INTEGRAL OPERATORS ASSOCIATED WITH THE GENERALIZED FUGLEDE PROPERTY

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Abstract. We consider a class of integral operators $T_{K,\mu}$ that arise from various problems studied in the past, one of which is the generalized Fuglede commutation property. We show that under a rather general growth condition on the measure μ, the operator $T_{K,\mu}$ belongs to the Lorentz ideal $\mathcal{C}_1^+$. This naturally leads to the challenge of computing the Dixmier trace of $T_{K,\mu}$. In the case where μ is the restriction of the Lebesgue measure m_n to a bounded Borel set in $\mathbf{R}^n$, we show that the Dixmier trace of $T_{K,\mu}$ is 0.

1. Introduction

In this paper we study integral operators of the type

$$(T_{K,\mu}f)(x) = \int K(x-y)f(y)d\mu(y),$$

where K is a homogeneous function of degree 0. Ordinarily, one might call $T_{K,\mu}$ a *singular* integral operator. But since we are considering homogeneous kernel K of degree 0, the word "singular" is not quite appropriate here. So we will just call $T_{K,\mu}$ an integral operator.

Throughout the paper, we will always assume that K is a C^∞ function on $\mathbf{R}^n\backslash\{0\}$. The assumption that K is a homogeneous function of degree 0 means that $K(\lambda u) = K(u)$ for all $u \in \mathbf{R}^n\backslash\{0\}$ and $0 < \lambda < \infty$.

Our consideration of $T_{K,\mu}$ is primarily motivated by its connection with the generalized Fuglede commutation property [1,5,9-11,15]. We will leave the discussion of the Fuglede property and its connection with $T_{K,\mu}$ to a latter part of the Introduction. Let us first focus on $T_{K,\mu}$, which turns out to be interesting in its own right.

Our first result is the membership of this operator in the Lorentz ideal $\mathcal{C}_1^+$. Before we discuss the membership, let us recall the definition of this class of ideals. Let $\mathcal{H}$ be a Hilbert space. Consider any $1 \le p < \infty$. Then the formula

$$\|A\|_p^+ = \sup_{j\ge1} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

defines a norm for bounded operators on $\mathcal{H}$. Here and in what follows, we write $s_1(A)$, $s_2(A)$, $\ldots$, $s_j(A)$, $\ldots$ for the s-numbers [6] of the operator A. It is well known that the collection of operators

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

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form a norm ideal, for which we cite [6] as our primary reference. In particular, we mention the well-known fact that $\mathcal{C}_p^+$ is not separable with respect to the norm $\|\cdot\|_p^+$. The ideal $\mathcal{C}_1^+$ deserves special attention, for it is the domain of every Dixmier trace [2,4].

For $x \in \mathbf{R}^n$ and $r > 0$, we write $B(x,r) = \{y \in \mathbf{R}^n : |x-y| < r\}$ as usual. Here is our first result:

Theorem 1.1. *Let μ be a compactly-supported regular Borel measure on $\mathbf{R}^n$. Suppose that there are constants $0 < \alpha < \infty$ and $0 < C < \infty$ such that*

$$\mu(B(x,r)) \le Cr^\alpha \tag{1.1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Let K be a homogeneous function of degree 0 on $\mathbf{R}^n\backslash\{0\}$. Define the operator $T_{K,\mu}$ on $L^2(d\mu)$ by the formula

$$(T_{K,\mu}f)(x) = \int K(x-y)f(y)d\mu(y), \tag{1.2}$$

$f \in L^2(d\mu)$. Then we have $T_{K,\mu} \in \mathcal{C}_1^+$.

The point of Theorem 1.1 is that in the case of homogeneous kernel of degree 0, no matter what $\alpha \in (0,\infty)$ is, one always ends up with the same membership $T_{K,\mu} \in \mathcal{C}_1^+$. This is completely different from the result for homogeneous kernels of degree -1, which is well-known:

Theorem 1.2. [3] *Let $1 < p < \infty$. Let μ be a compactly-supported regular Borel measure on $\mathbf{R}^n$. Suppose that there is a constant $0 < C < \infty$ such that*

$$\mu(B(x,r)) \le Cr^p \tag{1.3}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Let K be a homogeneous function of degree -1 on $\mathbf{R}^n\backslash\{0\}$. Define the singular integral operator T on $L^2(d\mu)$ by the formula

$$(Tf)(x) = \int K(x-y)f(y)d\mu(y),$$

$f \in L^2(d\mu)$. Then $T \in \mathcal{C}_q^+$, where $q = p/(p-1)$.

See [17] for related results.

Let us now explain where the integral operators $T_{K,\mu}$ come from. This involves both historical and mathematical background. For convenience of discussion, all operators are assumed to be bounded for the rest of the paper.

In [15], Weiss proved the remarkable identity

$$\|[N,X]\|_2 = \|[N^*,X]\|_2, \tag{1.4}$$

where N is any normal operator and $\|\cdot\|_2$ is the Hilbert-Schmidt norm. In particular, for a normal operator N, if $[N,X]$ is a Hilbert-Schmidt operator, then so is $[N^*,X]$. This is called the Fuglede commutation property modulo the Hilbert-Schmidt class. In light of this, it makes perfectly good sense to consider the Fuglede commutation property modulo the trace class. Thus Weiss raised the following:

Question 1.3. [15] If N is a normal operator and X is a compact operator such that the commutator $[N, X]$ is in the trace class, must the trace of $[N, X]$ be zero?

Immediately after raising this question, Weiss explained its connection with the generalized Fuglede commutation property:

Proposition 1.4. [15, page 15] *Let X be a compact operator and let T be any operator such that $[T, X]$ is in the trace class. If $\mathrm{tr}[T, X] \neq 0$, then $[T^*, X]$ is not in the trace class.*

Proof. Write $T = A + iB$, where A, B are self-adjoint operators. If it were true that $[T^*, X]$ is also in the trace class, then the commutators $[A, X]$ and $[B, X]$ would be in the trace class. But since X is compact and A, B are self-adjoint, by a well-know result of Helton and Howe [7, Lemma 1.3], we would have $\mathrm{tr}[A, X] = 0$ and $\mathrm{tr}[B, X] = 0$. Since $T = A + iB$, this contradicts the condition that $\mathrm{tr}[T, X] \neq 0$. □

Shortly after [15], this issue was revisited in the form

Question 1.5. [9, page 524] Does there exist a normal operator N and a compact operator X such that $[N, X]$ is in the trace class but $[N^*, X]$ is not?

Both questions were answered by Shulman and Turowska:

Example 1.6. [11, Example 8.5] Consider the Hilbert space $L^2(D, dA)$, where $D = \{z \in \mathbf{C} : |z| < 1\}$, the unit disc in $\mathbf{C}$, and dA is the area measure on $\mathbf{C}$. Let N be the normal operator on $L^2(D, dA)$ defined by the formula

$$(Nf)(z) = zf(z), \tag{1.5}$$

$f \in L^2(D, dA)$. Define the operator

$$(Xf)(z) = \int_D \frac{f(w)}{z - w} dA(w), \tag{1.6}$$

$f \in L^2(D, dA)$. Then X is in the Schatten p-class for every $p > 2$. (In fact, Theorem 1.2 tells us that X is in the Lorentz ideal $\mathcal{C}_2^+$). It is obvious that the commutator $[N, X]$ is the rank-one operator $1 \otimes 1$ on $L^2(D, dA)$. Thus $\mathrm{tr}[N, X] \neq 0$, which answers Question 1.3 in the negative. Furthermore, by Proposition 1.4, $[N^*, X]$ is not in the trace class, which also answers Question 1.5. □

For this pair of N and X, what intrigues us is the commutator $[N^*, X]$. Knowing that $[N^*, X]$ is not in the trace class, what else can we say about this commutator?

Note that for the operators N and X defined by (1.5) and (1.6), we have

$$([N^*, X]f)(z) = \int_D \frac{\bar{z} - \bar{w}}{z - w} f(w) dA(w), \tag{1.7}$$

$f \in L^2(D, dA)$. What stands out is the fact that $(\bar{z}-\bar{w})/(z-w)$ is a homogeneous kernel of degree 0. That is, $[N^*, X]$ is an example of the kind of integral operators in Theorem 1.1. In fact, (1.7) is the motivating example for our study of $T_{K,\mu}$ in this paper. Another connection of $T_{K,\mu}$ will be explained in Section 4.

Since the conclusion of Theorem 1.1 is $T_{K,\mu} \in \mathcal{C}_1^+$, a followup question naturally presents itself: can we compute the Dixmier trace of $T_{K,\mu}$?

As of this writing, we are only able to compute the Dixmier trace of $T_{K,\mu}$ under some restrictions on the measure μ. In other words, at least for now, condition (1.1) seems to be too general for the purpose of computing the Dixmier trace of $T_{K,\mu}$. A moment of thought tells us that condition (1.1) allows strange behaviors by μ. The reader will see that, in comparison with the proof of Theorem 1.1, computation of the Dixmier trace of $T_{K,\mu}$ requires additional estimates of the $\|\cdot\|_1^+$-norm. At the moment, we are not able to do the additional estimates without restrictions on μ. But fortunately, what we can do at the moment covers the connection with the Fuglede commutation property.

We write m_n for the Lebesgue measure on $\mathbf{R}^n$. Even though our techniques can handle more than just the Lebesgue measure, for simplicity we only present our Dixmier-trace result in the following case:

Theorem 1.7. *Let E be a bounded Borel set in $\mathbf{R}^n$. Let K be a homogeneous function of degree 0 on $\mathbf{R}^n\backslash\{0\}$. Define the operator $T_{K,E}$ on $L^2(E, dm_n)$ by the formula*

$$(T_{K,E}f)(x) = \int_E K(x-y)f(y)dm_n(y), \tag{1.8}$$

$f \in L^2(E, dm_n)$. Then for every Dixmier trace Tr_ω, we have $\mathrm{Tr}_\omega(T_{K,E}) = 0$.

The rest of the paper is organized as follows:

We prove Theorems 1.1 and 1.7 in Sections 2 and 3 respectively.

In Section 4 we discuss the applications of Theorems 1.1 and 1.7 to the motivating examples of $T_{K,\mu}$.

In Section 5, we conclude the paper with some thoughts on the Fuglede commutation property itself.

2. Membership in $\mathcal{C}_1^+$

The proof of Theorem 1.1 will be based on the duality between $\mathcal{C}_1^+$ and the Mačaev ideal $\mathcal{C}_\infty^-$. First of all, recall that $\mathcal{C}_\infty^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\infty^- < \infty\}$, where

$$\|A\|_\infty^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j}.$$

This duality is better explained, and more conveniently used in proofs, if we bring the relevant symmetric gauge functions (also called symmetric norming functions) into the discussion.

For this we follow the approach in [6]. Let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A symmetric gauge function is a map

$$\Phi : \hat{c} \to [0,\infty)$$

that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \ldots, 0, \ldots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [6, page 71]. The domain of such a Φ is then extended to include *every* sequence $\xi = \{\xi_j\}$ by the formula

$$\Phi(\xi) = \sup_{j\geq 1} \Phi(\{\xi_1, \ldots, \xi_j, 0, \ldots, 0, \ldots\}).$$

In the same spirit, each symmetric gauge function Φ gives rise to the symmetric norm

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \ldots, s_j(A), 0, \ldots, 0, \ldots\})$$

for bounded operators.

Given any $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, define

$$\Phi_1^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\geq 1} \frac{|a_{\pi(1)}| + |a_{\pi(2)}| + \cdots + |a_{\pi(j)}|}{1^{-1} + 2^{-1} + \cdots + j^{-1}} \quad \text{and} \quad \Phi_\infty^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|a_{\pi(j)}|}{j}$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \cdots \geq |a_{\pi(j)}| \geq \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. Then Φ_1^+ and Φ_∞^- are symmetric gauge functions on $\hat{c}$ with the properties

$$\|A\|_{\Phi_1^+} = \|A\|_1^+ \quad \text{and} \quad \|A\|_{\Phi_\infty^-} = \|A\|_\infty^-$$

for operators. The symmetric gauge functions Φ_1^+ and Φ_∞^- are dual to each other in the sense that for every $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, we have

$$\Phi_\infty^-(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^\infty a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi_1^+(\{b_j\}_{j\in\mathbf{N}}) \leq 1\right\} \quad \text{and}$$
$$\Phi_1^+(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^\infty a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi_\infty^-(\{b_j\}_{j\in\mathbf{N}}) \leq 1\right\}.$$

See [6, pages 148, 149 and 125]. Thus it follows that for every $T \in \mathcal{B}(\mathcal{H})$,

$$\|T\|_1^+ = \sup\{|\mathrm{tr}(TF)| : \|F\|_\infty^- \leq 1 \text{ and } \mathrm{rank}(F) < \infty\}. \tag{2.1}$$

The proof of Theorem 1.1 uses the system of dyadic decomposition in [17], which we now recall. Denote

$$Q = [0,1)^n = [0,1) \times \cdots \times [0,1),$$

the unit cube in $\mathbf{R}^n$. For each $\ell \in \mathbf{N}$, we let W_ℓ be the set of words of length ℓ with $\{1, 2, 3, \dots, 2^n\}$ being the set of alphabet. That is,

$$W_\ell = \{w_1 \dots w_\ell : w_1, \dots, w_\ell \in \{1, 2, 3, \dots, 2^n\}\}.$$

Let $\Gamma = \{(s_1, \dots, s_n) : s_1, \dots, s_n \in \{0,1\}\}$ and let $\gamma_1, \dots, \gamma_{2^n}$ be an enumeration of the elements in Γ. Given $w = w_1 \dots w_\ell \in W_\ell$, we define

$$Q_w = Q_{w_1 \dots w_\ell} = [0, 2^{-\ell})^n + 2^{-1}\gamma_{w_1} + \cdots + 2^{-\ell}\gamma_{w_\ell}.$$

It is clear that $\cup_{w \in W_\ell} Q_w = Q$ and that $Q_w \cap Q_{w'} = \emptyset$ for $w \neq w'$ in W_ℓ. Define

$$\mathcal{W} = \cup_{\ell=1}^{\infty} W_\ell.$$

By the homogeneity of the kernel $K(x-y)$ and an easy scaling transformation, it suffices to prove Theorem 1.1 in the case where the measure μ is concentrated on Q. Therefore, for the rest of the section we assume that μ is a regular Borel measure on $\mathbf{R}^n$ such that $\mu(\mathbf{R}^n \backslash Q) = 0$. To avoid total triviality, we assume $\mu(Q) \neq 0$.

For each $w \in \mathcal{W}$, we define the element $e_w \in L^2(d\mu)$ by the formula

$$e_w = \begin{cases} (\mu(Q_w))^{-1/2}\chi_{Q_w} & \text{if} \quad \mu(Q_w) > 0 \\ \\ 0 & \text{if} \quad \mu(Q_w) = 0 \end{cases}.$$

Let

$$\Lambda = \{(s_1, \dots, s_n) : s_1, \dots, s_n \in \{-1, 0, 1\}\}.$$

Given $w \in W_\ell$ and $\lambda \in \Lambda$, we have either $Q_w + 2^{-\ell}\lambda = Q_{w'}$ for some $w' \in W_\ell$ or $Q_w + 2^{-\ell}\lambda \subset \mathbf{R}^n \backslash Q$. Thus for $w \in W_\ell$ and $\lambda \in \Lambda$, we define the element $e(w,\lambda) \in L^2(d\mu)$ as follows:

$$e(w,\lambda) = \begin{cases} e_{w'} & \text{if} \quad Q_w + 2^{-\ell}\lambda = Q_{w'}, \ w' \in W_\ell \\ \\ 0 & \text{if} \qquad Q_w + 2^{-\ell}\lambda \subset \mathbf{R}^n \backslash Q \end{cases}.$$

Similarly, for $w \in W_\ell$ and $\lambda \in \Lambda$, we define

$$\mu(w,\lambda) = \begin{cases} \mu(Q_{w'}) & \text{if} \quad Q_w + 2^{-\ell}\lambda = Q_{w'}, \ w' \in W_\ell \\ \\ 0 & \text{if} \qquad Q_w + 2^{-\ell}\lambda \subset \mathbf{R}^n \backslash Q \end{cases}.$$

By our assumption, K is a C^∞ homogeneous function of degree 0 on $\mathbf{R}^n \backslash \{0\}$. Let $0 \le \tilde{\eta} \le 1$ be a C^∞-function on $[0,\infty)$ such that $\tilde{\eta} = 1$ on $[0, 1/2]$ and $\tilde{\eta} = 0$ on $[5/8, \infty)$. Define $\eta(r) = \tilde{\eta}(r) - \tilde{\eta}(2r)$, $r \in [0,\infty)$. It is easy to see that

$$\eta = 0 \text{ on } [0, 1/4] \cup [5/8, \infty) \quad \text{and} \quad \eta = 1 \text{ on } [1/3, 1/2]. \tag{2.2}$$

Let $\ell_0 \in \mathbf{N}$ be such that $2^{\ell_0-1} > \sqrt{n}$. Now $\sum_{\ell=-k}^{k'} \eta(2^\ell r) = \tilde{\eta}(2^{-k}r) - \tilde{\eta}(2^{k'+1}r)$. Since $|u| \leq \sqrt{n} < 2^{\ell_0-1}$ for every $u \in [-1,1]^n$, we have

$$\sum_{\ell=-\ell_0}^{\infty} \eta(2^\ell |u|) = 1 \quad \text{if } u \in [-1,1]^n \text{ and } u \neq 0. \tag{2.3}$$

(2.2) implies that $K(u)\eta(|u|) = 0$ if $0 < |u| \leq 1/4$. Hence there is a periodic C^∞-function φ on $\mathbf{R}^n$ with $(2^{\ell_0+2}\mathbf{Z})^n$ as its period lattice such that

$$\varphi(u) = K(u)\eta(|u|) \quad \text{if } u \in [-2^{\ell_0}, 2^{\ell_0}]^n \text{ and } u \neq 0. \tag{2.4}$$

Such a φ has a Fourier expansion

$$\varphi(u) = \sum_{z\in\mathbf{Z}^n} c_z \exp(2^{-\ell_0-1} i\pi\langle u, z\rangle) \quad \text{with} \quad \sum_{z\in\mathbf{Z}^n} |c_z| < \infty. \tag{2.5}$$

For $w \in W_\ell$ and $z \in \mathbf{Z}^n$, we define $f_w^z(x) = \exp(2^{\ell-\ell_0-1} i\pi\langle x, z\rangle)$. For $\ell \in \mathbf{N}$, $z \in \mathbf{Z}^n$ and $\lambda \in \Lambda$, we then define the operator

$$B_{\ell;z,\lambda} = \sum_{w\in W_\ell} \{\mu(w,\lambda)\mu(Q_w)\}^{1/2} (f_w^z e(w,\lambda)) \otimes (f_w^z e_w) \tag{2.6}$$

on $L^2(d\mu)$.

Lemma 2.1. *Under condition* (1.1), *there are constants* $0 < C_{2.1} < \infty$ *and* $N \in \mathbf{N}$ *such that for all* $\ell \in \mathbf{N}$, $z \in \mathbf{Z}^n$ *and* $\lambda \in \Lambda$ *and for every bounded operator* F, *we have*

$$|\mathrm{tr}(B_{\ell;z,\lambda}F)| \leq C_{2.1} 2^{-\alpha\ell} \sum_{j=1}^{N[2^{\alpha\ell}]} s_j(F), \tag{2.7}$$

where $[2^{\alpha\ell}]$ *denotes the integer part of* $2^{\alpha\ell}$.

Proof. Since $\mathrm{card}(W_\ell) = 2^{n\ell}$, we have $\mathrm{rank}(B_{\ell;z,\lambda}) \leq 2^{n\ell}$. Therefore

$$|\mathrm{tr}(B_{\ell;z,\lambda}F)| \leq \sum_{j=1}^{2^{n\ell}} s_j(B_{\ell;z,\lambda}) s_j(F).$$

Note that $f_w^z e_w \perp f_{w'}^z e_{w'}$ for all $w \neq w'$ in W_ℓ and that $f_w^z e(w,\lambda) \perp f_{w'}^z e(w',\lambda)$ for all $w \neq w'$ in W_ℓ and any given λ. Thus the nonzero s-numbers of $B_{\ell;z,\lambda}$ are just a descending arrangement of the nonzero values among $\{\mu(w,\lambda)\mu(Q_w)\}^{1/2}$, $w \in W_\ell$. Hence there is an enumeration

$$w(1), w(2), \ldots, w(2^{n\ell})$$

of the elements in W_ℓ such that

$$|\mathrm{tr}(B_{\ell;z,\lambda}F)| \le \sum_{j=1}^{2^{n\ell}} \{\mu(w(j),\lambda)\mu(Q_{w(j)})\}^{1/2} s_j(F). \tag{2.8}$$

For each $w \in W_\ell$, there is an $x(w) \in Q$ such that $Q_w \subset B(x(w), \sqrt{n}2^{-\ell})$. By (1.1), we have $\mu(Q_w) \le Cn^{\alpha/2}2^{-\alpha\ell}$ for every $w \in W_\ell$. Consequently, writing $C_1 = Cn^{\alpha/2}$, we have

$$\{\mu(w,\lambda)\mu(Q_w)\}^{1/2} \le C_1 2^{-\alpha\ell} \tag{2.9}$$

for every $w \in W_\ell$. If it so happens that

$$\sum_{j=1}^{2^{n\ell}} \{\mu(w(j),\lambda)\mu(Q_{w(j)})\}^{1/2} \le C_1 2^{-\alpha\ell}, \tag{2.10}$$

then (2.7) certainly follows from (2.8). Suppose that (2.10) does not hold. Then by (2.9), there is a partition

$$\{1, 2, \ldots, 2^{n\ell}\} = I_0 \cup I_1 \cup \cdots \cup I_{\nu(\ell)},$$

where I_0 may be empty, such that

$$\sum_{j \in I_0} \{\mu(w(j),\lambda)\mu(Q_{w(j)})\}^{1/2} < C_1 2^{-\alpha\ell} \tag{2.11}$$

and

$$C_1 2^{-\alpha\ell} \le \sum_{j \in I_i} \{\mu(w(j),\lambda)\mu(Q_{w(j)})\}^{1/2} < 2C_1 2^{-\alpha\ell} \tag{2.12}$$

for every $1 \le i \le \nu(\ell)$. Combining (2.8), (2.11) and (2.12), we find that

$$|\mathrm{tr}(B_{\ell;z,\lambda}F)| \le 2C_1 2^{-\alpha\ell} \sum_{i=0}^{\nu(\ell)} \max_{j \in I_i} s_j(F) \le 2C_1 2^{-\alpha\ell} \sum_{i=0}^{\nu(\ell)} s_{i+1}(F). \tag{2.13}$$

On the other hand, we have

$$\begin{aligned} \mu(Q) &\ge \Big(\sum_{w \in W_\ell} \mu(w,\lambda)\Big)^{1/2} \Big(\sum_{w \in W_\ell} \mu(Q_w)\Big)^{1/2} \ge \sum_{w \in W_\ell} \{\mu(w,\lambda)\mu(Q_w)\}^{1/2} \\ &\ge C_1 2^{-\alpha\ell}\nu(\ell), \end{aligned}$$

where the last $\ge$ follows from (2.12) and the fact that $I_i \cap I_{i'} = \emptyset$ for $i \ne i'$. This gives us $\nu(\ell) \le \{\mu(Q)/C_1\}2^{\alpha\ell}$. Substituting this in (2.13), the proof is complete. $\square$

For $z \in \mathbf{Z}^n$ and $\lambda \in \Lambda$, we denote

$$A_{z,\lambda} = \sum_{\ell=1}^{\infty} B_{\ell;z,\lambda}. \tag{2.14}$$

It is easy to see that (1.1) implies the operator-norm bound $\|B_{\ell;z,\lambda}\| \leq C_0 2^{-\alpha\ell}$. Therefore the above sum converges with respect to operator norm.

Lemma 2.2. *Under condition* (1.1), *there is a constant* $0 < C_{2.2} < \infty$ *such that*

$$\|A_{z,\lambda}\|_1^+ \leq C_{2.2} \tag{2.15}$$

for all $z \in \mathbf{Z}^n$ *and* $\lambda \in \Lambda$.

Proof. Applying Lemma 2.1, for any finite-rank operator F we have

$$\begin{aligned} |\mathrm{tr}(A_{z,\lambda}F)| &\leq \sum_{\ell=1}^{\infty} |\mathrm{tr}(B_{\ell;z,\lambda}F)| \leq C_{2.1} \sum_{\ell=1}^{\infty} 2^{-\alpha\ell} \sum_{j=1}^{N[2^{\alpha\ell}]} s_j(F) \\ &= C_{2.1} \sum_{j=1}^{\infty} s_j(F) \sum_{N[2^{\alpha\ell}] \geq j} 2^{-\alpha\ell}. \end{aligned}$$

Obviously,

$$\sum_{N[2^{\alpha\ell}] \geq j} 2^{-\alpha\ell} \leq \sum_{N2^{\alpha\ell} \geq j} 2^{-\alpha\ell} = \sum_{N/j \geq 2^{-\alpha\ell}} 2^{-\alpha\ell} \leq \frac{N}{j} \sum_{\nu=0}^{\infty} 2^{-\alpha\nu} = \frac{N}{1-2^{-\alpha}} \cdot \frac{1}{j}.$$

From these two estimates we obtain

$$|\mathrm{tr}(A_{z,\lambda}F)| \leq \frac{NC_{2.1}}{1-2^{-\alpha}} \sum_{j=1}^{\infty} \frac{s_j(F)}{j} = \frac{NC_{2.1}}{1-2^{-\alpha}} \|F\|_\infty^-.$$

Since this holds for every finite-rank operator F, from (2.1) we see that (2.15) holds for the constant $C_{2.2} = (1-2^{-\alpha})^{-1} N C_{2.1}$. □

Lemma 2.3. *The integral operator* (1.2) *admits a decomposition*

$$T_{K,\mu} = R_{K,\mu} + \sum_{\lambda \in \Lambda} \sum_{z \in \mathbf{Z}^n} c_z A_{z,\lambda},$$

where $R_{K,\mu}$ *is a trace class operator and* $A_{z,\lambda}$ *and* c_z *are given in* (2.14) *and* (2.5) *respectively.*

Proof. For each $\ell \geq 1$, define $V_\ell(u) = K(u)\eta(2^\ell|u|)$, $u \in \mathbf{R}^n \backslash \{0\}$. Then define

$$U(u) = \sum_{\ell=-\ell_0}^{0} K(u)\eta(2^\ell|u|) \quad \text{and} \quad V(u) = \sum_{\ell=1}^{\infty} V_\ell(u).$$

Since $\mu(\mathbf{R}^n \backslash Q) = 0$, we only need to consider $x, y \in Q$. But for any $x, y \in Q$, we have $x - y \in (-1, 1)^n$. Hence it follows from (2.3) that

$$T_{K,\mu} = R_{K,\mu} + S_{K,\mu},$$

where the operators $R_{K,\mu}$ and $S_{K,\mu}$ are respectively defined by the formulas

$$(R_{K,\mu}f)(x) = \int U(x-y)f(y)d\mu(y) \quad \text{and} \quad (S_{K,\mu}f)(x) = \int V(x-y)f(y)d\mu(y),$$

$f \in L^2(d\mu)$. Since K is a homogeneous function of degree 0, we have $K(u)\eta(2^\ell|u|) = K(2^\ell u)\eta(2^\ell|u|)$. By (2.4), we have

$$(R_{K,\mu}f)(x) = \sum_{\ell=-\ell_0}^{0} \int \varphi(2^\ell(x-y))f(y)d\mu(y),$$

$f \in L^2(d\mu)$. Applying (2.5), we see that $R_{K,\mu}$ is in the trace class.

For each $\ell \geq 1$, consider the operator

$$(S_\ell f)(x) = \int V_\ell(x-y)f(y)d\mu(y), \quad f \in L^2(d\mu).$$

By (2.2), for each $\ell \in \mathbf{N}$, $\eta(2^\ell|x-y|) \neq 0$ only if $2^\ell(x-y) \in (-1,1)^n$, i.e., only if $x \in y + (-2^{-\ell}, 2^{-\ell})^n$. Hence if $y \in Q_w$, $w \in W_\ell$, then $\eta(2^\ell|x-y|) \neq 0$ only if $x \in \cup_{\lambda\in\Lambda}(Q_w + 2^{-\ell}\lambda)$. On the other hand, if $y \in Q_w$, $w \in W_\ell$, and $x \in \cup_{\lambda\in\Lambda}(Q_w + 2^{-\ell}\lambda)$, then $2^\ell(x-y) \in [-2,2]^n$ and, therefore, $K(2^\ell(x-y))\eta(2^\ell|x-y|) = \varphi(2^\ell(x-y))$. By this analysis and the homogeneity of K, for $x \neq y$ in Q we have

$$\begin{aligned}
V_\ell(x-y) &= K(x-y)\eta(2^\ell|x-y|) = K(2^\ell(x-y))\eta(2^\ell|x-y|) \\
&= \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \chi_{Q_w+2^{-\ell}\lambda}(x)K(2^\ell(x-y))\eta(2^\ell|x-y|)\chi_{Q_w}(y) \\
&= \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \chi_{Q_w+2^{-\ell}\lambda}(x)\varphi(2^\ell(x-y))\chi_{Q_w}(y) \\
&= \sum_{z\in\mathbf{Z}^n} c_z \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \exp(2^{-\ell_0-1}i\pi\langle 2^\ell(x-y), z\rangle)\chi_{Q_w+2^{-\ell}\lambda}(x)\chi_{Q_w}(y)
\end{aligned}$$

where the last $=$ follows from (2.5). Combining this with (2.6), we have

$$S_\ell = \sum_{z\in\mathbf{Z}^n} c_z \sum_{\lambda\in\Lambda} B_{\ell;z,\lambda},$$

$\ell \in \mathbf{N}$. Recalling (2.14), we obtain

$$S_{K,\mu} = \sum_{\ell=1}^{\infty} S_\ell = \sum_{z\in\mathbf{Z}^n} c_z \sum_{\lambda\in\Lambda}\sum_{\ell=1}^{\infty} B_{\ell;z,\lambda} = \sum_{\lambda\in\Lambda}\sum_{z\in\mathbf{Z}^n} c_z A_{z,\lambda}$$

as promised. $\square$

Proof of Theorem 1.1. Applying Lemmas 2.3, 2.2 and (2.5), we have

$$\|T_{K,\mu}\|_1^+ \le \|R_{K,\mu}\|_1^+ + \sum_{\lambda\in\Lambda}\sum_{z\in\mathbf{Z}^n}|c_z|\|A_{z,\lambda}\|_1^+ \le \|R_{K,\mu}\|_1^+ + 3^n\sum_{z\in\mathbf{Z}^n}|c_z|C_{2.2} < \infty.$$

$\square$

3. Computing Dixmier trace

Before we compute the Dixmier trace of integral operators, it is appropriate to first review the basics of Dixmier trace. First, let us cite [2,4] as general references. To define the Dixmier trace, one starts with a Banach limit ω on $\ell^\infty(\mathbf{N})$. But in addition to the properties that Banach limits possess in general, ω is required to have the following "doubling" property:

(D) For each $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, $\omega(\{a_k\}_{k\in\mathbf{N}}) = \omega(\{a_1, a_1, a_2, a_2, \dots, a_k, a_k, \dots\})$.
Such an ω can be easily constructed. One way to achieve this is to use the doubling operator $D : \ell^\infty(\mathbf{N}) \to \ell^\infty(\mathbf{N})$. That is, we define

$$D\{a_1, a_2, \dots, a_k, \dots\} = \{a_1, a_1, a_2, a_2, \dots, a_k, a_k, \dots\}$$

for $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$. Take any Banach limits L_1 and L_2, distinct or identical. Then an elementary exercise shows that the formula

$$\omega(a) = L_2\left(\left\{\frac{1}{k}\sum_{j=1}^{k} L_1(D^j a)\right\}_{k\in\mathbf{N}}\right),$$

$a \in \ell^\infty(\mathbf{N})$, defines a Banach limit that has the doubling property (D).

With such an ω, for any *positive* operator $A \in \mathcal{C}_1^+$, its Dixmier trace is defined to be

$$\mathrm{Tr}_\omega(A) = \omega\left(\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} s_j(A)\right\}_{k\in\mathbf{N}}\right).$$

The doubling property of ω ensures the additivity $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for positive operators $A, B \in \mathcal{C}_1^+$. Thus Tr_ω naturally extends to a linear functional on $\mathcal{C}_1^+$. This definition guarantees unitary invariance: $\mathrm{Tr}_\omega(U^*TU) = \mathrm{Tr}_\omega(T)$ for every $T \in \mathcal{C}_1^+$ and every unitary operator U. Since UT is unitarily equivalent to TU, we have $\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU)$. From this it follows that $\mathrm{Tr}_\omega(XT) = \mathrm{Tr}_\omega(TX)$ for every $T \in \mathcal{C}_1^+$ and every bounded operator X, which is what one expects of a trace.

One basic property of Dixmier trace is that if A is a trace-class operator, then $\mathrm{Tr}_\omega(A) = 0$. In addition to this, we also need the following vanishing principles:

Lemma 3.1. [8, Lemma 7.1] *Let $A \in \mathcal{C}_1^+$. If the kernel of A contains its range, then* $\mathrm{Tr}_\omega(A) = 0$.

Lemma 3.2. [8, Lemma 7.2] *Let $Y_1, \dots, Y_j, \dots$ be operators in $\mathcal{C}_1^+$ such that $\sum_{j=1}^{\infty} \|Y_j\|_1^+ < \infty$. Define $Y = \sum_{j=1}^{\infty} Y_j$. If $\mathrm{Tr}_\omega(Y_j) = 0$ for every $j \in \mathbf{N}$, then $\mathrm{Tr}_\omega(Y) = 0$.*

We also need the following facts:

Lemma 3.3. *Let A, B be bounded operators such that $A^*A \in \mathcal{C}_1^+$ and $B^*B \in \mathcal{C}_1^+$. Then we have $AB \in \mathcal{C}_1^+$, $BA \in \mathcal{C}_1^+$ and $\mathrm{Tr}_\omega(AB) = \mathrm{Tr}_\omega(BA)$ for every Dixmier trace Tr_ω.*

Proof. First of all, since A^*A and AA^* have the same s-numbers, we have $\|A^*A\|_1^+ = \|AA^*\|_1^+$ and $\mathrm{Tr}_\omega(A^*A) = \mathrm{Tr}_\omega(AA^*)$ for every Dixmier trace Tr_ω.

We have $\|AB\|_1^+ \le \{\|A^*A\|_1^+ \|B^*B\|_1^+\}^{1/2}$ and $\|BA\|_1^+ \le \{\|B^*B\|_1^+ \|A^*A\|_1^+\}^{1/2}$ by [8, Lemma 4.4]. From these inequalities one easily deduces that if $A^*A \in \mathcal{C}_1^+$ and $B^*B \in \mathcal{C}_1^+$, then the correspondences

$$t \mapsto (A + e^{it}B^*)(A^* + e^{-it}B) \quad \text{and} \quad t \mapsto (A^* + e^{-it}B)(A + e^{it}B^*)$$

are continuous maps from $[-\pi, \pi]$ into $\mathcal{C}_1^+$ with respect to the norm $\|\cdot\|_1^+$. We have

$$AB = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it}(A + e^{it}B^*)(A^* + e^{-it}B)dt \quad \text{whereas}$$
$$BA = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it}(A^* + e^{-it}B)(A + e^{it}B^*)dt.$$

Taking Tr_ω on both sides of these two identities, the above-mentioned $\|\cdot\|_1^+$-continuity allows us to move Tr_ω inside $\int_{-\pi}^{\pi}$. Since $\mathrm{Tr}_\omega((A + e^{it}B^*)(A^* + e^{-it}B)) = \mathrm{Tr}_\omega((A^* + e^{-it}B)(A + e^{it}B^*))$ for every $t \in [-\pi, \pi]$, the equality $\mathrm{Tr}_\omega(AB) = \mathrm{Tr}_\omega(BA)$ follows. □

Lemma 3.4. [8, Lemma 7.3] *Suppose that B is a set and that A is a subset of B. Let $h : A \to B$ be an injective map which has the property that $h(a) \neq a$ for every $a \in A$. Then there is a partition $A = E_1 \cup E_2 \cup E_3$ such that for every $i \in \{1, 2, 3\}$, we have $h(E_i) \cap E_i = \emptyset$.*

Our starting point of computation is Lemma 2.3. By that lemma and (2.5), for every Dixmier trace Tr_ω we have

$$\mathrm{Tr}_\omega(T_{K,\mu}) = \sum_{\lambda \in \Lambda} \sum_{z \in \mathbf{Z}^n} c_z \mathrm{Tr}_\omega(A_{z,\lambda}). \tag{3.1}$$

Thus our task is reduced to the computation of $\mathrm{Tr}_\omega(A_{z,\lambda})$ for each pair of $\lambda \in \Lambda$ and $z \in \mathbf{R}^n$. As we mentioned in the Introduction, at the moment, we can do this computation only for certain μ's.

For the rest of the section, μ will denote the measure defined by the formula

$$\mu(B) = m_n(B \cap E) \quad \text{for Borel sets } B \subset \mathbf{R}^n, \tag{3.2}$$

where E is the set mentioned in Theorem 1.7. As it was the case in Section 2, by the homogeneity of K and elementary scaling transformations, we only need to consider the case where $E \subset Q$. Thus we can and will use all the notations from Section 2, with the proviso that μ is now given by (3.2).

To compute $\mathrm{Tr}_\omega(A_{z,\lambda})$, we need to decompose $A_{z,\lambda}$. To do that, we pick an orthonormal set $\{u_w : w \in \mathcal{W}\}$. For each pair of $\lambda \in \Lambda$ and $z \in \mathbf{Z}^n$, we define the operator

$$G_{z,\lambda} = \sum_{w \in \mathcal{W}} \{\mu(w,\lambda)\}^{1/2}(f_w^z e(w,\lambda)) \otimes u_w. \tag{3.3}$$

We further define

$$H_z = \sum_{w \in \mathcal{W}} \{\mu(Q_w)\}^{1/2} u_w \otimes (f_w^z e_w) \tag{3.4}$$

for each $z \in \mathbf{Z}^n$. It follows from (2.6) and (2.14) that

$$A_{z,\lambda} = G_{z,\lambda} H_z$$

for all $\lambda \in \Lambda$ and $z \in \mathbf{Z}^n$. Obviously, $H_z^* H_z = A_{z,0}$. Thus by Lemma 2.2, we have $H_z^* H_z \in \mathcal{C}_1^+$. Recall from Section 2 that for $w \in W_\ell$ and $z \in \mathbf{Z}^n$, $f_w^z(x) = \exp(2^{\ell-\ell_0-1} i\pi\langle x, z\rangle)$. That is, f_w^z depends only on the length of w, not on the particular word w that has the particular length. Hence a review of the relevant definitions gives us the operator inequality $G_{z,\lambda} G_{z,\lambda}^* \le A_{z,0}$. Thus from Lemma 2.2 we also deduce $G_{z,\lambda}^* G_{z,\lambda} \in \mathcal{C}_1^+$ for $\lambda \in \Lambda$ and $z \in \mathbf{Z}^n$. Applying Lemma 3.3, for every Dixmier trace Tr_ω we have

$$\mathrm{Tr}_\omega(A_{z,\lambda}) = \mathrm{Tr}_\omega(G_{z,\lambda} H_z) = \mathrm{Tr}_\omega(H_z G_{z,\lambda}). \tag{3.5}$$

It is $\mathrm{Tr}_\omega(H_z G_{z,\lambda})$ that we will actually compute.

To compute $\mathrm{Tr}_\omega(H_z G_{z,\lambda})$, we need to decompose $G_{z,\lambda}$ and H_z. For each $\ell \in \mathbf{N}$, define

$$G_{\ell;z,\lambda} = \sum_{w \in W_\ell} \{\mu(w,\lambda)\}^{1/2}(f_w^z e(w,\lambda)) \otimes u_w \quad \text{and}$$
$$H_{\ell;z} = \sum_{w \in W_\ell} \{\mu(Q_w)\}^{1/2} u_w \otimes (f_w^z e_w),$$

$\lambda \in \Lambda$ and $z \in \mathbf{Z}^n$. By (3.3) and (3.4), we now have

$$H_z G_{z,\lambda} = Y_{z,\lambda}^{(0)} + \sum_{k=1}^{\infty} \left(Y_{z,\lambda}^{(k)} + Z_{z,\lambda}^{(k)} \right), \tag{3.6}$$

where

$$Y_{z,\lambda}^{(k)} = \sum_{\ell=1}^{\infty} H_{\ell+k;z} G_{\ell;z,\lambda} \quad \text{and} \quad Z_{z,\lambda}^{(k)} = \sum_{\ell=1}^{\infty} H_{\ell;z} G_{\ell+k;z,\lambda}$$

for $k \ge 0$. Our next lemma is a version of Lemma 2.1 improved for the μ given by (3.2). It shows a “decay” with respect to k, which makes (3.6) useful for our purpose.

Lemma 3.5. *For all $k \in \mathbf{Z}_+$, $\ell \in \mathbf{N}$, $z \in \mathbf{Z}^n$ and $\lambda \in \Lambda$ and for every bounded operator F, we have*

$$|\mathrm{tr}(H_{\ell+k;z}G_{\ell;z,\lambda}F)| \le 2^{-nk/2} \cdot 2^{-n\ell} \sum_{j=1}^{2^{n\ell}} s_j(F) \quad \text{and} \tag{3.7}$$

$$|\mathrm{tr}(H_{\ell;z}G_{\ell+k;z,\lambda}F)| \le 2^{-nk/2} \cdot 2^{-n\ell} \sum_{j=1}^{2^{n\ell}} s_j(F).$$

Proof. Since the proofs for these two inequalities are similar, we will only present the proof of (3.7). It is easy to see that for $k \in \mathbf{Z}_+$, $\ell \in \mathbf{N}$, $z \in \mathbf{Z}^n$ and $\lambda \in \Lambda$,

$$H_{\ell+k;z}G_{\ell;z,\lambda} = \sum_{v \in W_{\ell+k}} \sum_{w \in W_\ell} a_{k;\ell;z,\lambda}(v,w) u_v \otimes u_w, \tag{3.8}$$

where

$$a_{k;\ell;z,\lambda}(v,w) = \{\mu(Q_v)\}^{1/2}\{\mu(w,\lambda)\}^{1/2}\langle f_w^z e(w,\lambda), f_v^z e_v\rangle.$$

A review of the definitions in Section 2 tells us that $a_{k;\ell;z,\lambda}(v,w) \ne 0$ only if $Q_v \subset Q_w + 2^{-\ell}\lambda$. Thus we can rewrite (3.8) in the form

$$H_{\ell+k;z}G_{\ell;z,\lambda} = \sum_{w \in W_\ell} h_{k;\ell;z,\lambda}(w) \otimes u_w,$$

where

$$h_{k;\ell;z,\lambda}(w) = \sum_{v \in W_{\ell+k} \,:\, Q_v \subset Q_w + 2^{-\ell}\lambda} a_{k;\ell;z,\lambda}(v,w) u_v.$$

For $v \in W_{\ell+k}$ and $w \in W_\ell$, (3.2) gives us the bound

$$|a_{k;\ell;z,\lambda}(v,w)| \le \mu(Q_v) \le m_n(Q_v) = 2^{-n(\ell+k)}.$$

Moreover, for each $w \in W_\ell$, the cardinality of the set $\{v \in W_{\ell+k} : Q_v \subset Q_w + 2^{-\ell}\lambda\}$ is either 2^{nk} or 0. Since $\{u_v : v \in \mathcal{W}\}$ is an orthonormal set, this gives us

$$\|h_{k;\ell;z,\lambda}(w)\| \le 2^{-nk/2} \cdot 2^{-n\ell}. \tag{3.9}$$

For $w \ne w'$ in W_ℓ, if $v, v' \in W_{\ell+k}$, $Q_v \subset Q_w + 2^{-\ell}\lambda$ and $Q_{v'} \subset Q_{w'} + 2^{-\ell}\lambda$, then $v \ne v'$. Thus $h_{k;\ell;z,\lambda}(w) \perp h_{k;\ell;z,\lambda}(w')$ for $w \ne w'$ in W_ℓ.

By this orthogonality, there is an enumeration

$$w(1), w(2), \ldots, w(2^{n\ell})$$

of the elements in W_ℓ such that for any bounded operator F,

$$|\mathrm{tr}(H_{\ell+k;z}G_{\ell;z,\lambda}F)| \le \sum_{j=1}^{2^{n\ell}} \|h_{k;\ell;z,\lambda}(w(j))\| s_j(F).$$

Substituting (3.9) in this inequality, the proof is complete. $\square$

Lemma 3.6. *For all* $k \in \mathbf{Z}_+$, $z \in \mathbf{Z}^n$ *and* $\lambda \in \Lambda$, *we have*

$$\|Y^{(k)}_{z,\lambda}\|_1^+ \leq (1-2^{-n})^{-1}2^{-nk/2} \quad \textit{and} \quad \|Z^{(k)}_{z,\lambda}\|_1^+ \leq (1-2^{-n})^{-1}2^{-nk/2}.$$

Proof. This lemma is derived from Lemma 3.5 in the same way Lemma 2.2 is derived from Lemma 2.1. Indeed for any finite-rank operator F, Lemma 3.5 gives us

$$\begin{aligned}|\mathrm{tr}(Y^{(k)}_{z,\lambda}F)| &\leq \sum_{\ell=1}^{\infty}|\mathrm{tr}(H_{\ell+k;z}G_{\ell;z,\lambda}F)| \leq 2^{-nk/2}\sum_{\ell=1}^{\infty}2^{-n\ell}\sum_{j=1}^{2^{n\ell}}s_j(F)\\ &= 2^{-nk/2}\sum_{j=1}^{\infty}s_j(F)\sum_{2^{n\ell}\geq j}2^{-n\ell} \leq 2^{-nk/2}\sum_{j=1}^{\infty}\frac{s_j(F)}{j}\sum_{\nu=0}^{\infty}2^{-n\nu} = \frac{2^{-nk/2}}{1-2^{-n}}\|F\|_\infty^-.\end{aligned}$$

By (2.1), this proves the lemma for $Y^{(k)}_{z,\lambda}$. The case for $Z^{(k)}_{z,\lambda}$ is similar and will be omitted. $\square$

Lemma 3.7. *Let* Tr_ω *be any Dixmier trace.*
(a) *If* $k \geq 1$, *then* $\mathrm{Tr}_\omega(Y^{(k)}_{z,\lambda}) = 0$ *and* $\mathrm{Tr}_\omega(Z^{(k)}_{z,\lambda}) = 0$ *for all* $z \in \mathbf{Z}^n$ *and* $\lambda \in \Lambda$.
(b) *If* $\lambda \in \Lambda\backslash\{0\}$, *then* $\mathrm{Tr}_\omega(Y^{(0)}_{z,\lambda}) = 0$ *for every* $z \in \mathbf{Z}^n$.

Proof. (a) Let $k \geq 1$ be given. Then $h(\ell) = \ell + k$ is an injective map from $\mathbf{N}$ into itself that has the property that $h(\ell) \neq \ell$ for every $\ell \in \mathbf{N}$. By Lemma 3.4, there is a partition $\mathbf{N} = N_1 \cup N_2 \cup N_3$ such that

$$\{\ell + k : \ell \in N_i\} \cap N_i = \emptyset \tag{3.10}$$

for $i = 1, 2, 3$. Now define the orthogonal projections

$$P_i = \sum_{\ell \in N_i}\sum_{w \in W_\ell} u_w \otimes u_w,$$

$i = 1, 2, 3$. Then we have $Y^{(k)}_{z,\lambda} = Y^{(k)}_{z,\lambda}(P_1 + P_2 + P_3)$. From (3.8) we see that (3.10) implies $P_iY^{(k)}_{z,\lambda}P_i = 0$ for each $i \in \{1, 2, 3\}$. Hence

$$\mathrm{Tr}_\omega(Y^{(k)}_{z,\lambda}) = \sum_{i=1}^{3}\mathrm{Tr}_\omega(Y^{(k)}_{z,\lambda}P_i) = \sum_{i=1}^{3}\mathrm{Tr}_\omega(P_iY^{(k)}_{z,\lambda}P_i) = \sum_{i=1}^{3}\mathrm{Tr}_\omega(0) = 0.$$

Similarly, $Z^{(k)}_{z,\lambda} = (P_1 + P_2 + P_3)Z^{(k)}_{z,\lambda}$, and (3.10) implies $P_iZ^{(k)}_{z,\lambda}P_i = 0$ for $i = 1, 2, 3$. Therefore $\mathrm{Tr}_\omega(Z^{(k)}_{z,\lambda}) = 0$ as well.

(b) Let $\lambda \in \Lambda\backslash\{0\}$ be given. For each $\ell \in \mathbf{N}$, define $W_\ell^{(\lambda)} = \{w \in W_\ell : Q_w + 2^{-\ell}\lambda = Q_{w'}$ for some $w' \in W_\ell\}$. Furthermore, for each $w \in W_\ell^{(\lambda)}$ we define $\lambda(w) = w'$ for the

$w' \in W_\ell$ with the property that $Q_{w'} = Q_w + 2^{-\ell}\lambda$. Then another chase of the definitions reveals that

$$Y^{(0)}_{z,\lambda} = \sum_{\ell=1}^{\infty} \sum_{w \in W^{(\lambda)}_\ell} \mu(Q_{\lambda(w)}) u_{\lambda(w)} \otimes u_w. \tag{3.11}$$

Since $\lambda \neq 0$, for each $\ell \in \mathbf{N}$, the injective map $w \mapsto \lambda(w)$ from $W^{(\lambda)}_\ell$ into W_ℓ has the property that $\lambda(w) \neq w$ for every $w \in W^{(\lambda)}_\ell$. Thus Lemma 3.4 provides, for each $\ell \in \mathbf{N}$, a partition $W^{(\lambda)}_\ell = W^{(\lambda,1)}_\ell \cup W^{(\lambda,2)}_\ell \cup W^{(\lambda,3)}_\ell$ such that

$$\{\lambda(w) : w \in W^{(\lambda,i)}_\ell\} \cap W^{(\lambda,i)}_\ell = \emptyset \tag{3.12}$$

for $i = 1, 2, 3$. This time, we define the orthogonal projections

$$Q_i = \sum_{\ell=1}^{\infty} \sum_{w \in W^{(\lambda,i)}_\ell} u_w \otimes u_w,$$

$i = 1, 2, 3$. Then from (3.11) we see that $Y^{(0)}_{z,\lambda} = Y^{(0)}_{z,\lambda}(Q_1 + Q_2 + Q_3)$, and (3.12) translates to $Q_i Y^{(0)}_{z,\lambda} Q_i = 0$ for $i = 1, 2, 3$. By the reasoning in (a), these facts imply $\mathrm{Tr}_\omega(Y^{(0)}_{z,\lambda}) = 0$. This completes the proof. □

Finally, we consider the case where $k = 0$ and $\lambda = 0$:

Lemma 3.8. *Given a Dixmier trace* Tr_ω, *there is a non-negative number* $a = a(E, \omega)$ *such that* $\mathrm{Tr}_\omega(Y^{(0)}_{z,0}) = a$ *for every* $z \in \mathbf{Z}^n$.

Proof. Recall from Section 2 that for every $z \in \mathbf{Z}^n$, $f^z_w(x) = \exp(2^{\ell-\ell_0-1} i\pi\langle x, z\rangle)$ if $w \in W_\ell$. Thus $\langle f^z_w e_w, f^z_w e_w\rangle = \|e_w\|^2$ for all $w \in \mathcal{W}$ and $z \in \mathbf{Z}^n$. Consequently,

$$Y^{(0)}_{z,0} = \sum_{w \in \mathcal{W}} \mu(Q_w) u_w \otimes u_w.$$

That is, $Y^{(0)}_{z,0}$ is actually independent of $z \in \mathbf{Z}^n$. Hence the conclusion is trivial. □

Proof of Theorem 1.7. First of all, it follows from Lemma 3.6 that

$$\|Y^{(0)}_{z,\lambda}\|^+_1 + \sum_{k=1}^{\infty} \left(\|Y^{(k)}_{z,\lambda}\|^+_1 + \|Z^{(k)}_{z,\lambda}\|^+_1 \right) < \infty \tag{3.13}$$

for every pair of $z \in \mathbf{Z}^n$ and $\lambda \in \Lambda$. Let Tr_ω be any Dixmier trace. If $\lambda \neq 0$, Lemma 3.7 tells us that $\mathrm{Tr}_\omega(Y^{(k)}_{z,\lambda}) = 0$ for every $k \geq 0$ and $\mathrm{Tr}_\omega(Z^{(k)}_{z,\lambda}) = 0$ for every $k \geq 1$. Thus for $\lambda \in \Lambda\backslash\{0\}$, we can apply Lemma 3.2 to (3.6) to conclude that

$$\mathrm{Tr}_\omega(H_z G_{z,\lambda}) = \mathrm{Tr}_\omega(Y^{(0)}_{z,\lambda}) + \mathrm{Tr}_\omega\left(\sum_{k=1}^{\infty} \left(Y^{(k)}_{z,\lambda} + Z^{(k)}_{z,\lambda} \right) \right) = 0 \tag{3.14}$$

for every $z \in \mathbf{Z}^n$. Lemma 3.7 also tells us that if $k \geq 1$, then $\mathrm{Tr}_\omega(Y^{(k)}_{z,0}) = 0$ and $\mathrm{Tr}_\omega(Z^{(k)}_{z,0}) = 0$. Hence by (3.13) and Lemma 3.2, we have

$$\mathrm{Tr}_\omega\left(\sum_{k=1}^{\infty}\left(Y^{(k)}_{z,0} + Z^{(k)}_{z,0}\right)\right) = 0$$

for every $z \in \mathbf{Z}^n$. Combining this with (3.6) and Lemma 3.8, we obtain

$$\mathrm{Tr}_\omega(H_z G_{z,0}) = \mathrm{Tr}_\omega(Y^{(0)}_{z,0}) + \mathrm{Tr}_\omega\left(\sum_{k=1}^{\infty}\left(Y^{(k)}_{z,0} + Z^{(k)}_{z,0}\right)\right) = a + 0 = a \tag{3.15}$$

for every $z \in \mathbf{Z}^n$. According to (3.5), we can rewrite (3.14) and (3.15) as

$$\mathrm{Tr}_\omega(A_{z,\lambda}) = \begin{cases} 0 & \text{if} \quad \lambda \neq 0 \\ \\ a & \text{if} \quad \lambda = 0 \end{cases},$$

$z \in \mathbf{Z}^n$. Substituting this in (3.1), we find that

$$\mathrm{Tr}_\omega(T_{K,\mu}) = a \sum_{z \in \mathbf{Z}^n} c_z.$$

By (2.2) and (2.4), the C^∞ function φ vanishes in a neighborhood of 0. Thus from (2.5) we obtain

$$\sum_{z \in \mathbf{Z}^n} c_z = \varphi(0) = 0.$$

Therefore for the measure μ given by (3.2), we have $\mathrm{Tr}_\omega(T_{K,\mu}) = 0$ as promised. This completes the proof. □

We end this section with one of the implications of Theorem 1.7:

Corollary 3.9. *For the operator $T_{K,E}$ defined by* (1.8), *we have*

$$\mathrm{Tr}_\omega(T_{K,E} M_g) = 0$$

for every bounded measurable function g on E and every Dixmier trace Tr_ω.

Proof. Let Δ be any Borel subset of E. Then note that $M_{\chi_\Delta} T_{K,E} M_{\chi_\Delta} = T_{K,\Delta}$, where $T_{K,\Delta}$ is the operator on the subspace $L^2(\Delta, dm_n)$ given by the formula

$$(T_{K,\Delta} f)(x) = \int_\Delta K(x-y) f(y) dm_n(y).$$

Applying Theorem 1.7 to $T_{K,\Delta}$, we have

$$\mathrm{Tr}_\omega(T_{K,E} M_{\chi_\Delta}) = \mathrm{Tr}_\omega(M_{\chi_\Delta} T_{K,E} M_{\chi_\Delta}) = \mathrm{Tr}_\omega(T_{K,\Delta}) = 0.$$

Now it suffices to mention the elementary fact that if g is a bounded measurable function on E, then g can be approximated with respect to $\|\cdot\|_\infty$ by functions in the linear span of $\{\chi_\Delta : \Delta \text{ is any Borel subset of } E\}$. $\square$

4. Applications to examples

Recall that for the operators N and X defined by (1.5) and (1.6), we have

$$([N^*, X]f)(z) = \int_D \frac{\bar{z} - \bar{w}}{z - w} f(w)dA(w), \tag{4.1}$$

$f \in L^2(D, dA)$. As we have already explained, (4.1) is the motivating example for our study of $T_{K,\mu}$ in this paper. As a result, Theorems 1.1 and 1.7 give us

Corollary 4.1. *For the N and X defined by* (1.5) *and* (1.6), *we have $[N^*, X] \in \mathcal{C}_1^+$ and $\mathrm{Tr}_\omega[N^*, X] = 0$ for every Dixmier trace Tr_ω.*

More can be said about the pair N, X defined by (1.5) and (1.6). Note that for any bounded measurable function φ on D, we have $\varphi(N) = M_\varphi$, the operator of multiplication by the function φ on $L^2(D, dA)$.

Proposition 4.2. *Let X be operator defined by* (1.6) *and let $\varphi \in C^\infty(\mathbf{C})$. Then $[M_\varphi, X] \in \mathcal{C}_1^+$. Moreover, we have $\mathrm{Tr}_\omega([M_\varphi, X]) = 0$ for every Dixmier trace Tr_ω.*

Proof. If $\varphi \in C^\infty(\mathbf{C})$, then there is a $\psi \in C_c^\infty(\mathbf{C})$ such that $\varphi = \psi$ on D. Therefore we only need to consider $\varphi \in C_c^\infty(\mathbf{C})$. For such a φ, let $\hat{\varphi}$ be its Fourier transform. Let N be the operator defined by (1.5). Then it follows from the Fourier inversion formula that

$$M_\varphi = \varphi(N) = \frac{1}{2\pi} \int_{\mathbf{C}} \hat{\varphi}(w) e^{i\mathrm{Re}(\bar{w}N)} dA(w),$$

where $\mathrm{Re}(\bar{w}N) = (1/2)(\bar{w}N + wN^*)$. Consequently,

$$[M_\varphi, X] = \frac{1}{2\pi} \int_{\mathbf{C}} \hat{\varphi}(w)[e^{i\mathrm{Re}(\bar{w}N)}, X]dA(w). \tag{4.2}$$

We have

$$\begin{aligned}
[e^{i\mathrm{Re}(\bar{w}N)}, X] &= \int_0^1 \frac{d}{dt} e^{it\mathrm{Re}(\bar{w}N)} X e^{i(1-t)\mathrm{Re}(\bar{w}N)} dt \\
&= \frac{i}{2} \int_0^1 e^{it\mathrm{Re}(\bar{w}N)} [\bar{w}N + wN^*, X] e^{i(1-t)\mathrm{Re}(\bar{w}N)} dt.
\end{aligned} \tag{4.3}$$

Since $[N^*, X] \in \mathcal{C}_1^+$ and $\mathrm{rank}([N, X]) = 1$, the above gives us the bound $\|[e^{i\mathrm{Re}(\bar{w}N)}, X]\|_1^+ \le C|w|$ for $w \in \mathbf{C}$. Combining this bound with the rapid decay of $|\hat{\varphi}|$, from (4.2) we obtain the membership $[M_\varphi, X] \in \mathcal{C}_1^+$.

Furthermore, by (4.3), the correspondence $w \mapsto [e^{i\mathrm{Re}(\bar{w}N)}, X]$ is a map from $\mathbf{C}$ into $\mathcal{C}_1^+$ that is continuous with respect to the norm $\|\cdot\|_1^+$. Thus from (4.2) we also obtain

$$\mathrm{Tr}_\omega([M_\varphi, X]) = \frac{1}{2\pi}\int_{\mathbf{C}} \hat{\varphi}(w)\mathrm{Tr}_\omega([e^{i\mathrm{Re}(\bar{w}N)}, X])dA(w) \tag{4.4}$$

for every Dixmier trace Tr_ω. Applying Tr_ω on both sides of (4.3), we have

$$\begin{aligned}\mathrm{Tr}_\omega([e^{i\mathrm{Re}(\bar{w}N)}, X]) &= \frac{i}{2}\int_0^1 \mathrm{Tr}_\omega(e^{it\mathrm{Re}(\bar{w}N)}[\bar{w}N + wN^*, X]e^{i(1-t)\mathrm{Re}(\bar{w}N)})dt \\ &= (iw/2)\mathrm{Tr}_\omega([N^*, X]e^{i\mathrm{Re}(\bar{w}N)}).\end{aligned} \tag{4.5}$$

For each $w \in \mathbf{C}$, $e^{i\mathrm{Re}(\bar{w}N)}$ is the operator of multiplication by the function $e^{i\mathrm{Re}(\bar{w}z)}$ on $L^2(D, dA)$. Hence it follows from (4.5), (4.1) and Corollary 3.9 that $\mathrm{Tr}_\omega([e^{i\mathrm{Re}(\bar{w}N)}, X]) = 0$ for every $w \in \mathbf{B}$. Substituting this in (4.4), we obtain $\mathrm{Tr}_\omega([M_\varphi, X]) = 0$ as promised. $\square$

In light of Corollary 4.1, perhaps we should consider the following:

Question 4.3. Let N be a normal operator and let X be any operator. If $[N, X] \in \mathcal{C}_1^+$, does it follow that $[N^*, X] \in \mathcal{C}_1^+$? In other words, does the ideal $\mathcal{C}_1^+$ have the generalized Fuglede property?

The integral operator $T_{K,\mu}$ studied in Sections 1-3 also has connection to the problem of diagonalization of tuples of operators modulo ideals [3,12-14,16-18]. More precisely, it is related to the *obstruction* to such diagonalization. Recall the following result due to David and Voiculescu:

Example 4.4. [3] Let μ be a compactly-supported regular Borel measure on $\mathbf{R}^n$. Suppose that μ satisfies (1.3) and that $\mu(\mathbf{R}^n) \neq 0$. For each $j \in \{1, \dots, n\}$, define the self-adjoint operator

$$(N_j f)(x) = x_j f(x), \quad f \in L^2(d\mu).$$

Also define the singular integral operators

$$(X_j f)(x) = \int \frac{x_j - y_j}{|x-y|^2} f(y)d\mu(y), \quad f \in L^2(d\mu),$$

$j = 1, \dots, n$. Then $\sum_{j=1}^n [N_j, X_j]$ is the rank-one operator $1 \otimes 1$ on $L^2(d\mu)$. Theorem 1.2 provides the membership $X_j \in \mathcal{C}_{p/(p-1)}^+$, $1 \le j \le n$. This and the fact $\mathrm{tr}(\sum_{j=1}^n [N_j, X_j]) \neq 0$ together imply that the commuting tuple of self-adjoint operators $(N_1, \dots, N_n)$ cannot be diagonalized modulo the Lorentz ideal $\mathcal{C}_p^-$ [14]. $\square$

The relevance of Example 4.4 to Sections 1-3 is this: Note that each individual commutator $[N_j, X_j]$, $1 \le j \le n$, is an integral operator with a homogeneous kernel of degree 0, and so is any partial sum

$$S_J = \sum_{j \in J}[N_j, X_j],$$

$J \subset \{1, \ldots, n\}$. In other words, these are natural examples of the $T_{K,\mu}$ studied in Sections 1-3. Thus we learn quite a bit about these commutators from Theorems 1.1 and 1.7:

Corollary 4.5. *For the N_j and X_j in Example 4.4, we have $[N_j, X_j] \in \mathcal{C}_1^+$, $1 \le j \le n$. If μ is given by (3.2), then $\mathrm{Tr}_\omega([N_j, X_j]) = 0$ for every Dixmier trace Tr_ω, $1 \le j \le n$.*

5. More on the Fuglede commutation property

Since the Fuglede commutation property is one of the original motivations for this paper, we conclude the paper with some thoughts on this property itself.

Weiss's identity (1.4) was later generalized to an inequality for Schatten p-norms for $1 < p < \infty$. That is, for each $1 < p < \infty$, there is a constant C_p such that

$$\|[N^*, X]\|_p \le C_p \|[N, X]\|_p$$

whenever N is a normal operator. This was proved by Abdessemed and Davies for the case $2 < p < \infty$ in [1] and by Shulman for the case $1 < p < 2$ in [10]. Furthermore, Example 1.6 tells us that for $\|\cdot\|_1$, the norm of the trace class, such an inequality does not hold. But it is actually more interesting to take a look at the other end of the scale, namely the operator norm $\|\cdot\|$.

Question 5.1. Is there a finite constant C such that the operator-norm inequality

$$\|[N^*, X]\| \le C \|[N, X]\| \tag{5.1}$$

holds whenever N is a normal operator?

In a way, Fuglede's original theorem [5] is really about the operator norm, which makes (5.1) very relevant as a question. At the moment, we do not have an answer to Question 5.1, although we are inclined to think that its answer should be negative.

Inequality (5.1) is reminiscent of a "Lipschitz condition". By this analogy, a weaker version of (5.1) would be a "uniform continuity", i.e., an ϵ-δ condition, which does hold:

Theorem 5.2. *Let $\mathcal{N}$ be the collection of normal operators N satisfying the condition $\|N\| \le 1$ and let $\mathcal{X}$ be the collection of all operators X satisfying the condition $\|X\| \le 1$. For every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if $N \in \mathcal{N}$ and $X \in \mathcal{X}$ satisfy the condition $\|[N, X]\| \le \delta$, then $\|[N^*, X]\| \le \epsilon$.*

To prove Theorem 5.2, we first establish

Lemma 5.3. *Let $a > 0$ and $b > 0$ be any positive numbers. Suppose that there are $N \in \mathcal{N}$ and $X \in \mathcal{X}$ such that*

$$\|[N, X]\| \le a \quad \textit{and} \quad \|[N^*, X]\| \ge b. \tag{5.2}$$

Then there are finite-rank operators $M \in \mathcal{N}$ and $Y \in \mathcal{X}$ such that

$$\|[M, Y]\| \le 2a \quad \textit{and} \quad \|[M^*, Y]\| \ge b/2.$$

Proof. Denote $c = (1/6)\min\{a, b\}$. By the spectral decomposition of N, there is a normal operator $N' \in \mathcal{N}$ with a finite spectrum such that $\|N - N'\| \le c$. Thus from (5.2) we obtain

$$\|[N', X]\| \le 2a \quad \text{and} \quad \|[(N')^*, X]\| \ge 2b/3.$$

Since the spectrum of N' consists of a finite number of eigenvalues, there is a finite-rank orthogonal projection F such that $[F, N'] = 0$ and $\|F[(N')^*, X]F\| \ge (3/4)\|[(N')^*, X]\|$. Define $M = N'F$ and $Y = FXF$. By the condition $[F, N'] = 0$, M is a normal operator, and we have $[M^*, Y] = F[(N')^*, X]F$ and $[M, Y] = F[N', X]F$. Thus

$$\|[M^*, Y]\| = \|F[(N')^*, X]F\| \ge (3/4)\|[(N')^*, X]\| \ge (3/4) \cdot (2b/3) = b/2$$

and

$$\|[M, Y]\| = \|F[N', X]F\| \le \|[N', X]\| \le 2a.$$

This completes the proof. □

Proof of Theorem 5.2. Suppose that there were an $\epsilon_0 > 0$ for which no desired $\delta = \delta(\epsilon_0)$ exists. We will show that this leads to a contradiction. The non-existence of $\delta(\epsilon_0)$ in particular means that for each $k \in \mathbf{N}$, $1/k$ is not a $\delta(\epsilon_0)$. That is, for each $k \in \mathbf{N}$, there are $N_k \in \mathcal{N}$ and $X_k \in \mathcal{X}$ such that

$$\|[N_k, X_k]\| \le 1/k \quad \text{while} \quad \|[N_k^*, X_k]\| > \epsilon_0.$$

Applying Lemma 5.3, we obtain finite-rank $M_k \in \mathcal{N}$ and $Y_k \in \mathcal{X}$ such that

$$\|[M_k, Y_k]\| \le 2/k \quad \text{and} \quad \|[M_k^*, Y_k]\| \ge \epsilon_0/2$$

for each $k \in \mathbf{N}$. Define

$$M = \bigoplus_{k=1}^{\infty} M_k \quad \text{and} \quad Y = \bigoplus_{k=1}^{\infty} Y_k.$$

Obviously, M is a normal operator. By the memberships $M_k \in \mathcal{N}$ and $Y_k \in \mathcal{X}$, $k \in \mathbf{N}$, we have $\|M\| \le 1$ and $\|Y\| \le 1$. Furthermore,

$$[M, Y] = \bigoplus_{k=1}^{\infty} [M_k, Y_k].$$

Since $\mathrm{rank}([M_k, Y_k]) < \infty$ for every $k \in \mathbf{N}$ and since $\|[M_k, Y_k]\| \to 0$ as $k \to \infty$, the commutator $[M, Y]$ is compact. On the other hand, we have

$$[M^*, Y] = \bigoplus_{k=1}^{\infty} [M_k^*, Y_k].$$

Since $\|[M_k^*, Y_k]\| \ge \epsilon_0/2$ for every $k \in \mathbf{N}$, the commutator $[M^*, Y]$ is not compact.

Let $\hat{M}$ and $\hat{Y}$ respectively be the images of M and Y in the Calkin algebra. Then the compactness of $[M, Y]$ implies $[\hat{M}, \hat{Y}] = 0$, while the non-compactness of $[M^*, Y]$ implies $[\hat{M}^*, \hat{Y}] \neq 0$. Since $\hat{M}$ is a normal element, this contradicts Fuglede's theorem for the Calkin algebra. $\square$

Note added in proof. Since the acceptance of the paper, the author has learned that the answer to Question 5.1 is negative, which was a result due to B. Johnson and J. Williams, *Pacific J. Math.* **58** (1975), 105-122. The author has shown that the answer to Question 4.3 is also negative, and details will appear elsewhere.

References

1. A. Abdessemed and E. Davies, Some commutator estimates in the Schatten classes, J. London Math. Soc. (2) **39** (1989), 2, 299-308.
2. A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.
3. G. David and D. Voiculescu, s-numbers of singular integrals for the invariance of absolutely continuous spectra in fractional dimensions, J. Funct. Anal. **94** (1990), 14-26.
4. J. Dixmier, Existence de traces non normales, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1107-A1108.
5. B. Fuglede, A commutativity theorem for normal operators, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 35-40.
6. I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
7. J. Helton and R. Howe, Traces of commutators of integral operators, Acta Math. **135** (1975), 271-305.
8. L. Jiang, Y. Wang and J. Xia, Toeplitz operators associated with measures and the Dixmier trace on the Hardy space, Complex Anal. Oper. Theory **14** (2020), no. 2, Paper No. 30.
9. R. Moore and G. Weiss, The metric Fuglede property and normality, Canad. J. Math. **35** (1983), 516-525.
10. V. Shulman, Some remarks on the Fuglede-Weiss theorem, Bull. London Math. Soc. **28** (1996), 385-392.
11. V. Shulman and L. Turowska, Operator synthesis. II. Individual synthesis and linear operator equations, J. Reine Angew. Math. **590** (2006), 143-187.
12. D. Voiculescu, Some results on norm-ideal perturbations of Hilbert space operators, J. Operator Theory **2** (1979), 3-37.
13. D. Voiculescu, Some results on norm-ideal perturbations of Hilbert space operators. II, J. Operator Theory **5** (1981), 77-100.
14. D. Voiculescu, On the existence of quasicentral approximate units relative to normed ideals. I, J. Funct. Anal. **91** (1990), 1-36.
15. G. Weiss, The Fuglede commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators. II, J. Operator Theory **5** (1981), 3-16.
16. J. Xia, Diagonalization and unitary equivalence modulo Schatten p-classes, J. Funct. Anal. 175 (2000), 279-307.

17. J. Xia, Singular integral operators associated with measures of varying density, J. Operator Theory **49** (2003), 311-324.
18. J. Xia, Singular integral operators and norm ideals satisfying a quantitative variant of Kuroda's condition, J. Funct. Anal. **228** (2005), 369-393.

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