LOCALIZATION AND THE TOEPLITZ ALGEBRA ON THE BERGMAN SPACE

Jingbo Xia

Abstract. Let T_f denote the Toeplitz operator with symbol function f on the Bergman space $L^2_a(\mathbf{B}, dv)$ of the unit ball in $\mathbf{C}^n$. It is a natural problem in the theory of Toeplitz operators to determine the norm closure of the set $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ in $\mathcal{B}(L^2_a(\mathbf{B}, dv))$. We show that the norm closure of $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ actually coincides with the Toeplitz algebra $\mathcal{T}$, i.e., the C^*-algebra generated by $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$. A key ingredient in the proof is the class of weakly localized operators recently introduced by Isralowitz, Mitkovski and Wick. Our approach simultaneously gives us the somewhat surprising result that $\mathcal{T}$ also coincides with the C^*-algebra generated by the class of weakly localized operators.

1. Introduction

We begin with a discussion of localized operators. Let $\mathbf{B}$ denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. The Bergman metric on $\mathbf{B}$ is given by the formula

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B},$$

where φ_z is the Möbius transform of the ball given on page 25 in [10]. For each $z \in \mathbf{B}$ and each $r > 0$, the corresponding β-ball will be denoted by $D(z, r)$. That is,

$$D(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}.$$

Let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. Then the formula

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

gives us the standard Möbius-invariant measure on $\mathbf{B}$.

Recall that the Bergman space $L^2_a(\mathbf{B}, dv)$ is the subspace

$$\{h \in L^2(\mathbf{B}, dv) : h \text{ is analytic on } \mathbf{B}\}$$

of $L^2(\mathbf{B}, dv)$. It is well known that the normalized reproducing kernel for the Bergman space is given by the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{(n+1)/2}}{(1 - \langle \zeta, z \rangle)^{n+1}}, \quad z, \zeta \in \mathbf{B}. \tag{1.1}$$

Keywords: Localization, Bergman space, Toeplitz algebra, Fock space.

It was first discovered in [14] that *localization* is a powerful tool for analyzing operators on reproducing-kernel Hilbert spaces (more on this in Section 4). Recently, this idea was further explored in [6]. More specifically, in [6] Isralowitz, Mitkovski and Wick introduced the notion of *weakly localized operators* on the Bergman space. Below we give a slightly more refined version of their definition. Our refinement lies in the realization that we can define a class of localized operators for each given localization parameter s.

Definition 1.1. Let a positive number $(n-1)/(n+1) < s < 1$ be given.
(a) A bounded operator B on the Bergman space $L^2_a(\mathbf{B}, dv)$ is said to be s-weakly localized if it satisfies the conditions

$$\sup_{z\in\mathbf{B}} \int |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z\in\mathbf{B}} \int |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0 \quad \text{and}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let $\mathcal{A}_s$ denote the collection of s-weakly localized operators defined as above.
(c) Let $C^*(\mathcal{A}_s)$ denote the C^*-algebra generated by $\mathcal{A}_s$.

For each $(n-1)/(n+1) < s < 1$, the simplest examples of s-weakly localized operators are the Toeplitz operators, which, as we recall, are defined as follows. Let $P : L^2(\mathbf{B}, dv) \to L^2_a(\mathbf{B}, dv)$ be the orthogonal projection. Then for $f \in L^\infty(\mathbf{B}, dv)$, the formula

$$T_f h = P(fh), \quad h \in L^2_a(\mathbf{B}, dv),$$

defines the Toeplitz operator T_f. Also recall that the *Toeplitz algebra* $\mathcal{T}$ on $L^2_a(\mathbf{B}, dv)$ is the C^*-algebra generated by the collection of Toeplitz operators

$$\{T_f : f \in L^\infty(\mathbf{B}, dv)\}.$$

It was shown in [6] that $\mathcal{A}_s \supset \{T_f : f \in L^\infty(\mathbf{B}, dv)\}$, hence $C^*(\mathcal{A}_s) \supset \mathcal{T}$.

In [13], Suárez showed that for $A \in \mathcal{T}$, the condition

$$\lim_{|z|\uparrow 1} \langle Ak_z, k_z\rangle = 0 \tag{1.2}$$

implies that A is compact. In [6], Isralowitz, Mitkovski and Wick showed that for $A \in C^*(\mathcal{A}_s)$, condition (1.2) also implies that A is compact. Moreover, the introduction of the notion of weakly localized operators in [6] has the added virtue that it significantly simplifies the work necessary to obtain the above result. Indeed the approach in [6] explains why such results should hold true.

The results in [6,13] certainly inspire further examinations of the inclusion relation

$$\mathcal{T} \subset C^*(\mathcal{A}_s). \tag{1.3}$$

Given what we know about Toeplitz operators (see, e.g., [1-5,7,9,12,15]), the C^*-algebra $\mathcal{T}$ is certainly much better understood than $C^*(\mathcal{A}_s)$. It is known, for example, that $\mathcal{T}$ coincides with its commutator ideal [11,8]. Thus an obvious question is, is the C^*-algebra $C^*(\mathcal{A}_s)$ structurally different from $\mathcal{T}$? In fact, one may raise the even more basic

Question 1.2. Is the inclusion in (1.3) proper for any $(n-1)/(n+1) < s < 1$? Is there any difference between $C^*(\mathcal{A}_s)$ and $C^*(\mathcal{A}_t)$ for $s \neq t$ in the interval $((n-1)/(n+1), 1)$?

The answer, as it turns out, is somewhat surprising:

Theorem 1.3. *For every $(n-1)/(n+1) < s < 1$ we have $C^*(\mathcal{A}_s) = \mathcal{T}$.*

An immediate consequence of Theorem 1.3 is, of course, that $C^*(\mathcal{A}_s) = C^*(\mathcal{A}_t)$ for all $s, t \in ((n-1)/(n+1), 1)$. We emphasize that this equality at the level of C^*-algebras is obtained without knowing whether there is any kind of inclusion relation between the classes $\mathcal{A}_s$ and $\mathcal{A}_t$ in the case $s \neq t$.

Although Question 1.2 was the original motivation for this paper, our approach to this problem naturally leads us to a stronger result, a result that simultaneously settles a much older question. Let us introduce

Definition 1.4. Let $\mathcal{T}^{(1)}$ denote the closure of $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ with respect to the operator norm.

Below is our main result, which not only answers Question 1.2, but also tells us something significant about the Toeplitz algebra $\mathcal{T}$ itself.

Theorem 1.5. *For every $(n-1)/(n+1) < s < 1$ we have $\mathcal{T}^{(1)} = C^*(\mathcal{A}_s)$. Consequently, $\mathcal{T}^{(1)} = \mathcal{T} = C^*(\mathcal{A}_s)$.*

The documented history of interest in $\mathcal{T}^{(1)}$ can be traced at least back to [3,4], where Engliš showed that it contains all the compact operators on $L^2_a(\mathbf{B}, dv)$. In retrospect, this was really a hint at the things to come.

Later in [12], Suárez took another look at $\mathcal{T}^{(1)}$. There he introduced a sequence of higher Berezin transforms $B_1, \ldots, B_k, \ldots$, which are generalizations of the original Berezin transform B_0. At the end of the paper, Suárez expressed his belief that every operator S in $\mathcal{T}$ is the limit in operator norm of the sequence of Toeplitz operators $\{T_{B_k(S)}\}$. If this is true, then it certainly implies that $\mathcal{T}^{(1)} = \mathcal{T}$. One can only speculate that, perhaps, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ was what Suárez had in mind all along, and the higher Berezin transforms were his tools to try to prove it. While we still do not know if it is true that

$$\lim_{k\to\infty} \|T_{B_k(S)} - S\| = 0$$

for every $S \in \mathcal{T}$, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ is now proven using completely different ideas. From the proof of Theorem 1.5, the reader will see that the approximation of a general $S \in \mathcal{T}$ by Toeplitz operators is quite complicated: it takes several stages.

Let us give an outline for the proof of Theorem 1.5. Since each $\mathcal{A}_s$ is known to be a $*$-algebra that contains $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ [6], it suffices to show that $\mathcal{A}_s \subset \mathcal{T}^{(1)}$. An elementary C^*-algebraic argument further reduces this to the proof of the inclusion

$$T_\Phi \mathcal{A}_s T_\Phi \subset \mathcal{T}^{(1)}$$

for a suitably chosen Toeplitz operator T_Φ that is both positive and invertible. We can pick the function Φ in such a way that for every $B \in \mathcal{A}_s$, the operator $T_\Phi B T_\Phi$ is "resolved" in the form

$$T_\Phi B T_\Phi = \iint_{D(0,2)\times D(0,2)} E_w B E_z d\lambda(w) d\lambda(z),$$

where each E_z is a sum of rank-one operators over a lattice:

$$E_z = \sum_{u\in\mathcal{L}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}.$$

A crucial ingredient in the proof is the norm estimate in Lemma 2.6 below. This estimate has a number of implications, and one of the implications is that the map $(w, z) \mapsto E_w B E_z$ is continuous with respect to the operator norm. This norm continuity immediately implies that $T_\Phi B T_\Phi$ is contained in the norm closure of the linear span of

$$\{E_w B E_z : w, z \in \mathbf{B}\}.$$

Thus we can complete the proof by showing that $E_w B E_z \in \mathcal{T}^{(1)}$ for all $z, w \in \mathbf{B}$. One can think of $E_w B E_z$ as an infinite matrix. The localization condition for B ensures that the terms in $E_w B E_z$ that are "far from the diagonal" form an operator of small norm. The rest of the terms in $E_w B E_z$ are a linear combination of operators in a special class $\mathcal{D}_0$ (see Definition 3.1). In other words, $E_w B E_z$ can be approximated in norm by operators in the linear span of $\mathcal{D}_0$. Then, with several applications of the estimate in Lemma 2.6, we are able to show that $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$, accomplishing our goal.

The rest of the paper is organized as follows. In Sections 2 and 3 we give the technical details of the argument outlined above. In Section 4, we discuss the analogue of Theorem 1.5 on the Fock space.

2. Separated sets and norm estimates

The technical details begin with

Definition 2.1. A subset Γ of $\mathbf{B}$ is said to be *separated* if there is a $\delta = \delta(\Gamma) > 0$ such that the inequality $\beta(u, v) \geq \delta$ holds for all $u \neq v$ in Γ.

Recall that for each $z \in \mathbf{B}\backslash\{0\}$, the Möbius transform φ_z is given by the formula

$$\varphi_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle} \left\{ z - \frac{\langle \zeta, z\rangle}{|z|^2} z - (1 - |z|^2)^{1/2} \left(\zeta - \frac{\langle \zeta, z\rangle}{|z|^2} z \right) \right\}$$

[10,page 25]. Also, we define $\varphi_0(\zeta) = -\zeta$. Recall that each φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$ [10,Theorem 2.2.2]. Let us list some of the elementary properties of separated sets that will be used repeatedly in the sequel.

Lemma 2.2. *Let Γ be a separated set in $\mathbf{B}$.*
(a) *For each $0 < R < \infty$, there is a natural number $N = N(\Gamma, R)$ such that* $\mathrm{card}\{v \in \Gamma : \beta(u,v) \le R\} \le N$ *for every $u \in \Gamma$.*
(b) *For every pair of $z \in \mathbf{B}$ and $\rho > 0$, there is a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $i \in \{1, \dots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta(\varphi_u(z), \varphi_v(z)) > \rho$.*

Proof. By definition, there is a $\delta > 0$ such that $\beta(u,v) \ge \delta$ for all $u \ne v$ in Γ. Thus

$$D(u, \delta/2) \cap D(v, \delta/2) = \emptyset \quad \text{for all} \;\; u \ne v \;\; \text{in} \;\; \Gamma.$$

Let $R > 0$ be given. Then for every pair of $u, v \in \Gamma$, the condition $\beta(u,v) \le R$ implies $D(v, \delta/2) \subset D(u, R + (\delta/2))$. By the Möbius invariance of the Bergman metric β and the the measure $d\lambda$, we have

$$\lambda(D(v, \delta/2)) = \lambda(\varphi_v(D(0, \delta/2))) = \lambda(D(0, \delta/2)).$$

Therefore if we write $N(u)$ for the cardinality of the set $\{v \in \Gamma : \beta(u,v) \le R\}$, then

$$N(u)\lambda(D(0,\delta/2)) = \sum_{\substack{v \in \Gamma \\ \beta(u,v) \le R}} \lambda(D(v, \delta/2)) \le \lambda(D(u, R + (\delta/2))) = \lambda(D(0, R + (\delta/2))).$$

That is, $N(u) \le \lambda(D(0, R + (\delta/2)))/\lambda(D(0, \delta/2))$, which proves (a).

To prove (b), let $z \in \mathbf{B}$ and $\rho > 0$ be given, and set $r = \rho + 2\beta(z, 0)$. By (a), there is an $m \in \mathbf{N}$ such that $\mathrm{card}\{v \in \Gamma : \beta(u,v) \le r\} \le m$ for every $u \in \Gamma$. By a standard maximality argument, there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $i \in \{1, \dots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta(u,v) > r$. But if u, v satisfy the condition $\beta(u,v) > r$, then by the Möbius invariance of β we have

$$\begin{aligned} \beta(\varphi_u(z), \varphi_v(z)) &\ge \beta(u,v) - \beta(\varphi_u(z), u) - \beta(v, \varphi_v(z)) \\ &= \beta(u,v) - \beta(\varphi_u(z), \varphi_u(0)) - \beta(\varphi_v(0), \varphi_v(z)) \\ &= \beta(u,v) - \beta(z, 0) - \beta(0, z) > r - 2\beta(z, 0) = \rho. \end{aligned}$$

This completes the proof. □

Lemma 2.3. *For all $u, v, x, y \in \mathbf{B}$ we have*

$$\frac{(1 - |\varphi_u(x)|^2)^{1/2}(1 - |\varphi_v(y)|^2)^{1/2}}{|1 - \langle \varphi_u(x), \varphi_v(y) \rangle|} \le 2e^{\beta(x,0) + \beta(y,0)} \frac{(1 - |u|^2)^{1/2}(1 - |v|^2)^{1/2}}{|1 - \langle u, v \rangle|}.$$

Proof. For $a, b \in \mathbf{B}$, we have $1 - |\varphi_a(b)|^2 = (1 - |a|^2)(1 - |b|^2)/|1 - \langle a, b \rangle|^2$ [10,Theorem 2.2.2]. Thus if we write

$$\alpha = \frac{(1 - |a|^2)^{1/2}(1 - |b|^2)^{1/2}}{|1 - \langle a, b \rangle|},$$

then

$$\log \frac{1}{\alpha} \leq \frac{1}{2} \log \frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|} \leq \log \frac{2}{\alpha}.$$

Consequently

$$e^{-\beta(a,b)} \leq \frac{(1-|a|^2)^{1/2}(1-|b|^2)^{1/2}}{|1-\langle a,b\rangle|} \leq 2e^{-\beta(a,b)}. \tag{2.1}$$

For $u, v, x, y \in \mathbf{B}$, by the Möbius invariance of the Bergman metric, we have

$$\beta(\varphi_u(x), \varphi_v(y)) \geq \beta(u,v) - \beta(\varphi_u(x), u) - \beta(\varphi_v(y), v) = \beta(u,v) - \beta(x,0) - \beta(y,0).$$

Combining (2.1) with this inequality, we find that

$$\begin{aligned}\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_v(y)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_v(y)\rangle|} &\leq 2e^{-\beta(\varphi_u(x),\varphi_v(y))} \leq 2e^{\beta(x,0)+\beta(y,0)}e^{-\beta(u,v)} \\ &\leq 2e^{\beta(x,0)+\beta(y,0)} \frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}.\end{aligned}$$

This proves the lemma. □

Lemma 2.4. *Let Γ be a separated set in $\mathbf{B}$. Then there is a $0 < C(\Gamma) < \infty$ such that*

$$\sum_{v\in\Gamma} \left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|} \right)^{n+1} (1-|v|^2)^{(4n+1)/8} \leq C(\Gamma)(1-|\xi|^2)^{(4n+1)/8}$$

for every $\xi \in \mathbf{B}$.

Proof. If Γ is a separated set in $\mathbf{B}$, then there is a $\delta > 0$ such that $\beta(u,v) \geq \delta$ for all $u \neq v$ in Γ. Thus $D(u,\delta/2) \cap D(v,\delta/2) = \emptyset$ for all $u \neq v$ in Γ. If $w \in D(v,\delta/2)$, then $v \in D(w,\delta/2) = \varphi_w(D(0,\delta/2))$. Thus if $w \in D(v,\delta/2)$, then there is a $v' \in D(0,\delta/2)$ such that $v = \varphi_w(v')$. Let $\xi \in \mathbf{B}$. Since $\xi = \varphi_\xi(0)$, we can apply Lemma 2.4 to obtain

$$\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|} \leq 2e^{\delta/2} \frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|} \tag{2.2}$$

for every $w \in D(v,\delta/2)$. Also, since $v = \varphi_w(v')$ and $v' \in D(0,\delta/2)$, we have

$$\begin{aligned}1-|v|^2 = 1-|\varphi_w(v')|^2 &= \frac{(1-|v'|^2)(1-|w|^2)}{|1-\langle v',w\rangle|^2} \leq \frac{4}{1-|v'|^2}(1-|w|^2) \\ &\leq 4e^{2\beta(v',0)}(1-|w|^2) \leq 4e^{\delta}(1-|w|^2).\end{aligned} \tag{2.3}$$

Set $C_1 = (2e^{\delta/2})^{n+1}(4e^{\delta})^{(4n+1)/8}$. Then it follows from (2.2) and (2.3) that

$$\begin{aligned}&\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|} \right)^{n+1} (1-|v|^2)^{(4n+1)/8} \\ &\quad \leq C_1 \left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|} \right)^{n+1} (1-|w|^2)^{(4n+1)/8}\end{aligned}$$

for every $w \in D(v, \delta/2)$. Hence for each $\xi \in \mathbf{B}$ we have

$$\sum_{v\in\Gamma}\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+1}(1-|v|^2)^{(4n+1)/8}$$

$$\leq \sum_{v\in\Gamma}\frac{C_1}{\lambda(D(v,\delta/2))}\int_{D(v,\delta/2)}\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w)$$

(2.4)

$$\leq \frac{C_1}{\lambda(D(0,\delta/2))}\int\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w).$$

To estimate the last integral, note that

$$\frac{(1-|\xi|^2)^{1/2}(1-|\varphi_\xi(\zeta)|^2)^{1/2}}{|1-\langle\xi,\varphi_\xi(\zeta)\rangle|} = (1-|\zeta|^2)^{1/2}.$$

Thus, making the substitution $w = \varphi_\xi(\zeta)$ and using the Möbius invariance of $d\lambda$, we have

$$\begin{aligned}
&\int\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w)\\
&=\int(1-|\zeta|^2)^{(n+1)/2}(1-|\varphi_\xi(\zeta)|^2)^{(4n+1)/8}d\lambda(\zeta)\\
&=\int(1-|\zeta|^2)^{(n+1)/2}\left(\frac{(1-|\xi|^2)(1-|\zeta|^2)}{|1-\langle\xi,\zeta\rangle|^2}\right)^{(4n+1)/8}d\lambda(\zeta)\\
&=(1-|\xi|^2)^{(4n+1)/8}\int\frac{dv(\zeta)}{|1-\langle\xi,\zeta\rangle|^{n+(1/4)}(1-|\zeta|^2)^{3/8}} = (*).
\end{aligned}$$

To further estimate $(*)$, let $d\sigma$ be the standard spherical measure on the unit sphere $\{x \in \mathbf{C}^n : |x| = 1\}$. There is a constant C_2 such that

$$\int\frac{d\sigma(x)}{|1-\langle z,x\rangle|^{n+(1/4)}} \leq \frac{C_2}{(1-|z|^2)^{1/4}}$$

for every $z \in \mathbf{B}$ [10,Proposition 1.4.10]. Combining this with the radial-spherical decomposition $dv = 2nr^{2n-1}drd\sigma$ of the volume measure, we have

$$\int\frac{dv(\zeta)}{|1-\langle\xi,\zeta\rangle|^{n+(1/4)}(1-|\zeta|^2)^{3/8}} \leq \int_0^1\frac{C_2 2nr^{2n-1}dr}{(1-r^2)^{(1/4)+(3/8)}} \leq nC_2\int_0^1\frac{dt}{(1-t)^{5/8}} = \frac{8}{3}nC_2.$$

Therefore

$$(*) \leq 3nC_2(1-|\xi|^2)^{(4n+1)/8}.$$

Substituting this in (2.4), we conclude that the desired inequality holds for the constant

$$C(\Gamma) = \frac{3nC_1C_2}{\lambda(D(0,\delta/2))}.$$

This completes the proof. $\square$

Recall that each Toeplitz operator has an "integral representation" in terms of the normalized reproducing kernel $\{k_w : w \in \mathbf{B}\}$. Indeed for each $f \in L^\infty(\mathbf{B}, dv)$, we have

$$T_f = \int f(w) k_w \otimes k_w d\lambda(w). \tag{2.5}$$

This formula is obtained through direct verification.

Let $\mathcal{L}$ be a subset of $\mathbf{B}$ which is maximal with respect to the property that

$$D(u,1) \cap D(v,1) = \emptyset \quad \text{for all} \quad u \neq v \quad \text{in} \quad \mathcal{L}. \tag{2.6}$$

This $\mathcal{L}$ will be fixed for the rest of the paper. The maximality of $\mathcal{L}$ implies that

$$\bigcup_{u \in \mathcal{L}} D(u,2) = \mathbf{B}. \tag{2.7}$$

Now, for each $z \in \mathbf{B}$, define

$$E_z = \sum_{u \in \mathcal{L}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}. \tag{2.8}$$

Define the function

$$\Phi = \sum_{u \in \mathcal{L}} \chi_{D(u,2)} \tag{2.9}$$

on $\mathbf{B}$. By (2.6) and Lemma 2.2(a), there is a natural number $\mathcal{N} \in \mathbf{N}$ such that

$$\text{card}\{v \in \mathcal{L} : D(u,2) \cap D(v,2) \neq \emptyset\} \leq \mathcal{N}$$

for every $u \in \mathcal{L}$. This and (2.7) together tell us that the inequality

$$1 \leq \Phi \leq \mathcal{N} \tag{2.10}$$

holds on the unit ball $\mathbf{B}$. By (2.5) and the Möbius invariance of β and $d\lambda$, we have

$$T_\Phi = \int \Phi(w) k_w \otimes k_w d\lambda(w) = \sum_{u \in \mathcal{L}} \int_{D(u,2)} k_w \otimes k_w d\lambda(w) = \sum_{u \in \mathcal{L}} \int_{D(0,2)} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} d\lambda(z).$$

That is, we have

$$T_\Phi = \int_{D(0,2)} E_z d\lambda(z). \tag{2.11}$$

Lemma 2.5. *There is a constant $0 < C_{2.5} < \infty$ such that $\|E_z\| \le C_{2.5}$ for every $z \in D(0,2)$.*

Proof. By Lemma 2.3, for $u, v, z \in \mathbf{B}$ we have

$$|\langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle| = \left(\frac{(1-|\varphi_v(z)|^2)^{1/2}(1-|\varphi_u(z)|^2)^{1/2}}{|1-\langle \varphi_u(z), \varphi_v(z)\rangle|}\right)^{n+1}$$

$$\le (2e^{2\beta(z,0)})^{n+1}\left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1}. \tag{2.12}$$

Let $\{\epsilon_u : u \in \mathcal{L}\}$ be an orthonormal set. For each $z \in \mathbf{B}$, define the operator

$$F_z = \sum_{u\in\mathcal{L}} \epsilon_u \otimes k_{\varphi_u(z)}. \tag{2.13}$$

Since $E_z = F_z^* F_z$ and $\|F_z^* F_z\| = \|F_z F_z^*\|$, it suffices to estimate the later. We have

$$F_z F_z^* = \sum_{u,v\in\mathcal{L}} \langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle \epsilon_u \otimes \epsilon_v.$$

Now suppose that $z \in D(0,2)$ and write $C_1 = (2e^4)^{n+1}$. By (2.12), for every vector $x = \sum_{u\in\mathcal{L}} x_u \epsilon_u$ we have

$$\langle F_z F_z^* x, x\rangle \le \sum_{u,v\in\mathcal{L}} |\langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle||x_u||x_v|$$

$$\le C_1 \sum_{u,v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} |x_u||x_v|$$

$$= C_1 \sum_{u\in\mathcal{L}} |x_u| y_u, \tag{2.14}$$

where

$$y_u = \sum_{v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} |x_v|$$

for each $u \in \mathcal{L}$. Next we apply the Schur test. Indeed by the Cauchy-Schwarz inequality and Lemma 2.4, we have

$$y_u^2 \le C(\mathcal{L})(1-|u|^2)^{\frac{4n+1}{8}} \sum_{v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}}.$$

Applying Lemma 2.4 again, we have

$$\sum_{u\in\mathcal{L}} y_u^2 \le C(\mathcal{L}) \sum_{v\in\mathcal{L}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} \sum_{u\in\mathcal{L}} (1-|u|^2)^{\frac{4n+1}{8}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1}$$

$$\le C^2(\mathcal{L}) \sum_{v\in\mathcal{L}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} (1-|v|^2)^{\frac{4n+1}{8}} = C^2(\mathcal{L}) \sum_{v\in\mathcal{L}} |x_v|^2.$$

Combining this with (2.14), we find that

$$\langle F_z F_z^* x, x\rangle \le C_1 C(\mathcal{L}) \sum_{v\in\mathcal{L}} |x_v|^2 = C_1 C(\mathcal{L})\|x\|^2.$$

Since the vector x is arbitrary, we conclude that $\|E_z\| = \|F_z F_z^*\| \le C_1 C(\mathcal{L})$ for every $z \in D(0,2)$. This completes the proof. □

Recall that for each $z \in \mathbf{B}$, the formula

$$(U_z h)(\zeta) = k_z(\zeta) h(\varphi_z(\zeta)), \quad \zeta \in \mathbf{B} \ \text{ and } \ h \in L_a^2(\mathbf{B}, dv), \tag{2.15}$$

defines a unitary operator. These unitary operators will play an essential role in this paper.

As usual, we write $H^\infty(\mathbf{B})$ for the collection of bounded analytic functions on $\mathbf{B}$. Also, we write $\|h\|_\infty = \sup_{\zeta\in\mathbf{B}} |h(\zeta)|$ for $h \in H^\infty(\mathbf{B})$. Naturally, we consider $H^\infty(\mathbf{B})$ as a subset of the Bergman space $L_a^2(\mathbf{B}, dv)$.

Lemma 2.6. *Given any separated set* Γ in $\mathbf{B}$, *there exists a constant* $0 < B(\Gamma) < \infty$ *such that the following estimate holds*: *Let* $\{h_u : u \in \Gamma\}$ *be functions in* $H^\infty(\mathbf{B})$ *such that* $\sup_{u\in\Gamma} \|h_u\|_\infty < \infty$, *and let* $\{e_u : u \in \Gamma\}$ *be any orthonormal set. Then*

$$\left\| \sum_{u\in\Gamma} (U_u h_u) \otimes e_u \right\| \le B(\Gamma) \sup_{u\in\Gamma} \|h_u\|_\infty.$$

Proof. Given Γ, $\{h_u : u \in \Gamma\}$ and $\{e_u : u \in \Gamma\}$ as in the statement, let us write

$$A = \sum_{u\in\Gamma} (U_u h_u) \otimes e_u$$

for convenience. By (2.10), the self-adjoint Toeplitz operator T_Φ is invertible with $\|T_\Phi^{-1}\| \le 1$. Therefore $\|A\| = \|T_\Phi^{-1} T_\Phi A\| \le \|T_\Phi A\|$. Combining this with (2.11), we see that

$$\|A\| \le \lambda(D(0,2)) \sup_{z\in D(0,2)} \|E_z A\|. \tag{2.16}$$

Thus it suffices to estimate $\|E_z A\|$ for $z \in D(0,2)$. Let F_z be the operator defined by (2.13). Then Lemma 2.5 implies that $\|F_z^*\| \le C_{2.5}^{1/2}$ for $z \in D(0,2)$. Hence

$$\|E_z A\| \le C_{2.5}^{1/2} \|F_z A\|, \quad z \in D(0,2). \tag{2.17}$$

Consequently, we only need to estimate $\|F_z A\|$.

To estimate $\|F_z A\|$, let us denote

$$H = \sup_{u\in\Gamma} \|h_u\|_\infty.$$

Let $z \in D(0,2)$. Then note that

$$F_z A = \sum_{u \in \mathcal{L}} \sum_{v \in \Gamma} \langle U_v h_v, k_{\varphi_u(z)} \rangle \epsilon_u \otimes e_v. \tag{2.18}$$

Since $U_v h_v = k_v \cdot h_v \circ \varphi_v$, the reproducing property of $k_{\varphi_u(z)}$ gives us

$$\langle U_v h_v, k_{\varphi_u(z)} \rangle = h_v(\varphi_v(\varphi_u(z))) \langle k_v, k_{\varphi_u(z)} \rangle,$$

which is one of the key facts on which this paper depends. Thus

$$|\langle U_v h_v, k_{\varphi_u(z)} \rangle| \le H |\langle k_v, k_{\varphi_u(z)} \rangle| = H \left(\frac{(1-|v|^2)^{1/2}(1-|\varphi_u(z)|^2)^{1/2}}{|1-\langle \varphi_u(z), v \rangle|} \right)^{n+1}.$$

Since $v = \varphi_v(0)$ and $z \in D(0,2)$, an application of Lemma 2.3 gives us

$$|\langle U_v h_v, k_{\varphi_u(z)} \rangle| \le C_1 H \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle v, u \rangle|} \right)^{n+1}, \tag{2.19}$$

where $C_1 = (2e^2)^{n+1}$. Now consider vectors

$$x = \sum_{v \in \Gamma} x_v e_v \quad \text{and} \quad y = \sum_{u \in \mathcal{L}} y_u \epsilon_u.$$

It follows from (2.18) and (2.19) that

$$\begin{aligned} |\langle F_z A x, y \rangle| &\le C_1 H \sum_{u \in \mathcal{L}} \sum_{v \in \Gamma} \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle v, u \rangle|} \right)^{n+1} |x_v||y_u| \\ &= C_1 H \sum_{u \in \mathcal{L}} b_u |y_u|, \end{aligned} \tag{2.20}$$

where

$$b_u = \sum_{v \in \Gamma} \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle v, u \rangle|} \right)^{n+1} |x_v|,$$

$u \in \mathcal{L}$. We apply the Schur test as we did in the proof of Lemma 2.5. By the Cauchy-Schwarz inequality and the bound given in Lemma 2.4, we have

$$b_u^2 \le C(\Gamma)(1-|u|)^{\frac{4n+1}{8}} \sum_{v \in \Gamma} \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle v, u \rangle|} \right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}},$$

$u \in \mathcal{L}$. Applying Lemma 2.4 again, we obtain

$$\begin{aligned} \sum_{u \in \mathcal{L}} b_u^2 &\le C(\Gamma) \sum_{v \in \Gamma} \sum_{u \in \mathcal{L}} (1-|u|)^{\frac{4n+1}{8}} \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle v, u \rangle|} \right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} \\ &\le C(\Gamma)C(\mathcal{L}) \sum_{v \in \Gamma} (1-|v|^2)^{\frac{4n+1}{8}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} = C(\Gamma)C(\mathcal{L})\|x\|^2. \end{aligned}$$

Combining this with (2.20), we obtain

$$|\langle F_z A x, y\rangle| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H\|x\|\|y\|.$$

Since the vectors x and y are arbitrary, this means

$$\|F_z A\| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H$$

for $z \in D(0,2)$. Recalling (2.16) and (2.17), we see that the lemma holds for the constant

$$B(\Gamma) = \lambda(D(0,2))C_{2.5}^{1/2} C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2}.$$

This completes the proof. □

Proposition 2.7. *Suppose that* Γ *is a separated set in* $\mathbf{B}$. *Furthermore, suppose that* $\{c_u : u \in \Gamma\}$ *are complex numbers satisfying the condition*

$$\sup_{u\in\Gamma} |c_u| < \infty. \tag{2.21}$$

Then for each $z \in \mathbf{B}$, *the operator*

$$Y_z = \sum_{u\in\Gamma} c_u k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} \tag{2.22}$$

is bounded on the Bergman space. Moreover, the map $z \mapsto Y_z$ *from* $\mathbf{B}$ *into* $\mathcal{B}(L^2_a(\mathbf{B}, dv))$ *is continuous with respect to the operator norm.*

Proof. For $u, z \in \mathbf{B}$, simple computation shows that

$$U_u k_z = \left(\frac{|1-\langle u,z\rangle|}{1-\langle u,z\rangle}\right)^{n+1} k_{\varphi_u(z)}. \tag{2.23}$$

Therefore

$$k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} = (U_u k_z) \otimes (U_u k_z).$$

Let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then for every $z \in \mathbf{B}$ we have the factorization

$$Y_z = A_z B_z^*,$$

where

$$A_z = \sum_{u\in\Gamma} c_u (U_u k_z) \otimes e_u \quad \text{and} \quad B_z = \sum_{u\in\Gamma} (U_u k_z) \otimes e_u.$$

Applying Lemma 2.6 to the case $h_u = c_u k_z$, $u \in \Gamma$, we see that each A_z is a bounded operator. Similarly, each B_z is also bounded. Hence $Y_z = A_z B_z^*$ is bounded.

To show that the map $z \mapsto Y_z$ is continuous with respect to the operator norm, it suffices to show that the maps $z \mapsto A_z$ and $z \mapsto B_z$ are continuous with respect to the operator norm. Since B_z is just a special case of A_z, it suffices to consider the map $z \mapsto A_z$.

For any $z, w \in \mathbf{B}$, we have

$$A_z - A_w = \sum_{u\in\Gamma} c_u\{U_u(k_z - k_w)\} \otimes e_u.$$

Applying Lemma 2.6 to the case where $h_u = c_u(k_z - k_w)$, $u \in \Gamma$, we find that

$$\|A_z - A_w\| \leq B(\Gamma)C\|k_z - k_w\|_\infty,$$

where $C = \sup_{u\in\Gamma} |c_u|$. For each $z \in \mathbf{B}$, it is elementary that

$$\lim_{w\to z} \|k_z - k_w\|_\infty = 0.$$

Hence the map $z \mapsto A_z$ is continuous with respect to operator norm. This completes the proof. □

Let us recall two known facts about $\mathcal{A}_s$. First, for each given $(n-1)/(n+1) < s < 1$, we have $\mathcal{A}_s \supset \{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ [6,Proposition 2.2]. Indeed by (2.5), this is a consequence of the fact

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\backslash D(z,r)} \int |\langle k_z, k_x\rangle||\langle k_x, k_w\rangle| d\lambda(x) \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) = 0.$$

To prove this limit, the idea in [6] is to split the inner x-integral above as the sum of the part on $D(z, r/2)$ and the part on $\mathbf{B}\backslash D(z, r/2)$. With such split, this limit follows from the Rudin-Forelli estimate [6,Lemma 2.1].

Second, each $\mathcal{A}_s$ is a $*$-algebra [6,Proposition 2.3]. In this case, the gist of the matter is the limit

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\backslash D(z,r)} \int |\langle Tk_z, k_x\rangle||\langle k_x, S^*k_w\rangle| d\lambda(x) \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) = 0 \tag{2.24}$$

for $S, T \in \mathcal{A}_s$. To prove this, [6] splits the inner x-integral in the same way as above. Then it is easy to see that (2.24) follows from the localization condition for S and T.

Next comes the most crucial step in the proof of Theorem 1.5:

Proposition 2.8. *Let $(n-1)/(n+1) < s < 1$. If $B \in \mathcal{A}_s$, then $E_w B E_z \in \mathcal{T}^{(1)}$ for all $z, w \in \mathbf{B}$.*

The proof of Proposition 2.8 will be the task of Section 3. But assuming Proposition 2.8, we have

Proof of Theorem 1.5. Let $(n-1)/(n+1) < s < 1$ be given. By the fact that $\mathcal{A}_s$ is a $*$-algebra mentioned above, $C^*(\mathcal{A}_s)$ is just the norm closure of $\mathcal{A}_s$. Since we also know that $\mathcal{A}_s \supset \{T_f : f \in L^\infty(\mathbf{B}, dv)\}$, Theorem 1.5 will follow if we can show that $\mathcal{A}_s \subset \mathcal{T}^{(1)}$. We prove this inclusion into two steps.

(1) Let $B \in \mathcal{A}_s$ be given. As the first step, let us show that $T_\Phi B T_\Phi \in \mathcal{T}^{(1)}$. Indeed it follows from (2.11) that

$$T_\Phi B T_\Phi = \iint_{D(0,2)\times D(0,2)} E_w B E_z d\lambda(w) d\lambda(z). \tag{2.25}$$

Consider the map

$$(w,z) \mapsto E_w B E_z \tag{2.26}$$

from $\mathbf{B} \times \mathbf{B}$ into $\mathcal{B}(L_a^2(\mathbf{B}, dv))$. Proposition 2.8 tells us that the range of map (2.26) is contained in $\mathcal{T}^{(1)}$. Hence every Riemann sum corresponding to the integral in (2.25) belongs to $\mathcal{T}^{(1)}$. On the other hand, by Proposition 2.7, the map $z \mapsto E_z$ is continuous with respect to the operator norm. Hence map (2.26) is also continuous with respect to the operator norm. Since the closure of $D(0,2) \times D(0,2)$ is a compact subset of $\mathbf{B} \times \mathbf{B}$, the norm continuity of (2.26) means that the integral in (2.25) is the limit with respect to the operator norm of a sequence of Riemann sums $s_1, s_2, \dots, s_k, \dots$. Since each s_k belongs to $\mathcal{T}^{(1)}$, so does $T_\Phi B T_\Phi$.

(2) Given $B \in \mathcal{A}_s$, we will now show that $B \in \mathcal{T}^{(1)}$. Since $T_\Phi \in \mathcal{A}_s$ and since $\mathcal{A}_s$ is an algebra, we have $T_\Phi^j B T_\Phi^k \in \mathcal{A}_s$ for all $j, k \in \mathbf{Z}_+$. Thus it follows from (1) that

$$T_\Phi^{j+1} B T_\Phi^{k+1} \in \mathcal{T}^{(1)} \quad \text{for all integers} \ \ j \ge 0 \ \ \text{and} \ \ k \ge 0. \tag{2.27}$$

Let $C^*(T_\Phi)$ be the unital C^*-algebra generated by T_Φ. Since T_Φ is self-adjoint, (2.27) implies that

$$T_\Phi X B T_\Phi X \in \mathcal{T}^{(1)} \quad \text{for every} \ \ X \in C^*(T_\Phi).$$

We again use the invertibility of T_Φ, which is guaranteed by (2.10). It is elementary that the inverse T_Φ^{-1}, once it exists, must belong to the C^*-algebra $C^*(T_\Phi)$. Thus, letting $X = T_\Phi^{-1}$ in the above, we obtain $B \in \mathcal{T}^{(1)}$. This completes the proof of Theorem 1.5. $\square$

3. Membership in $\mathcal{T}^{(1)}$

As we already mentioned, our goal for this section is to prove Proposition 2.8. For convenience, let us introduce

Definition 3.1. (a) Let $\mathcal{D}_0$ denote the collection of operators of the form

$$\sum_{u\in\Gamma} c_u k_u \otimes k_{\gamma(u)},$$

where Γ is any separated set in $\mathbf{B}$, $\{c_u : u \in \Gamma\}$ is any bounded set of complex coefficients, and $\gamma : \Gamma \to \mathbf{B}$ is any map for which there is a $0 < C < \infty$ such that

$$\beta(u, \gamma(u)) \le C \tag{3.1}$$

for every $u \in \Gamma$.

(b) Let $\mathcal{D}$ denote the operator-norm closure of the linear span of $\mathcal{D}_0$.

With $\mathcal{D}_0$ and $\mathcal{D}$ we can divide the proof of Proposition 2.8 into two independent parts:

Proposition 3.2. *Let $(n-1)/(n+1) < s < 1$. If $B \in \mathcal{A}_s$, then for every pair of $z, w \in \mathbf{B}$ we have $E_w B E_z \in \mathcal{D}$.*

Proposition 3.3. *We have $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$.*

Since $\mathcal{T}^{(1)}$ is a norm closed linear subspace of $\mathcal{B}(L^2_a(\mathbf{B}, dv))$, Proposition 2.8 follows immediately from Propositions 3.2 and 3.3.

We will see that the proofs of these two propositions are based on different ideas. More specifically, the proof of Proposition 3.3 relies on the estimate provided by Lemma 2.6, whereas the proof of Proposition 3.2 takes advantage of the localization condition of the operators in $\mathcal{A}_s$. The proof of Proposition 3.2 begins with

Lemma 3.4. *Let $(n-1)/(n+1) < s < 1$ be given. If $B \in \mathcal{A}_s$, then for every separated set Γ in $\mathbf{B}$ and every pair of $z, w \in \mathbf{B}$ we have*

$$\lim_{R\to\infty} \sup_{u\in\Gamma} \sum_{\substack{v\in\Gamma \\ \beta(u,v)>R}} |\langle B k_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} = 0 \quad \textit{and} \tag{3.2}$$

$$\lim_{R\to\infty} \sup_{u\in\Gamma} \sum_{\substack{v\in\Gamma \\ \beta(u,v)>R}} |\langle k_{\varphi_u(z)}, B k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} = 0. \tag{3.3}$$

Proof. Given such s and $B \in \mathcal{A}_s$, by Definition 1.1 we have

$$\lim_{r\to\infty} \sup_{x\in\mathbf{B}} \int_{\mathbf{B}\setminus D(x,r)} |\langle B k_x, k_\zeta\rangle| \left(\frac{1-|\zeta|^2}{1-|x|^2}\right)^{s(n+1)/2} d\lambda(\zeta) = 0 \quad \text{and} \tag{3.4}$$

$$\lim_{r\to\infty} \sup_{x\in\mathbf{B}} \int_{\mathbf{B}\setminus D(x,r)} |\langle B^* k_x, k_\zeta\rangle| \left(\frac{1-|\zeta|^2}{1-|x|^2}\right)^{s(n+1)/2} d\lambda(\zeta) = 0. \tag{3.5}$$

Let Γ, z and w also be given as in the lemma. Denote $G = D(0,1)$ and $G_w = \varphi_w(G)$. Then it is easy to see that $G_w \subset D(0, 1 + \beta(w, 0))$. For $h \in L^2_a(\mathbf{B}, dv)$ and $v \in \Gamma$, we have

$$\begin{aligned} h(\varphi_v(w)) &= (h\circ\varphi_v\circ\varphi_w)(0) = \frac{1}{\lambda(G)} \int_G h\circ\varphi_v\circ\varphi_w d\lambda = \frac{1}{\lambda(G)} \int_{(\varphi_v\circ\varphi_w)(G)} h d\lambda \\ &= \frac{1}{\lambda(G)} \int_{\varphi_v(G_w)} h d\lambda = \frac{1}{\lambda(G)} \int_{\varphi_v(G_w)} \frac{\langle h, k_\zeta\rangle}{(1-|\zeta|^2)^{(n+1)/2}} d\lambda(\zeta). \end{aligned}$$

Thus

$$\langle h, k_{\varphi_v(w)}\rangle = \frac{1}{\lambda(G)} \int_{\varphi_v(G_w)} \langle h, k_\zeta\rangle \left(\frac{1-|\varphi_v(w)|^2}{1-|\zeta|^2}\right)^{(n+1)/2} d\lambda(\zeta).$$

If $\zeta \in \varphi_v(G_w)$, then $\zeta = \varphi_v(\xi)$ for some $\xi \in G_w \subset D(0, 1+\beta(w,0))$, which means

$$1 - |\zeta|^2 = 1 - |\varphi_v(\xi)|^2 = \frac{(1-|v|^2)(1-|\xi|^2)}{|1-\langle \xi, v\rangle|^2} \geq \frac{1}{4}(1-|\xi|^2)(1-|v|^2).$$

On the other hand,

$$1 - |\varphi_v(w)|^2 = \frac{(1-|v|^2)(1-|w|^2)}{|1-\langle w, v\rangle|^2} \leq \frac{2}{1-|w|}(1-|v|^2).$$

Hence there is a $0 < C_1 < \infty$ which depends only on n and w such that

$$|\langle h, k_{\varphi_v(w)}\rangle|(1-|v|^2)^{s(n+1)/2} \leq \frac{C_1}{\lambda(G)} \int_{\varphi_v(G_w)} |\langle h, k_\zeta\rangle|(1-|\zeta|^2)^{s(n+1)/2} d\lambda(\zeta)$$

for all $h \in L^2_a(\mathbf{B}, dv)$ and $v \in \Gamma$. Applying this inequality to the case where $h = Bk_{\varphi_u(z)}$, $u \in \Gamma$, we have

$$\begin{aligned} &|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} \\ &\qquad \leq \frac{C_1}{\lambda(G)} \int_{\varphi_v(G_w)} |\langle Bk_{\varphi_u(z)}, k_\zeta\rangle| \left(\frac{1-|\zeta|^2}{1-|u|^2}\right)^{s(n+1)/2} d\lambda(\zeta), \end{aligned}$$

$v \in \Gamma$. Since

$$1 - |\varphi_u(z)|^2 = \frac{(1-|u|^2)(1-|z|^2)}{|1-\langle z, u\rangle|^2} \leq \frac{2}{1-|z|}(1-|u|^2),$$

there is a $0 < C_2 < \infty$ which depends only on n and z such that

$$\begin{aligned} &|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} \\ &\qquad \leq \frac{C_1 C_2}{\lambda(G)} \int_{\varphi_v(G_w)} |\langle Bk_{\varphi_u(z)}, k_\zeta\rangle| \left(\frac{1-|\zeta|^2}{1-|\varphi_u(z)|^2}\right)^{s(n+1)/2} d\lambda(\zeta), \end{aligned} \tag{3.6}$$

$u, v \in \Gamma$. Set $L = 1 + \beta(w,0) + \beta(z,0)$ and consider any $R > L$. If $u, v \in \Gamma$ are such that $\beta(u,v) > R$, then for every $\zeta \in \varphi_v(G_w) \subset \varphi_v(D(0, 1+\beta(w,0)))$ we have

$$\beta(\varphi_u(z), \zeta) \geq \beta(u,v) - \beta(v,\zeta) - \beta(u, \varphi_u(z)) > R - 1 - \beta(w,0) - \beta(z,0) = R - L. \tag{3.7}$$

Thus the combination of (3.6) and (3.7) gives us

$$\begin{aligned} &\sum_{\substack{v\in\Gamma \\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} \\ &\leq \frac{C_1 C_2}{\lambda(G)} \int_{\beta(\varphi_u(z),\zeta)>R-L} \sum_{v\in\Gamma} \chi_{\varphi_v(G_w)}(\zeta) |\langle Bk_{\varphi_u(z)}, k_\zeta\rangle| \left(\frac{1-|\zeta|^2}{1-|\varphi_u(z)|^2}\right)^{s(n+1)/2} d\lambda(\zeta), \end{aligned} \tag{3.8}$$

$u \in \Gamma$. By the Möbius invariance of β and the fact that $G_w \subset D(0, 1 + \beta(w, 0))$, we have $\varphi_v(G_w) \subset D(v, 1 + \beta(w, 0))$. Since Γ is separated, it follows from Lemma 2.2(a) that there is an $N \in \mathbf{N}$ which depends only on Γ and w such that the inequality

$$\sum_{v \in \Gamma} \chi_{\varphi_v(G_w)} \leq N$$

holds on $\mathbf{B}$. Substituting this in (3.8), we conclude that

$$\sum_{\substack{v \in \Gamma \\ \beta(u,v) > R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle| \left(\frac{1 - |v|^2}{1 - |u|^2} \right)^{s(n+1)/2}$$
$$\leq \frac{C_1 C_2 N}{\lambda(G)} \int_{\beta(\varphi_u(z), \zeta) > R - L} |\langle Bk_{\varphi_u(z)}, k_\zeta \rangle| \left(\frac{1 - |\zeta|^2}{1 - |\varphi_u(z)|^2} \right)^{s(n+1)/2} d\lambda(\zeta)$$

for every $u \in \Gamma$. By this inequality, (3.2) follows from (3.4). Since

$$\langle k_{\varphi_u(z)}, Bk_{\varphi_v(w)} \rangle = \langle B^* k_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle,$$

(3.3) follows from (3.5) by the same argument. This completes the proof. □

Proof of Proposition 3.2. Let $(n-1)/(n+1) < s < 1$. For $B \in \mathcal{A}_s$ and $z, w \in \mathbf{B}$, we have

$$\begin{aligned} E_w B E_z &= \sum_{u,v \in \mathcal{L}} k_{\varphi_v(w)} \otimes k_{\varphi_v(w)} \cdot B \cdot k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} \\ &= \sum_{u,v \in \mathcal{L}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)}. \end{aligned}$$

Thus for any $R > 0$, we can write $E_w B E_z = V_R + W_R$, where

$$\begin{aligned} V_R &= \sum_{\substack{u,v \in \mathcal{L} \\ \beta(u,v) \leq R}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)} \quad \text{and} \\ W_R &= \sum_{\substack{u,v \in \mathcal{L} \\ \beta(u,v) > R}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)}. \end{aligned}$$

Obviously, the proposition will follow if we can prove the following two statements:

(1) $\lim_{R \to \infty} \|W_R\| = 0$.

(2) $V_R \in \text{span}(\mathcal{D}_0)$ for every $R > 0$.

To prove (1), note that by (2.23) and Lemma 2.6, there are constants C_1, C_2 such that

$$(3.9) \qquad \sum_{u \in \mathcal{L}} |\langle h, k_{\varphi_u(z)} \rangle|^2 \leq C_1 \|h\|^2 \quad \text{and} \quad \sum_{v \in \mathcal{L}} |\langle h, k_{\varphi_v(w)} \rangle|^2 \leq C_2 \|h\|^2$$

for every $h \in L_a^2(\mathbf{B}, dv)$. Given $h, g \in L_a^2(\mathbf{B}, dv)$, we have

$$|\langle W_R h, g\rangle| \le \sum_{\substack{u,v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| s_u t_v, \tag{3.10}$$

where

$$s_u = |\langle h, k_{\varphi_u(z)}\rangle| \quad \text{and} \quad t_v = |\langle k_{\varphi_v(w)}, g\rangle|.$$

We apply the Schur test one more time. Indeed for each $u \in \mathcal{L}$, let us write

$$y_u = \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| t_v. \tag{3.11}$$

Then for each $u \in \mathcal{L}$, the Cauchy-Schwarz inequality gives us

$$\begin{aligned} y_u^2 &\le \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| (1-|v|^2)^{\frac{s(n+1)}{2}} \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \frac{t_v^2}{(1-|v|^2)^{\frac{s(n+1)}{2}}} \\ &\le H(R) \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}} t_v^2, \end{aligned}$$

where

$$H(R) = \sup_{\xi\in\mathcal{L}} \sum_{\substack{v\in\mathcal{L}\\ \beta(\xi,v)>R}} |\langle Bk_{\varphi_\xi(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|v|^2}{1-|\xi|^2}\right)^{\frac{s(n+1)}{2}}.$$

Therefore

$$\begin{aligned} \sum_{u\in\mathcal{L}} y_u^2 &\le H(R) \sum_{u\in\mathcal{L}} \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}} t_v^2 \\ &= H(R) \sum_{v\in\mathcal{L}} t_v^2 \sum_{\substack{u\in\mathcal{L}\\ \beta(u,v)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}} \\ &\le H(R)G(R) \sum_{v\in\mathcal{L}} t_v^2, \end{aligned}$$

where

$$G(R) = \sup_{\xi\in\mathcal{L}} \sum_{\substack{u\in\mathcal{L}\\ \beta(u,\xi)>R}} |\langle Bk_{\varphi_u(z)}, k_{\varphi_\xi(w)}\rangle| \left(\frac{1-|u|^2}{1-|\xi|^2}\right)^{\frac{s(n+1)}{2}}.$$

By (3.10) and (3.11), we now have

$$|\langle W_R h, g\rangle| \le \sum_{u\in\mathcal{L}} s_u y_u \le \left(\sum_{u\in\mathcal{L}} s_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} y_u^2\right)^{1/2}$$
$$\le \{H(R)G(R)\}^{1/2} \left(\sum_{u\in\mathcal{L}} s_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} t_u^2\right)^{1/2}.$$

Combining this with (3.9), we find that

$$|\langle W_R h, g\rangle| \le \{C_1 C_2 H(R)G(R)\}^{1/2}\|h\|\|g\|.$$

Since $h, g \in L_a^2(\mathbf{B}, dv)$ are arbitrary, this means

$$\|W_R\| \le \{C_1 C_2 H(R)G(R)\}^{1/2}.$$

Applying Lemma 3.4, we have $\lim_{R\to\infty} H(R) = 0$ and $\lim_{R\to\infty} G(R) = 0$. Therefore $\lim_{R\to\infty} \|W_R\| = 0$ as promised.

We now turn to the proof of (2). First of all, given an $R > 0$, for each $v \in \mathcal{L}$ we define

$$F_v = \{u \in \mathcal{L} : \beta(u, v) \le R\}.$$

By Lemma 2.2(a), there is an $N \in \mathbf{N}$ such that

$$\mathrm{card}(F_v) \le N$$

for every $v \in \mathcal{L}$. Also, by Lemma 2.2(b), for the given $w \in \mathbf{B}$, there is a partition

$$\mathcal{L} = L_1 \cup \cdots \cup L_m$$

such that for each $i \in \{1, \dots, m\}$, if $v, v' \in L_i$ and $v \ne v'$, then $\beta(\varphi_v(w), \varphi_{v'}(w)) \ge 1$. That is, for each $i \in \{1, \dots, m\}$, the set

$$K_i = \{\varphi_v(w) : v \in L_i\}$$

is separated. We have $V_R = X_1 + \cdots + X_m$, where

$$X_i = \sum_{\varphi_v(w)\in K_i} \sum_{u\in F_v} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)},$$

$i \in \{1, \dots, m\}$. To prove (2), if suffices to show that $X_i \in \mathrm{span}(\mathcal{D}_0)$ of every $i \in \{1, \dots, m\}$. For this purpose we further decompose each K_i. Indeed for each pair of $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, N\}$, we define

$$L_{i,j} = \{v \in L_i : \mathrm{card}(F_v) = j\} \quad \text{and} \quad K_{i,j} = \{\varphi_v(w) : v \in L_{i,j}\}.$$

Then $X_i = X_{i,1} + \cdots + X_{i,N}$, where

$$X_{i,j} = \sum_{\varphi_v(w) \in K_{i,j}} \sum_{u \in F_v} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)},$$

$i \in \{1, \dots, m\}$ and $j \in \{1, \dots, N\}$. Thus it suffices to show that $X_{i,j} \in \mathrm{span}(\mathcal{D}_0)$ for every such pair of i, j. But it is obvious that given a pair of such i, j, we can define maps

$$\gamma_{i,j}^{(1)}, \cdots, \gamma_{i,j}^{(j)} : K_{i,j} \to \mathbf{B}$$

such that

$$\{\varphi_u(z) : u \in F_v\} = \{\gamma_{i,j}^{(1)}(\varphi_v(w)), \cdots, \gamma_{i,j}^{(j)}(\varphi_v(w))\}$$

for every $v \in L_{i,j}$. Thus $X_{i,j} = X_{i,j}^{(1)} + \cdots + X_{i,j}^{(j)}$, where for each $\nu \in \{1, \dots, j\}$ we have

$$X_{i,j}^{(\nu)} = \sum_{\xi \in K_{i,j}} \langle Bk_{\gamma_{i,j}^{(\nu)}(\xi)}, k_\xi \rangle k_\xi \otimes k_{\gamma_{i,j}^{(\nu)}(\xi)}.$$

Hence the proof will be complete if we can show that $X_{i,j}^{(\nu)} \in \mathcal{D}_0$ for every triple of indices $i \in \{1, \dots, m\}$, $j \in \{1, \dots, N\}$ and $\nu \in \{1, \dots, j\}$.

By the above definitions, for every such triple of i, j, ν, if $\xi \in K_{i,j}$, then there exist $v \in L_{i,j}$ and $u \in F_v$ such that $\xi = \varphi_v(w)$ and $\gamma_{i,j}^{(\nu)}(\xi) = \varphi_u(z)$. Therefore

$$\begin{aligned}\beta(\xi, \gamma_{i,j}^{(\nu)}(\xi)) &= \beta(\varphi_v(w), \varphi_u(z)) \\ &\le \beta(\varphi_v(w), v) + \beta(v, u) + \beta(u, \varphi_u(z)) \le \beta(w, 0) + R + \beta(0, z).\end{aligned}$$

This shows that the map $\gamma_{i,j}^{(\nu)} : K_{i,j} \to \mathbf{B}$ satisfies condition (3.1). By Definition 3.1(a), we have $X_{i,j}^{(\nu)} \in \mathcal{D}_0$. This completes the proof of Proposition 3.2. $\square$

Next we turn to the proof of Proposition 3.3, which involves a few steps.

Proposition 3.5. *Suppose that* Γ *is a separated set in* $\mathbf{B}$. *Furthermore, suppose that* $\{c_u : u \in \Gamma\}$ *are complex numbers for which* (2.21) *holds. Then for each* $z \in \mathbf{B}$, *the operator* Y_z *defined by* (2.22) *belongs to* $\mathcal{T}^{(1)}$.

Proof. (1) Let us first show that $Y_0 \in \mathcal{T}^{(1)}$. Since Γ is separated, there is a $\delta > 0$ such that $\beta(u, v) \ge \delta$ for all $u \ne v$ in Γ. That is, if $u, v \in \Gamma$ and $u \ne v$, then $D(u, \delta/2) \cap D(v, \delta/2) = \emptyset$. For each $0 < \epsilon < \delta/2$, define the operator

$$A_\epsilon = \frac{1}{\lambda(D(0, \epsilon))} \int_{D(0,\epsilon)} Y_z d\lambda(z).$$

By the norm continuity of the map $z \mapsto Y_z$ provided by Proposition 2.7, we have

$$\lim_{\epsilon \downarrow 0} \|Y_0 - A_\epsilon\| = 0.$$

Thus to prove the membership $Y_0 \in \mathcal{T}^{(1)}$, it suffices to show that each A_ϵ is a Toeplitz operator with a bounded symbol. Indeed by the Möbius invariance of β and $d\lambda$, we have

$$\begin{aligned} A_\epsilon &= \frac{1}{\lambda(D(0,\epsilon))} \sum_{u \in \Gamma} c_u \int_{D(0,\epsilon)} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} d\lambda(z) \\ &= \frac{1}{\lambda(D(0,\epsilon))} \sum_{u \in \Gamma} c_u \int_{D(u,\epsilon)} k_w \otimes k_w d\lambda(w) = \int f_\epsilon(w) k_w \otimes k_w d\lambda(w), \end{aligned}$$

where

$$f_\epsilon = \frac{1}{\lambda(D(0,\epsilon))} \sum_{u \in \Gamma} c_u \chi_{D(u,\epsilon)}.$$

Since, $0 < \epsilon < \delta/2$, we have $D(u,\epsilon) \cap D(v,\epsilon) = \emptyset$ for $u \neq v$ in Γ. Hence $f_\epsilon \in L^\infty(\mathbf{B}, dv)$. By (2.5), we have $A_\epsilon = T_{f_\epsilon}$. This proves the membership $Y_0 \in \mathcal{T}^{(1)}$.

(2) Now consider an arbitrary $z \in \mathbf{B}$. By Lemma 2.2(b), there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $i \in \{1, \dots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \neq v$ imply $\beta(\varphi_u(z), \varphi_v(z)) \geq 1$. That is, for each $i \in \{1, \dots, m\}$, the set

$$G_i = \{\varphi_u(z) : u \in \Gamma_i\}$$

is separated. Obviusly, $Y_z = Y_{z,1} + \cdots + Y_{z,m}$, where

$$Y_{z,i} = \sum_{\varphi_u(z) \in G_i} c_u k_{\varphi_u(z)} \otimes k_{\varphi_u(z)},$$

$i = 1, \dots, m$. By (1) we have $Y_{z,i} \in \mathcal{T}^{(1)}$ for every $i \in \{1, \dots, m\}$. Hence $Y_z \in \mathcal{T}^{(1)}$. $\square$

In addition to the normalized reproducing kernel k_z given by (1.1), it will be convenient for our next step to use the *unnormalized* reproducing kernel

$$K_z(\zeta) = \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1}}, \quad z, \zeta \in \mathbf{B},$$

and other kernel-like functions. This involves monomials in the complex variables $\zeta_1, \dots, \zeta_n$ and the standard multi-index convention (see, e.g., [10,page 3]). For each pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we define

$$K_{z;\alpha}(\zeta) = \frac{\zeta^\alpha}{(1 - \langle \zeta, z \rangle)^{n+1+|\alpha|}}, \tag{3.12}$$

$\zeta \in \mathbf{B}$. Note that $K_z = K_{z;0}$ for every $z \in \mathbf{B}$.

Proposition 3.6. *Let Γ be a separated set in $\mathbf{B}$ and suppose that $\{c_u : u \in \Gamma\}$ is a bounded set of complex coefficients. Then for every pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we have*

$$\sum_{u \in \Gamma} c_u (U_u K_z) \otimes (U_u K_{z;\alpha}) \in \mathcal{T}^{(1)}.$$

Proof. We prove the proposition by an induction on $|\alpha|$. If $|\alpha| = 0$, i.e. $\alpha = 0$, then

$$(U_u K_z) \otimes (U_u K_{z;0}) = (U_u K_z) \otimes (U_u K_z) = \frac{1}{(1-|z|^2)^{n+1}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}.$$

Hence the case where $|\alpha| = 0$ follows from Proposition 3.5. Suppose that $k \in \mathbf{Z}_+$ and that the proposition holds true for every $\alpha \in \mathbf{Z}_+^n$ satisfying the condition $|\alpha| \le k$. Now consider the case where $\alpha \in \mathbf{Z}_+^n$ is such that $|\alpha| = k+1$. Then we can decompose α in the form

$$\alpha = a + b,$$

where $|a| = k$ and $|b| = 1$. That is, there is some $\nu \in \{1, \ldots, n\}$ such that the ν-th component of b is 1 and the other components of b are all 0. We will also consider b as a vector in $\mathbf{C}^n$. By the induction hypothesis, we have

$$\sum_{u\in\Gamma} c_u (U_u K_z) \otimes (U_u K_{z;a}) \in \mathcal{T}^{(1)} \quad \text{for every} \quad z \in \mathbf{B}. \tag{3.13}$$

Let $z \in \mathbf{B}$ be given. Then there is an $\epsilon = \epsilon(z) > 0$ such that $z + c \in \mathbf{B}$ for every $c \in \mathbf{C}^n$ satisfying the condition $|c| \le \epsilon$. For each $t \in [0, \epsilon]$, define the operators

$$A_t = \sum_{u\in\Gamma} c_u (U_u K_{z+tb}) \otimes (U_u K_{z+tb;a}) \quad \text{and} \quad B_t = \sum_{u\in\Gamma} c_u (U_u K_{z+itb}) \otimes (U_u K_{z+itb;a}).$$

Also, we define

$$X = \sum_{u\in\Gamma} c_u \{(n+1+k)(U_u K_z) \otimes (U_u K_{z;\alpha}) + (n+1)(U_u K_{z;b}) \otimes (U_u K_{z;a})\} \quad \text{and}$$
$$Y = \sum_{u\in\Gamma} c_u \{(n+1+k)(U_u K_z) \otimes (U_u K_{z;\alpha}) - (n+1)(U_u K_{z;b}) \otimes (U_u K_{z;a})\}.$$

We will show that

$$\lim_{t\downarrow 0} \left\| \frac{1}{t}(A_t - A_0) - X \right\| = 0 \quad \text{and} \tag{3.14}$$

$$\lim_{t\downarrow 0} \left\| \frac{1}{it}(B_t - B_0) - Y \right\| = 0. \tag{3.15}$$

Before getting to their proofs, let us first see the consequence of these limits. By (3.13) we have $A_t \in \mathcal{T}^{(1)}$ and $B_t \in \mathcal{T}^{(1)}$ for all $t \in [0, \epsilon]$. Hence it follows from (3.14) and (3.15) that $X, Y \in \mathcal{T}^{(1)}$. Thus

$$\sum_{u\in\Gamma} c_u (U_u K_z) \otimes (U_u K_{z;\alpha}) = \frac{1}{2(n+1+k)}(X+Y) \in \mathcal{T}^{(1)},$$

completing the induction on $|\alpha|$.

Let us now turn to the proof of (3.14). Note that $t^{-1}(A_t - A_0) = G_t + H_t$, where

$$H_t = \frac{1}{t}\sum_{u\in\Gamma} c_u(U_u K_{z+tb}) \otimes \{U_u(K_{z+tb;a} - K_{z;a})\} \quad \text{and}$$
$$G_t = \frac{1}{t}\sum_{u\in\Gamma} c_u\{U_u(K_{z+tb} - K_z)\} \otimes (U_u K_{z;a}).$$

Similarly, we write $X = V + W$, where

$$V = \sum_{u\in\Gamma} c_u(n+1+k)(U_u K_z) \otimes (U_u K_{z;\alpha}) \quad \text{and}$$
$$W = \sum_{u\in\Gamma} c_u(n+1)(U_u K_{z;b}) \otimes (U_u K_{z;a}).$$

Since $\|t^{-1}(A_t - A_0) - X\| \le \|H_t - V\| + \|G_t - W\|$, (3.14) will follow if we can show

(3.16) $$\lim_{t\downarrow 0} \|H_t - V\| = 0 \quad \text{and}$$

(3.17) $$\lim_{t\downarrow 0} \|G_t - W\| = 0.$$

To prove (3.16), for $0 < t \le \epsilon$ we write $H_t - V = S_t + T_t$, where

$$S_t = \sum_{u\in\Gamma} c_u(U_u K_{z+tb}) \otimes \{U_u(t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)K_{z;\alpha})\} \quad \text{and}$$
$$T_t = (n+1+k)\sum_{u\in\Gamma} c_u\{U_u(K_{z+tb} - K_z)\} \otimes (U_u K_{z;\alpha}).$$

Thus the proof of (3.16) is reduced to the proof of the fact that $\|S_t\| \to 0$ and $\|T_t\| \to 0$ as t descends to 0. To prove this, we pick an orthonormal set $\{e_u : u \in \Gamma\}$ and factor S_t in the form $S_t = S_t^{(1)} S_t^{(2)*}$, where

$$S_t^{(1)} = \sum_{u\in\Gamma} c_u(U_u K_{z+tb}) \otimes e_u \quad \text{and}$$
$$S_t^{(2)} = \sum_{u\in\Gamma} \{U_u(t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)K_{z;\alpha})\} \otimes e_u.$$

Set $C = \sup_{u\in\Gamma} |c_u|$. Then it follows from Lemma 2.6 that

$$\|S_t^{(1)}\| \le CB(\Gamma)\|K_{z+tb}\|_\infty \quad \text{and}$$
$$\|S_t^{(2)}\| \le B(\Gamma)\|t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)K_{z;\alpha}\|_\infty.$$

Since $a + b = \alpha$ and $k = |a|$, by (3.12) and elementary algebra, we have

$$\lim_{t\downarrow 0} \|t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)K_{z;\alpha}\|_\infty = 0.$$

Also, it is trivial that $\|K_{z+tb}\|_\infty$ remains bounded as t descends to 0. Hence

$$\begin{aligned}\|S_t\| &\le \|S_t^{(1)}\|\|S_t^{(2)}\| \\ &\le C(B(\Gamma))^2\|K_{z+tb}\|_\infty\|t^{-1}(K_{z+tb;a}-K_{z;a})-(n+1+k)K_{z;\alpha}\|_\infty \to 0\end{aligned}$$

as t descends to 0. For T_t, we have the factorization $T_t = T_t^{(1)}T^{(2)*}$, where

$$\begin{aligned}T_t^{(1)} &= (n+1+k)\sum_{u\in\Gamma} c_u\{U_u(K_{z+tb}-K_z)\}\otimes e_u \quad \text{and} \\ T^{(2)} &= \sum_{u\in\Gamma}(U_uK_{z;\alpha})\otimes e_u.\end{aligned}$$

By Lemma 2.6, $\|T_t^{(1)}\| \le (n+1+k)CB(\Gamma)\|K_{z+tb}-K_z\|_\infty$, and $T^{(2)}$ is a bounded operator. It is obvious that

$$\lim_{t\downarrow 0}\|K_{z+tb}-K_z\|_\infty = 0.$$

Hence $\|T_t\| \le \|T_t^{(1)}\|\|T^{(2)}\| \to 0$ as t descends to 0. This completes the proof of (3.16).

To prove (3.17), note that

$$G_t - W = \sum_{u\in\Gamma} c_u\{U_u(t^{-1}(K_{z+tb}-K_z)-(n+1)K_{z;b})\}\otimes(U_uK_{z;a}) = Z_tT^{(2)*},$$

where

$$Z_t = \sum_{u\in\Gamma} c_u\{U_u(t^{-1}(K_{z+tb}-K_z)-(n+1)K_{z;b})\}\otimes e_u.$$

Applying Lemma 2.6 again, we have

$$\|Z_t\| \le CB(\Gamma)\|t^{-1}(K_{z+tb}-K_z)-(n+1)K_{z;b}\|_\infty.$$

Another easy exercise shows that

$$\lim_{t\downarrow 0}\|t^{-1}(K_{z+tb}-K_z)-(n+1)K_{z;b}\|_\infty = 0.$$

Hence $\|G_t - W\| \le \|Z_t\|\|T^{(2)}\| \to 0$ as t descends to 0, proving (3.17). Thus we have completed the proof of (3.14).

The proof of (3.15) uses essentially the same argument as above, and the only additional care that needs to be taken is the following: The rank-one operator $f\otimes g$ is linear with respect to f and conjugate linear with respect to g. Moreover, the inner product $\langle\zeta, z\rangle$ on $\mathbf{C}^n$ is conjugate linear with respect to z. These are the properties that determine the + and − signs in each term $c_u\{\cdots\}$ in the sum that defines the operator Y. This completes the proof of the proposition. □

Proposition 3.7. *Let Γ be a separated set in* $\mathbf{B}$ *and let* $\{c_u : u \in \Gamma\}$ *be a bounded set of complex coefficients. Then for every* $w \in \mathbf{B}$ *we have*

$$\sum_{u\in\Gamma} c_u k_u \otimes k_{\varphi_u(w)} \in \mathcal{T}^{(1)}. \tag{3.18}$$

Proof. For each $\alpha \in \mathbf{Z}_+^n$, define the monomial function

$$p_\alpha(\zeta) = \zeta^\alpha$$

on $\mathbf{B}$. Given a $w \in \mathbf{B}$, let us define

$$d_u(w) = c_u \left(\frac{1 - \langle w, u\rangle}{|1 - \langle w, u\rangle|}\right)^{n+1},$$

$u \in \Gamma$. Note that $K_{0;\alpha} = p_\alpha$ for every $\alpha \in \mathbf{Z}_+^n$. Also, $U_u K_0 = U_u 1 = k_u$ for every $u \in \Gamma$. Thus, applying Proposition 3.6 to the case where $z = 0$, we have

$$\sum_{u\in\Gamma} d_u(w) k_u \otimes (U_u p_\alpha) \in \mathcal{T}^{(1)} \tag{3.19}$$

for every $\alpha \in \mathbf{Z}_+^n$. Define the function

$$g_w(\zeta) = \langle \zeta, w\rangle, \quad \zeta \in \mathbf{B}.$$

For each $j \in \mathbf{Z}_+$, define the operator

$$A_j = \sum_{u\in\Gamma} d_u(w) k_u \otimes (U_u g_w^j).$$

Since each g_w^j is in the linear span of $\{p_\alpha : \alpha \in \mathbf{Z}_+^n\}$, (3.19) implies that $A_j \in \mathcal{T}^{(1)}$ for every $j \in \mathbf{Z}_+$. Let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then we have the factorization $A_j = TB_j^*$ for each $j \in \mathbf{Z}_+$, where

$$T = \sum_{u\in\Gamma} d_u(w) k_u \otimes e_u \quad \text{and} \quad B_j = \sum_{u\in\Gamma} (U_u g_w^j) \otimes e_u.$$

Lemma 2.6 tells us that T is a bounded operator. Define

$$G = \sum_{u\in\Gamma} (U_u K_w) \otimes e_u.$$

It also follows from Lemma 2.6 that

$$\left\| G - \sum_{j=0}^{k} \frac{(n+j)!}{n!j!} B_j \right\| \le B(\Gamma) \left\| K_w - \sum_{j=0}^{k} \frac{(n+j)!}{n!j!} g_w^j \right\|_\infty \tag{3.20}$$

for every $k \in \mathbf{Z}_+$. By the expansion formula

$$\frac{1}{(1-c)^{n+1}} = \sum_{j=0}^{\infty} \frac{(n+j)!}{n!j!} c^j, \quad |c| < 1,$$

and the fact that $|w| < 1$, we have

$$\lim_{k\to\infty} \left\| K_w - \sum_{j=0}^{k} \frac{(n+j)!}{n!j!} g_w^j \right\|_\infty = 0.$$

Combining this with (3.20), we obtain

$$\lim_{k\to\infty} \left\| TG^* - \sum_{j=0}^{k} \frac{(n+j)!}{n!j!} A_j \right\| = \lim_{k\to\infty} \left\| TG^* - T \sum_{j=0}^{k} \frac{(n+j)!}{n!j!} B_j^* \right\| = 0.$$

Since each A_j belongs to $\mathcal{T}^{(1)}$, we conclude that

$$\sum_{u\in\Gamma} d_u(w) k_u \otimes (U_u K_w) = TG^* \in \mathcal{T}^{(1)}.$$

Since $k_w = (1-|w|^2)^{(n+1)/2} K_w$, this implies

$$\sum_{u\in\Gamma} d_u(w) k_u \otimes (U_u k_w) \in \mathcal{T}^{(1)}. \tag{3.21}$$

Recalling the definition of $d_u(w)$ and (2.23), we see that (3.21) implies (3.18). □

Proof of Proposition 3.3. Let Γ be a separated set in $\mathbf{B}$, let $\{c_u : u \in \Gamma\}$ be a bounded set of coefficients, and let $\gamma : \Gamma \to \mathbf{B}$ be a map satisfying (3.1). Let $K = \{w \in \mathbf{B} : \beta(0, w) \le C\}$, where C is the constant that appears in (3.1). We want to show that the operator

$$T = \sum_{u\in\Gamma} c_u k_u \otimes k_{\gamma(u)}$$

belongs to $\mathcal{T}^{(1)}$. For this purpose, define

$$\psi(u) = \varphi_u(\gamma(u)), \quad u \in \Gamma.$$

Since $\beta(u, \gamma(u)) \le C$, by the Möbius invariance of β and the fact $\varphi_u(u) = 0$, we have $\beta(0, \psi(u)) = \beta(u, \gamma(u)) \le C$ for every $u \in \Gamma$. That is, $\psi(u) \in K$ for every $u \in \Gamma$. Since $\varphi_u(\psi(u)) = \gamma(u)$, $u \in \Gamma$, by (2.23) we have

$$T = \sum_{u\in\Gamma} d_u k_u \otimes (U_u k_{\psi(u)}),$$

where $|d_u| = |c_u|$ for every $u \in \Gamma$. Let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then we have the factorization $T = AB^*$, where

$$A = \sum_{u \in \Gamma} d_u k_u \otimes e_u \quad \text{and} \quad B = \sum_{u \in \Gamma} (U_u k_{\psi(u)}) \otimes e_u.$$

We again use the fact that the map $z \mapsto k_z$ is $\| \cdot \|_\infty$-continuous. That is,

$$\lim_{w \to z} \|k_z - k_w\|_\infty = 0 \quad \text{for every} \quad z \in \mathbf{B}.$$

Let $\epsilon > 0$ be given. Since K is compact, there are non-empty open sets $\Omega_1, \ldots, \Omega_m$ in $\mathbf{B}$ and $z_i \in \Omega_i$, $i = 1, \ldots, m$, such that

$$\Omega_1 \cup \cdots \cup \Omega_m \supset K \tag{3.22}$$

and

$$\|k_{z_i} - k_w\|_\infty < \epsilon \quad \text{whenever} \quad w \in \Omega_i,$$

$i = 1, \ldots, m$. From the open cover (3.22) we obtain a partition

$$K = E_1 \cup \cdots \cup E_m$$

such that $E_i \subset \Omega_i$ for every $i \in \{1, \ldots, m\}$. We now define

$$\Gamma_i = \{u \in \Gamma : \psi(u) \in E_i\},$$

$i = 1, \ldots, m$. Then $\|k_{z_i} - k_{\psi(u)}\|_\infty < \epsilon$ if $u \in \Gamma_i$. For every $i \in \{1, \ldots, m\}$, we also define

$$B_i = \sum_{u \in \Gamma_i} (U_u k_{z_i}) \otimes e_u.$$

For each $i \in \{1, \ldots, m\}$ we have

$$AB_i^* = \sum_{u \in \Gamma_i} d_u k_u \otimes (U_u k_{z_i}) = \sum_{u \in \Gamma_i} d_{u,i} k_u \otimes k_{\varphi_u(z_i)},$$

where $|d_{u,i}| = |d_u|$ for $u \in \Gamma_i$. Thus it follows from Proposition 3.7 that

$$\{AB_1^*, \ldots, AB_m^*\} \subset \mathcal{T}^{(1)}. \tag{3.23}$$

On the other hand, we have

$$B - (B_1 + \cdots + B_m) = \sum_{i=1}^m \sum_{u \in \Gamma_i} \{U_u(k_{\psi(u)} - k_{z_i})\} \otimes e_u.$$

Since the sets $\Gamma_1, \dots, \Gamma_m$ form a partition of Γ, i.e., $\Gamma_i \cap \Gamma_j = \emptyset$ whenever $i \neq j$, Lemma 2.6 tells us that

$$\|B - (B_1 + \cdots + B_m)\| \le B(\Gamma) \max_{1 \le i \le m} \sup_{u \in \Gamma_i} \|k_{\psi(u)} - k_{z_i}\|_\infty \le B(\Gamma)\epsilon.$$

Lemma 2.6 also tells us that A is a bounded operator. Hence

$$\begin{aligned}\|T - (AB_1^* + \cdots + AB_m^*)\| &= \|AB^* - (AB_1^* + \cdots + AB_m^*)\| \\ &\le \|A\|\|B^* - (B_1^* + \cdots + B_m^*)\| = \|A\|\|B - (B_1 + \cdots + B_m)\| \\ &\le \|A\|B(\Gamma)\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, combining this inequality with (3.23), we conclude that $T \in \mathcal{T}^{(1)}$. This completes the proof of Proposition 3.3. □

4. Analogue on the Fock space

The analogue of Theorem 1.5 also holds in the setting of the Fock space. To discuss the details, let us first recall the necessary definitions.

Let $d\mu$ be the Gaussian measure on $\mathbf{C}^n$. It is well known that, in terms of the standard volume measure dV on $\mathbf{C}^n$, we have

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z).$$

Recall that the Fock space $H^2(\mathbf{C}^n, d\mu)$ is defined to be the subspace $\{h \in L^2(\mathbf{C}^n, d\mu) : h$ is analytic on $\mathbf{C}^n\}$ of $L^2(\mathbf{C}^n, d\mu)$. In this section, the symbol k_z will denote the normalized reproducing kernel for $H^2(\mathbf{C}^n, d\mu)$. That is,

$$k_z(\zeta) = e^{\langle \zeta, z \rangle} e^{-|z|^2/2}, \qquad z, \zeta \in \mathbf{C}^n.$$

In [14], the notion of *sufficiently localized operators* was introduced:

Definition 4.1. A bounded operator B on $H^2(\mathbf{C}^n, d\mu)$ is said to be sufficiently localized if there exist constants $2n < \beta < \infty$ and $0 < C < \infty$ such that

$$|\langle Bk_z, k_w \rangle| \le \frac{C}{(1 + |z - w|)^\beta}$$

for all $z, w \in \mathbf{C}^n$.

Let $C^*(\mathcal{SL})$ be the C^*-algebra generated by the collection of sufficiently localized operators on $H^2(\mathbf{C}^n, d\mu)$. Combining localization properties with a new approach, it was shown in [14] that for $A \in C^*(\mathcal{SL})$,

$$\text{(4.1)} \qquad \text{the condition } \lim_{|z| \to \infty} \langle Ak_z, k_z \rangle = 0 \text{ implies that } A \text{ is compact.}$$

This was the result that motivated Isralowitz, Mitkovski and Wick to introduce the notion of weakly localized operators in [6]. On the Fock space, weakly localized operators are defined as follows.

Definition 4.2. [6] A bounded operator T on $H^2(\mathbf{C}^n, d\mu)$ is said to be weakly localized if it satisfies the conditions

$$\sup_{z\in\mathbf{C}^n}\int |\langle Tk_z, k_w\rangle| dV(w) < \infty, \quad \sup_{z\in\mathbf{C}^n}\int |\langle T^*k_z, k_w\rangle| dV(w) < \infty,$$

and

$$\lim_{r\to\infty}\sup_{z\in\mathbf{C}^n}\int_{|z-w|\geq r} |\langle Tk_z, k_w\rangle| dV(w) = 0, \quad \lim_{r\to\infty}\sup_{z\in\mathbf{C}^n}\int_{|z-w|\geq r} |\langle T^*k_z, k_w\rangle| dV(w) = 0.$$

It is easy to see that any sufficiently localized operator is weakly localized. Moreover, it was shown in [6] that (4.1) also holds true if A is in the C^*-algebra generated by the weakly localized operators on $H^2(\mathbf{C}^n, d\mu)$.

Replacing the class $\mathcal{A}_s$ by the class of operators defined in Definition 4.2, one can prove the analogue of Theorem 1.5 on the Fock space $H^2(\mathbf{C}^n, d\mu)$. The proof is in fact easier in the Fock space case. This is because, compared with the Bergman space, the structure of the Fock space is much simpler, and one generally gets much better "decaying rate" in estimates.

For example, instead of general separated sets, in the Fock space setting we only need to be concerned with the standard lattice

$$\mathbf{Z}^{2n} = \{(j_1 + ik_1, \dots, j_n + ik_n) : j_1, \dots, j_n, k_1, \dots, k_n \in \mathbf{Z}\}$$

and its subsets. What replaces $D(0, 2)$ is the fundamental cube

$$S = \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, \dots, x_n, y_1, \dots, y_n \in [0, 1)\}$$

in $\mathbf{C}^n$. With $\mathbf{Z}^{2n}$ and S we have

$$\bigcup_{u\in\mathbf{Z}^{2n}} \{u - S\} = \mathbf{C}^n, \tag{4.2}$$

which is a *tiling* of the space, meaning that there is no overlap between $u - S$ and $v - S$ for $u \neq v$ in $\mathbf{Z}^{2n}$. Compared with the *covering* scheme (2.7), the tiling scheme (4.2) offers considerable advantages. For example, the Toeplitz operator T_Φ used in the proof of Theorem 1.5 can simply be replaced by the identity operator 1 in the case of Fock space.

There is, however, one technical issue in the Fock space case that warrants mentioning. This stems from the fact that there are no bounded analytic functions on $\mathbf{C}^n$ other than constants. Thus the straightforward analogue of Lemma 2.6 on $H^2(\mathbf{C}^n, d\mu)$, while true, is not very useful. In the Fock-space setting, the supremum norm $\|\cdot\|_\infty$ must be replaced by something else.

Definition 4.3. For an analytic function h on $\mathbf{C}^n$, we write

$$\|h\|_* = \left(\int |h(\zeta)|^2 e^{-(1/2)|\zeta|^2} dV(\zeta)\right)^{1/2}.$$

Let $\mathcal{H}_*$ be the collection of analytic functions h on $\mathbf{C}^n$ satisfying the condition $\|h\|_* < \infty$.

For each $z \in \mathbf{C}^n$, let U_z be the unitary operator defined by the formula

$$(U_z f)(\zeta) = f(z-\zeta)k_z(\zeta), \quad \zeta \in \mathbf{C}^n, \tag{4.3}$$

$f \in H^2(\mathbf{C}^n, d\mu)$. The following is what replaces Lemma 2.6 in the Fock-space setting:

Lemma 4.4. *There is a constant* $0 < C_{4.4} < \infty$ *such that the following estimate holds: Let* $\{e_u : u \in \mathbf{Z}^{2n}\}$ *be any orthonormal set and let* $h_u \in \mathcal{H}_*$, $u \in \mathbf{Z}^{2n}$, *be functions satisfying the condition* $\sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_* < \infty$. *Then*

$$\left\|\sum_{u\in\mathbf{Z}^{2n}} (U_u h_u) \otimes e_u\right\| \le C_{4.4} \sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_*.$$

Proof. Let us first estimate $|\langle U_u h_u, U_v h_v\rangle|$. By (4.3), for $u, v \in \mathbf{Z}^{2n}$ we have

$$\langle U_u h_u, U_v h_v\rangle = \int h_u(u-\zeta)\overline{h_v(v-\zeta)}k_u(\zeta)\overline{k_v(\zeta)}e^{-|\zeta|^2} dV(\zeta). \tag{4.4}$$

Moreover,

$$\left|k_u(\zeta)\overline{k_v(\zeta)}\right| e^{-|\zeta|^2} = e^{-(1/2)(|u-\zeta|^2+|v-\zeta|^2)}, \tag{4.5}$$

$\zeta \in \mathbf{C}^n$. Observe that

$$|u-\zeta|^2 + |v-\zeta|^2 \ge \frac{1}{2}(|u-\zeta| + |v-\zeta|)^2 \ge \frac{1}{2}|u-v|^2.$$

Thus, splitting the 1/2 in (4.5) as $(1/4) + (1/4)$, we find that

$$\left|k_u(\zeta)\overline{k_v(\zeta)}\right| e^{-|\zeta|^2} \le e^{-(1/8)|u-v|^2} e^{-(1/4)|u-\zeta|^2} e^{-(1/4)|v-\zeta|^2}.$$

Combining this with (4.4) and applying the Cauchy-Schwarz inequality, we obtain

$$|\langle U_u h_u, U_v h_v\rangle| \le e^{-(1/8)|u-v|^2}\|h_u\|_*\|h_v\|_* \le e^{-(1/8)|u-v|^2} H_*^2, \tag{4.6}$$

where

$$H_* = \sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_*.$$

Write

$$A = \sum_{u\in\mathbf{Z}^{2n}} (U_u h_u) \otimes e_u$$

and consider any vector $x = \sum_{u\in\mathbf{Z}^{2n}} x_u e_u$. By (4.6), we have

$$\|Ax\|^2 \le \sum_{u,v\in\mathbf{Z}^{2n}} |\langle U_v h_v, U_u h_u\rangle||x_u||x_v| \le H_*^2 \sum_{u,v\in\mathbf{Z}^{2n}} e^{-(1/8)|u-v|^2}|x_u||x_v|.$$

Applying the Schur test to the right-hand side, we find that

$$\|Ax\|^2 \le CH_*^2 \sum_{u\in\mathbf{Z}^{2n}} |x_u|^2 = CH_*^2\|x\|^2,$$

where $C = \sum_{z\in\mathbf{Z}^{2n}} e^{-(1/8)|z|^2}$, which is finite. Since the vector x is arbitrary, we conclude that $\|A\| \le C^{1/2}H_*$. Thus the lemma holds for the constant $C_{4.4} = C^{1/2}$. $\square$

In the proof of the Fock-space analogue of Theorem 1.5, the $\|\cdot\|_\infty$-continuities of the previous sections are replaced by the corresponding $\|\cdot\|_*$-continuities. For example, for the normalized reproducing kernel of the Fock space one easily verifies that

$$\lim_{w\to z} \|k_z - k_w\|_* = 0$$

for every $z \in \mathbf{C}^n$. Thus, using Lemma 4.4 in place of Lemma 2.6, the analogue of Theorem 1.5 on the Fock space can be obtained by following the argument in the previous sections.

References

1. C. Berger and L. Coburn, Toeplitz operators and quantum mechanics. J. Funct. Anal, **68** (1986), 273-299.

2. C. Berger and L. Coburn, Heat flow and Berezin-Toeplitz estimates. Amer. J. Math. **116** (1994), 563-590.

3. M. Engliš, Toeplitz operators on Bergman-type spaces, Ph.D. Dissertation, Prague 1991.

4. M. Engliš, Density of algebras generated by Toeplitz operator on Bergman spaces, Ark. Mat. **30** (1992), 227-243.

5. M. Engliš, K. Guo and G. Zhang, Toeplitz and Hankel operators and Dixmier traces on the unit ball of $\mathbf{C}^n$, Proc. Amer. Math. Soc. **137** (2009), 3669-3678.

6. J. Isralowitz, M. Mitkovski and B. Wick, Localization and compactness in Bergman and Fock spaces, to appear in Indiana Univ. Math. J., arXiv:1306.0316

7. C. Jiang and D. Zheng, Similarity of analytic Toeplitz operators on the Bergman spaces, J. Funct. Anal. **258** (2010), 2961-2982.

8. T. Le, On the commutator ideal of the Toeplitz algebra on the Bergman space of the unit ball in $\mathbf{C}^n$, J. Operator Theory **60** (2008), 149-163.
9. K. Nam, D. Zheng and C. Zhong, m-Berezin transform and compact operators, Rev. Mat. Iberoam. **22** (2006), 867-892.
10. W. Rudin, Function theory in the unit ball of $\mathbf{C}^n$, Springer-Verlag, New York-Berlin, 1980.
11. D. Suárez, The Toeplitz algebra on the Bergman space coincides with its commutator ideal, J. Operator Theory **51** (2004), 105-114.
12. D. Suárez, Approximation and the n-Berezin transform of operators on the Bergman space, J. Reine Angew. Math. **581** (2005), 175-192.
13. D. Suárez, The essential norm of operators in the Toeplitz algebra on $A^p(\mathbf{B}_n)$, Indiana Univ. Math. J. **56** (2007), 2185-2232.
14. J. Xia and D. Zheng, Localization and Berezin transform on the Fock space, J. Funct. Anal. **264** (2013), 97-117.
15. K. Zhu, Operator theory in function spaces, 2nd ed., Mathematical Surveys and Monographs, **138**, American Mathematical Society, Providence, RI, 2007.

Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260

E-mail: jxia@acsu.buffalo.edu