A HIERARCHY OF VON NEUMANN INEQUALITIES?

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Abstract. The well-known von Neumann inequality for commuting row contractions can be interpreted as saying that the tuple $(M_{z_1}, \ldots, M_{z_n})$ on the Drury-Arveson space H^2_n dominates every other commuting row contraction $(A_1, \ldots, A_n)$. We show that a similar domination relation exists among certain pairs of "lessor" row contractions $(B_1, \ldots, B_n)$ and $(A_1, \ldots, A_n)$. This hints at a possible hierarchical structure among the family of commuting row contractions.

1. Introduction

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H^2_n is the reproducing-kernel Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$\frac{1}{1 - \langle \zeta, z \rangle}$$

as its reproducing kernel [3,4,10]. Using the standard multi-index notation [17,page 3], one can alternately describe H^2_n as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle f, g \rangle = \sum_{\alpha \in \mathbf{Z}^n_+} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}$$

for

$$f(\zeta) = \sum_{\alpha \in \mathbf{Z}^n_+} c_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}^n_+} d_\alpha \zeta^\alpha.$$

An important role in operator theory is played by by the commuting tuple $(M_{z_1}, \ldots, M_{z_n})$ of multiplication on H^2_n by the coordinate functions $z_1, \ldots, z_n$.

Recall from [3,4] that a commuting tuple of bounded operators $(A_1, \ldots, A_n)$ on a Hilbert space H is said to be a *row contraction* if it satisfies the inequality

$$A_1 A_1^* + \cdots + A_n A_n^* \leq 1.$$

The tuple $(M_{z_1}, \ldots, M_{z_n})$ on H^2_n is, of course, an example of row contraction. In fact, it is *the* "master" row contraction in the sense that for each polynomial $p \in \mathbf{C}[z_1, \ldots, z_n]$, the von Neumann inequality

$$\|p(A_1, \ldots, A_n)\| \leq \|p(M_{z_1}, \ldots, M_{z_n})\| \tag{1.1}$$

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holds whenever the commuting tuple $(A_1, \ldots, A_n)$ is a row contraction [3,10]. In this sense, one might say that the tuple $(M_{z_1}, \ldots, M_{z_n})$ "dominates" *every* row contraction.

Because of their obvious importance in operator theory, the Drury-Arveson space H_n^2 and the von Neumann inequality (1.1) have been the subject of countless papers, of which we cite [1-14] as a sample. What we will do in this paper is to look at the kind of "domination" relation illustrated above at a more refined level. One might consider the following question. Suppose that we have two row contractions, $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$. It seems fair to say that $(B_1, \ldots, B_n)$ dominates $(A_1, \ldots, A_n)$ if the inequality

$$\|p(A_1, \ldots, A_n)\| \le \|p(B_1, \ldots, B_n)\|$$

holds for every polynomial $p \in \mathbf{C}[z_1, \ldots, z_n]$. Or, perhaps one can relax this condition slightly: if there is a constant $0 < C < \infty$ such that

$$\|p(A_1, \ldots, A_n)\| \le C\|p(B_1, \ldots, B_n)\|$$

for every polynomial $p \in \mathbf{C}[z_1, \ldots, z_n]$, one might still say that the tuple $(B_1, \ldots, B_n)$ dominates the tuple $(A_1, \ldots, A_n)$.

The main point is this: we are asking the rather restricted question whether a given tuple $(B_1, \ldots, B_n)$ dominates (whatever the word means) a particular $(A_1, \ldots, A_n)$, *not* the question whether it dominates a general class of $(A_1, \ldots, A_n)$'s. In other words, the tuple $(B_1, \ldots, B_n)$ may not be as dominating as the tuple $(M_{z_1}, \ldots, M_{z_n})$ on H_n^2, but does it dominate $(A_1, \ldots, A_n)$ nonetheless?

Obviously, this is an attempt to establish some sort of hierarchy, albeit partially, among commuting tuples of operators. Equally obviously, such a general task is a monumental undertaking, and perhaps requires the efforts of many researchers over many years. What we actually manage to do in this paper is quite limited: we will give some interesting examples of such a hierarchy.

The first hint of a possible hierarchical structure comes from the fact that the Drury-Arveson space H_n^2 is really "the head of a family" of reproducing-kernel Hilbert spaces. For each real number $-n \le t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\|\cdot\|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \ne \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)} \tag{1.2}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Obviously, $H_n^2 = \mathcal{H}^{(-n)}$. Also, $\mathcal{H}^{(-1)}$ is the Hardy space $H^2(S)$, and $\mathcal{H}^{(0)}$ is the Bergman space $L_a^2(\mathbf{B}, dv)$.

For each $-n \le t < \infty$, let $(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})$ denote the tuple of multiplication by the coordinate functions $z_1, \ldots, z_n$ on $\mathcal{H}^{(t)}$. Then an easy calculation using (1.2) shows that

$$M_{z_1}^{(t)} M_{z_1}^{(t)*} + \cdots + M_{z_n}^{(t)} M_{z_n}^{(t)*} = N(n+t+N)^{-1},$$

where N is the number operator introduced by Arveson [3]; i.e., $Nz^\alpha = |\alpha| z^\alpha$. This tells us that each tuple $(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})$ is a row contraction. Thus each $(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})$ is dominated by the tuple $(M_{z_1}^{(-n)}, \ldots, M_{z_n}^{(-n)}) = (M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}^{(-n)} = H_n^2$.

An obvious question at this point is, what about the "lessor" tuples $(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})$, $-n < t < \infty$. What do they dominate? In the rest of the paper, we will attempt to answer this question, and more.

2. Some General Results

We begin with some necessary notation. If $A = (A_1, \ldots, A_n)$ is a commuting tuple of operators and if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, we denote

$$A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n} \quad \text{and} \quad A^{*\alpha} = A_1^{*\alpha_1} \cdots A_n^{*\alpha_n},$$

which extends the standard multi-index convention [17,page 3]. Also, we have

$$M_{z^\alpha}^{(t)} = M_{z_1}^{(t)\alpha_1} \cdots M_{z_n}^{(t)\alpha_n} \quad \text{and} \quad M_{z^\alpha}^{(t)*} = M_{z_1}^{(t)*\alpha_1} \cdots M_{z_n}^{(t)*\alpha_n}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ and $-n \le t < \infty$. For each $-n \le t < \infty$, we define

$$u(\alpha; t) = \frac{1}{\alpha!} \prod_{j=1}^{|\alpha|} (n+t+j) \quad \text{for} \quad \alpha \in \mathbf{Z}_+^n \backslash \{0\}$$

and $u(0;t) = 1$. Since the case $t = -n$ is special, let us also write $u(\alpha)$ for $u(\alpha; -n)$, just as we write M_{z_j} for $M_{z_j}^{(-n)}$. In other words, we have $u(\alpha) = |\alpha|!/\alpha!$ for each $\alpha \in \mathbf{Z}_+^n$.

With $u(\alpha;t)$ defined as above, the standard orthonormal basis $\{e_\alpha^{(t)} : \alpha \in \mathbf{Z}_+^n\}$ for $\mathcal{H}^{(t)}$ can now be expressed by the formula

$$e_\alpha^{(t)}(z) = u^{1/2}(\alpha; t) z^\alpha.$$

Using this, it is straightforward to verify that for each pair of $\alpha, \beta \in \mathbf{Z}_+^n$, we have

$$M_{z^\alpha}^{(t)*} z^{\alpha+\beta} = \frac{u(\beta;t)}{u(\alpha+\beta;t)} z^\beta. \tag{2.1}$$

Moreover, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbf{Z}_+^n$, if there is a j such that $\alpha_j > \gamma_j$, then $M_{z^\alpha}^{(t)*} z^\gamma = 0$.

Recall from [10] that if $A = (A_1, \ldots, A_n)$ is a commuting tuple of operators on a Hilbert space H for which there is an $r \in (0,1)$ such that

$$\|A_1^*h\|^2 + \cdots + \|A_n^*h\|^2 \le r^2\|h\|^2$$

for every $h \in H$, then the operator identity

$$\sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha)A^\alpha(1 - A_1A_1^* - \cdots - A_nA_n^*)A^{*\alpha} = 1 \tag{2.2}$$

holds on H. Perhaps, the correct way to think of (2.2) is that it is a "resolution" of the identity operator 1. In [10], Drury showed that this resolution of the identity operator immediately leads to the von Neumann inequality (1.1).

Our starting point is to try to replace the coefficients $u(\alpha) = |\alpha|!/\alpha!$ in (2.2) by $u(\alpha; s)$. If $u(\alpha)$ is replaced by $u(\alpha; s)$ for some $-n < s < \infty$, then obviously the defect operator

$$D = 1 - A_1A_1^* - \cdots - A_nA_n^*$$

in (2.2) also needs to be replaced in order for the sum to converge. But what replaces D? This is obviously a wild card in the game. With these replacements, one may only obtain what we call a "quasi-resolution" of the identity operator. But, as we will now show, such a quasi-resolution suffices for certain purposes.

Theorem 2.1. *Let $-n \le s < \infty$, and let $A = (A_1, \ldots, A_n)$ be a commuting tuple of bounded operators on a Hilbert space H. Suppose that there is a positive self-adjoint operator W on H for which the sum*

$$\sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s)A^\alpha W A^{*\alpha}$$

converges in the weak operator topology to a bounded, positive, self-adjoint operator Y on H. Then the operator $Z : H \to \mathcal{H}^{(s)} \otimes H$ given by the formula

$$(Zh)(z) = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s)W^{1/2}A^{*\alpha}hz^\alpha, \quad h \in H, \tag{2.3}$$

is bounded and has the properties that $Z^*Z = Y$ and that

$$Zp(A_1^*, \ldots, A_n^*) = (p(M_{z_1}^{(s)*}, \ldots, M_{z_n}^{(s)*}) \otimes 1)Z \tag{2.4}$$

for every polynomial $p \in \mathbf{C}[z_1, \ldots, z_n]$. Moreover, if there are scalars $0 < c \le C < \infty$ such that the operator inequality $c \le Y \le C$ holds on H, then the operator Z has the property that $c^{1/2}\|h\| \le \|Zh\| \le C^{1/2}\|h\|$ for every $h \in H$.

Proof. First of all, the space $\mathcal{H}^{(s)} \otimes H$ is the collection of H-valued $\mathcal{H}^{(s)}$-functions. That is, $\mathcal{H}^{(s)} \otimes H$ consists of functions of the form

$$f(z) = \sum_{\alpha \in \mathbf{Z}_+^n} h_\alpha z^\alpha,$$

where $h_\alpha \in H$ for each $\alpha \in \mathbf{Z}_+^n$, with its norm given by the formula

$$\|f\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \|h_\alpha\|^2 \|z^\alpha\|_s^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\|h_\alpha\|^2}{u(\alpha; s)}. \tag{2.5}$$

For each $h \in H$, it follows from (2.5) that

$$\|Zh\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\|u(\alpha; s) W^{1/2} A^{*\alpha} h\|^2}{u(\alpha; s)} = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s) \langle A^\alpha W A^{*\alpha} h, h\rangle = \langle Yh, h\rangle.$$

Thus if Y is bounded, then Z is also bounded and has the property that $Z^*Z = Y$. For each $\alpha \in \mathbf{Z}_+^n$, we apply (2.1) to obtain

$$\begin{aligned}
(ZA^{*\alpha}h)(z) &= \sum_{\beta \in \mathbf{Z}_+^n} u(\beta; s) W^{1/2} A^{*\beta} A^{*\alpha} h z^\beta \\
&= \sum_{\beta \in \mathbf{Z}_+^n} \frac{u(\beta; s)}{u(\alpha + \beta; s)} u(\alpha + \beta; s) W^{1/2} A^{*\alpha+\beta} h z^\beta \\
&= (M_{z^\alpha}^{(s)*} \otimes 1) \sum_{\gamma \in \mathbf{Z}_+^n} u(\gamma; s) W^{1/2} A^{*\gamma} h z^\gamma \\
&= (M_{z^\alpha}^{(s)*} \otimes 1)(Zh)(z),
\end{aligned}$$

$h \in H$. This clearly implies (2.4). Lastly, because of the relation $Z^*Z = Y$, if Y satisfies the inequality $c \le Y \le C$ on H, then $c^{1/2}\|h\| \le \|Zh\| \le C^{1/2}\|h\|$ for every $h \in H$. □

Note that in Theorem 2.1, it is not necessary to assume the commuting tuple $A = (A_1, \dots, A_n)$ to be a row contraction. But we will need to assume $A = (A_1, \dots, A_n)$ to be a row contraction if we consider functional calculus beyond that for polynomials.

If f is an analytic function on $\mathbf{B}$, for each $0 \le r < 1$ we define the analytic function

$$f_r(z) = f(rz), \quad z \in \mathbf{B}.$$

Suppose that commuting tuple $A = (A_1, \dots, A_n)$ is a row contraction on a Hilbert space H. Then (1.1) implies that for each $\xi = (\xi_1, \dots, \xi_n)$ in the unit sphere S, we have

$$\|\bar{\xi}_1 A_1 + \cdots + \bar{\xi}_n A_n\| \le \|\bar{\xi}_1 M_{z_1} + \cdots + \bar{\xi}_n M_{z_n}\| = \|M_{\langle z, \xi\rangle}\| = |\xi| = 1. \tag{2.6}$$

This inequality allows us to define $f_r(A)$ for all $f \in H^\infty(S)$ and $0 \le r < 1$. Indeed for any given pair of $f \in H^\infty(S)$ and $0 \le r < 1$, by the Cauchy integral formula

$$f(z) = \int \frac{f(\xi)}{(1 - \langle z, \xi\rangle)^n} d\sigma(\xi),$$

where $d\sigma$ is the spherical measure on S, we have

$$f_r = \sum_{j=0}^{\infty} c_j r^j \psi_{f,j},$$

where

$$\psi_{f,j}(z) = \int f(\xi)\langle z, \xi\rangle^j d\sigma(\xi) \quad \text{and} \quad c_j = \frac{(j+n-1)!}{j!(n-1)!}.$$

It follows from (2.6) that $\|\psi_{f,j}(A)\| \le \|f\|_\infty$ for every $j \ge 0$. Since $0 \le r < 1$, the limit

$$f_r(A) = \lim_{J\to\infty} \sum_{j=0}^{J} c_j r^j \psi_{f,j}(A) \tag{2.7}$$

exists in the operator-norm topology.

Definition 2.2. For any commuting row contraction $A = (A_1, \ldots, A_n)$, $f \in H^\infty(S)$ and $0 \le r < 1$, the operator $f_r(A)$ will henceforth be defined by (2.7).

For each $-n \le t < \infty$, we denote the collection of multipliers for the space $\mathcal{H}^{(t)}$ by $\mathcal{M}^{(t)}$. The collection of multipliers for the Drury-Arveson space H^2_n will also be denoted by $\mathcal{M}$. That is, $\mathcal{M}^{(-n)} = \mathcal{M}$.

Lemma 2.3. *Let $-n \le t < \infty$ and $f \in \mathcal{M}^{(t)}$. Then for each $0 \le r < 1$, we have $f_r \in \mathcal{M}^{(t)}$ and $\|M^{(t)}_{f_r}\| \le \|M^{(t)}_f\|$.*

Proof. Let $\mathbf{T}^n$ denote the n-dimensional torus $\{(\tau_1, \ldots, \tau_n) : |\tau_j| = 1, 1 \le j \le n\}$. Let dm_n be the Lebesgue measure on $\mathbf{T}^n$ with the normalization $m_n(\mathbf{T}^n) = 1$. For each $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{T}^n$, define the unitary transformation U_τ on $\mathbf{C}^n$ by the formula

$$U_\tau(z_1, \ldots, z_n) = (\tau_1 z_1, \ldots, \tau_n z_n).$$

Let $f \in \mathcal{M}^{(t)}$. Then we obviously have $\|M^{(t)}_f\| = \|M^{(t)}_{f\circ U_\tau}\|$, $\tau \in \mathbf{T}^n$. For each $0 \le r < 1$, define the function

$$P_r(\tau_1, \ldots, \tau_n) = \prod_{j=1}^{n} \frac{1 - r^2}{|1 - r\bar{\tau}_j|^2}$$

on $\mathbf{T}^n$. By the well-known properties of the Poisson kernel, we have

$$M^{(t)}_{f_r} = \int M^{(t)}_{f\circ U_\tau} P_r(\tau) dm_n(\tau).$$

Since the integral of P_r on $\mathbf{T}^n$ equals 1 and $P_r \geq 0$, the lemma follows. $\square$

If $-n \leq t < \infty$ and $f \in \mathcal{M}^{(t)}$, then we obviously have

$$\langle fp, q\rangle_t = \lim_{r\uparrow 1}\langle f_r p, q\rangle_t$$

for all polynomials $p, q \in \mathbf{C}[z_1, \ldots, z_n]$. Combining this with the norm bound provided by Lemma 2.3, we have

Corollary 2.4. *For $-n \leq t < \infty$ and $f \in \mathcal{M}^{(t)}$, we have the weak convergence*

$$\lim_{r\uparrow 1} M^{(t)}_{f_r} = M^{(t)}_f$$

on $\mathcal{H}^{(t)}$.

Proposition 2.5. *Let $-n \leq s < \infty$, and let $A = (A_1, \ldots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive self-adjoint operator W on H such that the sum*

$$Y = \sum_{\alpha\in\mathbf{Z}^n_+} u(\alpha; s)A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \leq Y \leq C$ on H for some scalars $0 < c \leq C < \infty$. Then for each $f \in \mathcal{M}^{(s)}$, the limit

$$f(A) = \lim_{r\uparrow 1} f_r(A) \tag{2.8}$$

exists in the weak operator topology. Moreover, the identity

$$f(A)Z^* = Z^*(M^{(s)}_f \otimes 1) \tag{2.9}$$

holds for every $f \in \mathcal{M}^{(s)}$, where $Z : H \to \mathcal{H}^{(s)} \otimes H$ is the operator given by (2.3).

Proof. Let $f \in \mathcal{M}^{(s)}$. Then by Theorem 2.1 we have

$$\psi_{f,j}(A)Z^* = Z^*(M^{(s)}_{\psi_{f,j}} \otimes 1)$$

for each $j \geq 0$. Combining this with (2.7), we have

$$f_r(A)Z^* = Z^*(M^{(s)}_{f_r} \otimes 1) \tag{2.10}$$

for every $0 \leq r < 1$. Since $Z^*Z = Y$ and since we assume $c \leq Y \leq C$ on H for some $0 < c \leq C < \infty$, the range of Z^* contains $Z^*ZH = YH = H$. That is, the operator $Z^* : \mathcal{H}^{(s)} \otimes H \to H$ is surjective. Thus given any $h_1 \in H$, there is a $g_1 \in \mathcal{H}^{(s)} \otimes H$ such that $h_1 = Z^*g_1$. Thus if $h_2 \in H$, then

$$\langle f_r(A)h_1, h_2\rangle = \langle f_r(A)Z^*g_1, h_2\rangle = \langle Z^*(M^{(s)}_{f_r} \otimes 1)g_1, h_2\rangle = \langle (M^{(s)}_{f_r} \otimes 1)g_1, Zh_2\rangle.$$

This equality and Corollary 2.4 together tell us that the weak limit (2.8) exists. Once this is established, (2.9) follows from (2.10), (2.8) and another application of Corollary 2.4. □

Theorem 2.6. *Let* $-n \le s < \infty$, *and let* $A = (A_1, \dots, A_n)$ *be a commuting row contraction on a Hilbert space* H. *Suppose that there is a positive self-adjoint operator* W *on* H *such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s) A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y *satisfies the operator inequality* $c \le Y \le C$ *on* H *for some scalars* $0 < c \le C < \infty$. *Then the inequality*

$$\|f(A)\| \le (C/c)\|M_f^{(s)}\| \tag{2.11}$$

holds for every $f \in \mathcal{M}^{(s)}$.

Proof. Again, by Theorem 2.1 and the assumption on Y, we have $Z^*ZH = YH = H$. Thus for each $h \in H$, there is an $\tilde{h} \in H$ such that $Z^*Z\tilde{h} = h$. By (2.9), for each $f \in \mathcal{M}^{(s)}$ we have

$$\|f(A)h\| = \|f(A)Z^*Z\tilde{h}\| = \|Z^*(M_f^{(s)} \otimes 1)Z\tilde{h}\| \le \|Z^*\|\|M_f^{(s)}\|\|Z\|\|\tilde{h}\|.$$

Since $\|Z^*\|\|Z\| = \|Z\|^2 = \|Y\|$, we have

$$\|f(A)h\| \le C\|M_f^{(s)}\|\|\tilde{h}\|. \tag{2.12}$$

But $c\|\tilde{h}\|^2 \le \langle Y\tilde{h}, \tilde{h}\rangle = \langle h, \tilde{h}\rangle$. An application of the Cauchy-Schwarz inequality gives us $c\|\tilde{h}\| \le \|h\|$, i.e., $\|\tilde{h}\| \le (1/c)\|h\|$. Combining this with (2.12), (2.11) follows. □

Recall that the *essential norm* of a bounded operator B on a Hilbert space H is

$$\|B\|_{\mathcal{Q}} = \inf\{\|B + K\| : K \in \mathcal{K}(H)\},$$

where $\mathcal{K}(H)$ is the collection of compact operators on H. Alternately, $\|B\|_{\mathcal{Q}} = \|\pi(B)\|$, where π denotes the quotient homomorphism from $\mathcal{B}(H)$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(H)/\mathcal{K}(H)$. If H is a *separable* Hilbert space, then for each $B \in \mathcal{B}(H)$ there exists a sequence $\{x_k\}$ of unit vectors in H with the property that

$$\lim_{k\to\infty} \langle x_k, y\rangle = 0 \quad \text{for every} \quad y \in H$$

such that

$$\|B\|_{\mathcal{Q}} = \lim_{k\to\infty} \|Bx_k\|.$$

Theorem 2.7. *Let* $-n \le s < \infty$, *and let* $A = (A_1, \dots, A_n)$ *be a commuting row contraction on a separable Hilbert space* H. *Suppose that there is a positive, compact, self-adjoint operator* W *on* H *such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s) A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the operator Y has the following two properties:

(a) *There are scalars $0 < c \le C < \infty$ such that the operator inequality $c \le Y \le C$ holds on H;*

(b) $Y = 1 + K$, *where K is a compact operator on H.*

Then the inequality

$$\|f(A)\|_{\mathcal{Q}} \le \|M_f^{(s)}\|_{\mathcal{Q}}$$

holds for every $f \in \mathcal{M}^{(s)}$.

Proof. Let $f \in \mathcal{M}^{(s)}$. First of all, to prove the theorem, it suffices to prove that

$$\|f(A)^*\|_{\mathcal{Q}} \le \|M_f^{(s)*}\|_{\mathcal{Q}}. \tag{2.13}$$

To prove this, note that since H is assumed to be separable, there is a sequence of unit vectors $\{h_k\}$ in H that converges to 0 weakly such that

$$\|f(A)^*\|_{\mathcal{Q}} = \lim_{k\to\infty} \|f(A)^* h_k\|.$$

Obviously, the weak convergence $h_k \to 0$ implies that the sequence $\{f(A)^* h_k\}$ also converges to 0 weakly. Recall from Theorem 2.1 that $Z^*Z = Y$. Since we now assume $Y = 1 + K$, where K is compact, the weak convergence $f(A)^* h_k \to 0$ gives us

$$\begin{aligned}\lim_{k\to\infty} \|Zf(A)^* h_k\|^2 &= \lim_{k\to\infty} \langle Y f(A)^* h_k, f(A)^* h_k\rangle = \lim_{k\to\infty} \langle (1+K) f(A)^* h_k, f(A)^* h_k\rangle \\ &= \lim_{k\to\infty} \|f(A)^* h_k\|^2 = \|f(A)^*\|_{\mathcal{Q}}^2.\end{aligned}$$

Thus (2.13) will follow if we can prove that

$$\lim_{k\to\infty} \|Zf(A)^* h_k\| \le \|M_f^{(s)*}\|_{\mathcal{Q}}. \tag{2.14}$$

To prove this, we proceed as follows.

For each $\ell \in \mathbf{N}$, let $E_\ell^{(s)}$ denote the orthogonal projection from $\mathcal{H}^{(s)}$ onto the linear span of $\{z^\alpha : |\alpha| \le \ell\}$. By (2.3), for each $\ell \in \mathbf{N}$ we have

$$((E_\ell^{(s)} \otimes 1)Zh)(z) = \sum_{|\alpha|\le\ell} u(\alpha; s) W^{1/2} A^{*\alpha} h z^\alpha,$$

$h \in H$. Because the operator W is now assumed to be compact, each $W^{1/2}A^{*\alpha}$ is also compact. Thus the weak convergence $h_k \to 0$ gives us

$$\lim_{k\to\infty} \|(E_\ell^{(s)} \otimes 1)Zh_k\| = 0$$

for every $\ell \in \mathbf{N}$. This clearly implies that for each compact operator L on $\mathcal{H}^{(s)}$, we have

$$\lim_{k\to\infty} \|(L \otimes 1)Zh_k\| = 0.$$

Combining this with (2.9), we have

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| = \lim_{k\to\infty} \|(M_f^{(s)*} \otimes 1)Zh_k\| = \lim_{k\to\infty} \|(\{M_f^{(s)*} + L\} \otimes 1)Zh_k\|$$

whenever $L \in \mathcal{K}(\mathcal{H}^{(s)})$. Thus if $L \in \mathcal{K}(\mathcal{H}^{(s)})$, then

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| \le \|M_f^{(s)*} + L\| \limsup_{k\to\infty} \|Zh_k\|.$$

Since this holds for every compact operator L on $\mathcal{H}^{(s)}$, it follows that

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| \le \|M_f^{(s)*}\|_{\mathcal{Q}} \limsup_{k\to\infty} \|Zh_k\|. \tag{2.15}$$

Using the weak convergence $h_k \to 0$ and the compactness of K again, we have

$$\limsup_{k\to\infty} \|Zh_k\|^2 = \limsup_{k\to\infty} \langle Yh_k, h_k\rangle = \limsup_{k\to\infty} \langle (1+K)h_k, h_k\rangle = \limsup_{k\to\infty} \langle h_k, h_k\rangle = 1.$$

Combining this with (2.15), we obtain (2.14). This completes the proof. □

3. A Family of Examples

The purpose of this section is to give some non-trivial examples of pairs of H and $A = (A_1, \dots, A_n)$ to which the general results in Section 2 are applicable. In other words, we want to show that, in a non-trivial sense, the results in Section 2 are not vacuous.

For this purpose, let us introduce another family of Hilbert spaces of analytic functions on $\mathbf{B}$. First of all, for each real number $-n < t < \infty$, there is a natural number $m(t) \ge 4$ such that

$$\frac{\log(1 + (3/x))}{\log(2 + x)} \le \frac{n+t}{x} \quad \text{whenever} \quad x \ge m(t). \tag{3.1}$$

For each real value $-n < t < \infty$, define the inner product $\langle \cdot, \cdot \rangle_{[t]}$ according to the following rules: $\langle z^\alpha, z^\beta \rangle_{[t]} = 0$ for $\alpha \ne \beta$ in $\mathbf{Z}_+^n$,

$$\langle z^\alpha, z^\alpha \rangle_{[t]} = \frac{\alpha! \log(3 + |\alpha|)}{\prod_{j=1}^{|\alpha|}(n+t+j)} \quad \text{when} \quad |\alpha| \ge m(t),$$
$$\langle z^\alpha, z^\alpha \rangle_{[t]} = \frac{\alpha! \log(3 + m(t))}{\prod_{j=1}^{|\alpha|}(n+t+j)} \quad \text{when} \quad 0 < |\alpha| < m(t),$$

and $\langle 1, 1 \rangle_{[t]} = \log(3 + m(t))$. In other words, $\langle \cdot, \cdot \rangle_{[t]}$ is a modified version of the inner product defined by (1.2). Let $\mathcal{L}^{[t]}$ be the completion of $\mathbf{C}[z_1, \dots, z_n]$ with respect to the norm

$$\|f\|_{[t]} = \sqrt{\langle f, f \rangle_{[t]}}.$$

Furthermore, denote

$$\mu(\alpha;t) = \frac{1}{\langle z^\alpha, z^\alpha\rangle_{[t]}}$$

for $\alpha \in \mathbf{Z}_+^n$. Then $\mathcal{L}^{[t]}$ has an orthonormal basis $\{f_\alpha^{[t]} : \alpha \in \mathbf{Z}_+^n\}$, where

$$f_\alpha^{[t]}(z) = \mu^{1/2}(\alpha;t)z^\alpha.$$

Note that

$$\mu(\alpha;t) = \frac{u(\alpha;t)}{\log(3+|\alpha|)} \quad \text{if } \ |\alpha| \geq m(t) \quad \text{and}$$
$$\mu(\alpha;t) = \frac{u(\alpha;t)}{\log(3+m(t))} \quad \text{if } \ 0 \leq |\alpha| < m(t).$$

Keep in mind that the spaces $\mathcal{L}^{[t]}$ are only defined for the real values $-n < t < \infty$. For each such value t, let

$$M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]}$$

denote the operators of multiplication by the coordinate functions $z_1, \dots, z_n$ on $\mathcal{L}^{[t]}$. We will denote the number operator on $\mathcal{L}^{[t]}$ again by N. That is,

$$N f_\alpha^{[t]} = |\alpha| f_\alpha^{[t]}, \quad \alpha \in \mathbf{Z}_+^n.$$

Proposition 3.1. *For each $-n < t < \infty$, the commuting tuple $(M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]})$ on $\mathcal{L}^{[t]}$ is a row contraction.*

Proof. For each $i \in \{1, \dots, n\}$, let ϵ_i be the element in $\mathbf{Z}_+^n$ whose i-th component is 1 and whose other components are 0. Then easy calculations show that

$$M_{z_i}^{[t]} M_{z_i}^{[t]*} f_\alpha^{[t]} = \frac{\mu(\alpha - \epsilon_i;t)}{\mu(\alpha;t)} f_\alpha^{[t]} \quad \text{if the } i\text{-th component of } \alpha \text{ is not 0 and}$$
$$M_{z_i}^{[t]} M_{z_i}^{[t]*} f_\alpha^{[t]} = 0 \quad \text{if the } i\text{-th component of } \alpha \text{ is 0.}$$

Suppose that $\alpha = (\alpha_1, \dots \alpha_n)$. Then form the above we obtain

$$M_{z_i}^{[t]} M_{z_i}^{[t]*} f_\alpha^{[t]} = \frac{\alpha_i}{n+t+|\alpha|} \cdot \frac{\log(3+|\alpha|)}{\log(2+|\alpha|)} f_\alpha^{[t]} \quad \text{if } \ |\alpha| \geq m(t)+1 \ \text{ and}$$
$$M_{z_i}^{[t]} M_{z_i}^{[t]*} f_\alpha^{[t]} = \frac{\alpha_i}{n+t+|\alpha|} \cdot f_\alpha^{[t]} \quad \text{if } \ 0 \leq |\alpha| \leq m(t).$$

Hence

$$M_{z_1}^{[t]} M_{z_1}^{[t]*} + \dots + M_{z_n}^{[t]} M_{z_n}^{[t]*} = N(n+t+N)^{-1} G_t(N),$$

where G_t is the function on $[0, \infty)$ defined by the formula

$$G_t(x) = \begin{cases} \frac{\log(3+x)}{\log(2+x)} & \text{if} \quad x \geq m(t)+1 \\ \\ 1 & \text{if} \quad 0 \leq x < m(t)+1 \end{cases}.$$

If $x \geq m(t) + 1$, then

$$\begin{aligned} \frac{x}{n+t+x} G_t(x) &= \frac{1}{1 + \{(n+t)/x\}} \cdot \left(1 + \frac{\log\left(\frac{3+x}{2+x}\right)}{\log(2+x)} \right) \\ &\leq \frac{1}{1 + \{(n+t)/x\}} \cdot \left(1 + \frac{\log(1 + (3/x))}{\log(2+x)} \right) \leq 1, \end{aligned}$$

where the last $\leq$ follows from (3.1). Since we obviously have $xG_t(x)/(n+t+x) \leq 1$ for $0 \leq x < m(t)+1$, the lemma is proved. □

For $-n < t < \infty$ and $p \in \mathbf{C}[z_1, \ldots, z_n]$, we will write $M_p^{[t]}$ for the operator of multiplication by p on $\mathcal{L}^{[t]}$. Note that for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, we have

$$M_{z^\alpha}^{[t]} = M_{z_1}^{[t]\alpha_1} \cdots M_{z_n}^{[t]\alpha_n}.$$

The main result of this section is that if $-n \leq s < t < \infty$, then the tuple $(M_{z_1}^{[t]}, \ldots, M_{z_n}^{[t]})$ is an example of $A = (A_1, \ldots, A_n)$ to which Theorems 2.6 and 2.7 can be applied, if one considers the operator $W = (1+N)^{-n-s-1}$ on $\mathcal{L}^{[t]}$.

Proposition 3.2. *Suppose that $-n \leq s < t < \infty$. Then the sum*

$$Y_{s,t} = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s) M_{z^\alpha}^{[t]} (1+N)^{-n-s-1} M_{z^\alpha}^{[t]*}$$

converges in the weak operator topology. Moreover, the following two statements hold true:
(a) *There exist constants $0 < c \leq C < \infty$ such that the operator inequality $c \leq Y_{s,t} \leq C$ holds on $\mathcal{L}^{[t]}$.*
(b) *There is a scalar $y_{s,t} \in (0, \infty)$ such that $Y_{s,t} = y_{s,t} + K$, where K is a compact operator.*

The proof of Proposition 3.2 needs some preparation. First of all, we need a crude asymptotic formula for $r(r+1)\cdots(r+k)$, $r > 0$. This is derived in the same way as Stirling's formula for factorial. Indeed from the identity

$$\frac{1}{2}\{f(1) + f(0)\} = \int_0^1 f(x)dx - \frac{1}{2}\int_0^1 (x^2 - x) f''(x) dx$$

for C^2-functions we obtain

$$\sum_{j=0}^{k} \log(r+j) = \frac{1}{2}\{\log r + \log(r+k)\} + \int_0^k \log(r+x)dx + \frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \frac{x^2 - x}{(r+j+x)^2} dx,$$

$k \in \mathbf{N}$. Evaluating the integral $\int_0^k$ and then exponentiating both sides, we find that

$$\prod_{j=0}^{k}(r+j) = (r+k)^{r+k+(1/2)}e^{-k}e^{c(r;k)}, \tag{3.2}$$

where $c(r;k)$ has a finite limit (which depends on r) as $k \to \infty$.

In addition, the proof of Proposition 3.2 requires the following combinatorial lemma:

Lemma 3.3. *Let $\gamma \in \mathbf{Z}_+^n$. Then for each integer $0 \le k \le |\gamma|$ we have*

$$\sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=k}} \frac{\gamma!}{\alpha!\beta!} = \frac{|\gamma|!}{k!(|\gamma|-k)!}. \tag{3.3}$$

Proof. Let $\gamma = (\gamma_1, \dots, \gamma_n)$. Consider $\gamma_1 + \cdots + \gamma_n$ mutually distinguishable candies. Suppose that one divides these candies into n piles: the first pile has γ_1 candies, the second pile has γ_2 candies, ..., the n-th pile has γ_n candies. Then the left-hand side of (3.3) is exactly the number of ways of picking α_1 candies out of the first pile, α_2 candies out of the second pile, ..., α_n candies out the n-th pile, with the stipulation that $\alpha_1 + \cdots + \alpha_n = k$. This is obviously equal to the number of ways of simply picking k candies out of the entire collection of $\gamma_1 + \cdots + \gamma_n$, which is given by the right-hand side of (3.3). □

Lemma 3.4. *Given a pair of $-n \le s < t < \infty$, define*

$$a_{s,t}(\gamma) = \sum_{\alpha+\beta=\gamma} \frac{u(\alpha;s)\mu(\beta;t)}{\mu(\gamma;t)(1+|\beta|)^{n+s+1}} \tag{3.4}$$

for every $\gamma \in \mathbf{Z}_+^n$. Then there is a $y_{s,t} \in (0,\infty)$ such that

$$\lim_{|\gamma|\to\infty} a_{s,t}(\gamma) = y_{s,t}.$$

Proof. Define the function ρ_t on $[0,\infty)$ by the rules that $\rho_t(x) = x$ if $x \ge m(t)$ and that $\rho_t(x) = m(t)$ if $0 \le x < m(t)$. To prove the lemma, it suffices to consider γ with $|\gamma| > m(t)$. For such a γ, a chase of the definitions of u and μ gives us

$$a_{s,t}(\gamma) = \hat{a}_{s,t}(\gamma) + b_{s,t}(\gamma),$$

where

$$\hat{a}_{s,t}(\gamma) = \sum_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0, \alpha\neq\gamma}} \frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\beta|))} \cdot \frac{\gamma!}{\alpha!\beta!} \cdot \frac{\prod_{j=1}^{|\alpha|}(n+s+j)\prod_{j=1}^{|\beta|}(n+t+j)}{(1+|\beta|)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}$$

and

$$b_{s,t}(\gamma) = \frac{1}{(1+|\gamma|)^{n+s+1}} + \frac{\log(3+|\gamma|)}{\log(3+m(t))} \cdot \frac{\prod_{j=1}^{|\gamma|}(n+s+j)}{\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

In other words, $b_{s,t}(\gamma)$ is the sum of the terms $\alpha = 0$ and $\alpha = \gamma$ in $\sum_{\alpha+\beta=\gamma}$. Applying (3.2), we have

$$\begin{aligned} b_{s,t}(\gamma) &\leq \frac{1}{1+|\gamma|} + \frac{\log(3+|\gamma|)}{\log(3+m(t))} \cdot C\frac{(n+s+|\gamma|)^{n+s+|\gamma|+(1/2)}}{(n+t+|\gamma|)^{n+t+|\gamma|+(1/2)}} \\ &\leq \frac{1}{1+|\gamma|} + \frac{\log(3+|\gamma|)}{\log(3+m(t))} \cdot \frac{C}{(n+t+|\gamma|)^{t-s}}. \end{aligned}$$

Since $t-s>0$, we have $b_{s,t}(\gamma) \to 0$ as $|\gamma| \to \infty$. Thus what remains to be shown is that

$$\lim_{|\gamma|\to\infty} \hat{a}_{s,t}(\gamma) = y_{s,t} \tag{3.5}$$

for some $y_{s,t} \in (0,\infty)$.

To prove (3.5), note that an application of Lemma 3.3 gives us

$$\hat{a}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} \frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))} \cdot \frac{|\gamma|!}{k!(|\gamma|-k)!} \cdot \frac{\prod_{j=1}^{k}(n+s+j)\prod_{j=1}^{|\gamma|-k}(n+t+j)}{(1+|\gamma|-k)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

Applying the asymptotic expansion (3.2), we have

$$\begin{aligned} \hat{a}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))} \cdot \frac{|\gamma|^{|\gamma|+(1/2)}}{k^{k+(1/2)}(|\gamma|-k)^{|\gamma|-k+(1/2)}} \times \\ \frac{(n+s+k)^{n+s+k+(1/2)}(n+t+|\gamma|-k)^{n+t+|\gamma|-k+(1/2)}}{(1+|\gamma|-k)^{n+s+1}(n+t+|\gamma|)^{n+t+|\gamma|+(1/2)}}, \end{aligned}$$

where

$$E(|\gamma|,k) = \frac{e^{c(1,|\gamma|-1)+1}}{e^{c(1,k-1)+1}e^{c(1;|\gamma|-k-1)+1}} \cdot \frac{e^{c(n+s+1;k-1)+1}e^{c(n+t+1;|\gamma|-k-1)+1}}{e^{c(n+t+1;|\gamma|-1)+1}}. \tag{3.6}$$

We can further rewrite $\hat{a}_{s,t}(\gamma)$ as

$$\hat{a}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)F(|\gamma|,k)\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))} \cdot \frac{k^{n+s}}{|\gamma|^{n+t}(|\gamma|-k)^{1+s-t}}, \tag{3.7}$$

where

$$F(|\gamma|,k) = \frac{\left(\frac{n+s+k}{k}\right)^{n+s+k+(1/2)}\left(\frac{n+t+|\gamma|-k}{|\gamma|-k}\right)^{n+t+|\gamma|-k+(1/2)}}{\left(\frac{1+|\gamma|-k}{|\gamma|-k}\right)^{n+s+1}\left(\frac{n+t+|\gamma|}{|\gamma|}\right)^{n+t+|\gamma|+(1/2)}}. \tag{3.8}$$

A rearrangement of the powers in (3.7) then leads to
(3.9)

$$\hat{a}_{s,t}(\gamma) = \frac{1}{|\gamma|} \sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)F(|\gamma|,k) \frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))} \cdot \left(\frac{k}{|\gamma|}\right)^{n+s} \cdot \left(\frac{|\gamma|}{|\gamma|-k}\right)^{1+s-t},$$

which obviously suggests that we should treat it as some sort of "Riemann sum".

Next we define

$$G(m,k) = \frac{\log(3+m)}{\log(3+\rho_t(m-k))}$$

for natural numbers $1 \le k < m$. Then

$$G(m,k) = \frac{\log\left(\frac{3+m}{3+\rho_t(m-k)}\right) + \log(3+\rho_t(m-k))}{\log(3+\rho_t(m-k))} = 1 + \frac{\log\left(\frac{3+m}{3+\rho_t(m-k)}\right)}{\log(3+\rho_t(m-k))}. \tag{3.10}$$

Recall that if $j \ge m(t)$, then $\rho_t(j) = j$. Therefore for each pair of $0 < \eta < 1/8$ and $\epsilon > 0$, there exist a positive number $M(\eta,\epsilon)$ such that

$$|G(m,k) - 1| \le \epsilon \quad \text{if} \quad m \ge M(\eta,\epsilon) \quad \text{and} \quad 1 \le k \le (1-\eta)m. \tag{3.11}$$

Moreover, since $\rho_t(j) \ge j$ for all $j \in \mathbf{Z}_+$, from (3.10) we obtain

$$G(m,k) \le 1 + \log\left(\frac{3+m}{3+m-k}\right) \le 1 + \log\left(3 + \frac{m}{m-k}\right) \tag{3.12}$$

for all natural numbers $1 \le k < m$.

By (3.6) and (3.8), there exists a $w_{s,t} \in (0,\infty)$ such that the following statement holds true: For each pair of $0 < \eta < 1/8$ and $\epsilon > 0$, there exists a positive number $M_1(\eta,\epsilon)$ such that

$$|E(m,k)F(m,k) - w_{s,t}| \le \epsilon \quad \text{if} \quad m \ge M_1(\eta,\epsilon) \quad \text{and} \quad \eta m \le k \le (1-\eta)m. \tag{3.13}$$

On the other hand, it is obvious that there is a constant C_1 such that

$$E(m,k)F(m,k) \le C_1 \quad \text{for all} \quad 1 \le k < m. \tag{3.14}$$

Now let an $\eta \in (0,1/8)$ be given. By (3.9), for $\gamma \in \mathbf{Z}_+^n$ such that $|\gamma| > \min\{m(t), 1/\eta\}$, we can write

$$\hat{a}_{s,t}(\gamma) = \hat{a}_{s,t,\eta}(\gamma) + \hat{a}^{(0)}_{s,t,\eta}(\gamma) + \hat{a}^{(1)}_{s,t,\eta}(\gamma), \tag{3.15}$$

where

$$\hat{a}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{\eta|\gamma| \le k \le (1-\eta)|\gamma|} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k) \left(\frac{k}{|\gamma|}\right)^{n+s} \left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t},$$

$$\hat{a}^{(0)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{1 \le k < \eta|\gamma|} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k) \left(\frac{k}{|\gamma|}\right)^{n+s} \left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t},$$

$$\hat{a}^{(1)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{(1-\eta)|\gamma| < k \le |\gamma|-1} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k) \left(\frac{k}{|\gamma|}\right)^{n+s} \left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t}.$$

By (3.11) and (3.13), it is clear that

$$\lim_{|\gamma|\to\infty} \hat{a}_{s,t,\eta}(\gamma) = w_{s,t}\int_\eta^{1-\eta} \frac{x^{n+s}}{(1-x)^{1+s-t}}dx. \tag{3.16}$$

By (3.14) and (3.12), we have

$$\begin{aligned} \hat{a}^{(1)}_{s,t,\eta}(\gamma) &\le \frac{C_1}{|\gamma|}\sum_{(1-\eta)|\gamma|<k\le|\gamma|-1}\left\{1+\log\left(3+\frac{|\gamma|}{|\gamma|-k}\right)\right\}\left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t} \\ &\le C_1\int_{1-\eta}^1\left\{1+\log\left(3+\frac{1}{1-x}\right)\right\}\frac{1}{(1-x)^{1+s-t}}dx. \end{aligned} \tag{3.17}$$

Similarly,

$$\hat{a}^{(0)}_{s,t,\eta}(\gamma) \le C_1\int_0^{2\eta}\left\{1+\log\left(3+\frac{1}{1-x}\right)\right\}\frac{1}{(1-x)^{1+s-t}}dx. \tag{3.18}$$

Because of the condition $t > s$, we have

$$\int_0^1\left\{1+\log\left(3+\frac{1}{1-x}\right)\right\}\frac{1}{(1-x)^{1+s-t}}dx < \infty.$$

Thus the combination of (3.15), (3.16), (3.17) and (3.18) gives us

$$\lim_{|\gamma|\to\infty} \hat{a}_{s,t}(\gamma) = w_{s,t}\int_0^1 \frac{x^{n+s}}{(1-x)^{1+s-t}}dx.$$

This proves (3.5) and completes the proof of the lemma. □

Corollary 3.5. *For any given* $-n \le s < t < \infty$, *there exist* $0 < c \le C < \infty$ *such that*

$$c \le a_{s,t}(\gamma) \le C$$

for every $\gamma \in \mathbf{Z}^n_+$.

Proof. The upper bound follows immediately from Lemma 3.4. The lower bound follows from Lemma 3.4 and the obvious fact that $a_{s,t}(\gamma) > 0$ for every $\gamma \in \mathbf{Z}^n_+$. □

Proof of Proposition 3.2. Obviously, on the space $\mathcal{L}^{[t]}$ we have

$$(1+N)^{-n-s-1} = \sum_{\beta\in\mathbf{Z}^n_+}(1+|\beta|)^{-n-s-1}f^{[t]}_\beta \otimes f^{[t]}_\beta.$$

Therefore for each $\alpha \in \mathbf{Z}^n_+$,

$$\begin{aligned} M^{[t]}_{z^\alpha}(1+N)^{-n-s-1}M^{[t]*}_{z^\alpha} &= \sum_{\beta\in\mathbf{Z}^n_+}(1+|\beta|)^{-n-s-1}(M^{[t]}_{z^\alpha}f^{[t]}_\beta)\otimes(M^{[t]}_{z^\alpha}f^{[t]}_\beta) \\ &= \sum_{\beta\in\mathbf{Z}^n_+}\frac{\mu(\beta;t)}{\mu(\alpha+\beta;t)(1+|\beta|)^{n+s+1}}f^{[t]}_{\alpha+\beta}\otimes f^{[t]}_{\alpha+\beta}. \end{aligned}$$

Consequently

$$Y_{s,t} = \sum_{\alpha \in \mathbf{Z}_+^n} u(\alpha; s) M_{z^\alpha}^{[t]} (1+N)^{-n-s-1} M_{z^\alpha}^{[t]*} = \sum_{\gamma \in \mathbf{Z}_+^n} a_{s,t}(\gamma) f_\gamma^{[t]} \otimes f_\gamma^{[t]},$$

where $a_{s,t}(\gamma)$ is given by (3.4). Since $\{f_\gamma^{[t]} : \gamma \in \mathbf{Z}_+^n\}$ is an orthonormal basis for $\mathcal{L}^{[t]}$, statement (b) follows from Lemma 3.4 and statement (a) follows from Corollary 3.5. □

If f is a multiplier for the space $\mathcal{L}^{[t]}$, where $-n < t < \infty$, we will write $M_f^{[t]}$ for the operator of multiplication by f on $\mathcal{L}^{[t]}$.

Theorem 3.6. *Suppose that $-n \le s < t < \infty$. Then the following hold true*:
(1) *If f is a multiplier of $\mathcal{H}^{(s)}$, then f is also a multiplier of $\mathcal{L}^{[t]}$.*
(2) *There is a $0 < C_{3.6} < \infty$ such that $\|M_f^{[t]}\| \le C_{3.6} \|M_f^{(s)}\|$ for every multiplier f of $\mathcal{H}^{(s)}$.*
(3) *If f is a multiplier of $\mathcal{H}^{(s)}$, then $\|M_f^{[t]}\|_{\mathcal{Q}} \le \|M_f^{(s)}\|_{\mathcal{Q}}$.*

Proof. Obviously, (1) follows from Propositions 2.5 and 3.2(a), while (2) follows from Theorem 2.6 and Proposition 3.2(a). Note that the operator $(1+N)^{-n-s-1}$ is compact on $\mathcal{L}^{[t]}$. Thus (3) follows from Theorem 2.7 and Proposition 3.2(b). □

In very clear terms, Theorem 3.6 tells us that if $-n \le s < t < \infty$, then the row contraction $(M_{z_1}^{(s)}, \dots, M_{z_n}^{(s)})$ on $\mathcal{H}^{(s)}$ dominates the row contraction $(M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]})$ on $\mathcal{L}^{[t]}$. Next we will show that the roles of these two families can be reversed, so long as we keep the condition $s < t$. More precisely, in analogy to Theorem 3.6, we have

Theorem 3.7. *Suppose that $-n < s < t < \infty$. Then the following hold true*:
(1) *If f is a multiplier of $\mathcal{L}^{[s]}$, then f is also a multiplier of $\mathcal{H}^{(t)}$.*
(2) *There is a $0 < C_{3.7} < \infty$ such that $\|M_f^{(t)}\| \le C_{3.7} \|M_f^{[s]}\|$ for every multiplier f of $\mathcal{L}^{[s]}$.*
(3) *If f is a multiplier of $\mathcal{L}^{[s]}$, then $\|M_f^{(t)}\|_{\mathcal{Q}} \le \|M_f^{[s]}\|_{\mathcal{Q}}$.*

The proof of Theorem 3.7 will be given in Section 5, after we establish more general results in Section 4.

4. More General Results

Note that the general results in Section 2 tell us under what condition the tuple $(M_{z_1}^{(s)}, \dots, M_{z_n}^{(s)})$ on $\mathcal{H}^{(s)}$ dominates another commuting row contraction $(A_1, \dots, A_n)$. In this section we establish the analogous results for the tuple $(M_{z_1}^{[s]}, \dots, M_{z_n}^{[s]})$ on $\mathcal{L}^{[s]}$, $-n < s < \infty$. Then in Section 5 we give non-trivial applications of the results in this section, just as Section 3 gives non-trivial applications of the results in Section 2.

The reader will notice that the proofs in this section are very similar to the corresponding ones in Section 2. Although an omission of all the proofs in this section can be justified, we decide to retain most of them here. But the reader may choose to skip the proofs in this section, at least for a first reading.

Theorem 4.1. *Let $-n < s < \infty$, and let $A = (A_1, \ldots, A_n)$ be a commuting tuple of bounded operators on a Hilbert space H. Suppose that there is a positive self-adjoint operator W on H for which the sum*

$$\sum_{\alpha \in \mathbf{Z}_+^n} \mu(\alpha; s) A^\alpha W A^{*\alpha}$$

converges in the weak operator topology to a bounded, positive, self-adjoint operator Y on H. Then the operator $Z : H \to \mathcal{L}^{[s]} \otimes H$ given by the formula

$$(Zh)(z) = \sum_{\alpha \in \mathbf{Z}_+^n} \mu(\alpha; s) W^{1/2} A^{*\alpha} h z^\alpha, \quad h \in H, \tag{4.1}$$

is bounded and has the properties that $Z^* Z = Y$ and that

$$Zp(A_1^*, \ldots, A_n^*) = (p(M_{z_1}^{[s]*}, \ldots, M_{z_n}^{[s]*}) \otimes 1) Z \tag{4.2}$$

for every polynomial $p \in \mathbf{C}[z_1, \ldots, z_n]$. Moreover, if there are scalars $0 < c \leq C < \infty$ such that the operator inequality $c \leq Y \leq C$ holds on H, then the operator Z has the property that $c^{1/2} \|h\| \leq \|Zh\| \leq C^{1/2} \|h\|$ for every $h \in H$

Proof. Note that the space $\mathcal{L}^{[s]} \otimes H$ is the collection of functions of the form

$$f(z) = \sum_{\alpha \in \mathbf{Z}_+^n} h_\alpha z^\alpha,$$

where $h_\alpha \in H$ for each $\alpha \in \mathbf{Z}_+^n$, with its norm given by the formula

$$\|f\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \|h_\alpha\|^2 \|z^\alpha\|_{[s]}^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\|h_\alpha\|^2}{\mu(\alpha; s)}.$$

It follows that

$$\|Zh\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\|\mu(\alpha; s) W^{1/2} A^{*\alpha} h\|^2}{\mu(\alpha; s)} = \sum_{\alpha \in \mathbf{Z}_+^n} \mu(\alpha; s) \langle A^\alpha W A^{*\alpha} h, h \rangle = \langle Yh, h \rangle.$$

Thus for bounded Y, Z is also bounded and has the property that $Z^* Z = Y$. We have

$$M_{z^\alpha}^{[s]*} z^{\alpha+\beta} = \frac{\mu(\beta; s)}{\mu(\alpha + \beta; s)} z^\beta$$

for each $\alpha \in \mathbf{Z}_+^n$. Thus

$$\begin{aligned}
(ZA^{*\alpha} h)(z) &= \sum_{\beta \in \mathbf{Z}_+^n} \mu(\beta; s) W^{1/2} A^{*\beta} A^{*\alpha} h z^\beta \\
&= \sum_{\beta \in \mathbf{Z}_+^n} \frac{\mu(\beta; s)}{\mu(\alpha + \beta; s)} \mu(\alpha + \beta; s) W^{1/2} A^{*\alpha+\beta} h z^\beta \\
&= (M_{z^\alpha}^{[s]*} \otimes 1) \sum_{\gamma \in \mathbf{Z}_+^n} \mu(\gamma; s) W^{1/2} A^{*\gamma} h z^\gamma \\
&= (M_{z^\alpha}^{[s]*} \otimes 1)(Zh)(z),
\end{aligned}$$

$h \in H$. This clearly implies (4.2). Lastly, because of the relation $Z^*Z = Y$, if Y satisfies the inequality $c \le Y \le C$ on H, then $c^{1/2}\|h\| \le \|Zh\| \le C^{1/2}\|h\|$ for every $h \in H$. □

Just as in Theorem 2.1, the commuting tuple $A = (A_1, \dots, A_n)$ in Theorem 4.1 was not assumed to be a row contraction.

For each $-n < t < \infty$, let $\mathcal{M}^{[t]}$ denote the collection of multipliers for the space $\mathcal{L}^{[t]}$. Since $\mathcal{L}^{[t]}$ is a reproducing-kernel Hilbert space, it follows that $\mathcal{M}^{[t]} \subset H^\infty(S)$.

Lemma 4.2. *Let $-n < t < \infty$ and $f \in \mathcal{M}^{[t]}$. Then for each $0 \le r < 1$, we have $f_r \in \mathcal{M}^{[t]}$ and $\|M^{[t]}_{f_r}\| \le \|M^{[t]}_f\|$.*

The proof of this lemma is the same as the proof of Lemma 2.3, which will not be repeated here.

If $-n < t < \infty$ and $f \in \mathcal{M}^{[t]}$, then we obviously have

$$\langle fp, q\rangle_{[t]} = \lim_{r\uparrow 1}\langle f_r p, q\rangle_{[t]}$$

for all polynomials $p, q \in \mathbf{C}[z_1, \dots, z_n]$. Combining this with the norm bound provided by Lemma 4.2, we have

Corollary 4.3. *For $-n < t < \infty$ and $f \in \mathcal{M}^{[t]}$, we have the weak convergence*

$$\lim_{r\uparrow 1} M^{[t]}_{f_r} = M^{[t]}_f$$

on $\mathcal{L}^{[t]}$.

Proposition 4.4. *Let $-n < s < \infty$, and let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive self-adjoint operator W on H such that the sum*

$$Y = \sum_{\alpha\in\mathbf{Z}^n_+} \mu(\alpha; s)A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \le Y \le C$ on H for some scalars $0 < c \le C < \infty$. Then for each $f \in \mathcal{M}^{[s]}$, the limit

$$f(A) = \lim_{r\uparrow 1} f_r(A) \tag{4.3}$$

exists in the weak operator topology. Moreover, the identity

$$f(A)Z^* = Z^*(M^{[s]}_f \otimes 1) \tag{4.4}$$

holds for every $f \in \mathcal{M}^{[s]}$, where $Z : H \to \mathcal{L}^{[s]} \otimes H$ is the operator given by (4.1).

Proof. Let $f \in \mathcal{M}^{[s]}$. Then by Theorem 4.1 we have

$$\psi_{f,j}(A)Z^* = Z^*(M^{[s]}_{\psi_{f,j}} \otimes 1)$$

for each $j \geq 0$, where, as we recall,

$$\psi_{f,j}(z) = \int f(\xi)\langle z, \xi\rangle^j d\sigma(\xi).$$

Combining the above with (2.7), we have

$$f_r(A)Z^* = Z^*(M_{f_r}^{[s]} \otimes 1) \tag{4.5}$$

for every $0 \leq r < 1$. Since $Z^*Z = Y$ and since we assume $c \leq Y \leq C$ on H for some $0 < c \leq C < \infty$, the range of Z^* contains $Z^*ZH = YH = H$. That is, the operator $Z^* : \mathcal{L}^{[s]} \otimes H \to H$ is surjective. Hence given any $h_1 \in H$, there is a $g_1 \in \mathcal{L}^{[s]} \otimes H$ such that $h_1 = Z^*g_1$. Thus if $h_2 \in H$, then

$$\langle f_r(A)h_1, h_2\rangle = \langle f_r(A)Z^*g_1, h_2\rangle = \langle Z^*(M_{f_r}^{[s]} \otimes 1)g_1, h_2\rangle = \langle (M_{f_r}^{[s]} \otimes 1)g_1, Zh_2\rangle.$$

This equality and Corollary 4.3 together tell us that the weak limit (4.3) exists. Once this is established, (4.4) follows from (4.5), (4.3) and another application of Corollary 4.3. □

Theorem 4.5. *Let $-n < s < \infty$, and let $A = (A_1, \ldots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive self-adjoint operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} \mu(\alpha; s)A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \leq Y \leq C$ on H for some scalars $0 < c \leq C < \infty$. Then the inequality

$$\|f(A)\| \leq (C/c)\|M_f^{[s]}\| \tag{4.6}$$

holds for every $f \in \mathcal{M}^{[s]}$.

Proof. Again, by Theorem 4.1 and the assumption on Y, we have $Z^*ZH = YH = H$. Thus for each $h \in H$, there is an $\tilde{h} \in H$ such that $Z^*Z\tilde{h} = h$. By (4.4), for each $f \in \mathcal{M}^{[s]}$ we have

$$\|f(A)h\| = \|f(A)Z^*Z\tilde{h}\| = \|Z^*(M_f^{[s]} \otimes 1)Z\tilde{h}\| \leq \|Z^*\|\|M_f^{[s]}\|\|Z\|\|\tilde{h}\|.$$

Since $\|Z^*\|\|Z\| = \|Z\|^2 = \|Y\|$, we have

$$\|f(A)h\| \leq C\|M_f^{[s]}\|\|\tilde{h}\|. \tag{4.7}$$

But $c\|\tilde{h}\|^2 \leq \langle Y\tilde{h}, \tilde{h}\rangle = \langle h, \tilde{h}\rangle$. An application of the Cauchy-Schwarz inequality gives us $c\|\tilde{h}\| \leq \|h\|$, i.e., $\|\tilde{h}\| \leq (1/c)\|h\|$. Combining this with (4.7), (4.6) follows. □

Theorem 4.6. *Let $-n < s < \infty$, and let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a separable Hilbert space H. Suppose that there is a positive, compact, self-adjoint operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} \mu(\alpha; s) A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the operator Y has the following two properties:

(a) *There are scalars $0 < c \le C < \infty$ such that the operator inequality $c \le Y \le C$ holds on H;*

(b) $Y = 1 + K$, *where K is a compact operator on H.*

Then the inequality

$$\|f(A)\|_{\mathcal{Q}} \le \|M_f^{[s]}\|_{\mathcal{Q}}$$

holds for every $f \in \mathcal{M}^{[s]}$.

Proof. Let $f \in \mathcal{M}^{[s]}$. To prove the theorem, it suffices to show that

$$\|f(A)^*\|_{\mathcal{Q}} \le \|M_f^{[s]*}\|_{\mathcal{Q}}. \tag{4.8}$$

To prove this, note that since H is assumed to be separable, there is a sequence of unit vectors $\{h_k\}$ in H that converges to 0 weakly such that

$$\|f(A)^*\|_{\mathcal{Q}} = \lim_{k \to \infty} \|f(A)^* h_k\|.$$

The weak convergence $h_k \to 0$ implies that the sequence $\{f(A)^* h_k\}$ also converges to 0 weakly. Recall from Theorem 4.1 that $Z^*Z = Y$. Since we now assume $Y = 1 + K$, where K is compact, the weak convergence $f(A)^* h_k \to 0$ gives us

$$\begin{aligned}\lim_{k \to \infty} \|Zf(A)^* h_k\|^2 &= \lim_{k \to \infty} \langle Y f(A)^* h_k, f(A)^* h_k \rangle = \lim_{k \to \infty} \langle (1+K) f(A)^* h_k, f(A)^* h_k \rangle \\ &= \lim_{k \to \infty} \|f(A)^* h_k\|^2 = \|f(A)^*\|_{\mathcal{Q}}^2.\end{aligned}$$

Thus (4.8) will follow if we can prove that

$$\lim_{k \to \infty} \|Zf(A)^* h_k\| \le \|M_f^{[s]*}\|_{\mathcal{Q}}. \tag{4.9}$$

To prove this, we proceed as follows.

For each $\ell \in \mathbf{N}$, let $E_\ell^{[s]}$ denote the orthogonal projection from $\mathcal{L}^{[s]}$ onto the linear span of $\{z^\alpha : |\alpha| \le \ell\}$. By (4.1), for each $\ell \in \mathbf{N}$ we have

$$((E_\ell^{[s]} \otimes 1) Zh)(z) = \sum_{|\alpha| \le \ell} \mu(\alpha; s) W^{1/2} A^{*\alpha} h z^\alpha,$$

$h \in H$. Because the operator W is now assumed to be compact, each $W^{1/2}A^{*\alpha}$ is also compact. Thus the weak convergence $h_k \to 0$ gives us

$$\lim_{k\to\infty} \|(E_\ell^{[s]} \otimes 1)Zh_k\| = 0$$

for every $\ell \in \mathbf{N}$. This clearly implies that for each compact operator L on $\mathcal{L}^{[s]}$, we have

$$\lim_{k\to\infty} \|(L \otimes 1)Zh_k\| = 0.$$

Combining this with (4.4), we obtain

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| = \lim_{k\to\infty} \|(M_f^{[s]*} \otimes 1)Zh_k\| = \lim_{k\to\infty} \|(\{M_f^{[s]*} + L\} \otimes 1)Zh_k\|$$

whenever $L \in \mathcal{K}(\mathcal{L}^{[s]})$. Thus if $L \in \mathcal{K}(\mathcal{L}^{[s]})$, then

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| \le \|M_f^{[s]*} + L\| \limsup_{k\to\infty} \|Zh_k\|.$$

Since this holds for every compact operator L on $\mathcal{L}^{[s]}$, it follows that

$$\lim_{k\to\infty} \|Zf(A)^*h_k\| \le \|M_f^{[s]*}\|_{\mathcal{Q}} \limsup_{k\to\infty} \|Zh_k\|. \tag{4.10}$$

Using the weak convergence $h_k \to 0$ and the compactness of K again, we have

$$\limsup_{k\to\infty} \|Zh_k\|^2 = \limsup_{k\to\infty} \langle Yh_k, h_k\rangle = \limsup_{k\to\infty} \langle (1+K)h_k, h_k\rangle = \limsup_{k\to\infty} \langle h_k, h_k\rangle = 1.$$

Clearly, (4.9) follows from (4.10) and this equality. This completes the proof. □

5. Another Family of Examples

The purpose of this section is to use the results in Section 4 to prove Theorem 3.7. The reader will notice that this section parallels Section 3, just as Section 4 parallels Section 2. We begin with

Lemma 5.1. *Given $-n < s < t < \infty$, define*

$$g_{s,t}(\gamma) = \sum_{\alpha+\beta=\gamma} \frac{\mu(\alpha; s)u(\beta; t)\log(3 + |\beta|)}{u(\gamma; t)(1 + |\beta|)^{n+s+1}} \tag{5.1}$$

for every $\gamma \in \mathbf{Z}_+^n$. Then there is a $y_{s,t} \in (0, \infty)$ such that

$$\lim_{|\gamma|\to\infty} g_{s,t}(\gamma) = y_{s,t}.$$

Proof. Recall that $\rho_t : [0,\infty) \to [m(t),\infty)$ is defined by the rules that $\rho_t(x) = x$ if $x \geq m(t)$ and that $\rho_t(x) = m(t)$ if $0 \leq x < m(t)$. As in the proof of Lemma 3.4, it suffices to consider γ with $|\gamma| > m(t)$. Then a chase of the definitions of μ and u gives us

$$g_{s,t}(\gamma) = \hat{g}_{s,t}(\gamma) + h_{s,t}(\gamma),$$

where

$$\hat{g}_{s,t}(\gamma) = \sum_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0,\alpha\neq\gamma}} \frac{\log(3+|\beta|)}{\log(3+\rho_t(|\alpha|))} \cdot \frac{\gamma!}{\alpha!\beta!} \cdot \frac{\prod_{j=1}^{|\alpha|}(n+s+j)\prod_{j=1}^{|\beta|}(n+t+j)}{(1+|\beta|)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}$$

and

$$h_{s,t}(\gamma) = \frac{\log(3+|\gamma|)}{(1+|\gamma|)^{n+s+1}\log(3+m(t))} + \frac{\log 3}{\log(3+|\gamma|)} \cdot \frac{\prod_{j=1}^{|\gamma|}(n+s+j)}{\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

That is, $h_{s,t}(\gamma)$ is the sum of the terms $\alpha = 0$ and $\alpha = \gamma$ in $\sum_{\alpha+\beta=\gamma}$. Applying (3.2),

$$h_{s,t}(\gamma) \leq \frac{\log(3+|\gamma|)}{1+|\gamma|} + C\frac{(n+s+|\gamma|)^{n+s+|\gamma|+(1/2)}}{(n+t+|\gamma|)^{n+t+|\gamma|+(1/2)}} \leq \frac{\log(3+|\gamma|)}{1+|\gamma|} + \frac{C}{(n+t+|\gamma|)^{t-s}}.$$

Since $t - s > 0$, we have $h_{s,t}(\gamma) \to 0$ as $|\gamma| \to \infty$. What remains to be shown is that

$$\lim_{|\gamma|\to\infty} \hat{g}_{s,t}(\gamma) = y_{s,t} \tag{5.2}$$

for some $y_{s,t} \in (0,\infty)$.

To prove (5.2), we again apply Lemma 3.3, which gives us

$$\hat{g}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} \frac{\log(3+|\gamma|-k)}{\log(3+\rho_t(k))} \cdot \frac{|\gamma|!}{k!(|\gamma|-k)!} \cdot \frac{\prod_{j=1}^{k}(n+s+j)\prod_{j=1}^{|\gamma|-k}(n+t+j)}{(1+|\gamma|-k)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

Applying the asymptotic expansion (3.2), we have

$$\hat{g}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)\frac{\log(3+|\gamma|-k)}{\log(3+\rho_t(k))} \cdot \frac{|\gamma|^{|\gamma|+(1/2)}}{k^{k+(1/2)}(|\gamma|-k)^{|\gamma|-k+(1/2)}} \times \frac{(n+s+k)^{n+s+k+(1/2)}(n+t+|\gamma|-k)^{n+t+|\gamma|-k+(1/2)}}{(1+|\gamma|-k)^{n+s+1}(n+t+|\gamma|)^{n+t+|\gamma|+(1/2)}},$$

where $E(|\gamma|,k)$ is given by (3.6). We can further rewrite $\hat{g}_{s,t}(\gamma)$ as

$$\hat{g}_{s,t}(\gamma) = \sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)F(|\gamma|,k)\frac{\log(3+|\gamma|-k)}{\log(3+\rho_t(k))} \cdot \frac{k^{n+s}}{|\gamma|^{n+t}(|\gamma|-k)^{1+s-t}}, \tag{5.3}$$

where $F(|\gamma|, k)$ is given by (3.8). A rearrangement of the powers in (5.3) then leads to

$$(5.4) \quad \hat{g}_{s,t}(\gamma) = \frac{1}{|\gamma|}\sum_{k=1}^{|\gamma|-1} E(|\gamma|, k)F(|\gamma|, k)\frac{\log(3+|\gamma|-k)}{\log(3+\rho_t(k))} \cdot \left(\frac{k}{|\gamma|}\right)^{n+s} \cdot \left(\frac{|\gamma|}{|\gamma|-k}\right)^{1+s-t}.$$

As in the proof of Lemma 3.2, we will treat the above as some sort of "Riemann sum".

Next we define

$$V(m, k) = \frac{\log(3+m-k)}{\log(3+\rho_t(k))}$$

for natural numbers $1 \le k < m$. Dividing both the numerator and the denominator by $\log m$, we see that

$$V(m, k) = \left(1 + \frac{\log\left(\frac{3+m-k}{m}\right)}{\log m}\right) \cdot \left(1 + \frac{\log\left(\frac{3+\rho_t(k)}{m}\right)}{\log m}\right)^{-1}.$$

Recall that if $j \ge m(t)$, then $\rho_t(j) = j$. Therefore for each pair of $0 < \eta < 1/8$ and $\epsilon > 0$, there exist a positive number $M(\eta, \epsilon)$ such that

$$(5.5) \qquad |V(m, k) - 1| \le \epsilon \quad \text{if} \ \ m \ge M(\eta, \epsilon) \ \ \text{and} \ \ \eta m \le k \le (1-\eta)m.$$

Moreover, since $\rho_t(j) \ge j$ for all $j \in \mathbf{Z}_+$, we have

$$V(m, k) \le \frac{\log(3+m)}{\log(3+k)} = \frac{\log(3+m)}{\log m} \cdot \frac{\log m}{\log(3+k)} = \frac{\log(3+m)}{\log m}\left(1 + \frac{\log\left(\frac{m}{3+k}\right)}{\log(3+k)}\right)$$

$$(5.6) \qquad \le 2\left(1 + \log\left(\frac{m}{k+1}\right)\right)$$

if $m \ge 3$ and $1 \le k < m$.

Let an $\eta \in (0, 1/8)$ be given. By (5.4), for $\gamma \in \mathbf{Z}_+^n$ with $|\gamma| > \min\{m(t), 1/\eta\}$ we have

$$(5.7) \qquad \hat{g}_{s,t}(\gamma) = \hat{g}_{s,t,\eta}(\gamma) + \hat{g}^{(0)}_{s,t,\eta}(\gamma) + \hat{g}^{(1)}_{s,t,\eta}(\gamma),$$

where

$$\hat{g}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|}\sum_{\eta|\gamma| \le k \le (1-\eta)|\gamma|} E(|\gamma|, k)F(|\gamma|, k)V(|\gamma|, k)\left(\frac{k}{|\gamma|}\right)^{n+s}\left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t},$$

$$\hat{g}^{(0)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|}\sum_{1 \le k < \eta|\gamma|} E(|\gamma|, k)F(|\gamma|, k)V(|\gamma|, k)\left(\frac{k}{|\gamma|}\right)^{n+s}\left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t},$$

$$\hat{g}^{(1)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|}\sum_{(1-\eta)|\gamma| < k \le |\gamma|-1} E(|\gamma|, k)F(|\gamma|, k)V(|\gamma|, k)\left(\frac{k}{|\gamma|}\right)^{n+s}\left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t}.$$

By (5.5) and (3.13), it is clear that

$$\lim_{|\gamma|\to\infty} \hat{g}_{s,t,\eta}(\gamma) = w_{s,t} \int_{\eta}^{1-\eta} \frac{x^{n+s}}{(1-x)^{1+s-t}} dx. \tag{5.8}$$

By (3.14) and (5.6), we have

$$\begin{aligned} \hat{g}^{(1)}_{s,t,\eta}(\gamma) &\le \frac{C_1}{|\gamma|} \sum_{(1-\eta)|\gamma|<k\le|\gamma|-1} 2(1-\log(1-\eta)) \left(\frac{1}{1-(k/|\gamma|)}\right)^{1+s-t} \\ &\le 2C_1(1-\log(1-\eta)) \int_{1-\eta}^{1} \frac{1}{(1-x)^{1+s-t}} dx. \end{aligned} \tag{5.9}$$

Similarly,

$$\hat{g}^{(0)}_{s,t,\eta}(\gamma) \le \frac{C_1}{|\gamma|} \sum_{1\le k\le\eta|\gamma|} \frac{2(1+\log\{|\gamma|/(k+1)\})}{(1-(k/|\gamma|))^{1+s-t}} \le 2C_1 \int_0^{2\eta} \frac{1+\log(1/x)}{(1-x)^{1+s-t}} dx. \tag{5.10}$$

Because of the condition $t > s$, we have

$$\int_0^1 \frac{1}{(1-x)^{1+s-t}} dx < \infty.$$

Thus the combination of (5.7), (5.8), (5.9) and (5.10) gives us

$$\lim_{|\gamma|\to\infty} \hat{g}_{s,t}(\gamma) = w_{s,t} \int_0^1 \frac{x^{n+s}}{(1-x)^{1+s-t}} dx.$$

This proves (5.2) and completes the proof of the lemma. □

Corollary 5.2. *For any given* $-n < s < t < \infty$, *there exist* $0 < c \le C < \infty$ *such that*

$$c \le g_{s,t}(\gamma) \le C$$

for every $\gamma \in \mathbf{Z}_+^n$.

Proof. The upper bound follows immediately from Lemma 5.1. The lower bound follows from Lemma 5.1 and the obvious fact that $g_{s,t}(\gamma) > 0$ for every $\gamma \in \mathbf{Z}_+^n$. □

Proposition 5.3. *Suppose that* $-n < s < t < \infty$. *Define the operator*

$$W_{s,t} = \sum_{\beta\in\mathbf{Z}_+^n} \frac{\log(3+|\beta|)}{(1+|\beta|)^{n+s+1}} e_\beta^{(t)} \otimes e_\beta^{(t)} \tag{5.11}$$

on the space $\mathcal{H}^{(t)}$. *Furthermore, define*

$$\tilde{Y}_{s,t} = \sum_{\alpha\in\mathbf{Z}_+^n} \mu(\alpha; s) M_{z^\alpha}^{(t)} W_{s,t} M_{z^\alpha}^{(t)*}.$$

Then the following two statements hold true:
(a) *There exist constants* $0 < c \le C < \infty$ *such that the operator inequality* $c \le \tilde{Y}_{s,t} \le C$ *holds on* $\mathcal{H}^{(t)}$.
(b) *There is a scalar* $y_{s,t} \in (0, \infty)$ *such that* $\tilde{Y}_{s,t} = y_{s,t} + K$, *where* K *is a compact operator.*

Proof. For each $\alpha \in \mathbf{Z}_+^n$ we have

$$M_{z^\alpha}^{(t)} W_{s,t} M_{z^\alpha}^{(t)*} = \sum_{\beta \in \mathbf{Z}_+^n} \frac{\log(3 + |\beta|)}{(1 + |\beta|)^{n+s+1}} (M_{z^\alpha}^{(t)} e_\beta^{(t)}) \otimes (M_{z^\alpha}^{(t)} e_\beta^{(t)})$$
$$= \sum_{\beta \in \mathbf{Z}_+^n} \frac{u(\beta; t) \log(3 + |\beta|)}{u(\alpha + \beta; t)(1 + |\beta|)^{n+s+1}} e_{\alpha+\beta}^{(t)} \otimes e_{\alpha+\beta}^{(t)}.$$

Consequently

$$\tilde{Y}_{s,t} = \sum_{\gamma \in \mathbf{Z}_+^n} g_{s,t}(\gamma) e_\gamma^{(t)} \otimes e_\gamma^{(t)},$$

where $g_{s,t}(\gamma)$ is given by (5.1). Since $\{e_\gamma^{(t)} : \gamma \in \mathbf{Z}_+^n\}$ is an orthonormal basis for $\mathcal{H}^{(t)}$, statement (b) follows from Lemma 5.1 and statement (a) follows from Corollary 5.2. □

Proof of Theorem 3.7. Obviously, (1) follows from Propositions 4.4 and 5.3(a), while (2) follows from Theorem 4.5 and Proposition 5.3(a). Note that the operator $W_{s,t}$ defined by (5.11) is compact. Thus (3) follows from Theorem 4.6 and Proposition 5.3(b). □

6. Beyond Interpolation

An immediate consequence of the combination of Theorems 3.6 and 3.7 is the following:

Corollary 6.1. *Suppose that* $-n \le s < t < \infty$. *Then the following hold true*:
(a) *If* f *is a multiplier of* $\mathcal{H}^{(s)}$, *then* f *is also a multiplier of* $\mathcal{H}^{(t)}$.
(b) *There is a* $0 < C_{6.1} < \infty$ *such that* $\|M_f^{(t)}\| \le C_{6.1} \|M_f^{(s)}\|$ *for every multiplier of* $\mathcal{H}^{(s)}$.
(c) *If* f *is a multiplier of* $\mathcal{H}^{(s)}$, *then* $\|M_f^{(t)}\|_{\mathcal{Q}} \le \|M_f^{(s)}\|_{\mathcal{Q}}$.

Proof. Given a pair of $-n \le s < t < \infty$, pick an arbitrary $s' \in (s, t)$. Then Theorem 3.6 asserts $\mathcal{M}^{[s']} \subset \mathcal{M}^{(s)}$, and Theorem 3.7 asserts $\mathcal{M}^{(t)} \subset \mathcal{M}^{[s']}$. Combining the two inclusions, we have $\mathcal{M}^{(t)} \subset \mathcal{M}^{(s)}$, proving (a). Now let $f \in \mathcal{M}^{(s)}$. Then Theorem 3.6 gives us the inequalities

$$\|M_f^{[s']}\| \le C_{3.6} \|M_f^{(s)}\| \quad \text{and} \quad \|M_f^{[s']}\|_{\mathcal{Q}} \le \|M_f^{(s)}\|_{\mathcal{Q}},$$

while by Theorem 3.7 we have

$$\|M_f^{(t)}\| \le C_{3.7} \|M_f^{[s']}\| \quad \text{and} \quad \|M_f^{(t)}\|_{\mathcal{Q}} \le \|M_f^{[s']}\|_{\mathcal{Q}}.$$

Obviously, (b) and (c) follow from these two sets of inequalities. □

Obviously, Corollary 6.1 is just one of many consequences of Theorems 3.6 and 3.7. The reason that we single out Corollary 6.1 for mentioning is that we want to alert the reader to the fact that statements (a) and (b) in Corollary 6.1 can be alternately proved through interpolation in the family of spaces $\{\mathcal{H}^{(s)} : -n \le s < \infty\}$. Moreover, the fact that (a) and (b) in Corollary 6.1 can be obtained through interpolation was known long ago [15,16].

By contrast, it is not clear how one can obtain (c) through interpolation, particularly considering the fact that the "constant" in (c) is 1. In the literature, so far we have not seen any estimates of *essential norm* obtained through interpolation of underlying spaces.

More important, Theorems 3.6 and 3.7 themselves are not obtainable through interpolation, as each of these theorems involves two families of spaces, $\{\mathcal{H}^{(s)} : -n \le s < \infty\}$ and $\{\mathcal{L}^{(s)} : -n < s < \infty\}$. In fact, the introduction of $\{\mathcal{L}^{(s)} : -n < s < \infty\}$ was specifically intended to take interpolation out of the picture. Thus Theorems 3.6 and 3.7 demonstrate not only the fact that the general results in Sections 2 and 4 are not vacuous, but also that these general results are genuinely non-trivial in that they accomplish what interpolation does not.

References

1. C. Ambrozie and D. Timotin, *A von Neumann type inequality for certain domains in* $\mathbf{C}^n$, Proc. Amer. Math. Soc. **31** (2003), 859–869.

2. N. Arcozzi, R. Rochberg and E. Sawyer, *Carleson Measures for the Drury-Arveson Hardy space and other Besov-Sobolev Spaces on Complex Balls*, Advances in Math. **218** (2008), 1107-1180.

3. W. Arveson, *Subalgebras of* C^**-algebras.* III. *Multivariable operator theory*, Acta Math. **181** (1998), 159-228.

4. W. Arveson, *The curvature invariant of a Hilbert module over* $\mathbf{C}[z_1, \ldots, z_d]$, J. Reine Angew. Math. **522** (2000), 173-236.

5. J. Ball and V. Bolotnikov, *Interpolation problems for Schur multipliers on the Drury-Arveson space: from Nevanlinna-Pick to abstract interpolation problem*, Integr. Equ. Oper. Theory **62** (2008), 301-349.

6. J. Ball, V. Bolotnikov and Q. Fang, *Transfer-function realization for multipliers of the Arveson space*, J. Math. Anal. Appl. **333** (2007), 68-92.

7. J. Ball, V. Bolotnikov and Q. Fang, *Schur-class multipliers on the Arveson space: de Branges-Rovnyak reproducing kernel spaces and commutative transfer-function realizations*, J. Math. Anal. Appl. 341(2008), 519-539.

8. J. Ball, T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernels Hilbert spaces*, Operator Theory and Analysis: The M.A. Kaashoek Anniversary Volume (Workshop in Amsterdam, Nov. 1997), OT 122, Birkhäuser-Verlag, Basel-Boston, 2001, pp. 89–138.

9. S. Costea, E. Sawyer and B. Wick, *The Corona Theorem for the Drury-Arveson Hardy space and other holomorphic Besov-Sobolev spaces on the unit ball in* $\mathbf{C}^n$, Anal. PDE **4** (2011), 499-550.

10. S. W. Drury, *A generalization of von Neumanns inequality to the complex ball*, Proc. Amer. Math. Soc. **68** (1978), 300-304.

11. J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine Angew. Math. **542** (2002), 219-236.
12. Q. Fang and J. Xia, *Commutators and localization on the Drury-Arveson space*, J. Funct. Anal. **260** (2011), 639-673.
13. D. Greene, S. Richter and C. Sundberg, *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, J. Funct. Anal. **194** (2002), 311-331.
14. S. McCullough and T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. **178** (2000), 226-249.
15. J. Ortega and J. Fàbrega, *Pointwise multipliers and decomposition theorems in analytic Besov spaces*, Math. Z. **235** (2000), 53-81.
16 J. Ortega and J. Fàbrega, *Multipliers in Hardy-Sobolev spaces*, Integr. Equ. Oper. Theory **55** (2006), 535-560.
17. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York-Berlin, 1980.

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