BEREZIN TRANSFORM OF PRODUCTS OF TOEPLITZ OPERATORS ON THE HARDY SPACE

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Abstract. Let $H^2(S)$ be the Hardy space on the unit sphere in $\mathbf{C}^n$. We show that there are Toeplitz operators T_f and T_g on $H^2(S)$ such that the product T_fT_g is not compact and yet $\|T_fT_gk_z\|$ tends to 0 as $|z| \to 1$. Consequently, the Berezin transform $\langle T_fT_gk_z, k_z\rangle$ tends to 0 as $|z| \to 1$.

1. Introduction

The problem we consider in this paper has a long history. To motivate our result, we start from the beginning of the story.

In [2], Axler and Zheng showed that if A is an algebraic combination of Toeplitz operators on the Bergman space $L^2_a(D)$ of the unit disc D, then the condition

$$\lim_{|z|\uparrow 1} \langle Ak_z, k_z\rangle = 0 \tag{1.1}$$

implies that A is a compact operator on $L^2_a(D)$. Here and in what follows, we write k_z for the normalized reproducing kernel of the space in discussion. Later in [9], Suárez strengthened this result to the extent that for $A \in \mathcal{T}$, where $\mathcal{T}$ is the C^*-algebra generated by Toeplitz operators with bounded symbols on the Bergman space $L^2_a(\mathbf{B})$ of the unit ball $\mathbf{B}$ in $\mathbf{C}^n$, (1.1) implies that A is compact. The fact that Suárez was able to do this for every $A \in \mathcal{T}$ was viewed as a major breakthrough. Ever since, Suárez's result has inspired many generalizations [3,5,12]. In particular, this compactness criterion was recently shown to hold on the Bergman space of strongly pseudo-convex domains [10, Theorem 1.2] and even on certain quotient modules of the Hardy module [11, Theorem 1.3].

But it is a different story on the Hardy space itself. In [4], Guo and Zheng constructed Blashke products b and b_1 on D such that on the Hardy space H^2 of the unit circle $\mathbf{T}$, the Toeplitz operators $T_{\bar{b}}$, T_{b_1} and T_{b_1b} have the property that the operator

$$A = T_{b_1} - T_{b_1b}T_{\bar{b}} \tag{1.2}$$

satisfies the H^2-version of (1.1) but is not compact. This example uncovers an important difference between the Hardy space and other function spaces, namely Toeplitz operators on H^2 are not as "localized" as those on the Bergman space or Fock space.

This, however, is not the end of the story. An obvious question is, what about the Hardy space $H^2(S)$ on the unit sphere S in $\mathbf{C}^n$, $n \geq 2$? To be sure, for $n \geq 2$, the general

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expectation is that the $H^2(S)$-version of (1.1) is not sufficient for compactness. On the other hand, in the literature, one cannot find an example of the Guo-Zheng type for the case $n \geq 2$. In fact, if we analyze the Guo-Zheng example carefully, we will see why it is hard to come up with a straightforward generalization in dimensions $n \geq 2$. Let us go over the Guo-Zheng construction in [4].

Take, for example, any infinite sequence $\{a_k\}$ in $D\backslash\{0\}$ such that

$$\sum_{k=1}^{\infty} k(1-|a_k|) < \infty. \tag{1.3}$$

Then Guo and Zheng define the Blashke products

$$b(z) = \prod_{k=1}^{\infty} \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z} \quad \text{and} \quad b_1(z) = \prod_{k=1}^{\infty} \left(\frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z} \right)^k,$$

$z \in D$, for which (1.3) ensures convergence. Their main observation is that if $\{\gamma_j\}$ is any sequence in D such that $|\gamma_j| \to 1$ and $|b(\gamma_j)| \to \alpha$ for some $\alpha < 1$, then $|b_1(\gamma_j)| \to 0$. Thus it follows that

$$\lim_{|z|\uparrow 1} b_1(z)(1 - |b(z)|^2) = 0. \tag{1.4}$$

For the A given by (1.2), it is easy to see that $\langle Ak_z, k_z \rangle = b_1(z)(1 - |b(z)|^2)$. Therefore this A satisfies the H^2-version of (1.1). On the other hand, $A = T_{b_1}(1 - T_b T_{\bar{b}})$. Since b is an infinite Blashke product and T_{b_1} is an isometry, A is not compact.

From the above details, we see that it is very hard to mimic the Guo-Zheng construction in the case $n \geq 2$. First of all, when $n \geq 2$, there are no Blashke products. There are non-constant inner functions on the unit ball, thanks to the works of Løw [6] and Aleksandrov [1], but these only come in very limited supply. In particular, we do not know if one can produce inner functions b and b_1 on the unit ball $\mathbf{B}$ such that (1.4) holds. Thus, in the case $n \geq 2$, for algebraic combinations A of Toeplitz operators on $H^2(S)$, if we want to show that the $H^2(S)$-version of (1.1) is not sufficient to imply the compactness of A, then we need to come up with a significantly different kind of construction.

We are pleased to report that we have managed to find just such a construction. Moreover, our construction yields a stronger result than the statement that the $H^2(S)$-version of (1.1) is not sufficient for compactness:

Theorem 1.1. *Consider the unit sphere $S \subset \mathbf{C}^n$, $n \geq 2$. There exist $f, g \in L^\infty(S)$ such that for the Toeplitz operators T_f and T_g on the Hardy space $H^2(S)$, the product $T_f T_g$ is not compact while*

$$\lim_{|z|\uparrow 1} \|T_f T_g k_z\| = 0. \tag{1.5}$$

Obviously, (1.5) implies

$$\lim_{|z|\uparrow 1}\langle T_f T_g k_z, k_z\rangle = 0.$$

While it was fully anticipated that the $H^2(S)$-version of (1.1) is not sufficient for compactness, (1.5) somewhat exceeds expectation.

Theorem 1.1 should be contrasted with the well-known fact that for a single Toeplitz operator T_f on $H^2(S)$, $f \in L^\infty(S)$, the condition

$$\lim_{|z|\uparrow 1}\langle T_f k_z, k_z\rangle = 0$$

implies that f is the zero function, i.e., $T_f = 0$. Thus we see that the failure of the $H^2(S)$-version of (1.1) as a sufficient condition for compactness occurs at the first available opportunity. This is really an emphatic way to say that Toeplitz operators on the Hardy space are not localized.

The rest of the paper is organized as follows. In Section 2 we collect a number of preliminaries that are necessary for the construction of the desired f and g. The construction itself is carried out in Section 3.

2. Preliminaries

Let S and $\mathbf{B}$ respectively denote the unit sphere and the unit ball in $\mathbf{C}^n$. That is, $S = \{z \in \mathbf{C}^n : |z| = 1\}$ and $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$. We write $d\sigma$ for the standard spherical measure on S, with the normalization $\sigma(S) = 1$. Then the Hardy space $H^2(S)$ is just the closure of the analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S) = L^2(S, d\sigma)$.

Let $P : L^2(S) \to H^2(S)$ be the orthogonal projection. Recall that for $\varphi \in L^\infty(S)$, the Toeplitz operator T_φ is defined by the formula

$$T_\varphi h = P(\varphi h), \quad h \in H^2(S).$$

For the Hardy space $H^2(S)$, the normalized reproducing kernel k_z is given by the formula

$$k_z(w) = \frac{(1-|z|^2)^{n/2}}{(1-\langle w, z\rangle)^n}.$$

Let ψ be any bounded analytic function on $\mathbf{B}$. Then it is well known that

$$T_{\bar\psi} k_z = \overline{\psi(z)} k_z \quad \text{for} \quad z \in \mathbf{B} \tag{2.1}$$

and $T_\varphi T_\psi = T_{\varphi\psi}$, $T_{\bar\psi} T_\varphi = T_{\bar\psi\varphi}$ for $\varphi \in L^\infty(S)$. Our construction relies on these facts.

Recalling from [8, Section 2.2], for $z \in \mathbf{B}\backslash\{0\}$, we have the Möbius transform

$$\varphi_z(w) = \frac{1}{1-\langle w, z\rangle}\left\{z - \frac{\langle w, z\rangle}{|z|^2} z - (1-|z|^2)^{1/2}\left(w - \frac{\langle w, z\rangle}{|z|^2} z\right)\right\}.$$

For $z = 0$, we define $\varphi_0(w) = -w$. Then $\varphi_z \circ \varphi_z = \mathrm{id}$ for every $z \in \mathbf{B}$. Also recall that for each $a \in \mathbf{B}$, the formula

$$(U_a h)(w) = k_a(w) h(\varphi_a(w)), \quad h \in H^2(S),$$

defines a unitary operator on $H^2(S)$. In particular, for $a, z \in \mathbf{B}$, we have

$$(U_a k_z)(w) = k_a(w) k_z(\varphi_a(w)) = \left(\frac{|1 - \langle a, z \rangle|}{1 - \langle a, z \rangle} \right)^n k_{\varphi_a(z)}(w).$$

This implies that if A is any bounded operator on $H^2(S)$, then for each $a \in \mathbf{B}$,

$$\sup_{|z|<1} \|U_a A U_a^* k_z\| = \sup_{|z|<1} \|A k_{\varphi_a(z)}\| = \sup_{|z|<1} \|A k_z\|. \tag{2.2}$$

A similar identity holds for unitary rotations of $\mathbf{C}^n$. That is, any unitary transformation $V : \mathbf{C}^n \to \mathbf{C}^n$ induces a unitary operator U_V on $H^2(S)$ by the formula

$$(U_V h)(w) = h(Vw), \quad h \in H^2(S).$$

It is easy to see that if A is any bounded operator on $H^2(S)$, then

$$\sup_{|z|<1} \|U_V A U_V^* k_z\| = \sup_{|z|<1} \|A k_{Vz}\| = \sup_{|z|<1} \|A k_z\|. \tag{2.3}$$

Our construction in the next section will take advantage of (2.2) and (2.3).

For a bounded operator A on a Hilbert space $\mathcal{H}$, denote

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \text{ is any compact operator on } \mathcal{H}\},$$

which is the essential norm of A. We need the following fact:

Lemma 2.1. [7, Lemma 2.1] *Let $\{B_i\}$ be a sequence of compact operators on a Hilbert space $\mathcal{H}$ satisfying the following conditions*:
(a) *Both sequences $\{B_i\}$ and $\{B_i^*\}$ converge to 0 in the strong operator topology.*
(b) *The limit $\lim_{i\to\infty} \|B_i\|$ exists.*
Then there exist natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$ such that the sum

$$\sum_{m=1}^{\infty} B_{i(m)} = \lim_{N\to\infty} \sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology and we have

$$\left\| \sum_{m=1}^{\infty} B_{i(m)} \right\|_{\mathcal{Q}} = \lim_{i\to\infty} \|B_i\|.$$

3. The construction

On the unit sphere S, we define the functions

$$\xi(w_1, \ldots, w_n) = \begin{cases} 1 & \text{if } 1/8 \le |w_1| \le 3/8 \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$\eta(w_1, \ldots, w_n) = \begin{cases} 1 & \text{if } 5/8 \le |w_1| \le 7/8 \\ \\ 0 & \text{otherwise} \end{cases},$$

$(w_1, \ldots, w_n) \in S$. We define the subset

$$\Omega = \{(w_1, \ldots, w_n) \in S : |w_1| \le 15/16\}$$

of S. Note that the supports of both ξ and η are contained in Ω.

Lemma 3.1. *For the functions ξ and η defined above, we have $T_\xi T_\eta \neq 0$. Moreover, $M_\xi P M_\eta$ is a compact operator on $L^2(S)$.*

Proof. Since both functions depend only on the partial radial variable $|w_1|$, it is easy to see that there are $a > 0$ and $b > 0$ such that $T_\xi 1 = a$ and $T_\eta 1 = b$. Hence $T_\xi T_\eta \neq 0$. The compactness of $M_\xi P M_\eta$ follows from the obvious fact that the distance between the supports of ξ and η is greater than 0. □

We now take a non-constant inner function v on $\mathbf{B}$ from the construction of Løw [6] or Aleksandrov [1]. This v will be fixed for the rest of the paper.

Lemma 3.2. *For each $k \in \mathbf{N}$, there is an $m_k \in \mathbf{N}$ such that*

$$\sup_{|z|<1} \|T_\xi T_{\eta \bar{v}^{m_k}} k_z\| \le 2^{-k}.$$

Proof. By elementary estimates, for every $\varphi \in L^2(S)$, we have

$$\lim_{|z|\uparrow 1} \int |\varphi||k_z| d\sigma = 0.$$

Therefore

$$\lim_{|z|\uparrow 1} \sup_{m\in\mathbf{N}} |\langle \bar{v}^m k_z, \varphi\rangle| = 0$$

for each $\varphi \in L^2(S)$. Since $M_\xi P M_\eta$ is a compact operator, the above implies

$$\lim_{|z|\uparrow 1} \sup_{m\in\mathbf{N}} \|M_\xi P M_{\eta\bar{v}^m} k_z\| = \lim_{|z|\uparrow 1} \sup_{m\in\mathbf{N}} \|M_\xi P M_\eta \bar{v}^m k_z\| = 0.$$

Thus, given any $k \in \mathbf{N}$, there is a $0 < t_k < 1$ such that

$$\|M_\xi P M_{\eta\bar{v}^m} k_z\| \le 2^{-k} \quad \text{if} \ \ t_k \le |z| < 1 \ \ \text{and} \ \ m \in \mathbf{N}. \tag{3.1}$$

From (2.1) and the maximum modulus principle we see that the sequence of Toeplitz operators $\{T_{\bar{v}^m}\}$ converges to 0 strongly. Obviously, $\{T_\eta k_z : |z| \leq t_k\}$ is a compact subset of $H^2(S)$. Thus the fact that $\{T_{\bar{v}^m}\}$ converges to 0 strongly implies that

$$\lim_{m\to\infty} \sup_{|z|\leq t_k} \|T_{\eta\bar{v}^m} k_z\| = \lim_{m\to\infty} \sup_{|z|\leq t_k} \|T_{\bar{v}^m} T_\eta k_z\| = 0.$$

Therefore there is an $m_k \in \mathbf{N}$ such that $\|T_{\eta\bar{v}^{m_k}} k_z\| \leq 2^{-k}$ if $|z| \leq t_k$. Since $\|T_\xi\| \leq 1$, we conclude that

$$\|T_\xi T_{\eta\bar{v}^{m_k}} k_z\| \leq 2^{-k} \quad \text{if} \quad |z| \leq t_k. \tag{3.2}$$

Note that $T_\xi T_{\eta\bar{v}^{m_k}} k_z = PM_\xi PM_{\eta\bar{v}^{m_k}} k_z$ for $z \in \mathbf{B}$. Thus it follows from (3.1) that

$$\|T_\xi T_{\eta\bar{v}^{m_k}} k_z\| \leq 2^{-k} \quad \text{if} \quad t_k \leq |z| < 1.$$

Combining this with (3.2), the lemma is proved. □

For any $\zeta \in S$ and $a > 0$, we denote $B(\zeta, a) = \{x \in S : |\zeta - x| < a\}$. Next we pick a sequence $\{\zeta_k\}$ in S which has the property that there exists a sequence of positive numbers $\{\alpha_k\}$ such that

$$B(\zeta_j, 2\alpha_j) \cap B(\zeta_k, 2\alpha_k) = \emptyset \quad \text{for all} \quad j \neq k. \tag{3.3}$$

This obviously forces $\alpha_j \to 0$ as $j \to \infty$.

For the rest of the paper, the symbol e will denote the unit vector $(1, 0, \ldots, 0)$ in $\mathbf{C}^n$.

Lemma 3.3. *For each $k \in \mathbf{N}$, there is a $0 < \rho_k < 1$ such that $\{\varphi_{\rho_k e}(w) : w \in \Omega\} \subset B(e, \alpha_k)$.*

Proof. Given any $w = (w_1, \ldots, w_n) \in \Omega$, we have

$$\begin{aligned}\varphi_{\rho e}(w) - e &= \left(\frac{\rho - w_1}{1 - \rho w_1} - 1, 0, \ldots, 0\right) - \frac{(1-\rho^2)^{1/2}}{1 - \rho w_1}(0, w_2, \ldots, w_n) \\ &= \frac{(\rho - 1)(w_1 + 1)}{1 - \rho w_1}(1, 0, \ldots, 0) - \frac{(1-\rho^2)^{1/2}}{1 - \rho w_1}(0, w_2, \ldots, w_n),\end{aligned}$$

$0 < \rho < 1$. Since the membership $w \in \Omega$ imposes the upper bound 15/16 on $|w_1|$, the conclusion of the lemma is now obvious. □

For each $k \in \mathbf{N}$, since $\varphi_{\rho_k e} \circ \varphi_{\rho_k e} = \text{id}$, we can restate Lemma 3.3 in the form

$$\varphi_{\rho_k e}(B(e, \alpha_k)) \supset \Omega. \tag{3.4}$$

For each $k \in \mathbf{N}$, we pick a unitary transformation $V_k : \mathbf{C}^n \to \mathbf{C}^n$ such that $V_k \zeta_k = e$. We then define the Möbius transform

$$\psi_k = \varphi_{\rho_k e} \circ V_k,$$

$k \in \mathbf{N}$. Thus we can further rewrite (3.4) in the form $\psi_k(B(\zeta_k, \alpha_k)) \supset \Omega$, i.e.,

$$\psi_k^{-1}(\Omega) \subset B(\zeta_k, \alpha_k) \tag{3.5}$$

for every $k \in \mathbf{N}$.

We now define the functions

$$\xi_k = \xi \circ \psi_k \quad \text{and} \quad \eta_k = (\eta \bar{v}^{m_k}) \circ \psi_k \tag{3.6}$$

for each $k \in \mathbf{N}$. By (3.5), the supports of ξ_k and η_k are both contained in $B(\zeta_k, \alpha_k)$, $k \in \mathbf{N}$. In particular, for $j \neq k$, the supports of ξ_j and ξ_k do not intersect, and the same is true for η_j and η_k. Thus for any subset E of $\mathbf{N}$, we can define the functions

$$\xi_E = \sum_{k \in E} \xi_k \quad \text{and} \quad \eta_E = \sum_{k \in E} \eta_k.$$

Obviously, $\|\xi_E\|_\infty = 1$ and $\|\eta_E\|_\infty = 1$ if $E \neq \emptyset$. Using (3.3), easy estimates show that

$$\lim_{j \to \infty} \|T_{\xi_j} T_{\eta_k}\|_2 = 0 \quad \text{and} \quad \lim_{j \to \infty} \|T_{\xi_k} T_{\eta_j}\|_2 = 0$$

for every $k \in \mathbf{N}$, where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. From these two limits and a standard inductive selection process, we obtain

Lemma 3.4. *There is an infinite subset N of $\mathbf{N}$ such that for every $E \subset N$, we have*

$$T_{\xi_E} T_{\eta_E} = \sum_{k \in E} T_{\xi_k} T_{\eta_k} + K_E,$$

where K_E is a compact operator.

With the above preparation, we can now proceed with

Proof of Theorem 1.1. For each $k \in \mathbf{N}$, since ψ_k is a Möbius transform, $T_{\xi \circ \psi_k} T_{\eta \circ \psi_k}$ is unitarily equivalent to $T_\xi T_\eta$. Therefore $\|T_{\xi \circ \psi_k} T_{\eta \circ \psi_k}\| = \|T_\xi T_\eta\|$. Since v is an inner function, so is $(v \circ \psi_k)^{m_k}$. Thus from (3.6) we obtain

$$T_{\xi_k} T_{\eta_k} T_{(v \circ \psi_k)^{m_k}} = T_{\xi \circ \psi_k} T_{\eta \circ \psi_k}.$$

Consequently, we have the lower bound

$$\|T_{\xi_k} T_{\eta_k}\| \geq \|T_{\xi \circ \psi_k} T_{\eta \circ \psi_k}\| = \|T_\xi T_\eta\|$$

for every $k \in \mathbf{N}$. Recall from Lemma 3.1 that $\|T_\xi T_\eta\| > 0$.

By (3.6) and the definitions of ξ and η, for each $k \in \mathbf{N}$, the distance between the supports of ξ_k and η_k is greater than 0. Thus each $T_{\xi_k} T_{\eta_k}$ is a compact operator. From (3.3) we obtain the limits

$$\lim_{k \to \infty} T_{\xi_k} T_{\eta_k} = 0 \quad \text{and} \quad \lim_{k \to \infty} (T_{\xi_k} T_{\eta_k})^* = 0$$

in the strong operator topology. Thus we have verified the conditions necessary for applying Lemma 2.1 to $\{T_{\xi_k}T_{\eta_k} : k \in N\}$. Since $\mathrm{card}(N) = \infty$, by Lemma 2.1, there is a subset G of N such that the operator

$$\sum_{k\in G} T_{\xi_k}T_{\eta_k}$$

is not compact. We now define

$$f = \xi_G = \sum_{k\in G} \xi_k \quad \text{and} \quad g = \eta_G = \sum_{k\in G} \eta_k.$$

Then Lemma 3.4 gives us

$$T_f T_g = \sum_{k\in G} T_{\xi_k}T_{\eta_k} + K_G, \tag{3.7}$$

where K_G is compact. Hence the operator T_fT_g is not compact. What remains is to show that (1.5) holds.

To do that, pick any $\epsilon > 0$. By (3.6), (2.2), (2.3) and Lemma 3.2, we have

$$\sup_{|z|<1} \|T_{\xi_k}T_{\eta_k}k_z\| = \sup_{|z|<1} \|T_{\xi}T_{\eta\bar{v}^{m_k}}k_z\| \le 2^{-k}$$

for every $k \in \mathbf{N}$. Hence there is a partition $G = G_1 \cup G_2$ such that $\mathrm{card}(G_1) < \infty$ and

$$\sum_{k\in G_2} \|T_{\xi_k}T_{\eta_k}k_z\| \le \epsilon$$

for every $z \in \mathbf{B}$. Combining this with (3.7), we find that

$$\|T_fT_gk_z\| \le \sum_{k\in G_1} \|T_{\xi_k}T_{\eta_k}k_z\| + \|K_Gk_z\| + \epsilon,$$

$z \in \mathbf{B}$. Since $T_{\xi_k}T_{\eta_k}$ and K_G are compact and $\mathrm{card}(G_1)$ is finite, we see that

$$\limsup_{|z|\uparrow 1} \|T_fT_gk_z\| \le \limsup_{|z|\uparrow 1} \left(\sum_{k\in G_1} \|T_{\xi_k}T_{\eta_k}k_z\| + \|K_Gk_z\| + \epsilon \right) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (1.5) follows. This completes the proof. □

Remark. Even though the above construction was carried out in the case $n \ge 2$, it also works in the case $n = 1$. In the case $n = 1$, one can simply replace the ξ and η above by, for example, the functions

$$\xi(\tau) = \begin{cases} 1 & \text{if } \mathrm{Im}(\tau) \ge 1/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(\tau) = \begin{cases} 1 & \text{if } \mathrm{Im}(\tau) \le -1/2 \\ 0 & \text{otherwise} \end{cases},$$

$\tau \in \mathbf{T}$. For these two functions, we have $T_\xi T_\eta \neq 0$ for the simple reason that $\ker(T_\xi) = \{0\}$ and $\ker(T_\eta) = \{0\}$. One then replaces the $e = (1, 0, \ldots, 0) \in S$ by $1 \in \mathbf{T}$. Moreover, any non-constant inner function on $\mathbf{T}$ will do as v. It is easy to see that, with these replacements, the same construction works.

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