ESSENTIAL NORMALITY FOR QUOTIENT MODULES AND COMPLEX DIMENSIONS

YI WANG AND JINGBO XIA

Abstract. For analytic sets $\tilde{M}$ introduced in [9], we show that the corresponding quotient module $\mathcal{Q}$ of the Bergman module is p-essentially normal for all $p > \dim_{\mathbb{C}}\tilde{M}$, which verifies the Geometric Arveson-Douglas Conjecture in this case. This result makes it possible to study the Helton-Howe trace invariants on both $\mathcal{Q}$ and the corresponding submodule $\mathcal{R}$.

1. Introduction

In this paper we continue the study of the Geometric Arveson-Douglas Conjecture that began in [9]. In Theorem 1.6 in [9], the quotient module Q was proved to be p-essentially normal for $p > 2d$, where d is the complex dimension of the analytic set involved. But the Geometric Arveson-Douglas Conjecture predicts that Q is p-essentially normal for all $p > d$. Using different techniques, in this paper we will close that gap between $2d$ and d. That is, we will show that Q is indeed p-essentially normal for all $p > d$ as conjectured.

This not only improves what we can say about the quotient module, but also has implications for the corresponding *submodule*, as we will see. More specifically, this improved essential normality makes it possible for us to study the Helton-Howe trace invariants [16] for both the submodule and quotient module. This is significant because the ultimate goal of the Arveson-Douglas Conjecture is the study of such invariants for module operators.

Let us turn to the technical details of the paper. As usual, we write $\mathbb{B}_n$ for the unit ball $\{z : |z| < 1\}$ in $\mathbb{C}^n$, and we will assume $n \geq 2$ throughout the paper. Let $L^2_a(\mathbb{B}_n)$ denote the Bergman space of analytic functions on $\mathbb{B}_n$. With the natural multiplication, $L^2_a(\mathbb{B}_n)$ is a Hilbert module over the ring of analytic polynomials $\mathbb{C}[z_1, \dots, z_n]$. A closed linear subspace $\mathcal{S}$ of $L^2_a(\mathbb{B}_n)$ is said to be a submodule of the Bergman module if it is invariant under the multiplication by $\mathbb{C}[z_1, \dots, z_n]$. If $\mathcal{S}$ is a submodule, then

$$\mathcal{S}^\perp = \{f \in L^2_a(\mathbb{B}_n) : f \perp \mathcal{S}\}$$

is a quotient Hilbert module over $\mathbb{C}[z_1, \dots, z_n]$. This is because for all $f \in \mathcal{S}^\perp$ and $g, h \in \mathbb{C}[z_1, \dots, z_n]$, we have $P_{\mathcal{S}^\perp} ghf = P_{\mathcal{S}^\perp} g P_{\mathcal{S}^\perp} hf$, where $P_{\mathcal{S}^\perp}$ is the projection onto $\mathcal{S}^\perp$.

For any $\mathcal{M}$ that is either a submodule or a quotient module, we have the orthogonal projection $P_{\mathcal{M}} : L^2_a(\mathbb{B}_n) \to \mathcal{M}$. Of course, we can also view $P_{\mathcal{M}}$ as the projection from $L^2(\mathbb{B}_n)$ onto $\mathcal{M}$, and this is the crucial point on which our techniques are based. In any case, we have the module operators

$$\mathcal{Z}_{\mathcal{M},j} = P_{\mathcal{M}} M_{z_j} |\mathcal{M}, \quad j = 1, \dots, n.$$

For any $1 \leq p < \infty$, the module $\mathcal{M}$ is said to be *p-essentially normal* if the commutators

$$[Z^*_{\mathcal{M},i}, Z_{\mathcal{M},j}], \quad i, j \in \{1, \dots, n\},$$

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all belong to the Schatten class $\mathcal{C}_p$.

The study of essential normality began with [1], [4] and has become a very active research area (see, for example, [6], [8], [9], [20], [10], [11], [13], [15], [14], [17]). The famous Arveson Conjecture predicts that every graded submodule of the Drury-Arveson module is p-essentially normal for $p > n$. This was later refined by Douglas [5], who observed that in the case of the quotient module it should really be $p > d$, where d is the complex dimension of the variety involved. Simply stated, for the Bergman space we have

Geometric Arveson-Douglas Conjecture. For a variety V in $\mathbb{B}_n$, the quotient module

$$L_a^2(\mathbb{B}_n) \ominus \{f \in L_a^2(\mathbb{B}_n) : f = 0 \text{ on } V\}$$

is p-essentially normal for all $p > \dim_{\mathbb{C}} V$.

The challenge here is to get to $p > \dim_{\mathbb{C}} V$, which is more than just $p > n$. As we will show in this paper, reaching the lower limit $p > \dim_{\mathbb{C}} V$ leads to actual applications.

To tackle this conjecture, some mild conditions were imposed in [9]:

Assumption 1.1. Let $\tilde{M}$ be an analytic set in an open neighborhood of the closed ball $\overline{\mathbb{B}_n}$. Furthermore, $\tilde{M}$ satisfies the following conditions:
(1) $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversely.
(2) $\tilde{M}$ has no singular points on $\partial\mathbb{B}_n$.
(3) $\dim_{\mathbb{C}} \tilde{M} = d$, where $1 \le d \le n-1$.

We emphasize that Assumption 1.1 will always be in force for the rest of the paper. Given such an $\tilde{M}$, it will be convenient to fix certain notations:

Notation 1.2. (a) Let $M = \tilde{M} \cap \mathbb{B}_n$.
(b) Denote $\mathcal{R} = \{f \in L_a^2(\mathbb{B}_n) : f = 0 \text{ on } M\}$.
(c) Let R be the orthogonal projection from $L^2(\mathbb{B}_n)$ onto $\mathcal{R}$.
(d) Denote $\mathcal{Q} = L_a^2(\mathbb{B}_n) \ominus \mathcal{R}$.
(e) Let Q be the orthogonal projection from $L^2(\mathbb{B}_n)$ onto $\mathcal{Q}$.

As we have mentioned, our starting point is

Theorem 1.3. [9, Theorem 1.6] *The quotient module $\mathcal{Q}$ is p-essentially normal for all $p > 2d$.*

Here is our improvement:

Theorem 1.4. *The quotient module $\mathcal{Q}$ is p-essentially normal for all $p > d$.*

The improved essential normality in Theorem 1.4 has consequences. First of all, we know that the full Bergman module $L_a^2(\mathbb{B}_n)$ is p-essentially normal for all $p > n$. Since $d < n$, by Douglas's well-known matrix argument [4, page 119], we immediately have

Corollary 1.5. *The submodule $\mathcal{R}$ is p-essentially normal for all $p > n$.*

Second, once Theorem 1.4 brings p below n, it opens the door for the study of trace invariants. Suppose that $A_1, \ldots, A_k$ are bounded operators on a Hilbert space $\mathcal{H}$. In [16], Helton and Howe introduced the antisymmetric sum

$$[A_1, \ldots, A_k] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)},$$

which naturally generalizes the notion of commutator. This and [7] provided some of the early examples of non-commutative geometry [3]. What particularly motivate us are antisymmetric sums of Toeplitz operators. Let $P : L^2(\mathbb{B}_n) \to L^2_a(\mathbb{B}_n)$ be the orthogonal projection. For each $\varphi \in L^\infty(\mathbb{B}_n)$, we have the familiar Toeplitz operator T_φ defined by the formula

$$T_\varphi h = P(\varphi h), \quad h \in L^2_a(\mathbb{B}_n).$$

Recall the following classic result:

Theorem 1.6. [16, Theorem 7.2] *For $f_1, f_2, \dots, f_{2n} \in \mathbb{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, the antisymmetric sum $[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbb{B}_n} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}.$$

Obviously, this motivates the question, what about operators on $\mathcal{R}$ and $\mathcal{Q}$? Equally obviously, we can define "Toeplitz operators for modules": for any $\varphi \in L^\infty(\mathbb{B}_n)$, we define

$$R_\varphi h = R(\varphi h), \quad h \in \mathcal{R},$$

and

$$Q_\varphi h = Q(\varphi h), \quad h \in \mathcal{Q}.$$

On the submodule, the improved essential normality allows us to prove

Theorem 1.7. *For any $f_1, f_2, \dots, f_{2n} \in \mathbb{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, the antisymmetric sum*

$$[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}]$$

is in the trace class.

On the quotient module, the improved essential normality leads to

Theorem 1.8. *Let $m > d$. Then for any $f_1, f_2, \dots, f_{2m} \in \mathbb{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, the antisymmetric sum $[Q_{f_1}, Q_{f_2}, \dots, Q_{f_{2m}}]$ is in the trace class with zero trace.*

Next let us explain the main ideas in this paper. As it turns out, the key to the improvement from Theorem 1.3 to Theorem 1.4 was in [9] itself:

Theorem 1.9. [9, Theorem 4.3] *There exist a measure μ on M and $0 < c \le C < \infty$ such that*

$$c\|f\|^2 \le \int_M |f(w)|^2 d\mu(w) \le C\|f\|^2$$

for every $f \in \mathcal{Q}$.

On $L^2_a(\mathbb{B}_n)$, such a μ defines a Toeplitz operator via the formula

$$(T_\mu f)(z) = \int_M \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} d\mu(w).$$

As noted in [9], the upper bound in Theorem 1.9 means that μ is a Carleson measure for the Bergman space, consequently T_μ is a bounded operator on $L^2_a(\mathbb{B}_n)$. In fact the two bounds translate to the operator inequality

$$cQ \le T_\mu \le CQ$$

on $L^2_a(\mathbb{B}_n)$. This inequality turns Q into a function of T_μ: $Q = h(T_\mu)$ for some smooth function h. By the standard smooth functional calculus, the estimate of a commutator of the form $[A, Q]$, which is the object of interest in the study of essential normality, is reduced to the estimate of

$[A, T_\mu]$. The point is that T_μ has an explicit integral formula. Thus, for the purpose of proving essential normality, Theorem 1.9 practically endows Q with an explicit integral formula.

But to take p below n, one must avoid using the essential normality of the full Bergman module, because $L_a^2(\mathbb{B}_n)$ is p-essentially normal only for $p > n$. Our idea is to use the big space $L^2(\mathbb{B}_n)$, on which all multiplication operators and their adjoints freely commute. This requires extending T_μ to an operator on $L^2(\mathbb{B}_n)$. But obviously, an arbitrary function in $L^2(\mathbb{B}_n)$ cannot be integrated on M against μ. So how does one extend T_μ to $L^2(\mathbb{B}_n)$?

Observe that, using the reproducing kernel $K_w(z) = (1 - \langle z, w \rangle)^{-n-1}$ for the Bergman space, we can write the Toeplitz operator T_μ in the form

$$T_\mu = \int_M K_w \otimes K_w d\mu(w).$$

This automatically extends the Toeplitz operator T_μ to an operator on the big space $L^2(\mathbb{B}_n)$, by exactly the same integral formula! The key part of the proof of Theorem 1.4 is to show that on the big space $L^2(\mathbb{B}_n)$, the commutators $[M_{z_i}, T_\mu]$, $i = 1, \ldots, n$, are in the Schatten class $\mathcal{C}_p$ for $p > 2d$.

To accomplish that, we use the fact that μ can be approximated in the weak-* topology by point masses on M. This implies that T_μ is in the weak closure of the convex hull of operators of the form

$$D = \sum_{w \in \Gamma \cap M} c_w k_w \otimes k_w,$$

where Γ is a discrete set in $\mathbb{B}_n$ with certain separation properties, c_w are non-negative with an upper bound determined by μ, and k_w is the normalized reproducing kernel for the Bergman space. Thus it suffices to consider the commutators $[M_{z_i}, D]$. These commutators are further decomposed as follows. For each $k \geq 0$, consider the "strip" $M_k = \{w \in M : 1 - 2^{-2k} \leq |w| < 1 - 2^{-2(k+1)}\}$ in M. Accordingly, we have the operators

$$D_k = \sum_{w \in \Gamma \cap M_k} c_w k_w \otimes k_w \quad \text{and} \quad F_k = [M_{z_i}, D_k].$$

We will show that for any $\epsilon > 0$, we have

$$\|F_k\| \leq C_1 2^{-(1-\epsilon)k} \quad \text{and} \quad \operatorname{rank}(F_k) \leq C_2 2^{2dk},$$

$k \geq 0$. As we will see, these two estimates are sufficient to imply the essential normality promised in Theorem 1.4.

These two estimates themselves deserve some explanation. The first estimate, $\|F_k\| \leq C_1 2^{-(1-\epsilon)k}$, is simply a reflection of the properties of the ball and the Bergman space. More revealing is the second estimate, $\operatorname{rank}(F_k) \leq C_2 2^{2dk}$, which shows exactly how the dimension of the underlying variety enters into the essential normality of the quotient module. In fact, this is exactly the kind of structure that the Geometric Arveson-Douglas Conjecture is meant to uncover.

The rest of the paper is devoted to the proofs of our results. Specifically, Section 2 contains the technical preparations for the proofs. After that, we prove Theorem 1.4 in Section 3. Then the proofs of Theorems 1.7 and 1.8 are given in Sections 4 and 5 respectively.

2. Preliminaries

We begin with a lemma about commutators.

Lemma 2.1. *Suppose that $\mathcal{H}$ is a Hilbert space. Let A, B be bounded operators on $\mathcal{H}$, and let Q be an orthogonal projection on $\mathcal{H}$. Define $S = QAQ$ and $T = QBQ$. Then*

$$[S, T] = [Q, B](1 - Q)[A, Q] - [Q, A](1 - Q)[B, Q] + Q[A, B]Q.$$

As a consequence, if $[A, B] = 0$ and if $[Q, A]$, $[Q, B] \in \mathcal{C}_{2p}$ for some $1 \le p < \infty$, then $[S, T] \in \mathcal{C}_p$.

Proof. Since $Q(1 - Q) = 0$ and $(1 - Q)Q = 0$, simple algebra yields

$$\begin{aligned}[S, T] &= QAQBQ - QBQAQ \\ &= QB(1 - Q)AQ - QA(1 - Q)BQ + Q[A, B]Q \\ &= [Q, B](1 - Q)[A, Q] - [Q, A](1 - Q)[B, Q] + Q[A, B]Q.\end{aligned}$$

This completes the proof. □

Consider the case where $\mathcal{H} = L^2(\mathbb{B}_n)$, $\mathcal{Q}$ is the quotient module in Notation 1.2, and $Q : L^2(\mathbb{B}_n) \to \mathcal{Q}$ is the orthogonal projection. Let $\hat{M}_{z_i}$ be the operator of multiplication by the coordinate function z_i on the big space $L^2(\mathbb{B}_n)$, $i = 1, \ldots, n$. For $p > n$, since $L^2_a(\mathbb{B}_n)$ is p-essentially normal, if we know that every $[Q, \hat{M}_{z_i}]$ is in the Schatten class $\mathcal{C}_{2p}$, then by Proposition 4.1 in [1] we can conclude that the quotient module $\mathcal{Q}$ is p-essentially normal. But since the essential normality of the Bergman module $L^2_a(\mathbb{B}_n)$ is involved in this argument, it does not cover the case $p \le n$. That is where Lemma 2.1 comes in.

The advantage of Lemma 2.1 is that it allows us to bypass the Bergman module $L^2_a(\mathbb{B}_n)$. More to the point, it allows us to bypass Proposition 4.1 in [1]. For any $1 \le p < \infty$, Lemma 2.1 tells us that if we know that $[Q, \hat{M}_{z_i}] \in \mathcal{C}_{2p}$ for every $i \in \{1, \ldots, n\}$, then we can conclude that the quotient module $\mathcal{Q}$ is p-essentially normal.

In general, we write C, C_1, C_2, etc, for constants, and they may represent different values in different context. The notation $A \approx B$ means that there exist $0 < c < C < \infty$ such that $cA \le B \le CA$. Similarly, by $A \lesssim B$ we mean that there exists a $0 < C < \infty$ such that $A \le CB$.

2.1. **Bergman Metric and Carleson Measure.**

For $z \in \mathbb{B}_n$, write P_z for the orthogonal projection from $\mathbb{C}^n$ onto the subspace $\mathbb{C}z$ and $Q_z = 1 - P_z$. The Möbius transform

$$\varphi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2} Q_z(w)}{1 - \langle w, z \rangle}$$

is the (unique) automorphism of $\mathbb{B}_n$ that satisfies $\varphi_z \circ \varphi_z = \mathrm{id}$ and $\varphi_z(0) = z$.

Recall that the Bergman metric on the unit ball is given by the formula

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.$$

It is well know that β is Möbius invariant, and so is the metric

$$\rho(w, z) = |\varphi_z(w)|.$$

For $z \in \mathbb{B}_n$ and $r > 0$, denote

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(w, z) < r\} = \{w \in \mathbb{B}_n : \rho(w, z) < s_r\},$$

where $s_r = \tanh r$.

Lemma 2.2. [18, 2.2.7] *For $z \in \mathbb{B}_n$ and $r > 0$, the Bergman-metric ball $D(z,r)$ consists of all w that satisfy*

$$\frac{|P_z w - c|^2}{s_r^2 \rho^2} + \frac{|Q_z w|^2}{s_r^2 \rho} < 1,$$

where

$$c = \frac{(1-s_r^2)z}{1-s_r^2|z|^2}, \quad \rho = \frac{1-|z|^2}{1-s_r^2|z|^2}.$$

As a consequence, for a fixed r, $v(D(z,r)) \approx (1-|z|^2)^{n+1}$. One of the reasons that the Bergman metric is important is that it matches the analytic structure on the unit ball. From the properties of the Möbius transform φ_z (see [18, Section 2.2]) it is easy to deduce

Lemma 2.3. [22] *Given any $0 < r < \infty$, there exists a constant $0 < C_r < \infty$ such that for any $z, w \in \mathbb{B}_n$ satisfying $\beta(z,w) < r$ and any $\lambda \in \mathbb{B}_n$,*

(1) $C_r^{-1} \le \frac{1-|z|^2}{1-|w|^2} \le C_r$,
(2) $C_r^{-1} \le \frac{|1-\langle \lambda, z\rangle|}{|1-\langle \lambda, w\rangle|} \le C_r$.

Lemma 2.4. [19] *Let ν be a positive, finite, regular, Borel measure on $\mathbb{B}_n$ and $r > 0$. The following quantities are equivalent (with constants depending on n and r).*

(1) $\|\nu\|_* := \sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-\langle w,z\rangle|^{2(n+1)}} d\nu(w)$,
(2) $\inf\left\{C > 0 : \int |f|^2 d\nu \le C \int |f|^2 dv \text{ for } f \in L_a^2(\mathbb{B}_n)\right\}$,
(3) $\sup_{z\in\mathbb{B}_n} \frac{\nu(D(z,r))}{v(D(z,r))}$,
(4) $\|T_\nu\|_{L_a^2(\mathbb{B}_n)\to L_a^2(\mathbb{B}_n)}$.

Here the operator T_ν is defined by

$$(T_\nu f)(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1}} d\nu(w).$$

A Carleson measure for the Bergman space $L_a^2(\mathbb{B}_n)$ is a ν for which one of the above quantities is finite. In this paper, if we call a measure a *Carleson measure*, we mean it is one for the Bergman space $L_a^2(\mathbb{B}_n)$, as defined above.

Let us recall the definitions of the technical terms in Assumption 1.1.

Definition 2.5. [2] Let Ω be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of Ω if for each point $a \in \Omega$ there are a neighborhood $U \ni a$ and functions $f_1, \cdots, f_N$ holomorphic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood $U \ni a$ in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a *singular point* of A if it is not regular.

Definition 2.6. Let Y be a manifold and let X, Z be submanifolds of Y. We say that the submanifolds X and Z *intersect transversely* if for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.

The authors of [9] proved the following theorem.

Theorem 2.7. *Suppose $\tilde{M}$ is a complex analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$ satisfying the following conditions:*

(1) *$\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversely.*
(2) *$\tilde{M}$ has no singular point on $\partial\mathbb{B}_n$.*

Let $M = \tilde{M} \cap \mathbb{B}_n$. Then there exists a Carleson measure μ on M such that the $L^2(\mu)$ norm defines an equivalent norm for functions

$$f \in \mathcal{Q} = L_a^2(\mathbb{B}_n) \ominus \{h \in L_a^2(\mathbb{B}_n) : h = 0 \text{ on } M\} = \overline{\operatorname{span}}\{K_\lambda : \ \lambda \in M\}.$$

As a consequence, the projection operator Q onto $\mathcal{Q}$ is a C^∞ functional calculus of the positive operator T_μ.

With additional effort, the quotient module $\mathcal{Q}$ was shown in [9] to be p-essentially normal for all $p > 2d$, where $d = \dim_{\mathbb{C}} M$. As we explained in the Introduction, we will show that this is true for all $p > d$, fulfilling the prediction of the Geometric Arveson-Douglas Conjecture.

2.2. **The Class $\mathcal{C}_p^+$.** Recall that, for each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k \ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} \tag{2.1}$$

defines a symmetric norm for operators, where $s_1(A), \ldots, s_k(A), \ldots$ are the s-numbers of A. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [12].

It is well known that $\mathcal{C}_p^+$ contains the Schatten class $\mathcal{C}_p$ and that $\mathcal{C}_p^+ \neq \mathcal{C}_p$. Moreover, we have $\mathcal{C}_p^+ \subset \mathcal{C}_{p'}$ for all $1 \le p < p' < \infty$. A property of $\mathcal{C}_p^+$ that does not concern us in this paper, but is nonetheless interesting, it that this ideal is not separable with respect to the norm $\|\cdot\|_p^+$.

The reason for introducing $\mathcal{C}_p^+$ is that the norm $\|\cdot\|_p^+$ is particularly easy to handle in the essential normality problems for modules, as was demonstrated in [11]. Estimates in this paper will further show that the norm $\|\cdot\|_p^+$ is user-friendly indeed.

Lemma 2.8. *Suppose T is in the weak operator closure of a set of operators $\{T_\alpha\}_{\alpha \in I}$. Assume $T_\alpha \in \mathcal{C}_p^+$ and*

$$\sup_{\alpha \in I} \|T_\alpha\|_p^+ \le C < \infty.$$

Then $T \in \mathcal{C}_p^+$ and $\|T\|_p^+ \le C$.

Proof. Let us denote $\sigma_k(T) = s_1(T) + \cdots + s_k(T)$. It is well known that

$$\sigma_k(T) = \sup\{|\operatorname{tr}(TA_k)| : \|A_k\| \le 1 \text{ and } \operatorname{rank}(A_k) = k\}.$$

For each A_k, since its rank equals $k < \infty$, there is a sequence $\{\alpha_m\}$ in I such that $\operatorname{tr}(T_{\alpha_m} A_k) \to \operatorname{tr}(TA_k)$ as $m \to \infty$. Therefore

$$|\operatorname{tr}(TA_k)| = \lim_{m \to \infty} |\operatorname{tr}(T_{\alpha_m} A_k)| \le \sup_{\alpha \in I} \sigma_k(T_\alpha) \le C(1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}).$$

Taking supremum over all such A_k, we obtain

$$\sigma_k(T) \le C(1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}).$$

By (2.1), we have

$$\|T\|_p^+ = \sup_k \frac{\sigma_k(T)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} \le C.$$

This completes the proof. $\square$

The following lemma provides a key estimate.

Lemma 2.9. *Given any positive numbers $0 < a \le b < \infty$, there is a constant $0 < B(a,b) < \infty$ such that the following holds true: Let $\mathcal{H}$ be a Hilbert space, and suppose that $F_0, F_1, \dots, F_k, \dots$ are operators on $\mathcal{H}$ such that the following two conditions are satisfied for every k:*

(1) $\|F_k\| \le 2^{-ak}$,
(2) *rank*$(F_k) \le 2^{bk}$.

Then the operator $F = \sum_{k=0}^{\infty} F_k$ satisfies the estimate $\|F\|_{b/a}^{+} \le B(a,b)$. In particular, $F \in \mathcal{C}_{b/a}^{+}$.

Proof. Recall from [12] that for any bounded operator A and any $i \ge 1$,

$$s_i(A) = \inf\{\|A + K\| : \text{rank}(K) \le i - 1\}.$$

Obviously, condition (1) implies that F is a bounded linear operator on $\mathcal{H}$. By condition (2),

$$\text{rank}\left(\sum_{j=0}^{k} F_j\right) \le \sum_{j=0}^{k} 2^{bj} \le C_1 2^{bk}, \tag{2.2}$$

where $C_1 = (1 - 2^{-b})^{-1}$. For any integer $m > C_1$, let $k \ge 0$ be such that

$$C_1 2^{bk} < m \le C_1 2^{b(k+1)}.$$

Then from (2.2) we obtain

$$s_m(F) \le \left\| \sum_{j=k+1}^{\infty} F_j \right\| \le \sum_{j=k+1}^{\infty} 2^{-aj} \le C_2 2^{-ak},$$

where $C_2 = (1 - 2^{-a})^{-1}$. Therefore

$$s_m(F) m^{a/b} \le C_2 2^{-ak} \cdot (C_1 2^{b(k+1)})^{a/b} = 2^a C_2 C_1^{a/b}.$$

Set $B(a,b) = 2^a C_2 C_1^{a/b}$. Then the above translates to

$$s_m(F) \le B(a,b) m^{-a/b}$$

for every $m > C_1$. On the other hand, since $\|F\| \le C_2$, for $m \le C_1$ we have

$$s_m(F) \le C_2 = C_2 m^{a/b} m^{-a/b} \le C_2 C_1^{a/b} m^{-a/b} \le B(a,b) m^{-a/b}.$$

Combining these two estimates, we see that $s_m(F) \le B(a,b) m^{-a/b}$ for every $m \ge 1$. By (2.1), this means $\|F\|_{b/a}^{+} \le B(a,b)$. $\square$

2.3. **Other Tools.** The following lemma can be found in Appendix C to [3, Chapter IV].

Lemma 2.10. *Suppose $p \ge 1$, S, T are bounded linear operators on a Hilbert space $\mathcal{H}$ and $[S,T] \in \mathcal{C}_p$. If S is self-adjoint and if f is a C^∞ function on the spectrum of S, one has $[f(S),T] \in \mathcal{C}_p$.*

We will also need the well-known Schur test for boundedness:

Lemma 2.11. *Let $(X, d\mu)$ be a measure space and $R(x, y)$ a non-negative, measurable function on $X \times X$. Suppose that there exist a positive, measurable function h function on X and positive numbers C_1, C_2 such that*

$$\int_X R(x,y)h(y)d\mu(y) \le C_1 h(x) \quad \text{for } \mu\text{-a.e. } x$$

and

$$\int_X R(x,y)h(x)d\mu(x) \le C_2 h(y) \quad \text{for } \mu\text{-a.e. } y.$$

Then

$$(Tf)(x) = \int_X R(x,y)f(y)d\mu(y)$$

defines a bounded operator on $L^2(X, d\mu)$ with $\|T\| \le (C_1C_2)^{1/2}$.

3. Proof of Theorem 1.4

Suppose μ is a Carleson measure supported on M. Let $\hat{T}_\mu$ denote the operator on $L^2(\mathbb{B}_n)$ that sends $L_a^2(\mathbb{B}_n)^\perp$ to $\{0\}$ and coincides with T_μ on $L_a^2(\mathbb{B}_n)$. Our first observation is that we have the integral representation

$$\hat{T}_\mu = \int K_w \otimes K_w d\mu(w).$$

This is verified by direct calculation: for $f \in L^2(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$,

$$\int \langle f, K_w\rangle K_w(z)d\mu(w) = \int \frac{(Pf)(w)}{(1-\langle z,w\rangle)^{n+1}}d\mu(w) = (\hat{T}_\mu f)(z),$$

where $P : L^2(\mathbb{B}_n) \to L_a^2(\mathbb{B}_n)$ is the orthogonal projection.

For each $\varphi \in L^\infty(\mathbb{B}_n)$, let $\hat{M}_\varphi$ denote the operator of multiplication by φ on $L^2(\mathbb{B}_n)$. That is,

$$\hat{M}_\varphi f = \varphi f, \quad f \in L^2(\mathbb{B}_n).$$

The following theorem is the main step in the proof of Theorem 1.4.

Theorem 3.1. *Let μ be a Carleson measure supported on M. Then for every $j \in \{1, \dots, n\}$ and every $p > 2d$, we have $[\hat{T}_\mu, \hat{M}_{z_j}] \in \mathcal{C}_p^+$. As a consequence, $[\hat{T}_\mu, \hat{M}_{z_j}] \in \mathcal{C}_p$ for every $j \in \{1, \dots, n\}$ and every $p > 2d$.*

First, let us give the outline of our proof. The main idea is to approximate the operator $\hat{T}_\mu$ by a certain kind of discrete sums. Then we estimate the $\mathcal{C}_p^+$ norms of commutators of these discrete sums with $\hat{M}_{z_j}$. We break the commutators into parts and estimate the ranks and norms of these parts. Finally, an application of Lemma 2.9 will end the proof.

Now let us construct the discrete sums. Choose a subset $\mathcal{L} \subset M$ that is maximal with respect to the property that

$$D(z,1) \cap D(w,1) = \emptyset \quad \text{for all } \ z \ne w \ \text{ in } \ \mathcal{L}. \tag{3.1}$$

Obviously, such an $\mathcal{L}$ is countable, which allows us to write $\mathcal{L} = \{z_i\}_{i=1}^\infty$. It follows from the maximality of $\mathcal{L}$ that

$$\bigcup_{i=1}^\infty D(z_i, 2) \supset M.$$

There exist Borel sets $\Delta_1, \Delta_2, \dots, \Delta_i, \dots$ in $\mathbb{B}_n$ satisfying the following three requirements:

(1) $D(z_i, 1) \subset \Delta_i \subset D(z_i, 2)$ for every i.
(2) $\Delta_i \cap \Delta_{i'} = \emptyset$ for $i \neq i'$.
(3) $\cup_{i=1}^{\infty}\Delta_i = \cup_{i=1}^{\infty}D(z_i, 2) \supset M$.

The construction of these sets is standard. In fact, obviously there are pairwise disjoint Borel subsets $E_1, E_2, \ldots, E_i, \ldots$ of $\{\cup_{i=1}^{\infty}D(z_i, 2)\}\backslash\{\cup_{i=1}^{\infty}D(z_i, 1)\}$ such that

$$E_1 \cup E_2 \cup \cdots \cup E_i \cup \cdots = \{\cup_{i=1}^{\infty}D(z_i, 2)\}\backslash\{\cup_{i=1}^{\infty}D(z_i, 1)\}$$

and $E_i \subset D(z_i, 2)$ for every i. Then the sets $\Delta_i = D(z_i, 1) \cup E_i$, $i = 1, 2, 3, \ldots$, satisfy requirements (1)-(3) above.

Let μ be a Carleson measure supported on M. By Lemma 2.3,

$$c_i := \int_{\Delta_i} (1 - |w|^2)^{-(n+1)} d\mu(w) \lesssim (1 - |z_i|^2)^{-(n+1)}\mu(\Delta_i) \lesssim \frac{\mu(D(z_i, 2))}{v(D(z_i, 2))}.$$

By Lemma 2.4, there is a constant $0 < C < \infty$ such that $c_i \leq C$ for every i.

Define $N = \{i \in \mathbb{N} : \mu(\Delta_i) \neq 0\} = \{i \in \mathbb{N} : c_i > 0\}$. For each $i \in N$, we define the measure $d\mu_i$ to be the restriction of the measure $c_i^{-1}(1-|w|^2)^{-(n+1)}d\mu$ to the set Δ_i. Obviously, $\mu_i(\Delta_i) = 1$. Observe that

$$\begin{aligned} \hat{T}_\mu &= \int K_w \otimes K_w d\mu(w) = \sum_{i=1}^{\infty} \int_{\Delta_i} K_w \otimes K_w d\mu(w) \\ &= \sum_{i \in N} c_i \int_{\Delta_i} k_w \otimes k_w c_i^{-1}(1 - |w|^2)^{-(n+1)} d\mu(w) = \sum_{i \in N} c_i \int_{\Delta_i} k_w \otimes k_w d\mu_i(w), \end{aligned}$$

where $k_w = K_w/\|K_w\|$ is the normalized reproducing kernel. Since μ is a Carleson measure, the positive operator $\hat{T}_\mu$ is bounded. By the monotone convergence theorem, the above sums converge in the strong operator topology.

Since μ is supported on M, each probability measure μ_i can be approximated in the weak-* topology by measures of the form $\frac{1}{k}\sum_{j=1}^{k} \delta_{w_j}$, where $w_j \in \Delta_i \cap M$. Therefore each operator $\int_{\Delta_i} k_w \otimes k_w d\mu_i(w)$ can be approximated in the weak operator topology by operators of the form

$$\frac{1}{k}\sum_{j=1}^{k} k_{w_j} \otimes k_{w_j}, \quad w_j \in \Delta_i \cap M.$$

Hence $\hat{T}_\mu$ can be weakly approximated by operators of the form

$$\sum_{i \in F} c_i \frac{1}{k} \sum_{j=1}^{k} k_{w_{i,j}} \otimes k_{w_{i,j}} = \frac{1}{k}\sum_{j=1}^{k}\sum_{i \in F} c_i k_{w_{i,j}} \otimes k_{w_{i,j}},$$

where F is a finite subset of N, $k \in \mathbb{N}$, and $w_{i,j} \in \Delta_i \cap M$. We summarize the above arguments in the following lemma.

Lemma 3.2. *The operator $\hat{T}_\mu$ is in the weak closure of the convex hull of operators of the form*

$$\sum_{i \in F} c_i k_{w_i} \otimes k_{w_i}, \tag{3.2}$$

where F is any finite subset of N, $w_i \in \Delta_i \cap M$ and $0 < c_i \leq C$. Moreover, the finite bound C depends only on the Carleson measure μ on M.

It follows immediately that for every $1 \le m \le n$, the commutator $[\hat{T}_\mu, \hat{M}_{z_m}]$ is in the weak closure of the convex hull of operators of the form

$$\sum_{i \in F} c_i [k_{w_i} \otimes k_{w_i}, \hat{M}_{z_m}].$$

Thus to estimate $\|[\hat{T}_\mu, \hat{M}_{z_m}]\|_p^+$, it suffices to estimate the $\mathcal{C}_p^+$ norms of operators of the above form. To estimate the latter, we use Lemma 2.9. Conditions (1) and (2) in Lemma 2.9 will be verified in the following steps.

Let v_M denote the natural volume measure on the smooth part of $\tilde{M}$.

Lemma 3.3. *For $0 < s < t < 1$, define*

$$M_s^t = \{z \in \tilde{M} : s < |z| \le t\}.$$

Then for r sufficiently close to 1 and $r < s < t < 1$, we have $v_M(M_s^t) \lesssim t - s$.

Proof. Let $r(z) = |z|$ be the radius function. By Assumption 1.1, $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversely. Thus for each point $\zeta \in \tilde{M} \cap \partial\mathbb{B}_n$, $\tilde{M}$ has a real local coordinate system of the form $\Phi = (\phi_1, \ldots, \phi_{2d-1}, r(z))$ defined on a neighborhood $U_\zeta \cap \tilde{M}$, where U_ζ is an open set containing ζ in $\mathbb{C}^n$. Therefore the volume form locally can be expressed as $dv_M = g d\phi_1 \wedge \ldots \wedge d\phi_{2d-1} \wedge dr$. If we shrink the neighborhood U_ζ we can also assume that g is bounded and Φ maps $U_\zeta \cap \tilde{M}$ to a bounded set in $\mathbb{R}^{2d}$. By the compactness of $\tilde{M} \cap \partial\mathbb{B}_n$, it can be covered by finitely many such open sets U_{ζ_j}, $j = 1, \ldots, m$. Thus it suffices to show that

$$v_M(M_s^t \cap U_{\zeta_j}) \lesssim t - s$$

for each j and $s < t$ sufficiently close to 1. By direct computation,

$$v_M(M_s^t \cap U_{\zeta_j}) \lesssim \int_s^t 1 dr \lesssim t - s.$$

This completes the proof. ∎

Lemma 3.4. *There exists a $0 < r < 1$ such that $v_M(D(z,1) \cap M) \gtrsim (1 - |z|^2)^{d+1}$ for $z \in M$ satisfying the condition $r < |z| < 1$.*

Proof. There is a $0 < r < 1$ such that for each $z \in M$, $|z| > r$, there is a smooth map

$$p_z : M \cap D(z, 2) \mapsto TM|_z$$

defined on page 1513 in [9]. Using the formula for p_z given there and the property

$$\sup_{w \in D(z,2)} \beta(p_z(w), w) \to 0 \quad \text{as} \quad |z| \to 1,$$

it is straightforward to verify that $p_z(D(z,1) \cap M) \supset D(z, 1/2) \cap TM|_z$ when $|z|$ is close enough to 1. Therefore, writing v_d for the volume measure on $TM|_z = \mathbb{C}^d$, we have

$$v_M(D(z,1) \cap M) \gtrsim v_d(D(z,1/2) \cap TM|_z) \approx (1 - |z|^2)^{d+1}.$$

This completes the proof. ∎

Proposition 3.5. *Given any $0 < \epsilon < 1/2$, there is a $0 < C' < \infty$ such that the following estimate holds: Let F be any finite subset of N. Suppose that for every $i \in F$, $w_i \in \Delta_i \cap M$ and $0 \le c_i \le C$, where C is the constant in Lemma 3.2. Define $\nu = \sum_{i\in F} c_i(1-|w_i|^2)^{n+1}\delta_{w_i}$ and*

$$\hat{T}_\nu = \sum_{i\in F} c_i k_{w_i} \otimes k_{w_i}.$$

Then we have $\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C'$ for every $m \in \{1, \ldots, n\}$.

Proof. Let $0 < \epsilon < 1/2$ be given. For each $k \ge 0$, define

$$M_k = \{z \in M : 1 - 2^{-2k} \le |z| < 1 - 2^{-2(k+1)}\} \tag{3.3}$$

and

$$\nu_k = \nu|_{M_k} = \sum_{i\in F, w_i\in M_k} c_i(1-|w_i|^2)^{n+1}\delta_{w_i}.$$

Also, write

$$F_k = [\hat{T}_{\nu_k}, \hat{M}_{z_m}] = \sum_{i\in F, w_i\in M_k} c_i[k_{w_i}\otimes k_{w_i}, \hat{M}_{z_m}]$$

for $k \ge 0$. We will show that there are constants C_1 and C_2 such that

$$\|F_k\| \le C_1 2^{-(1-2\epsilon)k} \tag{3.4}$$

and

$$\text{rank}(F_k) \le C_2 2^{2dk} \tag{3.5}$$

for every $k \ge 0$. Since $\sum_{k=0}^\infty F_k = [\hat{T}_\nu, \hat{M}_{z_m}]$, it follows from these estimates and Lemma 2.9 that

$$\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C_1(1 + C_2)B(1 - 2\epsilon, 2d).$$

That is, the proposition holds for $C' = C_1(1+C_2)B(1-2\epsilon, 2d)$ provided that we find constants C_1 and C_2 such that (3.4) and (3.5) hold.

To find C_1, note that for any $f \in L^2(\mathbb{B}_n)$,

$$\begin{aligned}
&([K_{w_i}\otimes K_{w_i}, \hat{M}_{z_m}]f)(z)\\
&\quad= \int_{\mathbb{B}_n} \lambda_m f(\lambda)K_\lambda(w_i)dv(\lambda)K_{w_i}(z) - z_m\int_{\mathbb{B}_n} f(\lambda)K_\lambda(w_i)dv(\lambda)K_{w_i}(z)\\
&\quad= \int_{\mathbb{B}_n} (\lambda_m - w_{i,m})f(\lambda)K_\lambda(w_i)dv(\lambda)K_{w_i}(z) + \int_{\mathbb{B}_n} (w_{i,m} - z_m)f(\lambda)K_\lambda(w_i)dv(\lambda)K_{w_i}(z),
\end{aligned}$$

where $w_{i,m}$ denotes the m-th component of w_i. Since

$$F_k = \sum_{i\in F, w_i\in M_k} c_i(1-|w_i|^2)^{n+1}[K_{w_i}\otimes K_{w_i}, \hat{M}_{z_m}],$$

we have

$$\begin{aligned}
|(F_k f)(z)| \le &\sum_{i\in F, w_i\in M_k} c_i(1-|w_i|^2)^{n+1}\int_{\mathbb{B}_n} |\lambda - w_i||f(\lambda)||K_\lambda(w_i)|dv(\lambda)|K_{w_i}(z)|\\
&+ \sum_{i\in F, w_i\in M_k} c_i(1-|w_i|^2)^{n+1}\int_{\mathbb{B}_n} |w_i - z||f(\lambda)||K_\lambda(w_i)|dv(\lambda)|K_{w_i}(z)|.
\end{aligned}$$

Recalling the definition of ν, we have

$$
\begin{aligned}
|(F_k f)(z)| \le & \int_{M_k}\int_{\mathbb{B}_n} |\lambda - w||f(\lambda)||K_\lambda(w)||K_w(z)|dv(\lambda)d\nu(w) \\
& + \int_{M_k}\int_{\mathbb{B}_n} |w - z||f(\lambda)||K_\lambda(w)||K_w(z)|dv(\lambda)d\nu(w) \\
= & \int_{\mathbb{B}_n} |f(\lambda)| \int_{M_k} |\lambda - w||K_\lambda(w)||K_w(z)|d\nu(w)dv(\lambda) \\
& + \int_{\mathbb{B}_n} |f(\lambda)| \int_{M_k} |w - z||K_\lambda(w)||K_w(z)|d\nu(w)dv(\lambda) \\
= & \int_{\mathbb{B}_n} |f(\lambda)|G_k(z,\lambda)dv(\lambda) + \int_{\mathbb{B}_n} |f(\lambda)|H_k(z,\lambda)dv(\lambda).
\end{aligned}
$$

Here,

$$
\begin{aligned}
G_k(z,\lambda) &= \int_{M_k} |\lambda - w||K_\lambda(w)||K_w(z)|d\nu(w) \quad \text{and} \\
H_k(z,\lambda) &= \int_{M_k} |w - z||K_\lambda(w)||K_w(z)|d\nu(w).
\end{aligned}
$$

To estimate $\|F_k\|$, we apply the Schur test. Let $h(\lambda) = (1-|\lambda|^2)^{-1/2}$. Then

$$
\begin{aligned}
\int_{\mathbb{B}_n} G_k(z,\lambda)h(\lambda)dv(\lambda) &= \int_{M_k}\int_{\mathbb{B}_n} |\lambda - w||K_\lambda(w)|h(\lambda)dv(\lambda)|K_w(z)|d\nu(w) \\
&\lesssim \int_{M_k}\int_{\mathbb{B}_n} \frac{(1-|\lambda|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}dv(\lambda)\frac{1}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \\
&\lesssim \int_{M_k} \left(\log\frac{1}{1-|w|^2}\right)\frac{1}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \\
&\lesssim \int_{M_k} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \le C\sum_{i\in F, w_i\in M_k} \frac{(1-|w_i|^2)^{-\epsilon}}{|1-\langle z,w_i\rangle|^{n+1}}(1-|w_i|^2)^{n+1},
\end{aligned}
$$

where, as we recall, C is the constant in Lemma 3.2. By Lemma 2.3,

$$
\frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}} \approx \frac{(1-|w_i|^2)^{-\epsilon}}{|1-\langle z,w_i\rangle|^{n+1}}
$$

for any $z \in \mathbb{B}_n$ and $w \in \Delta_i \subset D(z_i,2) \subset D(w_i,4)$. Recall that $\Delta_i \supset D(z_i,1)$. Therefore the integral above is bounded, up to a constant, by

$$
\sum_{i\in F, w_i\in M_k}\int_{\Delta_i} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w) = \int_{\bigcup_{i\in F, w_i\in M_k}\Delta_i} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w).
$$

By Lemma 2.3, there is a constant $0 < A < \infty$ such that $\bigcup_{i\in F, w_i\in M_k}\Delta_i \subset W_k$, where

$$
W_k = \{w \in \mathbb{B}_n : |w| \ge 1 - 2^{-2(k-A)}\}.
$$

Therefore

$$\int_{\mathbb{B}_n} G_k(z,\lambda)h(\lambda)dv(\lambda) \lesssim \int_{\bigcup_{i\in F, w_i\in M_k}\Delta_i} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w) \leq \int_{W_k} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w)$$
$$\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon}\int_{\partial\mathbb{B}_n}\frac{1}{|1-\langle rz,\zeta\rangle|^{n+1}}d\sigma(\zeta)dr$$
$$\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon}(1-|rz|^2)^{-1}dr$$
$$\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon-(1/2)}(1-|z|^2)^{-1/2}dr$$
$$\lesssim \{1-(1-2^{-2(k-A)})\}^{(1/2)-\epsilon}h(z) \lesssim 2^{-(1-2\epsilon)k}h(z).$$

On the other hand, using the same method, we have

$$\int_{\mathbb{B}_n} G_k(z,\lambda)h(z)dv(z) \lesssim \int_{\mathbb{B}_n}\int_{M_k}\frac{1}{|1-\langle w,\lambda\rangle|^{n+1/2}}\frac{1}{|1-\langle z,w\rangle|^{n+1}}d\nu(w)(1-|z|^2)^{-1/2}dv(z)$$
$$\lesssim \int_{M_k}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{-1/2}}{|1-\langle z,w\rangle|^{n+1}}dv(z)\frac{1}{|1-\langle w,\lambda\rangle|^{n+1/2}}d\nu(w)$$
$$\lesssim \int_{M_k}\frac{(1-|w|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}d\nu(w) \lesssim \int_{W_k}\frac{(1-|w|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}dv(w)$$
$$\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-1/2}(1-|r\lambda|^2)^{-1/2}dr$$
$$\lesssim 2^{-k}h(\lambda) \leq 2^{-(1-2\epsilon)k}h(\lambda).$$

Combining the last two estimates with Lemma 2.11, we conclude that G_k defines an integral operator on $L^2(\mathbb{B}_n)$ whose norm is bounded by $B2^{-(1-2\epsilon)k}$, where the constant B depends only on ϵ, the complex dimension n and the bound $c_i \leq C$ in Lemma 3.2. Obviously, the same conclusion holds for H_k. Thus we have shown that there is a C_1 such that (3.4) holds.

Next we estimate the rank of F_k. Notice that $\mathrm{rank}([k_{w_i}\otimes k_{w_i}, \hat{M}_{z_m}]) \leq 2$. Therefore

$$\mathrm{rank}(F_k) \leq 2\mathrm{card}\{w_i : i\in F,\ w_i\in\Delta_i\cap M_k\}.$$

Since $\Delta_i \supset D(z_i,1)$, by Lemma 3.4, $v_M(\Delta_i\cap M) \gtrsim (1-|z_i|^2)^{d+1}$. For $w_i\in\Delta_i\cap M_k$, Lemma 2.3 gives us $1-|z_i|^2 \approx 1-|w_i|^2 \approx 2^{-2k}$. Consequently $v_M(\Delta_i\cap M)\gtrsim 2^{-2(d+1)k}$ if $w_i\in\Delta_i\cap M_k$. On the other hand, we saw in the above that if $w_i\in M_k$, then

$$\Delta_i\cap M \subset \{w\in M : 1-2^{-2(k-A)} \leq |w| < 1\}.$$

It follows from Lemma 3.3 that $v_M(\{w\in M : 1-2^{-2(k-A)} \leq |w| < 1\}) \lesssim 2^{-2k}$. Since $\Delta_i\cap\Delta_{i'}=\emptyset$ for $i\neq i'$, we conclude that

$$\mathrm{card}\{w_i : i\in F,\ w_i\in\Delta_i\cap M_k\} \lesssim \frac{2^{-2k}}{2^{-2(d+1)k}} = 2^{2kd}.$$

Thus we have shown that $\mathrm{rank}(F_k)\lesssim 2^{2dk}$, i.e., (3.5) holds for some C_2 that depends only on n and the analytic set $\tilde{M}$. This completes the proof. □

Proof of Theorem 3.1. By Lemma 3.2, the commutator $[\hat{T}_\mu, \hat{M}_{z_m}]$ is in the weak operator closure of the convex hull of operators of the form $[\hat{T}_\nu, \hat{M}_{z_m}]$, where ν is a discrete measure as in Proposition 3.5. Given any $p > 2d$, let $0 < \epsilon < 1/2$ be such that $2d/(1-2\epsilon) < p$. Now Proposition 3.5 provides the bound $\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C'$ for all such ν. From this we obtain $\|[\hat{T}_\mu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C'$ by applying Lemma 2.8. Thus $[\hat{T}_\mu, \hat{M}_{z_m}] \in \mathcal{C}^+_{2d/(1-2\epsilon)} \subset \mathcal{C}_p$ as promised. □

Theorem 3.6. *We have $[Q, \hat{M}_{z_j}] \in \mathcal{C}_p$ for all $p > 2d$ and $j \in \{1, \dots, n\}$.*

Proof. By Theorem 1.9, there exist a Carleson measure μ supported on M and $0 < c \le C < \infty$ such that

$$c\|f\|^2 \le \int_M |f(w)|^2 d\mu(w) \le C\|f\|^2$$

for every $f \in \mathcal{Q}$. If $w \in M$, then $K_w \in \mathcal{Q}$. Thus the above inequality implies

$$c\|Qg\|^2 \le \int_M |\langle g, K_w\rangle|^2 d\mu(w) \le C\|Qg\|^2$$

for every $g \in L^2(\mathbb{B}_n)$. This translates to the operator inequality $cQ \le \hat{T}_\mu \le CQ$ on $L^2(\mathbb{B}_n)$. Thus, by the spectral theory of self-adjoint operators, there is a C^∞ function h such that $Q = h(\hat{T}_\mu)$. Now the membership $[Q, \hat{M}_{z_j}] \in \mathcal{C}_p$, $p > 2d$, follows from Lemma 2.10 and Theorem 3.1. □

Proof of Theorem 1.4. The point is that on the big space $L^2(\mathbb{B}_n)$, we have $\hat{M}^*_{z_i} = \hat{M}_{\bar{z}_i}$, consequently $[\hat{M}^*_{z_i}, \hat{M}_{z_j}] = 0$. Applying Lemma 2.1 and Theorem 3.6, we have $[Z^*_{\mathcal{Q},i}, Z_{\mathcal{Q},j}] \in \mathcal{C}_p$ for $p > d$. □

The authors of [20] proved that for two varieties satisfying nice conditions, their union defines a essentially normal quotient module:

Theorem 3.7. *Suppose $\tilde{M}_1$ and $\tilde{M}_2$ are two analytic subsets of an open neighborhood of $\overline{\mathbb{B}_n}$. Let $\tilde{M}_3 = \tilde{M}_1 \cap \tilde{M}_2$. Assume that*

(i) *$\tilde{M}_1$ and $\tilde{M}_2$ intersect transversely with $\partial\mathbb{B}_n$ and have no singular points on $\partial\mathbb{B}_n$.*
(ii) *$\tilde{M}_3$ also intersects transversely with $\partial\mathbb{B}_n$ and has no singular points on $\partial\mathbb{B}_n$.*
(iii) *$\tilde{M}_1$ and $\tilde{M}_2$ intersect cleanly on $\partial\mathbb{B}_n$.*

Let $M_i = \tilde{M}_i \cap \mathbb{B}_n$ and $\mathcal{Q}_i = \overline{span}\{K_\lambda : \lambda \in M_i\}$ for $i = 1, 2, 3$, $M = M_1 \cup M_2$, and $\mathcal{Q} = \overline{\text{span}}\{K_\lambda :\ \lambda \in M\}$. Then $\mathcal{Q}_1 \cap \mathcal{Q}_2/\mathcal{Q}_3$ is finite dimensional and $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed. As a consequence, $\mathcal{Q}$ is p-essentially normal for $p > 2d$, where $d = \dim_{\mathbb{C}} M = \max\{\dim_{\mathbb{C}} M_1, \dim_{\mathbb{C}} M_2\}$.

As a consequence of the improved essential normality in Theorem 1.4, the essential normality in Theorem 3.7 can be improved accordingly.

Corollary 3.8. *Under the same assumption as in Theorem 3.7, the quotient module $\mathcal{Q}$ is p-essentially normal for all $p > d$.*

Once we know that $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed from Theorem 3.7, we have $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. Thus Corollary 3.8 follows from Theorem 3.6 and [17, Lemma 3.3].

4. Antisymmetric Sums on the Submodule

We now consider antisymmetric sums on the submodule $\mathcal{R}$.

Lemma 4.1. *Let $f, g \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. Then for every $p > d$ we have $[Q, \hat{M}_f] \in \mathcal{C}_{2p}$ and $[R, \hat{M}_f]Q[R, \hat{M}_g] \in \mathcal{C}_p$.*

Proof. For any A, B, we have $[Q, AB] = [Q, A]B + A[Q, B]$. Thus the first conclusion, $[Q, \hat{M}_f] \in \mathcal{C}_{2p}$ for $p > d$, is an obvious consequence of Theorem 3.6. Then note that since $RQ = 0 = QR$, we have

$$[R, \hat{M}_f]Q[R, \hat{M}_g] = -R\hat{M}_f Q\hat{M}_g R = R[\hat{M}_f, Q]Q[\hat{M}_g, Q]R.$$

Hence the second conclusion follows from the first. □

Lemma 4.2. *For $f_1, f_2, \ldots, f_{2n-1}, f_{2n} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the operator*

$$[R_{f_1}, R_{f_2}] \cdots [R_{f_{2n-1}}, R_{f_{2n}}] - [T_{f_1}, T_{f_2}] \cdots [T_{f_{2n-1}}, T_{f_{2n}}] \tag{4.1}$$

is in the trace class.

Proof. Let T be the subgroup of the symmetric group S_{2n} generated by the transpositions of the pairs $2j-1$ and $2j$, $j = 1, \ldots, n$. Then

$$\begin{aligned}
[R_{f_1}, R_{f_2}] \cdots [R_{f_{2n-1}}, R_{f_{2n}}] &= (-1)^n (R\hat{M}_{f_1}(1-R)\hat{M}_{f_2}R - R\hat{M}_{f_2}(1-R)\hat{M}_{f_1}R) \cdots \\
&= (-1)^n ([R, \hat{M}_{f_1}](1-R)[\hat{M}_{f_2}, R] - [R, \hat{M}_{f_2}](1-R)[\hat{M}_{f_1}, R]) \cdots \\
&= ([R, \hat{M}_{f_1}](1-R)[R, \hat{M}_{f_2}] - [R, \hat{M}_{f_2}](1-R)[R, \hat{M}_{f_1}]) \cdots \\
&= \sum_{\tau \in T} \operatorname{sgn}(\tau)[R, \hat{M}_{f_{\tau(1)}}](1-R)[R, \hat{M}_{f_{\tau(2)}}][R, \hat{M}_{f_{\tau(3)}}](1-R)[R, \hat{M}_{f_{\tau(4)}}] \cdots \ .
\end{aligned}$$

Recall that $P = R + Q$ is the projection onto the Bergman space $L^2_a(\mathbb{B}_n)$. Consider any product of the form

$$[R, \hat{M}_{f_{\tau(1)}}]X_1[R, \hat{M}_{f_{\tau(2)}}][R, \hat{M}_{f_{\tau(3)}}]X_2[R, \hat{M}_{f_{\tau(4)}}] \cdots [R, \hat{M}_{f_{\tau(2n-1)}}]X_n[R, \hat{M}_{f_{\tau(2n)}}], \tag{4.2}$$

where $X_1, \ldots, X_n$ are bounded operators. Since $[P, \hat{M}_{f_j}] \in \mathcal{C}_p$ for every $p > 2n$, by Lemma 4.1 we have $[R, \hat{M}_{f_j}] \in \mathcal{C}_p$ for every $p > 2n$. Also by Lemma 4.1, if $X_i = Q$ for any i, then

$$[R, \hat{M}_{f_{\tau(2i-1)}}]X_i[R, \hat{M}_{f_{\tau(2i)}}] \in \mathcal{C}_{d+\epsilon} \quad \text{for every} \quad \epsilon > 0.$$

Since $d < n$, if there is an i such that $X_i = Q$, then (4.2) is in the trace class. It is easy to see that the difference

$$\begin{aligned}
D = {} & [R_{f_1}, R_{f_2}] \cdots [R_{f_{2n-1}}, R_{f_{2n}}] - \\
& \sum_{\tau \in T} \operatorname{sgn}(\tau)[R, \hat{M}_{f_{\tau(1)}}](1-P)[R, \hat{M}_{f_{\tau(2)}}][R, \hat{M}_{f_{\tau(3)}}](1-P)[R, \hat{M}_{f_{\tau(4)}}] \cdots
\end{aligned}$$

is a linear combination of operators of the form (4.2) for which at least one X_i equals Q. Hence we conclude that D is in the trace class $\mathcal{C}_1$. Now consider any product of the form

$$[Y_1, \hat{M}_{f_{\tau(1)}}](1-P)[Y_2, \hat{M}_{f_{\tau(2)}}][Y_3, \hat{M}_{f_{\tau(3)}}](1-P)[Y_4, \hat{M}_{f_{\tau(4)}}] \cdots , \tag{4.3}$$

where $Y_1, Y_2, \ldots, Y_{2n-1}, Y_{2n}$ are R, Q or $R + Q = P$. Thus $[Y_j, \hat{M}_{f_k}] \in \mathcal{C}_p$ for $p > 2n$. Since there are $2n$ such commutators in the product (4.3), if there is an i such that $Y_i = Q$, then Lemma

4.1 guarantees that (4.3) is in the trace class. Since the D above is in the trace class, it follows that the difference

$$[R_{f_1}, R_{f_2}] \cdots [R_{f_{2n-1}}, R_{f_{2n}}] - \\ \sum_{\tau \in T} \operatorname{sgn}(\tau)[P, \hat{M}_{f_{\tau(1)}}](1-P)[P, \hat{M}_{f_{\tau(2)}}][P, \hat{M}_{f_{\tau(3)}}](1-P)[P, \hat{M}_{f_{\tau(4)}}] \cdots$$

is in the trace class. The same kind of algebra shows that

$$\sum_{\tau \in T} \operatorname{sgn}(\tau)[P, \hat{M}_{f_{\tau(1)}}](1-P)[P, \hat{M}_{f_{\tau(2)}}][P, \hat{M}_{f_{\tau(3)}}](1-P)[P, \hat{M}_{f_{\tau(4)}}] \cdots \\ = [T_{f_1}, T_{f_2}] \cdots [T_{f_{2n-1}}, T_{f_{2n}}].$$

Therefore (4.1) is in the trace class. $\square$

Proposition 4.3. *For $f_1, f_2, \ldots, f_{2n} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the difference*

$$[R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}] - [T_{f_1}, T_{f_2} \ldots, T_{f_{2n}}]$$

is in the trace class.

Proof. Let T be the same subgroup of the symmetric group S_{2n} generated by the transpositions of the pairs $2j-1$ and $2j$, $j = 1, \ldots, n$, as in the previous proof. Then S_{2n} is the disjoint union of T-cosets. Thus for antisymmetric sums, there is a subset C of S_{2n} such that

$$[R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}] = \sum_{\sigma \in C} \operatorname{sgn}(\sigma)[R_{f_{\sigma(1)}}, R_{f_{\sigma(2)}}] \cdots [R_{f_{\sigma(2n-1)}}, R_{f_{\sigma(2n)}}] \quad \text{and}$$

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] = \sum_{\sigma \in C} \operatorname{sgn}(\sigma)[T_{f_{\sigma(1)}}, T_{f_{\sigma(2)}}] \cdots [T_{f_{\sigma(2n-1)}}, T_{f_{\sigma(2n)}}].$$

Combining these identities with Lemma 4.2, the proposition follows. $\square$

Proof of Theorem 1.7. Given Proposition 4.3, it suffices to recall from Theorem 1.6 that for $f_1, f_2, \ldots, f_{2n} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the antisymmetric sum $[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}]$ is in the trace class. $\square$

5. Antisymmetric Sums on the Quotient Module

We now turn to the proof of Theorem 1.8. Since we have Theorem 3.6, for $m > d$ it is easy to show that the antisymmetric sum $[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2m}}]$ is in the trace class, $f_1, f_2, \ldots, f_{2m} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. The difficult part is to show that the trace of such an antisymmetric sum is zero; in fact, a lot of additional work is required to accomplish this goal. First of all, our proof of zero trace relies on the following principle:

Lemma 5.1. [16, Lemma 1.3] *Suppose that X is a self-adjoint operator and C is a compact operator. If $[X, C]$ is in the trace class, then* $\operatorname{tr}[X, C] = 0$.

The proof of Theorem 1.8 will be based on Lemma 5.1, Theorem 3.1 and

Proposition 5.2. *Let μ be any Carleson measure supported on M. Then for every pair of $f, g \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the double commutator*

$$[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu]]$$

belongs to the class $\mathcal{C}^+_{2d/(1+\epsilon)}$ *for every* $0 < \epsilon < 1/n$.

Most of the work in this section is taken up by the proof of Proposition 5.2, which requires quite a few steps. We begin with some basic estimates on $\mathbb{B}_n$.

Lemma 5.3. (1) *There is a constant* C_1 *such that*

$$\int \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle|^n|1-\langle z,\zeta\rangle|^n} \le \frac{C_1}{|1-\langle w,z\rangle|^{n-1}}\left(1+\log\frac{1}{1-\max\{|w|,|z|\}}\right)$$

for all $w, z \in \mathbb{B}_n$.
(2) *There is a constant* C_2 *such that*

$$\int \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle|^{n+(1/2)}|1-\langle z,\zeta\rangle|^{n+(1/2)}} \le \frac{C_2}{|1-\langle w,z\rangle|^{n}}\left(1+\log\frac{1}{1-\max\{|w|,|z|\}}\right)$$

for all $w, z \in \mathbb{B}_n$.

Proof. Given any $w, z \in \mathbb{B}_n$, define

$$A = \{\zeta \in \mathbb{B}_n : |1-\langle w,\zeta\rangle| \ge (1/4)|1-\langle w,z\rangle|\} \quad \text{and}$$
$$B = \{\zeta \in \mathbb{B}_n : |1-\langle z,\zeta\rangle| \ge (1/4)|1-\langle w,z\rangle|\}.$$

By [18, Proposition 5.1.2], the triangle inequality

$$|1-\langle a,b\rangle|^{1/2} \le |1-\langle a,c\rangle|^{1/2} + |1-\langle b,c\rangle|^{1/2}$$

holds for all $a, b, c \in \mathbb{B}_n$. Therefore $A \cup B = \mathbb{B}_n$. Obviously,

$$\int_A \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle|^n|1-\langle z,\zeta\rangle|^n} \le \frac{4^{n-1}}{|1-\langle w,z\rangle|^{n-1}}\int_A \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle||1-\langle z,\zeta\rangle|^n}.$$

By [18, Proposition 1.4.10] and Hölder's inequality,

$$\int \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle||1-\langle z,\zeta\rangle|^n} \le C\left(1+\log\frac{1}{1-|w|}\right)^{\frac{1}{n+1}}\left(1+\log\frac{1}{1-|z|}\right)^{\frac{n}{n+1}}.$$

Combining these two estimates, we obtain

$$\int_A \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle|^n|1-\langle z,\zeta\rangle|^n} \le \frac{4^{n-1}C}{|1-\langle w,z\rangle|^{n-1}}\left(1+\log\frac{1}{1-\max\{|w|,|z|\}}\right).$$

Obviously, this argument also works for the integral over B. Since $A \cup B = \mathbb{B}_n$, this proves (1). The proof of (2) is similar and will be omitted. $\square$

Recall that M_k was defined by (3.3).

Lemma 5.4. *There is a* $c \ge 1$ *such that if* $D(z,2) \cap M_k \ne \emptyset$ *for some* $k \ge 0$, *then*

$$1 - 2^{-2(k-c)} \le |z| \le 1 - 2^{-2(k+c)}.$$

Proof. This is a special case of Lemma 2.3(1). $\square$

Lemma 5.5. *Let* c *be the same as in Lemma* 5.4. *There is a* $0 < \beta_0 < \infty$ *such that for* $z, w \in \mathbb{B}_n$ *and* $k \ge 0$, *if the conditions* $\beta(z,w) \ge \beta_0$, $|z| \le 1 - 2^{-2(k+c)}$ *and* $|w| \le 1 - 2^{-2(k+c)}$ *are satisfied, then* $|1-\langle z,w\rangle| \ge 2^{-2k} \times 3 \times 2^{2c}$.

Proof. By the definition of the Bergman metric, these conditions imply

$$\beta_0 \le \frac{1}{2}\log\frac{4}{1-|\varphi_z(w)|^2} = \frac{1}{2}\log\frac{4|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)} \le \frac{1}{2}\log\frac{4|1-\langle z,w\rangle|^2}{(2^{-2(k+c)})^2}.$$

From this we obtain

$$(1/2)e^{\beta_0}2^{-2(k+c)} \le |1-\langle z,w\rangle|.$$

Thus it suffices to pick β_0 such that $(1/2)e^{\beta_0}2^{-2c} \ge 2^{2c}\times 3$. $\square$

By what we saw in Section 3, it is obvious that to prove Proposition 5.2, we need to again consider the discrete sum given by (3.2). But here we need to further decompose that sum. First of all, by (3.1) and [21, Lemma 2.2.], the set $\mathcal{L}$ admits a finite partition

$$\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_\ell$$

such that for each $r \in \{1,\dots,\ell\}$, the conditions $w, z \in \mathcal{L}_r$ and $z \ne w$ imply $\beta(z,w) > \beta_0$, where β_0 is the constant in Lemma 5.5.

Recall from Section 3 that $\mathcal{L} = \{z_i\}_{i=1}^\infty$. In particular, recall that $\Delta_i \subset D(z_i,2)$. Let S denote the unit sphere $\{z \in \mathbb{C}^n : |z| = 1\}$. For every $i \in F$, there is a $\xi_i \in S$ such that

$$z_i = |z_i|\xi_i. \tag{5.1}$$

Let F, $w_i \in \Delta_i \cap M$, etc, be the same as in Lemma 3.2. For every pair of $k \ge 0$ and $r \in \{1,\dots,\ell\}$, we define

$$\Gamma_{k,r} = \{w_i : i \in F,\ z_i \in \mathcal{L}_r,\ 1-2^{-2k} \le |w_i| < 1-2^{-2(k+1)}\}.$$

Lemma 5.6. *There is a constant $C_{5.6}$ such that for every pair of $k \ge 0$ and $r \in \{1,\dots,\ell\}$, every $w \in \Gamma_{k,r}$, and every $0 \le j \le k+1$, we have*

$$\mathrm{card}\{\zeta \in \Gamma_{k,r} : |1-\langle w,\zeta\rangle| \le 2^{2j}\cdot 2^{-2k}\} \le C_{5.6}(2^{2j})^n.$$

Proof. Consider any $w \ne w'$ in $\Gamma_{k,r}$. Then there are $i, i' \in F$ such that $w \in \Delta_i \subset D(z_i,2)$ and $w' \in \Delta_{i'} \subset D(z_{i'},2)$. Recalling how the sum was defined in (3.2), the condition $w \ne w'$ implies $i \ne i'$. Since $z_i, z_{i'} \in \mathcal{L}_r$, we have $\beta(z_i,z_{i'}) > \beta_0$. Also, since $w \in D(z_i,2)\cap M_k$ and $w' \in D(z_{i'},2)\cap M_k$, applying Lemma 5.4, we have

$$1-2^{-2(k-c)} \le |z_i| \le 1-2^{-2(k+c)} \quad \text{and} \quad 1-2^{-2(k-c)} \le |z_{i'}| \le 1-2^{-2(k+c)}.$$

Applying Lemma 5.5, we obtain

$$|1-\langle z_i,z_{i'}\rangle| \ge 2^{-2k}\times 3\times 2^{2c} = 2^{-2(k-c)}\times 3.$$

Recalling (5.1), we have

$$1-|z_i| + 1-|z_{i'}| + |1-\langle \xi_i,\xi_{i'}\rangle| \ge |1-\langle z_i,z_{i'}\rangle| \ge 2^{-2(k-c)}\times 3.$$

Recapping the above, we obtain

$$|1-\langle \xi_i,\xi_{i'}\rangle| \ge 2^{-2k} \tag{5.2}$$

if $w \in \Delta_i \subset D(z_i,2)$, $w' \in \Delta_{i'} \subset D(z_{i'},2)$, $w \ne w'$, and $w, w' \in \Gamma_{k,r}$.

On the other hand, for $\zeta \in \Delta_h \subset D(z_h,2)$, $h \in F$, we have

$$\frac{1}{2}\log\frac{1}{1-|\varphi_{z_h}(\zeta)|^2} \le \beta(\zeta,z_h) < 2,$$

which leads to

$$\frac{|1-\langle\zeta, z_h\rangle|^2}{(1-|z_h|^2)(1-|\zeta|^2)} \le e^4.$$

If we also have $\zeta \in M_k$, then Lemma 5.4 gives us $1-|z_h|^2 \le 2\times 2^{-2(k-c)}$, consequently

$$|1-\langle\zeta, z_h\rangle|^2 \le 4e^4 2^{2c}(2^{-2k})^2.$$

It is easy to see that for $0 \le \rho \le 1$ and complex number a with $|a| \le 1$, we have

$$|1-a| \le 2|1-\rho a|.$$

We can write $\zeta = |\zeta|\xi$ for some $\xi \in S$. Thus

$$|1-\langle\xi, \xi_h\rangle| \le 2|1-|\zeta||z_h|\langle\xi, \xi_h\rangle| = 2|1-\langle\zeta, z_h\rangle| \le C_1 2^{-2k}, \tag{5.3}$$

where $C_1 = 4e^2 2^c$. Now suppose that $|1-\langle w, \zeta\rangle| \le 2^{2j}\cdot 2^{-2k}$, where w is the same as in the first paragraph of the proof. Write $w = |w|\eta$, where $\eta \in S$. Then

$$|1-\langle\eta, \xi\rangle| \le 2|1-\langle w, \zeta\rangle| \le 2\times 2^{2j}\times 2^{-2k}. \tag{5.4}$$

Since $w \in \Delta_i \subset D(z_i, 2)$, (5.3) implies

$$|1-\langle\eta, \xi_i\rangle| \le C_1 2^{-2k}. \tag{5.5}$$

Since $d(x, y) = |1-\langle x, y\rangle|^{1/2}$ is a metric on S, from (5.3), (5.4) and (5.5) we obtain

$$|1-\langle\xi_i, \xi_h\rangle| \le \left(C_1^{1/2} + 2^{j+1} + C_1^{1/2}\right)^2 2^{-2k} \le C_2 2^{2j} 2^{-2k}.$$

Combining this with (5.2), by a standard estimate using the spherical measure on S [18, Proposition 5.1.4], the number of such ξ_h's does not exceed $C_3(C_2 2^{2j})^n = C_{5.6}(2^{2j})^n$. $\square$

For every pair of $k \ge 0$ and $r \in \{1, \ldots, \ell\}$, define the operator

$$T_{k,r} = \sum_{w\in\Gamma_{k,r}} c_w k_w \otimes k_w,$$

where $c_w = c_i$ (see Lemma 3.2) if $w = w_i$ for some $i \in F$. Let $f, g \in \mathbb{C}[z_1, \bar z_1, \ldots, z_n, \bar z_n]$ be given. We will now estimate the operator norm $\|[\hat M_f, [\hat M_g, T_{k,r}]]\|$, which is the main difficulty in the proof of Proposition 5.2. Obviously, we can decompose the double commutator in the form

$$[\hat M_f, [\hat M_g, T_{k,r}]] = A_1 - A_2 - A_3 + A_4,$$

where

$$A_1 = \sum_{w\in\Gamma_{k,r}} c_w\{(f-f(w))(g-g(w))k_w\} \otimes k_w,$$

$$A_2 = \sum_{w\in\Gamma_{k,r}} c_w\{(f-f(w))k_w\} \otimes \{\overline{(g-g(w))}k_w\},$$

$$A_3 = \sum_{w\in\Gamma_{k,r}} c_w\{(g-g(w))k_w\} \otimes \{\overline{(f-f(w))}k_w\} \quad \text{and}$$

$$A_4 = \sum_{w\in\Gamma_{k,r}} c_w k_w \otimes \{\overline{(g-g(w))(f-f(w))}k_w\}.$$

Since f, g are arbitrary in $\mathbb{C}[z_1, \bar z_1, \ldots, z_n, \bar z_n]$, A_4^* is just another A_1, and A_3 another A_2. Thus it suffices to estimate $\|A_1\|$ and $\|A_2\|$.

To do that, pick an orthonormal set $\{e_w : w \in \Gamma_{k,r}\}$. We then factor A_1 and A_2 in the form $A_1 = XY^*$ and $A_2 = Z_1 Z_2^*$, where

$$\begin{aligned} X &= \sum_{w \in \Gamma_{k,r}} \{(f - f(w))(g - g(w))k_w\} \otimes e_w, \\ Y &= \sum_{w \in \Gamma_{k,r}} c_w k_w \otimes e_w, \\ Z_1 &= \sum_{w \in \Gamma_{k,r}} c_w \{(f - f(w))k_w\} \otimes e_w \quad \text{and} \\ Z_2 &= \sum_{w \in \Gamma_{k,r}} \{\overline{(g - g(w))}k_w\} \otimes e_w. \end{aligned}$$

We have

$$X^*X = \sum_{w,z \in \Gamma_{k,r}} h(z,w) e_w \otimes e_z,$$

where

$$h(z,w) = \langle (f - f(z))(g - g(z))k_z, (f - f(w))(g - g(w))k_w \rangle.$$

Since $\{e_w : w \in \Gamma_{k,r}\}$ is an orthonormal set and $|h(w,z)| = |h(z,w)|$, by the simplest version of the Schur test, we have

$$\|X^*X\| \le \sup_{w \in \Gamma_{k,r}} \sum_{z \in \Gamma_{k,r}} |h(z,w)|.$$

Lemma 5.7. *For any $0 < \epsilon < 1/n$, there is a constant C that depends only on ϵ, n, M and $f, g \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ such that*

$$\sup_{w \in \Gamma_{k,r}} \sum_{z \in \Gamma_{k,r}} |h(z,w)| \le C 2^{-2(1+\epsilon)k}.$$

Consequently, $\|X\| = \|X^*X\|^{1/2} \le C^{1/2} 2^{-(1+\epsilon)k}$.

Proof. A key to this estimate is the following fact that we proved at the end of the proof of Proposition 3.5: There is a C such that $\operatorname{card}\{w_i : w_i \in M_k\} \le C 2^{2dk}$. Since $d \le n-1$, this implies that

$$\operatorname{card}(\Gamma_{k,r}) \le C 2^{2(n-1)k}. \tag{5.6}$$

Since $f, g \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, for $w \in \Gamma_{k,r}$ we have

$$|(f(\zeta) - f(w))(g(\zeta) - g(w))k_w(\zeta)| \le C_1 \frac{(1 - |w|^2)^{(n+1)/2}}{|1 - \langle \zeta, w \rangle|^n} \le \frac{C_2 2^{-(n+1)k}}{|1 - \langle \zeta, w \rangle|^n}.$$

Therefore

$$|h(z,w)| \le C_2^2 2^{-2(n+1)k} \int \frac{dv(\zeta)}{|1 - \langle w, \zeta \rangle|^n |1 - \langle z, \zeta \rangle|^n}$$

for $w, z \in \Gamma_{k,r}$. Applying Lemma 5.3(1), we obtain

$$|h(z,w)| \le \frac{C_3 (k+1) 2^{-2(n+1)k}}{|1 - \langle w, z \rangle|^{n-1}}.$$

Hence for each $w \in \Gamma_{k,r}$,

$$\sum_{z\in\Gamma_{k,r}} |h(z,w)| \le C_3(k+1)2^{-2k} \sum_{z\in\Gamma_{k,r}} \frac{2^{-2nk}}{|1-\langle w,z\rangle|^{n-1}} = C_3(k+1)2^{-2k} \sum_{j=0}^{k+1} \sum_{z\in G_j} \frac{2^{-2nk}}{|1-\langle w,z\rangle|^{n-1}},$$

where

$$G_0 = \{z \in \Gamma_{k,r} : |1-\langle w,z\rangle| \le 2^{-2k}\} \quad \text{and}$$
$$G_j = \{z \in \Gamma_{k,r} : 2^{2(j-1)} \cdot 2^{-2k} < |1-\langle w,z\rangle| \le 2^{2j} \cdot 2^{-2k}\}, \quad 1 \le j \le k+1.$$

For $z \in G_0$, $|1-\langle w,z\rangle| \ge 1-|z| \ge 2^{-2(k+1)}$. Thus $2^{-2k}/|1-\langle w,z\rangle| \le 2^2$ if $z \in G_0$. Hence for every $0 \le j \le k+1$ we have

$$\frac{2^{-2k}}{|1-\langle w,z\rangle|} \le 2^{-2(j-1)} \quad \text{if} \ \ z \in G_j. \tag{5.7}$$

Writing $C_4 = 2^{2(n-1)}C_3$, we now have

$$\sum_{z\in\Gamma_{k,r}} |h(z,w)| \le C_4(k+1)2^{-2k} \cdot 2^{-2k} \sum_{j=0}^{k+1} 2^{-2(n-1)j}\mathrm{card}(G_j). \tag{5.8}$$

We will show that

$$2^{-2k} \sum_{j=0}^{k+1} 2^{-2(n-1)j}\mathrm{card}(G_j) \le C_5 2^{-(2/n)k}. \tag{5.9}$$

Then, combining (5.8) and (5.9), we see that the lemma holds for any $0 < \epsilon < 1/n$.

To prove (5.9), let k_0 be the largest integer satisfying the condition $k_0 \le (n-1)k/n$. Then

$$2^{-2k} \sum_{j=0}^{k+1} 2^{-2(n-1)j}\mathrm{card}(G_j) = I + J,$$

where

$$I = 2^{-2k} \sum_{j=0}^{k_0} 2^{-2(n-1)j}\mathrm{card}(G_j) \quad \text{and} \quad J = 2^{-2k} \sum_{j=k_0+1}^{k+1} 2^{-2(n-1)j}\mathrm{card}(G_j).$$

For I, we apply Lemma 5.6, which tells us that $\mathrm{card}(G_j) \le C_6 2^{2nj}$. Thus

$$I \le C_6 2^{-2k} \sum_{j=0}^{k_0} 2^{-2(n-1)j} 2^{2nj} \le C_7 2^{-2k} 2^{2k_0} \le C_7 2^{-(2/n)k}.$$

For J, we use (5.6), which implies that $\mathrm{card}(G_j) \le C 2^{2(n-1)k}$. Hence

$$J \le C 2^{-2k} 2^{2(n-1)k} \sum_{j=k_0+1}^{\infty} 2^{-2(n-1)j} \le C_8 2^{-2k} 2^{2(n-1)k} 2^{-2(n-1)(k_0+1)}.$$

By definition, $k_0 + 1 > (n-1)k/n$. Therefore

$$J \le C_8 2^{-2k} 2^{2(n-1)k} 2^{-2(n-1)(n-1)k/n} = C_8 2^{-(2/n)k}.$$

Thus $I + J \le (C_7 + C_8)2^{-(2/n)k}$, which proves (5.9) and completes the proof of the lemma. $\square$

Lemma 5.8. *The norm $\|Y\|$ is bounded by a constant that depends only on n, M and μ.*

Proof. Obviously,
$$Y^*Y = \sum_{w,z\in\Gamma_{k,r}} c_w c_z \langle k_z, k_w\rangle e_w \otimes e_z.$$
Recall from Lemma 3.2 that c_w is bounded by a C determined by μ. Using the easy version of the Schur test mentioned earlier, we obtain
$$\|Y^*Y\| \le C^2 \sup_{w\in\Gamma_{k,r}} \sum_{z\in\Gamma_{k,r}} |\langle k_z, k_w\rangle|.$$
For $w, z \in \Gamma_{k,r}$, we have
$$|\langle k_z, k_w\rangle| \le \frac{2^{n+1}\cdot 2^{-2(n+1)k}}{|1-\langle w,z\rangle|^{n+1}}. \tag{5.10}$$
Given a $w \in \Gamma_{k,r}$, let G_j, $0 \le j \le k+1$, be the same as in the proof of lemma 5.7. Combining (5.10) with (5.7), we have
$$\sum_{z\in\Gamma_{k,r}} |\langle k_z, k_w\rangle| = \sum_{j=0}^{k+1}\sum_{z\in G_j} |\langle k_z, k_w\rangle| \le 2^{n+1}\sum_{j=0}^{k+1} 2^{-2(n+1)(j-1)}\mathrm{card}(G_j).$$
By Lemma 5.6, we have $\mathrm{card}(G_j) \le C_1 2^{2nj}$ for all $0 \le j \le k+1$. Thus
$$\sum_{z\in\Gamma_{k,r}} |\langle k_z, k_w\rangle| \le C_1 2^{3(n+1)} \sum_{j=0}^{k+1} 2^{-2(n+1)j}2^{2nj} \le 2C_1 2^{3(n+1)}.$$
This completes the proof. □

Lemma 5.9. *For any $\delta > 0$, there is a C that depends only on δ, n, M, μ and f such that $\|Z_1\| \le C2^{-(1-\delta)k}$. A similar estimate holds for $\|Z_2\|$.*

Proof. Writing $\psi(z,w) = \langle (f - f(z))k_z, (f - f(w))k_w\rangle$, we have
$$Z_1^* Z_1 = \sum_{w,z\in\Gamma_{k,r}} c_w c_z \psi(z,w) e_w \otimes e_z. \tag{5.11}$$
Since $f \in \mathbb{C}[z_1, \bar z_1, \dots, z_n, \bar z_n]$, for $w \in \Gamma_{k,r}$, we have
$$|(f(\zeta) - f(w))k_w(\zeta)| \le C_1 \frac{(1-|w|^2)^{(n+1)/2}}{|1-\langle \zeta, w\rangle|^{n+(1/2)}} \le \frac{C_2 2^{-(n+1)k}}{|1-\langle \zeta, w\rangle|^{n+(1/2)}}.$$
Therefore
$$|\psi(z,w)| \le C_2^2 2^{-2(n+1)k}\int \frac{dv(\zeta)}{|1-\langle w,\zeta\rangle|^{n+(1/2)}|1-\langle z,\zeta\rangle|^{n+(1/2)}}$$
for $w, z \in \Gamma_{k,r}$. Applying Lemma 5.3(2), we obtain
$$|\psi(z,w)| \le \frac{C_3(k+1)2^{-2(n+1)k}}{|1-\langle w,z\rangle|^n}.$$
Recalling (5.7) again, for each $w \in \Gamma_{k,r}$,
$$\sum_{z\in\Gamma_{k,r}} |\psi(z,w)| = \sum_{j=0}^{k+1}\sum_{z\in G_j} |\psi(z,w)| \le C_3(k+1)2^{-2k}\sum_{j=0}^{k+1} 2^{-2n(j-1)}\mathrm{card}(G_j).$$

Another application of Lemma 5.6 then leads to

$$\sum_{z\in\Gamma_{k,r}} |\psi(z,w)| \le C_4(k+1)^2 2^{-2k},$$

$w \in \Gamma_{k,r}$. Again, $|\psi(z,w)| = |\psi(w,z)|$. Thus by (5.11) and the Schur test, we have $\|Z_1^* Z_1\| \le C^2 C_4 (k+1)^2 2^{-2k}$, which implies the conclusion of the lemma. □

Corollary 5.10. *For any $0 < \epsilon < 1/n$, there is a C that depends only on ϵ, n, M and $f, g \in \mathbb{C}[z_1, \bar z_1, \ldots, z_n, \bar z_n]$ such that $\|[\hat M_f, [\hat M_g, T_{k,r}]]\| \le C2^{-(1+\epsilon)k}$.*

Proof. By Lemmas 5.7 and 5.8, we have $\|A_1\| \le \|X\| \cdot \|Y\| \le C_1 2^{-(1+\epsilon)k} \cdot C_2$. Take $\delta > 0$ small enough so that $2(1-\delta) \ge 1+\epsilon$. Then by Lemma 5.9 we have $\|A_2\| \le \|Z_1\|\|Z_2\| \le (C_3 2^{-(1-\delta)k})^2 \le C_3^2 2^{-(1+\epsilon)k}$. Similar estimates respectively hold for $\|A_4\|$ and $\|A_3\|$. Since $A_1 - A_2 - A_3 + A_4 = [\hat M_f, [\hat M_g, T_{k,r}]]$, our conclusion follows. □

Proof of Proposition 5.2. For each $k \ge 0$, denote

$$\Gamma_k = \{w_i : i \in F, \quad 1 - 2^{-2k} \le |w_i| < 1 - 2^{-2(k+1)}\},$$

where F and w_i are the same as in Lemma 3.2. Then define

$$T_k = \sum_{w\in\Gamma_k} c_w k_w \otimes k_w.$$

Given $0 < \epsilon < 1/n$, Corollary 5.10 tells us that $\|[\hat M_f, [\hat M_g, T_{k,r}]]\| \le C2^{-(1+\epsilon)k}$ for all $k \ge 0$ and $r \in \{1, \ldots, \ell\}$. Obviously, $T_k = T_{k,1} + \cdots + T_{k,\ell}$. Hence

$$\|[\hat M_f, [\hat M_g, T_k]]\| \le \ell C 2^{-(1+\epsilon)k}.$$

By the argument given at the end of the proof of Proposition 3.5, we have $\mathrm{card}(\Gamma_k) \le C_1 2^{2dk}$. Therefore

$$\mathrm{rank}([\hat M_f, [\hat M_g, T_k]]) \le 4\mathrm{card}(\Gamma_k) \le 4C_1 2^{2dk}.$$

Applying Lemma 2.9, we have

$$\|[\hat M_f, [\hat M_g, T]]\|^+_{2d/(1+\epsilon)} \le \ell C(1+4C_1)B(1+\epsilon, 2d), \tag{5.12}$$

where

$$T = \sum_{k=0}^\infty T_k = \sum_{k=0}^\infty \sum_{w\in\Gamma_k} c_w k_w \otimes k_w = \sum_{i\in F} c_i k_{w_i} \otimes k_{w_i}.$$

Lemma 3.2 tells us that $\hat T_\mu$ is in the weak closure of the convex hull of such T's. By Lemma 2.8, from (5.12) we deduce $\|[\hat M_f, [\hat M_g, \hat T_\mu]]\|^+_{2d/(1+\epsilon)} \le \ell C(1+4C_1)B(1+\epsilon, 2d)$. In particular, $[\hat M_f, [\hat M_g, \hat T_\mu]] \in \mathcal{C}^+_{2d/(1+\epsilon)}$ as promised. □

Proposition 5.11. *For $f, g \in \mathbb{C}[z_1, \bar z_1, \ldots, z_n, \bar z_n]$, the double commutator*

$$[\hat M_f, [\hat M_g, Q]]$$

belongs to the class $\mathcal{C}^+_{2d/(1+\epsilon)}$ for every $0 < \epsilon < 1/n$.

Proof. By Theorem 1.9, there exist a Carleson measure μ on M and $0 < c \le C < \infty$ for which the operator inequality $cQ \le \hat T_\mu \le CQ$ holds on $L^2(\mathbb{B}_n)$. This means that the spectrum of $\hat T_\mu$ is contained in $\{0\} \cup [c, C]$, and that the spectral projection of $\hat T_\mu$ corresponding to the interval $[c, C]$ equals Q.

Now let Γ be a simple Jordan curve in $\mathbb{C}\backslash(\{0\} \cup [c, C])$ whose winding number about every $x \in [c, C]$ is one and whose winding number about 0 is zero. By the above paragraph and the Riesz functional calculus, we have

$$Q = \frac{1}{2\pi i} \int_\Gamma (\lambda - \hat{T}_\mu)^{-1} d\lambda.$$

Therefore for any $f, g \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$,

$$[\hat{M}_f, [\hat{M}_g, Q]] = \frac{1}{2\pi i} \int_\Gamma [\hat{M}_f, [\hat{M}_g, (\lambda - \hat{T}_\mu)^{-1}]] d\lambda = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int_\Gamma (\lambda - \hat{T}_\mu)^{-1} [\hat{M}_f, \hat{T}_\mu] (\lambda - \hat{T}_\mu)^{-1} [\hat{M}_g, \hat{T}_\mu] (\lambda - \hat{T}_\mu)^{-1} d\lambda, \\ I_2 &= \frac{1}{2\pi i} \int_\Gamma (\lambda - \hat{T}_\mu)^{-1} [\hat{M}_f, [\hat{M}_g, \hat{T}_\mu]] (\lambda - \hat{T}_\mu)^{-1} d\lambda \quad \text{and} \\ I_3 &= \frac{1}{2\pi i} \int_\Gamma (\lambda - \hat{T}_\mu)^{-1} [\hat{M}_g, \hat{T}_\mu] (\lambda - \hat{T}_\mu)^{-1} [\hat{M}_f, \hat{T}_\mu] (\lambda - \hat{T}_\mu)^{-1} d\lambda. \end{aligned}$$

For every $0 < \epsilon < 1/n$, Proposition 5.2 gives us $I_2 \in \mathcal{C}^+_{2d/(1+\epsilon)}$. It follows from Theorem 3.1 that $I_1, I_3 \in \mathcal{C}_p$ for every $p > d$. Hence $[\hat{M}_f, [\hat{M}_g, Q]] \in \mathcal{C}^+_{2d/(1+\epsilon)}$, $0 < \epsilon < 1/n$. $\square$

Proposition 5.12. *Let $f, g, h \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$.*
(1) *If $d \geq 2$, then the double commutator $[Q_f, [Q_g, Q_h]]$ belongs to the Schatten class $\mathcal{C}_p$ for every $p > 2nd/(2n + 1)$.*
(2) *If $d = 1$, then the double commutator $[Q_f, [Q_g, Q_h]]$ belongs to the trace class.*

Proof. Consider any $1 < r < \infty$ and $1 < s < \infty$, and define t by the relation $1/t = (1/r) + (1/s)$. For $A \in \mathcal{C}_r$ and $B \in \mathcal{C}_s$, we have $AB \in \mathcal{C}_t$ if $t > 1$ and $AB \in \mathcal{C}_1$ if $t \leq 1$.

Now define p_0 by the formula $1/p_0 = (1/2d) + (1 + (1/n))/(2d)$. Then $p_0 = 2nd/(2n + 1)$. We have $p_0 > 1$ if $d \geq 2$, and $p_0 < 1$ if $d = 1$.

For any $f, g, h \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, simple algebra shows that

$$\begin{aligned} [Q_f, [Q_g, Q_h]] &= Q(\hat{M}_f [Q_g, Q_h] - [Q_g, Q_h] \hat{M}_f) Q = Q[\hat{M}_f, [Q_g, Q_h]] Q \\ &= Q[\hat{M}_f, [Q, \hat{M}_h](1 - Q)[\hat{M}_g, Q] - [Q, \hat{M}_g](1 - Q)[\hat{M}_h, Q]] Q. \end{aligned}$$

Then note that

$$\begin{aligned} &[\hat{M}_f, [Q, \hat{M}_h](1 - Q)[\hat{M}_g, Q]] \\ &\quad = [\hat{M}_f, [Q, \hat{M}_h]](1 - Q)[\hat{M}_g, Q] - [Q, \hat{M}_h][\hat{M}_f, Q][\hat{M}_g, Q] + [Q, \hat{M}_h](1 - Q)[\hat{M}_f, [\hat{M}_g, Q]] \\ &\quad = T_1 - T_2 + T_3. \end{aligned}$$

By Theorem 3.6 and Proposition 5.11, we have $T_1, T_3 \in \mathcal{C}_p$ for every $p > p_0 = 2nd/(2n+1)$ in the case $d \geq 2$, and $T_1, T_3 \in \mathcal{C}_1$ in the case $d = 1$. Also, Theorem 3.6 tells us that $T_2 \in \mathcal{C}_p$ for every $p > 2d/3$ if $d \geq 2$ and $T_2 \in \mathcal{C}_1$ if $d = 1$. This shows that the operator $[\hat{M}_f, [Q, \hat{M}_h](1-Q)[\hat{M}_g, Q]]$ belongs to $\mathcal{C}_p$ for every $p > 2nd/(2n + 1)$ or to $\mathcal{C}_1$ depending on $d \geq 2$ or $d = 1$. The same is true for $[\hat{M}_f, [Q, \hat{M}_g](1 - Q)[\hat{M}_h, Q]]$. This proves the proposition. $\square$

Proposition 5.13. Let $\nu \geq d$. *Then for any* $f, g, f_1, f_2, \ldots, f_{2\nu} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, *the operator*

$$[Q_f, Q_g[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2\nu}}]]$$

is in the trace class with zero trace.

Proof. First of all, it follows from Theorem 3.6 and Lemma 2.1 that if $\varphi, \psi \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, then the commutator $[Q_\varphi, Q_\psi]$ belongs to $\mathcal{C}_p$ for every $p > d$.

Let $f, g, f_1, f_2, \ldots, f_{2\nu} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ be given. For convenience, denote

$$Y = [Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2\nu}}].$$

As we mentioned in the proof of Proposition 4.3, there is a subset C of the symmetric group $S_{2\nu}$ such that

$$Y = \sum_{\sigma \in C} \mathrm{sgn}(\sigma)[Q_{f_{\sigma(1)}}, Q_{f_{\sigma(2)}}] \cdots [Q_{f_{\sigma(2\nu-1)}}, Q_{f_{\sigma(2\nu)}}].$$

Since $\nu \geq d$, we have $Y \in \mathcal{C}_p$ for every $p > 1$. Therefore $[Q_f, Q_g]Y \in \mathcal{C}_1$.

Next we show that $[Q_f, Y] \in \mathcal{C}_1$. If $d = 1$, then this is a direct consequence of Proposition 5.12(2). Suppose that $d \geq 2$. In this case, Proposition 5.12(1) tells us that $[Q_f, [Q_{f_{\sigma(2i-1)}}, Q_{f_{\sigma(2i)}}]] \in \mathcal{C}_p$ for every $p > 2nd/(2n+1)$, where $1 \leq i \leq \nu$ and $\sigma \in C$. Since $2nd/(2n+1) < d$, combining this with the Schatten-class membership of every $[Q_{f_{\sigma(2j-1)}}, Q_{f_{\sigma(2j)}}]$, $j \neq i$, mentioned in the first paragraph, we see that $[Q_f, Y] \in \mathcal{C}_1$.

Combining the last two paragraphs, we conclude that

$$[Q_f, Q_g Y] = [Q_f, Q_g]Y + Q_g[Q_f, Y] \in \mathcal{C}_1.$$

Since $\bar{f}$ also belongs to $\mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we similarly have $[Q_f^*, Q_g Y] = [Q_{\bar{f}}, Q_g Y] \in \mathcal{C}_1$. Since Y is compact, it follows from Lemma 5.1 that

$$\mathrm{tr}[Q_f + Q_f^*, Q_g Y] = 0 = \mathrm{tr}[Q_f - Q_f^*, Q_g Y].$$

From this we obtain $\mathrm{tr}[Q_f, Q_g Y] = 0$ as promised. □

Proof of Theorem 1.8. Let $m > d$ and $f_1, f_2, \ldots, f_{2m} \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. Since $2m$ is even, Proposition 1.1 in [16] tells us that the antisymmetric sum $[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2m}}]$ is a linear combination of terms of the form

$$[Q_{f_{\sigma(1)}}, Q_{f_{\sigma(2)}}[Q_{f_{\sigma(3)}}, Q_{f_{\sigma(4)}}, \cdots, Q_{f_{\sigma(2m)}}]],$$

where σ runs over a certain subset D of the symmetric group S_{2m}. Taking $\nu = m - 1$, Theorem 1.8 is a direct consequence of Proposition 5.13. □

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Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: yiwangfdu@gmail.com

Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260, USA
E-mail address: jxia@acsu.buffalo.edu