1, \infty)\), we have

\[
\int \frac{(1 - |z|^2)^x}{A(D(z))} \chi_{D(z)}(w) dA(z) \leq 2^{2\max\{x,0\}+5} (1 - |w|^2)^x.
\]

**Proof.** Let \(w \in D\), and let \(z \in D\) be such that \(w \in D(z)\). Then we have \(1 - |w| \leq 1 - |z| + |z - w| < (3/2)(1 - |z|)\). Also, \(1 - |z| \leq 1 - |w| + |w - z| \leq 1 - |w| + (1/2)(1 - |z|)\). After cancellation, we find \((1/2)(1 - |z|) \leq 1 - |w|\). Thus

\[
(2.1) \quad (2/3)(1 - |w|) \leq 1 - |z| \leq 2(1 - |w|) \quad \text{whenever} \quad w \in D(z).
\]
From this we obtain that for $w \in D(z)$ and $x \in (-1, \infty)$,

$$(1 - |z|^2)^x \leq \begin{cases} 
2^{2x}(1 - |w|^2)^x & \text{if } 0 \leq x < \infty \\
3(1 - |w|^2)^x & \text{if } -1 < x < 0 
\end{cases}.$$

Thus, to complete the proof, it suffices to show that

\begin{equation}
(2.2) \quad \int \frac{\chi_D(z)(w)}{A(D(z))} dA(z) \leq 9
\end{equation}

for every $w \in D$. For each $w \in D$, let $G(w) = \{ z \in D : w \in D(z) \}$. If $z \in G(w)$, then $|z - w| \leq (1/2)(1 - |z|) \leq 1 - |w|$ by (2.1). Hence $A(G(w)) \leq (1 - |w|)^2$. On the other hand, if $z \in G(w)$, then $A(D(z)) = (1/4)(1 - |z|^2) \geq (1/3)^2(1 - |w|)^2$, also by (2.1). Clearly, (2.2) follows from these two inequalities. □

**Proposition 2.3.** Suppose that $g$ is a one-variable polynomial of degree $K \geq 1$, and that $f$ is analytic on $D$. Then for all $k \in \mathbb{N}$ and $t \in (0, \infty)$ satisfying the condition $t - 2k > -1$,

$$\int |(\partial^k g)(z)f(z)|^2 (1 - |z|^2)^t dA(z) \leq 2^{6k+2\max\{t-2k,0\}+7}(K!)^2 \int |g(w)f(w)|^2 (1 - |w|^2)^{t-2k} dA(w).$$

**Proof.** Define $g_z(u) = g(z + (1/2)(1 - |z|)u)$ and $f_z(u) = f(z + (1/2)(1 - |z|)u)$ for each $z \in D$. Then $2^{-k}(1 - |z|)^k(\partial^k g)(z) = (\partial^k g_z)(0)$ and $f(z) = f_z(0)$. By Lemma 2.1,

$$|(\partial^k g)(z)f(z)|^2 = \frac{2^{2k}|(\partial^k g_z)(0)f_z(0)|^2}{(1 - |z|^2)^{2k}} \leq \frac{2^{4k+2}(K!)^2}{(1 - |z|^2)^{2k}} \int |g_z(u)f_z(u)|^2 dA(u)$$

$$= \frac{2^{4k+2}(K!)^2}{(1 - |z|^2)^{2k}} \cdot \frac{1}{A(D(z))} \int_{D(z)} |g(w)f(w)|^2 dA(w).$$

Therefore, if $t - 2k > -1$, then

$$\int |(\partial^k g)(z)f(z)|^2 (1 - |z|^2)^t dA(z)$$

$$\leq 2^{6k+2}(K!)^2 \int (1 - |z|^2)^{t-2k} \left( \int_{D(z)} |g(w)f(w)|^2 dA(w) \right) dA(z)$$

$$= 2^{6k+2}(K!)^2 \int \left\{ \int (1 - |z|^2)^{t-2k} \frac{\chi_D(z)(w)}{A(D(z))} dA(z) \right\} |g(w)f(w)|^2 dA(w)$$

$$\leq 2^{6k+2\max\{t-2k,0\}+7}(K!)^2 \int (1 - |w|^2)^{t-2k} |g(w)f(w)|^2 dA(w),$$

where the last step is an application of Lemma 2.2. This completes the proof. □
3. Derivatives on the Ball

Recall that there is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^n \leq \sigma(\{ \xi \in S : |1 - \langle u, \xi \rangle| < r \}) \leq A_0 r^n$$

for all $u \in S$ and $0 < r \leq 2$ [15, Proposition 5.1.4]. For each $z \in B$, define the subset

$$T(z) = \{ w \in B : |1 - \langle w, z \rangle| < 2(1 - |z|^2), \ 1 - |w|^2 > (1/2)(1 - |z|^2) \}$$

of the unit ball. We begin our estimates with the properties of the set $T(z)$.

Let $dv$ be the volume measure on $B$ with the normalization $v(B) = 1$.

**Lemma 3.1.** There is a constant $0 < C_{3.1} < \infty$ such that for all $\zeta \in B$ and $x \in (-1, \infty)$,

$$\int (1 - |\zeta|^2)^{x-n-1} \chi_{T(z)}(\zeta) dv(z) \leq C_{3.1} 2^{\max\{x,0\}} (1 - |\zeta|^2)^x.$$

**Proof.** Let $z \in B$ be such that $\zeta \in T(z)$. Then we have $1 - |\zeta|^2 \leq 2(1 - |\zeta|) \leq 2|1 - \langle \zeta, z \rangle| < 4(1 - |z|^2)$. Combining this with the condition $1 - |\zeta|^2 > (1/2)(1 - |z|^2)$, we have

$$\frac{1}{4} (1 - |\zeta|^2) \leq 1 - |z|^2 \leq 2(1 - |\zeta|^2).$$

Therefore, for $x \in (-1, \infty)$ we have

$$(1 - |z|^2)^x \leq \begin{cases} 2^x (1 - |\zeta|^2)^x & \text{if } 0 \leq x < \infty \\ 4(1 - |\zeta|^2)^x & \text{if } -1 < x < 0 \end{cases}.$$ 

Thus, to complete the proof, it suffices to show that there is a $0 < C < \infty$ such that

$$\int \frac{\chi_{T(z)}(\zeta)}{(1 - |\zeta|^2)^{n+1}} dv(z) \leq C$$

for every $\zeta \in B$. Given a $\zeta \in B$, consider the set $\Omega(\zeta) = \{ z \in B : \zeta \in T(z) \}$. Write $\zeta = \zeta \eta$ with $\eta \in S$. If $z = |z| \xi \in \Omega(\zeta)$, where $\xi \in S$, then $|1 - \langle \eta, \xi \rangle| \leq 2|1 - \langle \zeta, \eta \rangle| < 4(1 - |\zeta|^2) < 8(1 - |\zeta|^2)$. Also, $1 - |z| \leq |1 - \langle \zeta, z \rangle| < 4(1 - |\zeta|^2)$ if $z \in \Omega(\zeta)$. Hence

$$\Omega(\zeta) \subset \{ r \xi : 0 < r < 4(1 - |\zeta|^2); \ \xi \in S, \ |1 - \langle \eta, \xi \rangle| < 8(1 - |\zeta|^2) \}.$$ 

By (3.1) and the decomposition $dv = 2n r^{2n-1} dr d\sigma$, there is a $0 < C_1 < \infty$ such that $v(\Omega(\zeta)) \leq C_1 (1 - |\zeta|^2)^{n+1}$ for every $\zeta \in B$. By (3.2), $(1 - |\zeta|^2)^{-n-1} \leq 4^{n+1}(1 - |\zeta|^2)^{-n-1}$ when $z \in \Omega(\zeta)$. Clearly, (3.3) follows from these two inequalities. □

**Lemma 3.2.** There is a constant $0 < \epsilon < 1$ such that for each $0 \leq a < 1$, the set $T((a, 0, \ldots, 0))$ contains the polydisc

$$P_a = \{(a + u, \zeta_2, \ldots, \zeta_n) : |u| < \epsilon(1 - a^2), \ |\zeta_j| < \epsilon \sqrt{1 - a^2}, 2 \leq j \leq n \}.$$
Proof. Given an \( a \in [0, 1) \), write \( \alpha = (a, 0, \ldots, 0) \). Let \( 0 < \epsilon < 1 \), and suppose that \( u \) and \( \zeta_2, \ldots, \zeta_n \) satisfy the conditions \(|u| < \epsilon(1 - a^2)\) and \(|\zeta_j| < \epsilon \sqrt{1 - a^2}, \ 2 \leq j \leq n\). Then consider the vector \( w = (a + u, \zeta_2, \ldots, \zeta_n) \). We have \(|1 - \langle w, \alpha \rangle| = |1 - a^2 - au| < (1 + \epsilon)(1 - a^2)\). Moreover, \( 1 - |w|^2 = 1 - |a + u|^2 - (|\zeta_2|^2 + \cdots + |\zeta_n|^2) \geq 1 - |a + u|^2 - (n - 1)\epsilon^2(1 - a^2) \). On the other hand, \( 1 - |a + u|^2 = 1 - (a^2 + 2Re(au) + |u|^2) \geq 1 - a^2 - 3|u| \geq (1 - 3\epsilon)(1 - a^2) \). Hence \( 1 - |w|^2 \geq (1 - (n + 2)\epsilon)(1 - a^2) \). Thus \( \epsilon = (3(n + 2))^{-1} \) suffices for our purpose. \( \square \)

As usual, write \( \partial_1, \ldots, \partial_n \) for the differentiations with respect to the complex variables \( z_1, \ldots, z_n \). For each vector \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \), define the directional derivative \( \partial_b = b_1 \partial_1 + \cdots + b_n \partial_n \).

**Lemma 3.3.** There is a constant \( 0 < C_{3.3} < \infty \) such that the following estimate holds: Suppose that \( q \in \mathbb{C}[z_1, \ldots, z_n] \) and that \( \deg(q) = K \geq 1 \). Let \( f \in \mathbb{C}[z_1, \ldots, z_n] \). If \( z \) and \( b \) are vectors in \( \mathbb{B} \setminus \{0\} \) satisfying the relation \( \langle b, z \rangle = 0 \), then

\[
|\langle \partial_b q \rangle(z) f(z) |^2 \leq \frac{C_{3.3}(K!)^2}{(1 - |z|^2)^{n+2}} \int_{T(z)} |q f|^2 dA.
\]

**Proof.** Consider the special case where \( z = \alpha = (a, 0, \ldots, 0) \) for some \( 0 < a < 1 \). Let \( \epsilon \) be the constant provided by Lemma 3.2. Define the polydisc

\[
Y = \{(a + u, 0, \zeta_3, \ldots, \zeta_n) : |u| < \epsilon(1 - a^2), \ |\zeta_j| < \epsilon \sqrt{1 - a^2}, \ 3 \leq j \leq n\}.
\]

For each \( y \in Y \), we define the one-variable polynomial \( q_y(w) = q(y + \epsilon \sqrt{1 - a^2}we_2) \), where \( e_2 = (0, 1, 0, \ldots, 0) \). Similarly, define \( f_y(w) = f(y + \epsilon \sqrt{1 - a^2}we_2) \) on \( D \). Since \( \langle \partial q_y \rangle(0) = \epsilon \sqrt{1 - a^2} \langle \partial q \rangle(y) \) and \( f_y(0) = f(y) \), we apply Lemma 2.1 to obtain

\[
|\langle \partial q \rangle(y) f(y) |^2 \leq \frac{16(K!)^2}{e^2(1 - a^2)^2} \int |q_y(w) f_y(w)|^2 dA(w).
\]

Making the substitution \( \zeta_2 = \epsilon \sqrt{1 - a^2} w \), we find that

\[
|\langle \partial q \rangle(y) f(y) |^2 \leq \frac{16(K!)^2}{e^2(1 - a^2)^2} \int_{|\zeta| < \epsilon \sqrt{1 - a^2}} |q(y + \zeta_2 e_2) f(y + \zeta_2 e_2)|^2 dA(\zeta_2).
\]

Now, integrating both sides over \( Y \), we see that

\[
e^{2n-2(1 - a^2)} |\langle \partial q \rangle(\alpha) f(\alpha) |^2 \leq \int_Y |\langle \partial q \rangle(y) f(y) |^2 dy \leq \frac{16C(K!)^2}{e^4(1 - a^2)^2} \int_{T(\alpha)} |q f|^2 dA,
\]

where \( P_a \) is given by (3.4) and \( C \) accounts for the normalization constants for the measures involved. Since Lemma 3.2 tells us that \( P_a \subset T(\alpha) \), we have

\[
|\langle \partial q \rangle(\alpha) f(\alpha) |^2 \leq \frac{16e^{-(2n+2)} C(K!)^2}{(1 - a^2)^{n+2}} \int_{T(\alpha)} |q f|^2 dA.
\]
Obviously, the above inequality also holds if we replace $\partial_2$ by $\partial_j$ for any $2 \leq j \leq n$. Applying these and the Cauchy-Schwarz inequality, we see that

$$|(\partial_bq)(\alpha)f(\alpha)|^2 \leq (n-1)\frac{16e^{-(2n+2)C(K)!^2}}{(1-a^2)^{n+2}} \int_{T(\alpha)} |qf|^2 dv \quad \text{if } \langle b, \alpha \rangle = 0, \ b \in B.$$ 

This proves the lemma in the special case where $z = \alpha = (a, 0, \ldots, 0), \ 0 < a < 1$. The general case follows from this special case and the following easily-verified relations:

If $U$ is any unitary transformation on $C^n$ and $w, b \in B$, then $UT(w) = T(Uw)$ and $(\partial_b(q \circ U))(w) = (\partial_Ubq)(Uw)$. □

Following [9], for each pair of $i \neq j$ in $\{1, \ldots, n\}$ we define $L_{i,j} = \bar{z}_j \partial_i - \bar{z}_i \partial_j$.

**Proposition 3.4.** There is a constant $1 \leq C_{3,4} < \infty$ such that the following estimate holds: Suppose that $q \in C[z_1, \ldots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in C[z_1, \ldots, z_n]$. Then for every positive number $t > 0$ and all integers $i \neq j$ in $\{1, \ldots, n\}$, we have

$$\int |(L_{i,j}q)(z)f(z)|^2(1 - |z|^2)^t dv(z) \leq C_{3,4}2^t(K!)^2 \int |q(\zeta)f(\zeta)|^2(1 - |\zeta|^2)^{t-1} dv(\zeta).$$

**Proof.** It follows from Lemma 3.3 that

$$|(L_{i,j}q)(z)f(z)|^2 \leq \frac{C_{3,3}(K)!^2}{(1 - |z|^2)^{n+2}} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta),$$

$z \in B$. Multiplying both sides by $(1 - |z|^2)^t$ and integrating, we find that

$$\int |(L_{i,j}q)(z)f(z)|^2(1 - |z|^2)^t dv(z) \leq C_{3,3}(K)!^2 \int \left( (1 - |z|^2)^{t-n-2} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta) \right) dv(z)$$

$$= C_{3,3}(K)!^2 \int \left\{ (1 - |z|^2)^{t-n-2} \chi_{T(z)}(z) dv(z) \right\} |q(\zeta)f(\zeta)|^2 dv(\zeta).$$

Applying Lemma 3.1 with $x = t - 1$ to the $\{\cdots\}$ above, the proposition follows. □

Write $R = z_1 \partial_1 + \cdots + z_n \partial_n$, the radial derivative in $n$ variables. We will denote the one-variable radial derivative by $R$. For each polynomial $h$ and each $\xi \in S$, define the “slice” function $h_\xi(z) = h(z\xi), \ z \in D$. If $q$ is a polynomial in $n$ variables, then for every $\xi \in S$ we have the relation $(Rq_\xi)(z) = (Rq)\xi(z)$.

**Proposition 3.5.** There is a constant $1 \leq C_{3,5} < \infty$ such that the following estimate holds: Suppose that $q \in C[z_1, \ldots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in C[z_1, \ldots, z_n]$. Then for each pair of $k \in N$ and $t \in (0, \infty)$ satisfying the condition $t - 2k > -1$,

$$\int |(R^kq)(\zeta)f(\zeta)|^2(1 - |\zeta|^2)^t dv(\zeta) \leq C_{3,5}^{K(k+t)}(K!)^2 \int |q(\zeta)f(\zeta)|^2(1 - |\zeta|^2)^{t-2k} dv(\zeta).$$
Proof. As in [9], we need the following relation between \(dv, d\sigma\) and \(dA\): Since \(dv = 2nr^{2n-1}drd\sigma\), \(dA = 2rdrdm\), and \(d\sigma\) is invariant under rotation, we have

\[
(3.5) \quad \int gdv = n \int \left( \int g(z\xi)|z|^{2n-2}dA(z) \right) d\sigma(\xi).
\]

By Lemma 3.6 in [9], for each \(k \in \mathbb{N}\),

\[
(3.6) \quad \mathcal{R}^k = \sum_{j=1}^{k} a_j^{(k)} z^j \partial^j \quad \text{with} \quad |a_j^{(k)}| < (j+1)^k.
\]

Since the degree of \(q\) equals \(K\), for each \(\xi \in S\) we have

\[
(R^k q)_\xi(z) = (\mathcal{R}^k q)_\xi(z) = \sum_{j=1}^{\min\{k,K\}} a_j^{(k)} z^j (\partial^j q)_\xi(z).
\]

Given \(f\), for each \(\xi \in S\) we define the “rigid” slice function \(f(\xi)(z) = z^{n-1} f(z\xi), z \in D\). Applying first (3.6) and then Proposition 2.3, when \(t - 2k > -1\), we have

\[
\int |(R^k q)_\xi(z) f(\xi)(z)|^2 (1 - |z|^2)^t dA(z)
\]

\[
\leq K(K+1)^{2k} \sum_{j=1}^{\min\{k,K\}} \int |(\partial^j q)_\xi(z) f(\xi)(z)|^2 (1 - |z|^2)^t dA(z)
\]

\[
\leq K(K+1)^{2k} \sum_{j=1}^{\min\{k,K\}} 2^{6j+2 \max\{t-2j,0\}+7(K!)} 2^j \int |q_\xi(z) f(\xi)(z)|^2 (1 - |z|^2)^{t-2j} dA(z)
\]

\[
\leq K^2(K+1)^{2k} 2^{6k+2t+7(K!)} 2^j \int |q_\xi(z) f(\xi)(z)|^2 (1 - |z|^2)^{t-2k} dA(z)
\]

\[
\leq C_{3.5}^{K(k+t)} (K!)^2 \int |q_\xi(z) f(\xi)(z)|^2 (1 - |z|^2)^{t-2k} dA(z).
\]

By the relations \((R^k q)_\xi(z) = (R^k q)_\xi(z), f(\xi)(z) = z^{n-1} f(z\xi)\) and \(|z| = |z\xi|\), we now have

\[
\int |(R^k q)(z\xi)f(z\xi)|^2 (1 - |z\xi|^2)^t |z|^{2n-2} dA(z)
\]

\[
\leq C_{3.5}^{K(k+t)} (K!)^2 \int |q(z\xi)f(z\xi)|^2 (1 - |z\xi|^2)^{t-2k} |z|^{2n-2} dA(z).
\]

Integrating both sides with respect to the measure \(d\sigma\) on \(S\) and applying (3.5), the proposition follows. \(\square\)

**Proposition 3.6.** There is a constant \(1 \leq C_{3.6} < \infty\) such that the following estimate holds: Suppose that \(q \in C[z_1, \ldots, z_n]\) and that \(\deg(q) = K \geq 1\). Let \(f \in C[z_1, \ldots, z_n]\). Then for each \(t \in (1, \infty)\) and each \(j \in \{1, \ldots, n\}\), we have

\[
(3.7) \quad \int |(\partial_j q)(\zeta)f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \leq C_{3.6}^{K_t}(K!)^2 \int |q(\zeta)f(\zeta)|^2 (1 - |\zeta|^2)^{t-2} dv(\zeta).
\]
Proof. There is a $C$ such that for every analytic function $h$ on $B$ and every $t > 0$, we have
\begin{equation}
(3.8) \quad \int_{|\zeta|<1/2} |h(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \leq C \left( \frac{16}{7} \right)^t \int_{1/2 \leq |\zeta| < 3/4} |h(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta).
\end{equation}
Now apply Proposition 3.4 and the case $k = 1$ in Proposition 3.5: by the identity $|z|^2 \partial_j = \bar{z}_j R + \sum_{i \neq j} z_i L_{j,i}$, (3.7) obviously holds if $|\zeta|^2 (\partial_j q)(\zeta)$ is replaced by $|\zeta|^2 (\bar{\partial}_j q)(\zeta)$ on the left-hand side. The extra factor $|\zeta|^2$ is then removed by using (3.8). \qed

4. Embedding and Norm Ideals

For a bounded operator $A$, we write its $s$-numbers as $s_1(A), \ldots, s_k(A), \ldots$ as usual. Recall that, for each $1 \leq p < \infty$, the formula
\begin{equation}
(4.1) \quad \|A\|_p^+ = \sup_{k \geq 1} s_1(A) + s_2(A) + \cdots + s_k(A)
\end{equation}
defines a symmetric norm for operators [12,Section III.14]. On any Hilbert space $\mathcal{H}$, the set $C_p^+ = \{ A \in B(\mathcal{H}) : \|A\|_p^+ < \infty \}$ is a norm ideal [12,Section III.2]. It is well known that if $p < p'$, then $C_p^+$ is contained in the Schatten class $C_{p'}^+$.

For a non-increasing sequence of non-negative numbers $\{a_1, \ldots, a_k, \ldots\}$, if $a_1 + \cdots + a_k \leq C (1-1/p + \cdots + k^{-1/p})$, then $ka_k \leq C (1-1/p + \cdots + k^{-1/p})$. It follows that if $p > 1$ and if $T \in C_p^+$, then there is a $0 < C(T) < \infty$ such that $s_k(T) \leq C(T) k^{-1/p}$ for every $k \in \mathbb{N}$. Thus if $p > 1$ and if $B$ is a bounded operator such that $B^* B \in C_{p}^+$, then $B \in C_{2p}^+$.

Proposition 4.1. For each $t \geq -n$, let $I^{(t)} : \mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ be the natural embedding. Then $I^{(t)} I^{(t)} \in C_n^+$.

Proof. Expanding the reproducing kernel $(1 - \langle \zeta, z \rangle)^{-(n+1+t)}$, we see that the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\{ e_\alpha^{(t)} : \alpha \in \mathbb{Z}_n^+ \}$, where
\begin{equation}
(4.2) \quad e_\alpha^{(t)}(\zeta) = \left( \frac{1}{\alpha!} \prod_{j=1}^{\alpha} (n + t + j) \right)^{1/2} \zeta^\alpha, \quad \alpha \neq 0,
\end{equation}
and $e_0^{(t)}(\zeta) = 1$. Given these orthonormal bases, it is straightforward to verify that
\[ I^{(t)} I^{(t)} e_\alpha^{(t)} = \frac{n+1+t}{n+1+|\alpha|+t} e_\alpha^{(t)}, \quad \alpha \in \mathbb{Z}_n^+ \]
The formula gives us all the $s$-numbers for $I^{(t)} I^{(t)}$. By (4.1), $I^{(t)} I^{(t)} \in C_n^+$. \qed

Proposition 4.2. Suppose that $E$ is a linear subspace of $C[z_1, \ldots, z_n]$ and that $t \geq -n$. Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $E^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in B(\mathcal{H}^{(t)})$, and suppose that there is a $C$ such that
\begin{equation}
(4.3) \quad \|Ag\|_t \leq C\|g\|_{t+1}
\end{equation}
for every $g \in E$. Then $A\mathcal{E}^{(t)} \in C_{2n}^+.$

**Proof.** By (4.3), for each $g \in E$ we have

$$\langle A^* Ag, g \rangle_t = \|Ag\|_t^2 \leq C^2 \|g\|_{t+1}^2 = C^2 \|I^{(t)}g\|_{t+1}^2 = C^2 \langle I^{(t)}g, I^{(t)}g \rangle_{t+1} = C^2 \langle I^{(t)}I^{(t)}g, g \rangle_t.$$  

That is, the operator inequality $(A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)} \leq C^2 \mathcal{E}^{(t)} f^{(t)} f^{(t)} \mathcal{E}^{(t)}$ holds on $\mathcal{H}^{(t)}$. Thus $s_j((A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)}) \leq s_j(C^2 \mathcal{E}^{(t)} I^{(t)}I^{(t)} \mathcal{E}^{(t)})$ for each $j \in \mathbb{N}$ [12, Lemma II.1.1]. By Proposition 4.1, $(A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)} \in C_{2n}^+$. Since $n \geq 2$, this implies $A\mathcal{E}^{(t)} \in C_{2n}^+$.

### 5. Proof of Theorem 1.1

For each $t \geq -n$ and each polynomial $q$, we write $M_q^{(t)}$ for the operator of multiplication by $q$ on the space $\mathcal{H}^{(t)}$. Keep in mind that the notation “*” is $t$-specific: $M_q^{(t)*}$ means the adjoint of $M_q^{(t)}$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$.

**Proposition 5.1.** Let $q \in \mathbb{C}[z_1, \ldots, z_n]$, $1 \leq j \leq n$ and $t \geq -n$. For $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $f(0) = 0$, we have

$$M_{z_j}^{(t)*} M_q^{(t)} f - M_q^{(t)} M_{z_j}^{(t)*} f = \sum_{k=0}^{\infty} (N + 1 + n + t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1} q}^{(t)}) f.$$

**Proof.** The main idea is that both sides are linear with respect to both $q$ and $f$. Therefore the proof is a matter of straightforward verification in the special case of $q = z^\alpha$ and $f = z^\beta$, $\beta \neq 0$, using (4.2). The details of the verification are similar to the Bergman space case (see the proof of Proposition 2.1 in [9]).

**Proposition 5.2.** Let $t \geq -n$ and $\ell \in \mathbb{N}$. (1) For each $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell$ and each non-negative integer $k$, we have

$$\|(N + 1 + n + t)^{-k-1} f\|_t^2 \leq \frac{(n + 2k + 2 + t + \ell)^\ell}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+2+t}^2.$$

(2) For each $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell + 1$, each non-negative integer $k$ and each $1 \leq j \leq n$, we have

$$\|(N + 1 + n + t)^{-k-1} (M_{z_j}^{(t)*} - M_{z_j}^{(t+2k+2)*}) f\|_t^2 \leq (2k+4)^4 \frac{(n + 2k + 4 + t + \ell)^{2\ell}}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+4+t}^2.$$

**Proof.** For (1), it suffices to consider the case where $f$ is a homogeneous polynomial of degree $m \geq \ell$, as it was the case for the corresponding part in [9]. For such an $f$,

$$\|(N + 1 + n + t)^{-k-1} f\|_t^2 = \frac{\|f\|_t^2}{(m + 1 + n + t)^{2k+2}} \leq \frac{\|f\|_{2k+2+t}^2}{(m + 1 + n + t)^{2k+2}} \prod_{j=1}^{m} \frac{n + 2k + 2 + t + j}{n + t + j} \quad (\text{see } (4.2))$$

$$= \frac{\|f\|_{2k+2+t}^2}{(m + 1 + n + t)^{2k+2}} \prod_{j=1}^{2k+2} \frac{n + m + t + j}{n + t + j},$$
where the last $= \text{ is obtained by considering } \prod_{j=1}^{2k+2+m} (n + t + j)$. Since $m \geq \ell$, for each $j \geq 1$ we have $(n + m + t + j)/(m + 1 + n + t) \leq (n + \ell + t + j)/(\ell + 1 + n + t)$. Hence

$$
\| (N + 1 + n + t)^{-k-1} f \|^2_t \leq \| f \|^2_{2k+2+t} \prod_{j=1}^{2k+2} \frac{n + \ell + t + j}{(\ell + 1 + n + t)(n + t + j)}
$$

$$
= \frac{\| f \|^2_{2k+2+t}}{(n + 2k + 2 + t + j)/n + t + j} \leq \frac{(n + 2k + 2 + t + \ell)^{k}}{(n + 1 + n + t)^{2k+2+t}} \| f \|^2_{2k+2+t}.
$$

This proves (1).

Let $e_j$ be the element in $\mathbb{Z}^n_+$ whose $j$-th component is 1 and whose other components are 0. To prove (2), first note that (4.2) gives us

$$
M_{z_j}^{(t)} z^\alpha = \frac{\alpha_j}{n + t + |\alpha|} z^\alpha - e_j
$$

whenever the $j$-th component $\alpha_j$ of $\alpha$ is greater than 0. Hence

$$
(M_{z_j}^{(t)} - M_{z_j}^{(2k+2+t)*}) z^\alpha = \frac{\alpha_j (2k + 2)}{(n + t + |\alpha|)(n + 2k + 2 + t + |\alpha|)} z^\alpha - e_j
$$

(5.1)

For $f \in \mathbb{C}[z_1, \ldots, z_n]$ with $f(0) = 0$, $(N + n + t)^{-1} f$ is well defined. Thus we can define

$$
f_{t,k} = (N + 1 + n + 2k + 2 + t)(N + n + t)^{-1} f.
$$

We have $\| f_{t,k} \|_\tau \leq (2k + 4) \| f \|_\tau$ for every $\tau \geq -n$. Obviously, (5.1) implies

$$
(M_{z_j}^{(t)} - M_{z_j}^{(2k+2+t)*}) f = (2k + 2) M_{z_j}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k}
$$

Now suppose that $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell + 1$. Applying (1) twice, we have

$$
\| (N + 1 + n + t)^{-k} (M_{z_j}^{(t)} - M_{z_j}^{(2k+2+t)*}) f \|^2_t
$$

$$
= (2k + 2)^2 \| (N + 1 + n + t)^{-k} M_{z_j}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k} \|^2_t
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 2 + t + \ell)^{k}}{(n + 1 + n + t)^{2k+2+t}} \| M_{z_j}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k} \|^2_{2k+2+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 2 + t + \ell)^{k}}{(n + 1 + n + t)^{2k+2+t}} \| (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k} \|^2_{2k+2+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 4 + t + \ell)^{k}}{(n + 1 + n + 2k + 2 + t)^{2k+4+t}} \| f_{t,k} \|^2_{2k+4+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 4 + t + \ell)^{2k+4+t}}{(n + 1 + n + t)^{2k+2}} (2k + 4)^2 \| f \|^2_{2k+4+t}.
$$
This completes the proof of (2). □

For each real number $t > -1$, define

$$a_{n,t} = \frac{1}{n!} \prod_{j=1}^{n} (t + j).$$

Using (4.2) and [15, Proposition 1.4.9], it is straightforward to verify that

$$\langle f, g \rangle_t = a_{n,t} \int f(\zeta) \overline{g(\zeta)} (1 - |\zeta|^2)^t dv(\zeta)$$

for $f, g \in \mathcal{H}^{(t)}$, $t > -1$. In other words, if $t > -1$, then $\mathcal{H}^{(t)}$ is the weighted Bergman space $L^2_a(B, a_{n,t}(1 - |\zeta|^2)^t dv(\zeta))$.

Our next step requires the assumption that $t > -2$.

**Proposition 5.3.** Let real number $t > -2$ and integer $K \geq 1$ be given. Then there is a constant $C_{5.3} = C_{5.3}(n, K, t)$ such that the following estimate holds: Let $q \in \mathbb{C}[z_1, \ldots, z_n]$ be such that $\deg(q) = K$. Suppose that $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfies the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| \leq \ell + 1$, where $\ell \in \mathbb{N}$. Then for every integer $k \geq 0$ and every $j \in \{1, \ldots, n\},$

$$\|(N + 1 + n + t)^{-k-1}(M^{(t)}_{\partial_j R^k_q} - M^{(t)}_{R^{k+1}_q})f\|_t \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1} \|q f\|_{t+1}.$$

**Proof.** Since

$$M^{(t)}_{\partial_j R^k_q} - M^{(t)}_{R^{k+1}_q} = (M^{(t)}_{\partial_j R^k_q} - M^{(2k+2+t)}_{R^{k+1}_q}) - (M^{(t)}_{z_j} - M^{(2k+2+t)}_{z_j}) M^{(t)}_{R^{k+1}_q},$$

we have

$$\|(N + 1 + n + t)^{-k-1}(M^{(t)}_{\partial_j R^k_q} - M^{(t)}_{R^{k+1}_q} f\|_t \leq A + B,$$

where

$$A = \|(N + 1 + n + t)^{-k-1}(M^{(t)}_{\partial_j R^k_q} - M^{(2k+2+t)}_{R^{k+1}_q} f\|_t \quad \text{and}$$

$$B = \|(N + 1 + n + t)^{-k-1}(M^{(t)}_{z_j} - M^{(2k+2+t)}_{z_j}) M^{(t)}_{R^{k+1}_q} f\|_t.$$

We estimate $A$ and $B$ separately. For $A$, we apply Proposition 5.2(1), which gives us

$$A \leq \frac{(n + 2k + 2 + t + \ell)^{\ell}}{(\ell + 1 + n + t)^{k+1}} \|(M^{(t)}_{\partial_j R^k_q} - M^{(2k+2+t)}_{R^{k+1}_q}) f\|_{2k+2+t}$$

and

$$B = \frac{(n + 2k + 2 + t + \ell)^{\ell}}{(\ell + 1 + n + t)^{k+1}} \|(M^{(2k+2+t)}_{\partial_j R^k_q} - M^{(2k+2+t)}_{z_j}) M^{(2k+2+t)}_{R^{k+1}_q}) f\|_{2k+2+t}.$$
Since \( t > -2 \), we have \( 2k + 2 + t > 0 \) for each \( k \geq 0 \). Hence \( \mathcal{H}^{(2k+2+t)} \) is a weighted Bergman space. By (5.2), we have
\[
\| (M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)}* M_{R^{k+1} q}^{(2k+2+t)}) f \|_{2k+2+t} \\
\leq a_{n,2k+2+t}^{1/2} \left( \int \left| \{ (\partial_j R^k q)(z) - z_j (R^{k+1} q)(z) \} f(z) \right|^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.
\]

The identity \( \partial_j - z_j R = (1 - |z|^2) \partial_j + \sum_{i \neq j} z_i L_{j,i} \) then leads to
\[
\| (M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)}* M_{R^{k+1} q}^{(2k+2+t)}) f \|_{2k+2+t} \\
\leq a_{n,2k+2+t}^{1/2} \left( \int \left| (\partial_j R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+4+t} dv(z) \right)^{1/2} \\
+ a_{n,2k+2+t}^{1/2} \sum_{i \neq j} \left( \int \left| (L_{j,i} R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.
\]

Applying Propositions 3.6 and 3.5, we have
\[
\int \left| (\partial_j R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+4+t} dv(z) \\
\leq C_{3.6}^{(2k+4+t)} (K!)^2 \int \left| (R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+2+t} dv(z) \\
\leq (C_{3.6} C_{3.5}) K^{(3k+4+t)} (K!)^4 \int |q(z) f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z).
\]

Since \( 1 + t > -1 \), we can apply Propositions 3.4 and 3.5 to obtain
\[
\int \left| (L_{j,i} R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+2+t} dv(z) \\
\leq C_{3.4}^{2k+2+t} (K!)^2 \int \left| (R^k q)(z) f(z) \right|^2 (1 - |z|^2)^{2k+1+t} dv(z) \\
\leq C_{3.4} (2C_{3.5}) K^{(3k+2+t)} (K!)^4 \int |q(z) f(z)|^2 (1 - |z|^2)^{1+t} dv(z).
\]

By the assumption \( t > -2 \), we have \( a_{n,1+t} \geq (n!)^{-1} (2 + t)^n \). Also note that \( a_{n,2k+2+t} \leq (n!)^{-1} (n + 2k + 2 + t)^n \). Combining (5.5), (5.6), (5.7) and (5.2), we see that there is a \( C_1 \) that depends only on \( n, K \) and \( t (> -2) \) such that
\[
\| (M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)}* M_{R^{k+1} q}^{(2k+2+t)}) f \|_{2k+2+t} \leq C_1^{k+1} \| q f \|_{t+1}.
\]

Recalling (5.4), this gives us
\[
A \leq \frac{(n + 2k + 2 + t + \ell)^{\ell}}{(\ell + 1 + n + t)^{k+1} C_1^{k+1}} \| q f \|_{t+1}.
\]
It follows from Proposition 5.2(2) that
\[
B \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} \| M^{(t)}_{R^{k+1}q} f \|_{2k+4+t}.
\]

Applying (5.2) and Proposition 3.5, we obtain
\[
\| M^{(t)}_{R^{k+1}q} f \|_{2k+4+t}^2 = a_{n,2k+4+t} \int \left( |(R^{k+1}q)(z)f(z)|^2 (1 - |z|^2)^{2k+4+t} dv(z) \right)
\leq a_{n,2k+4+t} C_{3.5}^{K(3k+5+t)} (K!)^2 \int |q(z)f(z)|^2 (1 - |z|^2)^{2+t} dv(z).
\]

Thus there is a \( C_2 \) that depends only on \( n, K \) and \( t > -2 \) such that \( \| M^{(t)}_{R^{k+1}q} f \|_{2k+4+t} \leq C_2^{k+1} \| qf \|_{t+1} \). Consequently,
\[
B \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_2^{k+1} \| qf \|_{t+1}.
\]

Combining this with (5.8) and (5.3), the proof of the proposition is complete. \( \square \)

\textbf{Proof of Theorem 1.1.} Let \( q \in \mathbb{C}[z_1, \ldots, z_n] \) be such that \( \text{deg}(q) = K, K \geq 1 \). Let \( t > -2 \) also be given. For this pair of \( K \) and \( t \), let \( C_{5.3} = C_{5.3}(n, K, t) \) be the constant provided by Proposition 5.3. Let \( \ell \in \mathbb{N} \) satisfy the condition
\[
(\ell + 1 + n + t) > 2C_{5.3}.
\]

With this \( \ell \), we now define
\[
E = \{ qf : f \in \mathbb{C}[z_1, \ldots, z_n], \ (\partial^\alpha f)(0) = 0 \text{ for } |\alpha| \leq \ell + 1 \}.
\]

For the given \( q \), let \( Q^{(t)} \) denote the orthogonal projection from \( \mathcal{H}^{(t)} \) onto \( \mathcal{H}^{(t)} \ominus [q]^{(t)} \). Let \( j \in \{1, \ldots, n\} \), and let \( f \in \mathbb{C}[z_1, \ldots, z_n] \) be such that \( (\partial^\alpha f)(0) = 0 \) for \( |\alpha| \leq \ell + 1 \). Then
\[
Q^{(t)}M^{(t)*}_{z_j} qf = Q^{(t)}M^{(t)*}_{z_j} M^{(t)}_q f = Q^{(t)}(M^{(t)*}_{z_j} M^{(t)}_q - M^{(t)}_q M^{(t)*}_{z_j}) f.
\]

Applying Propositions 5.1 and 5.3, we have
\[
\| Q^{(t)}M^{(t)*}_{z_j} qf \|_t \leq \sum_{k=0}^{\infty} \| (N + 1 + n + t)^{-k-1} (M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j} M^{(t)}_{R^{k+1}q}) f \|_t
\leq \sum_{k=0}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1} \| qf \|_{t+1}.
\]

Set
\[
C = \sum_{k=0}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1}.
\]
Then (5.9) ensures that $C < \infty$. Thus (5.10) can be restated as

$$\|Q^{(t)} M_{z_j}^{(t)} g\|_t \leq C \|g\|_{t+1}$$

for every $g \in E$.

Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)} : \mathcal{H}^{(t)} \to E^{(t)}$ be the orthogonal projection. By Proposition 4.2, the above implies that

$$Q^{(t)} M_{z_j}^{(t)} \mathcal{E}^{(t)} E^{(t)} \in \mathcal{C}^{+}_{2n}.$$

Obviously, $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension. That is, if $P^{(t)}$ denotes the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$, then $\text{rank}(P^{(t)} - \mathcal{E}^{(t)}) < \infty$. Therefore

$$Q^{(t)} M_{z_j}^{(t)} P^{(t)} \in \mathcal{C}^{+}_{2n}.$$

Combining this with the well-known fact that $[M_{z_j}^{(t)}]^* M_{z_i}^{(t)} \in \mathcal{C}^{+}_n$, it follows from a routine argument that $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}] \in \mathcal{C}^{+}_n$, $i, j \in \{1, \ldots, n\}$. This completes the proof. □

References


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