INVARIANT SUBSPACES FOR CERTAIN FINITE-RANK PERTURBATIONS OF DIAGONAL OPERATORS

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Abstract. Suppose that $\{e_k\}$ is an orthonormal basis for a separable, infinite-dimensional Hilbert space $\mathcal{H}$. Let D be a diagonal operator with respect to the orthonormal basis $\{e_k\}$. That is, $D = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$, where $\{\lambda_k\}$ is a bounded sequence of complex numbers. Let

$$T = D + u_1 \otimes v_1 + \cdots + u_n \otimes v_n.$$

Improving a result [2] of Foias et al., we show that if the vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ satisfy an ℓ^1-condition with respect to the orthonormal basis $\{e_k\}$, and if T is not a scalar multiple of the identity operator, then T has a non-trivial hyperinvariant subspace.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. Throughout the paper, we fix an orthonormal basis $\{e_k\}$ in $\mathcal{H}$. Then an operator that is *diagonal* with respect to the chosen orthonormal basis $\{e_k\}$ has the form

$$D = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k,$$

where the sequence of complex numbers $\{\lambda_k\}$ will always be assumed to be bounded.

Diagonal operators are the simplest among all operators. Thus it is intuitively natural in operator theory to test difficult problems on operators that are "not too far from" diagonal operators. A good example of such problems is the well-known *invariant subspace problem*. Moreover, in the context of operator theory, the meaning of the phrase "not too far from" can be made rather precise: it is usually taken to mean perturbations of one particular kind or another. But even for operators that are "not too far from" diagonal operators, finding invariant subspaces is generally not an easy task. For example, the following specific problem goes back to [5]:

Problem 1.1. Consider a rank-one perturbation $D + u \otimes v$ of D. Suppose that $D + u \otimes v$ is not a scalar multiple of the identity operator. Then does it have a non-trivial invariant subspace? Does it have a non-trivial hyperinvariant subspace?

This problem, which had baffled investigators for a long time, was partially solved in 2007. In [2], Foias, Jung, Ko and Pearcy proved the following:

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Theorem 1.2. *Suppose that* $u = \sum_{k=1}^{\infty} \alpha_k e_k$ *and* $v = \sum_{k=1}^{\infty} \beta_k e_k$ *satisfy the condition*

$$\sum_{k=1}^{\infty} (|\alpha_k|^{2/3} + |\beta_k|^{2/3}) < \infty. \tag{1.1}$$

If the operator $D + u \otimes v$ *is not a scalar multiple of the identity operator, then it has a non-trivial hyperinvariant subspace.*

Given the importance of the problem, this theorem obviously inspires and motivates further investigations. The purpose of this paper is to improve Theorem 1.2 in two aspects. First, condition (1.1) will be significantly weakened. Second, instead of rank-one perturbations, we will consider perturbations of arbitrary finite rank. Before stating our result, let us introduce

Definition 1.3. Let $\ell^1(\{e_k\})$ denote the collection of vectors $u = \sum_{k=1}^{\infty} \alpha_k e_k$ in $\mathcal{H}$ satisfying the condition

$$\sum_{k=1}^{\infty} |\alpha_k| < \infty.$$

In other words, the notation $\ell^1(\{e_k\})$ is very suggestive: it means what one thinks it means. The following is the main result of the paper:

Theorem 1.4. *Let* $u_1, \ldots, u_n$ *and* $v_1, \ldots, v_n$ *be vectors in* $\ell^1(\{e_k\})$*. If the operator*

$$T = D + u_1 \otimes v_1 + \cdots + u_n \otimes v_n$$

is not a scalar multiple of the identity operator, then it has a non-trivial hyperinvariant subspace.

Before giving the proof of Theorem 1.4 in the subsequent sections, let us explain the basic idea here. Given what is known about hyperinvariant subspaces, it suffices to consider the case where D has at least two points in its essential spectrum and T has no eigenvalues. As it was the case in [2], we will prove our theorem through an unconventional kind of Riesz functional calculus. It is unconventional because it involves a contour Γ that has a troublesome segment

$$s_0 = \{x_0 + iy : |y| \le N\},$$

a segment that possibly passes through the spectrum of T as well as the spectrum of D. In Section 2, we will show that the condition $u_1, \ldots, u_n, v_1, \ldots, v_n \in \ell^1(\{e_k\})$ implies the existence of plenty of desired x_0 such that we have continuous maps

$$z \mapsto (D - z)^{-1} u_j \quad \text{and} \quad z \mapsto (D^* - \bar{z})^{-1} v_j, \tag{1.2}$$

$1 \le j \le n$, from s_0 into $\mathcal{H}$. In Section 3, we will derive an explicit formula for a finite-rank operator $K(z)$ which has the property that

$$(T - z)((D - z)^{-1} w - K(z) w) = w \tag{1.3}$$

for every w in the domain of $(D-z)^{-1}$. One can interpret (1.3) as a "right inversion formula" for $T-z$, even though z may belong to the spectrum of T. As it turns out, this "right inversion formula" is all that we need for the proof of Theorem 1.4.

The explicit formula for $K(z)$ enables us to show that the continuity of the maps given in (1.2) implies that the map $z \mapsto K(z)$ is continuous on s_0 with respect to the operator norm. This continuity allows us to integrate to obtain the formula

$$\frac{1}{2\pi i}\int_\Gamma ((D-z)^{-1}w - K(z)w)dz = -(P+K)w \tag{1.4}$$

for vectors w in a certain dense subset $\mathcal{W}$ of $\mathcal{H}$, where K is a compact operator and P is an orthogonal projection with the property that $\dim(P\mathcal{H}) = \infty$ and $\dim((1-P)\mathcal{H}) = \infty$. For such a pair of P and K, it is easy to show that $P+K$ has a non-trivial hyperinvariant subspace. But from (1.3) and (1.4) we can deduce that the commutant of T is contained in the commutant of $P+K$. Hence any hyperinvariant subspace for $P+K$ is also a hyperinvariant subspace for T. This is our strategy for proving Theorem 1.4.

We end the introduction with two conventions. First of all, in this paper the operator $u \otimes v$ is defined by the usual formula

$$(u \otimes v)h = \langle h, v\rangle u.$$

Second, for a bounded operator A on $\mathcal{H}$, we denote its commutant by $\{A\}'$. That is, $\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$.

2. Spectral consequence of the ℓ^1-condition

Recall that our basic setting is the following. Suppose that $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space and that $\{e_k\}$ is an orthonormal basis for $\mathcal{H}$. Throughout the paper, D is the diagonal operator given by the formula

$$D = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k, \tag{2.1}$$

where $\{\lambda_k\}$ is a bounded sequence of complex numbers.

Let m denote the standard Lebesgue measure on $\mathbf{R}$.

Lemma 2.1. *Let $\{\alpha_k\}$ be a sequence of complex numbers such that*

$$\sum_{k=1}^{\infty} |\alpha_k| < \infty. \tag{2.2}$$

Then for a.e. $x \in \mathbf{R}\backslash\{\mathrm{Re}(\lambda_k) : k \in \mathbf{N}\}$ we have

$$\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} < \infty. \tag{2.3}$$

Proof. Let Θ be the collection of $x \in \mathbf{R}\backslash\{\mathrm{Re}(\lambda_k) : k \in \mathbf{N}\}$ for which (2.3) fails. We need to show that $m(\Theta) = 0$. For this purpose, pick an arbitrary $\epsilon > 0$. Then by (2.2), there is a $\delta > 0$ such that

$$2\delta \sum_{k=1}^{\infty} |\alpha_k| \leq \epsilon. \tag{2.4}$$

For each $k \in \mathbf{N}$, define the closed interval

$$I_k = [\mathrm{Re}(\lambda_k) - \delta|\alpha_k|, \mathrm{Re}(\lambda_k) + \delta|\alpha_k|] \, .$$

Furthermore, for each $k \in \mathbf{N}$ define the function f_k by the rule that

$$f_k(x) = \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} \quad \text{if} \quad x \in \mathbf{R}\backslash I_k$$

and $f_k(x) = 0$ if $x \in I_k$. Observe that if $\alpha_k \neq 0$, then

$$\int f_k(x)dx = \int_{|\mathrm{Re}(\lambda_k)-x|>\delta|\alpha_k|} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} dx = |\alpha_k|^2 \cdot \frac{2}{\delta|\alpha_k|} = \frac{2}{\delta}|\alpha_k|.$$

If $\alpha_k = 0$, then, of course, we have

$$\int f_k(x)dx = 0 = \frac{2}{\delta}|\alpha_k|.$$

Define the function

$$F(x) = \sum_{k=1}^{\infty} f_k(x)$$

on $\mathbf{R}$. By the monotone convergence theorem and (2.2), we have

$$\int F(x)dx = \sum_{k=1}^{\infty} \int f_k(x)dx = \frac{2}{\delta} \sum_{k=1}^{\infty} |\alpha_k| < \infty.$$

This means in particular that $F(x) < \infty$ for a.e. $x \in \mathbf{R}$. That is, $m(\{x : F(x) = \infty\}) = 0$.

Let $\Omega = \cup_{k=1}^{\infty} I_k$. Then by (2.4),

$$m(\Omega) \leq \sum_{k=1}^{\infty} m(I_k) = 2\delta \sum_{k=1}^{\infty} |\alpha_k| \leq \epsilon. \tag{2.5}$$

By the definition of the functions f_k, if $x \in \mathbf{R}\backslash\Omega$, then

$$\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} = \sum_{k=1}^{\infty} f_k(x) = F(x).$$

This shows that $\Theta \subset \Omega \cup \{x : F(x) = \infty\}$. Since $m(\{x : F(x) = \infty\}) = 0$, it follows from (2.5) that $m(\Theta) \leq \epsilon$. Since this is true for every $\epsilon > 0$, we conclude that $m(\Theta) = 0$. This completes the proof. $\square$

We refer the reader to [1] for the spectral theory of normal operators. For any $z \in \mathbf{C}$ that is not an eigenvalue of D, $(D - z)^{-1}$ is a (not necessarily bounded) normal operator. In fact, $(D - z)^{-1}$ has the spectral decomposition

$$(D - z)^{-1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - z} e_k \otimes e_k.$$

The domain of $(D - z)^{-1}$, which by definition equals the range of $D - z$, consists of vectors $h = \sum_{k=1}^{\infty} h_k e_k$ in $\mathcal{H}$ satisfying the condition

$$\sum_{k=1}^{\infty} \frac{|h_k|^2}{|\lambda_k - z|^2} < \infty.$$

In particular, a vector h belongs to the domain of $(D - z)^{-1}$ if and only if it belongs to the domain of $(D^* - \bar{z})^{-1}$.

Lemma 2.2. *Let $u \in \ell^1(\{e_k\})$. Then there is a Borel subset E of $\mathbf{R}$ that has the following properties*:

(1) $\mathrm{Re}(\lambda_k) \in E$ *for every* $k \in \mathbf{N}$.

(2) $m(E) = 0$.

(3) *For each $x \in \mathbf{R}\backslash E$ and each $y \in \mathbf{R}$, u belongs to the domain of $(D - (x + iy))^{-1}$.*

(4) *For each $x \in \mathbf{R}\backslash E$, the maps $y \mapsto (D - (x + iy))^{-1}u$ and $y \mapsto (D^* - (x - iy))^{-1}u$ from $\mathbf{R}$ into $\mathcal{H}$ are continuous with respect to the norm topology on $\mathcal{H}$.*

Proof. Given a $u \in \ell^1(\{e_k\})$, we have $u = \sum_{k=1}^{\infty} \alpha_k e_k$, where the sequence of complex numbers $\{\alpha_k\}$ satisfies (2.2). By Lemma 2.1, there is a Borel subset E of $\mathbf{R}$ that has properties (1), (2) and the property that

$$\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} < \infty \quad \text{for every} \quad x \in \mathbf{R}\backslash E. \tag{2.6}$$

Note that for each pair of $x \in \mathbf{R}\backslash E$ and $y \in \mathbf{R}$, it follows from (2.6) that

$$\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{|\lambda_k - (x + iy)|^2} \leq \sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} < \infty.$$

Thus E has property (3). To prove (4), fix an $x \in \mathbf{R}\backslash E$ for the moment. For each $\ell \in \mathbf{N}$, we define the $\mathcal{H}$-valued functions

$$\varphi_\ell(y) = \sum_{1 \leq k \leq \ell} \frac{\alpha_k}{\lambda_k - (x + iy)} e_k \quad \text{and} \quad \gamma_\ell(y) = \sum_{k > \ell} \frac{\alpha_k}{\lambda_k - (x + iy)} e_k,$$

$y \in \mathbf{R}$. Then

$$(D - (x+iy))^{-1}u = \varphi_\ell(y) + \gamma_\ell(y)$$

for all $y \in \mathbf{R}$ and $\ell \in \mathbf{N}$. It is obvious that for each ℓ, the map $\varphi_\ell : \mathbf{R} \to \mathcal{H}$ is continuous with respect to the norm topology on $\mathcal{H}$. Note that for every $y \in \mathbf{R}$, we have

$$\|\gamma_\ell(y)\|^2 = \sum_{k>\ell} \frac{|\alpha_k|^2}{|\lambda_k - (x+iy)|^2} \le \sum_{k>\ell} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2}.$$

But (2.6) implies that

$$\lim_{\ell\to\infty} \sum_{k>\ell} \frac{|\alpha_k|^2}{(\mathrm{Re}(\lambda_k) - x)^2} = 0.$$

Therefore the sequence of $\mathcal{H}$-valued functions $\{\gamma_\ell\}$ converges to 0 uniformly on $\mathbf{R}$. Combining this uniform convergence with the continuity of each φ_ℓ, we see that the map $y \mapsto (D - (x+iy))^{-1}u$ is continuous on $\mathbf{R}$.

Similarly, we have $(D^* - (x-iy))^{-1}u = \tilde{\varphi}_\ell(y) + \tilde{\gamma}_\ell(y)$ for $y \in \mathbf{R}$, where

$$\tilde{\varphi}_\ell(y) = \sum_{1\le k\le \ell} \frac{\alpha_k}{\bar{\lambda}_k - (x-iy)} e_k \quad \text{and} \quad \tilde{\gamma}_\ell(y) = \sum_{k>\ell} \frac{\alpha_k}{\bar{\lambda}_k - (x-iy)} e_k.$$

Since $\|\tilde{\gamma}_\ell(y)\| = \|\gamma_\ell(y)\|$, we also have the uniform convergence $\tilde{\gamma}_\ell \to 0$ on $\mathbf{R}$. Since each $\tilde{\varphi}_\ell$ is again continuous, we similarly conclude that the map $y \mapsto (D^* - (x-iy))^{-1}u$ is continuous on $\mathbf{R}$. This proves (4) and completes the proof of the lemma. □

3. Finite-rank perturbation

We will now consider the operator

$$T = D + u_1 \otimes v_1 + \cdots + u_n \otimes v_n,$$

where D is given by (2.1), and to begin we only assume $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathcal{H}$.

We would like to repeat something that we mentioned in Section 2: Suppose that z is not an eigenvalue of D. Then a vector h belongs to the domain of $(D-z)^{-1}$ if and only if it belongs to the domain of $(D^* - \bar{z})^{-1}$.

Lemma 3.1. *Suppose that z is a complex number satisfying the following three conditions*:

(a) $z \ne \lambda_k$ *for every* $k \ge 1$.

(b) *The vectors $u_1, \ldots u_n$ and $v_1, \ldots v_n$ all belong to the domain of* $(D-z)^{-1}$.

(c) $\ker(T - z) = \{0\}$.

Then the $n \times n$ matrix

$$M(z) = \begin{bmatrix} 1 + \langle (D-z)^{-1}u_1, v_1\rangle & \langle (D-z)^{-1}u_2, v_1\rangle & \cdots & \langle (D-z)^{-1}u_n, v_1\rangle \\ \langle (D-z)^{-1}u_1, v_2\rangle & 1 + \langle (D-z)^{-1}u_2, v_2\rangle & \cdots & \langle (D-z)^{-1}u_n, v_2\rangle \\ & \cdots\cdots & & \\ \langle (D-z)^{-1}u_1, v_n\rangle & \langle (D-z)^{-1}u_2, v_n\rangle & \cdots & 1 + \langle (D-z)^{-1}u_n, v_n\rangle \end{bmatrix}$$

is invertible.

Proof. Because of (a) and (b), we can factor $T-z$ in the form

$$T-z=(D-z)(1+\{(D-z)^{-1}u_1\}\otimes v_1+\cdots+\{(D-z)^{-1}u_n\}\otimes v_n).$$

Utilizing the orthonormal basis $\{e_k\}$, we also have the factorization

$$\{(D-z)^{-1}u_1\}\otimes v_1+\cdots+\{(D-z)^{-1}u_n\}\otimes v_n=XY,$$

where

$$\begin{aligned}&X=\{(D-z)^{-1}u_1\}\otimes e_1+\cdots+\{(D-z)^{-1}u_n\}\otimes e_n \quad \text{and}\\ &Y=e_1\otimes v_1+\cdots+e_n\otimes v_n.\end{aligned}$$

Hence

$$T-z=(D-z)(1+XY).$$

Thus (c) implies that $\ker(1+XY)=\{0\}$. Since $\operatorname{rank}(XY)<\infty$, there is a finite-dimensional reducing subspace $\mathcal{E}$ for XY such that $XY|(\mathcal{H}\ominus\mathcal{E})=0$. That is, $(1+XY)|(\mathcal{H}\ominus\mathcal{E})$ equals the identity operator on $\mathcal{H}\ominus\mathcal{E}$. Therefore the condition $\ker(1+XY)=\{0\}$ implies that $1+XY$ is invertible. Hence $1+YX$ is also invertible, for it is well known that the operator

$$1-Y(1+XY)^{-1}X$$

is the inverse of $1+YX$ whenever $1+XY$ is invertible (see, e.g., [1,page 199]). But

$$1+YX=\sum_{1\le i,j\le n}\{\delta_{ij}+\langle (D-z)^{-1}u_j,v_i\rangle\}e_i\otimes e_j+\sum_{k=n+1}^{\infty}e_k\otimes e_k, \tag{3.1}$$

where δ_{ij} is Kronecker's delta. Thus the invertibility of $1+YX$ implies the invertibility of $M(z)$. □

Lemma 3.1 allows us to introduce

Definition 3.2. For any complex number z that satisfies conditions (a), (b) and (c) in Lemma 3.1, we set

$$K(z)=\frac{1}{\det(M(z))}\sum_{1\le i,j\le n}a_{i,j}(z)\{(D-z)^{-1}u_i\}\otimes\{(D^*-\bar z)^{-1}v_j\}, \tag{3.2}$$

where $a_{i,j}(z)=(-1)^{i+j}\det(M_{j,i}(z))$, where $M_{j,i}(z)$ is the $(n-1)\times(n-1)$ matrix obtained from $M(z)$ by deleting the j-th row and the i-th column. (In the event $n=1$, $a_{1,1}(z)$ is defined to be 1.)

Lemma 3.3. *Suppose that Δ is a compact subset of* $\mathbf{C}$ *such that every $z\in\Delta$ satisfies conditions* (a), (b) *and* (c) *in Lemma* 3.1. *Furthermore, suppose that for every $j\in\{1,\dots,n\}$, the maps*

$$z\mapsto (D-z)^{-1}u_j \quad \textit{and} \quad z\mapsto (D^*-\bar z)^{-1}v_j \tag{3.3}$$

are continuous on Δ with respect to the norm topology of $\mathcal{H}$. Then the map $z \mapsto K(z)$ is continuous on Δ with respect to the operator norm topology.

Proof. Obviously, the continuity of the maps given by (3.3) implies that the map

$$z \mapsto \sum_{1 \le i,j \le n} a_{i,j}(z)\{(D-z)^{-1}u_i\} \otimes \{(D^* - \bar{z})^{-1}v_j\} \tag{3.4}$$

is continuous on Δ with respect to the operator norm. The continuity of the maps given by (3.3) also implies that the function $\det(M(z))$ is continuous on Δ. Lemma 3.1 tells us that $\det(M(z))$ does not vanish on Δ. Since Δ is compact, it follows that the function $\{\det(M(z))\}^{-1}$ is also continuous on Δ. Combining this with the continuity of (3.4), the lemma is proved. $\square$

Lemma 3.4. *Let z be a complex number that satisfies conditions* (a), (b) *and* (c) *in Lemma* 3.1. *Then for every w in the domain of $(D-z)^{-1}$, we have*

$$(T-z)((D-z)^{-1}w - K(z)w) = w.$$

Proof. Define

$$L(z) = \frac{1}{\det(M(z))} \sum_{1 \le i,j \le n} a_{i,j}(z)\{(D-z)^{-1}u_i\} \otimes v_j.$$

If w is in the domain of $(D-z)^{-1}$, then it is easy to see that

$$K(z)w = L(z)(D-z)^{-1}w.$$

Therefore

$$(T-z)((D-z)^{-1}w - K(z)w) = (T-z)(1-L(z))(D-z)^{-1}w.$$

Using the operators X and Y introduced in the proof of Lemma 3.1, we have

$$(T-z)((D-z)^{-1}w - K(z)w) = (D-z)(1+XY)(1-L(z))(D-z)^{-1}w. \tag{3.5}$$

Recall from the proof of Lemma 3.1 that $1+XY$ and $1+YX$ are invertible. Moreover,

$$(1+XY)^{-1} = 1 - X(1+YX)^{-1}Y.$$

By (3.1) and Definition 3.2, we have

$$(1+YX)^{-1} = \frac{1}{\det(M(z))} \sum_{1 \le i,j \le n} a_{i,j}(z) e_i \otimes e_j + \sum_{k=n+1}^{\infty} e_k \otimes e_k.$$

Multiplying both sides by X on the left and by Y on the right, we obtain

$$X(1+YX)^{-1}Y = \frac{1}{\det(M(z))} \sum_{1 \le i,j \le n} a_{i,j}(z)\{(D-z)^{-1}u_i\} \otimes v_j = L(z).$$

Thus $1 - L(z) = (1 + XY)^{-1}$. Substituting this in (3.5), the lemma is proved. $\square$

4. Compactness and its implications

For the proof of our main result, we need to recall a few more general operator-theoretical facts. The content of this section should really be considered as well-known material. Nevertheless, we decide to include it here both for the self-containedness of the paper and for the convenience of the reader.

Lemma 4.1. *Suppose that P is an orthogonal projection on a separable Hilbert space H that has the property that both subspaces PH and $(1-P)H$ are infinite dimensional. Then for any compact operator K, the operator $P+K$ has a non-trivial hyperinvariant subspace.*

Proof. Write $G = P + K$. Then there are the following two possibilities:

(a) Suppose that $G^2 = G$. Since $\dim(PH) = \infty$ and $\dim((1-P)H) = \infty$, the essential spectrum of P is the two-point set $\{0, 1\}$. Since K is compact, the essential spectrum of G is also the two-point set $\{0, 1\}$. Hence $G \neq 0$ and $G \neq 1$. Thus from the equation $G(G - 1) = 0$ we deduce that both $\ker(G)$ and $\ker(G - 1)$ are non-trivial subspaces of H. But $\ker(G)$ and $\ker(G - 1)$ are obviously hyperinvariant for G.

(b) Suppose that $G^2 \neq G$. Then $G^2 - G \neq 0$. Since $P^2 = P$, we have

$$G^2 - G = PK + KP + K^2 - K,$$

which is a compact operator. Thus by the famous theorem of Lomonosov [1,4,6], $G^2 - G$ has a non-trivial hyperinvariant subspace. Since $\{G^2 - G\}' \supset \{G\}'$, we conclude that $G = P + K$ has a non-trivial hyperinvariant subspace. $\square$

Lemma 4.2. *Let $\{X, \mathcal{M}, \mu\}$ be a (finite or infinite) measure space. Let H be a separable Hilbert space and let $\mathcal{K}(H)$ denote the collection of compact operators on H. Suppose that $F : X \to \mathcal{K}(H)$ is a weakly $\mathcal{M}$-measurable map. If*

$$\int_X \|F(x)\| d\mu(x) < \infty, \tag{4.1}$$

then

$$K = \int_X F(x) d\mu(x)$$

is a compact operator on the Hilbert space H.

This lemma is, of course, is a well-known fact from the theory of Bochner integral. See, for example, [3,Theorem 3.5.2]. But here we would like to offer the following simple proof, which takes full advantage of our setting.

Proof of Lemma 4.2. Since H is a separable Hilbert space, there exists a sequence $\{E_j\}$ of finite-rank orthogonal projections on H such that $\lim_{j\to\infty} E_j = 1$ in the strong operator topology. The rank of each operator E_jK is, of course, also finite. But observe that

$$E_jK = \int_X E_jF(x) d\mu(x),$$

consequently

$$\|K - E_j K\| \le \int_X \|F(x) - E_j F(x)\| d\mu(x).$$

Since $F(x) \in \mathcal{K}(\mathcal{H})$, the strong convergence $E_j \to 1$ implies $\lim_{j\to\infty} \|F(x) - E_j F(x)\| = 0$ for every $x \in X$. Thus by (4.1) and the dominated convergence theorem, we have $\|K - E_j K\| \to 0$ as $j \to \infty$, proving the compactness of K. $\square$

5. Proof of the main result

With the preparations in the previous sections, we are now ready to prove Theorem 1.4. But before we get to the actual proof, let us review the various operators, vectors, conditions, symbols and notation one more time.

Recall that $\mathcal{H}$ is a separable Hilbert space, and that $\{e_k\}$ is an orthonormal basis for $\mathcal{H}$. The diagonal operator D is given by (2.1), where $\{\lambda_k\}$ is a bounded sequence of complex numbers. The object of our main interest, the operator T, is given by the formula

$$T = D + u_1 \otimes v_1 + \cdots + u_n \otimes v_n.$$

Theorem 1.4 assumes that the vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ all belong to $\ell^1(\{e_k\})$.

Proof of Theorem 1.4. As usual, we begin by eliminating some trivial cases.

(1) If T has an eigenvalue, and if T is not a scalar multiple of the identity operator, then, of course, T has a non-trivial hyperinvariant subspace.

(2) If the essential spectrum of the diagonal operator D consists of a single point λ, then $D = \lambda + K_0$, where K_0 is a compact operator on $\mathcal{H}$. Consequently, $T = \lambda + K_1$, where $K_1 = K_0 + u_1 \otimes v_1 + \cdots + u_n \otimes v_n$, which is also compact. If T is not a scalar multiple of the identity operator, then $K_1 \ne 0$. By Lomonosov's theorem [1,4,6], K_1 has a non-trivial hyperinvariant subspace. Since in this case $\{T\}' = \{K_1\}'$, it follows that T has a non-trivial hyperinvariant subspace.

(3) We now only need to prove the theorem under the following two additional assumptions:

(i) The operator T has no eigenvalues.

(ii) The essential spectrum of D contains as least two distinct points, A and B.

For any $\theta \in \mathbf{R}$, we have $e^{i\theta}T = e^{i\theta}D + (e^{i\theta}u_1) \otimes v_1 + \cdots + (e^{i\theta}u_n) \otimes v_n$ and $\{T\}' = \{e^{i\theta}T\}'$. Thus, replacing T by some $e^{i\theta}T$ if necessary, we may require that

(iii) if we set $a = \mathrm{Re}(A)$ and $b = \mathrm{Re}(B)$, then $a < b$.

We now apply Lemma 2.2 to the vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. By Lemma 2.2, we can pick an $x_0 \in (a, b)$ with the following three properties:

(α) $\mathrm{Re}(\lambda_k) \ne x_0$ for every k.

(β) For each $y \in \mathbf{R}$, the vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ all belong to the domain of $(D - (x_0 + iy))^{-1}$.

(γ) For each $j \in \{1, \ldots, n\}$, the maps $y \mapsto (D - (x_0 + iy))^{-1}u_j$ and $y \mapsto (D^* - (x_0 - iy))^{-1}v_j$ from $\mathbf{R}$ into $\mathcal{H}$ are continuous with respect to the norm topology.

With this x_0, we define the orthogonal projection

$$P = \sum_{\mathrm{Re}(\lambda_k) < x_0} e_k \otimes e_k. \tag{5.1}$$

Since $\mathrm{Re}(A) = a < x_0 < b = \mathrm{Re}(B)$ and since A, B belong to the essential spectrum of D, both subspaces $P\mathcal{H}$ and $(1 - P)\mathcal{H}$ are infinite dimensional. Thus Lemma 4.1 tells us that if K is any compact operator on $\mathcal{H}$, then $P + K$ has a non-trivial hyperinvariant subspace. Hence the proof of the theorem will be complete if we can show that

$$\text{there is a compact operator } K \text{ such that } \{P + K\}' \supset \{T\}'. \tag{5.2}$$

We will accomplish this by using contour integral.

Let N be a positive number such that the disc $\{z \in \mathbf{C} : |z| \le N - 1\}$ contains both the spectrum of T and the sequence $\{\lambda_k\}$. Let Γ be the rectangular contour in $\mathbf{C}$ that is made of the following four line segments:

$$\begin{aligned} s_0 &= \{x_0 + iy : -N \le y \le N\}, \\ s_1 &= \{x + iN : -N \le x \le x_0\}, \\ s_2 &= \{-N + iy : -N \le y \le N\}, \\ s_3 &= \{x - iN : -N \le x \le x_0\}. \end{aligned}$$

Let Γ be oriented in the usual counter-clockwise direction. By (α), (β) and (i), each $z \in s_0$ satisfies conditions (a), (b) and (c) in Lemma 3.1. By the choice of N, the segements s_1, s_2 and s_3 are outside the spectra of T and D. Therefore, for each $z \in \Gamma$ we have the finite-rank operator $K(z)$ given by (3.2).

Condition (γ) asserts that for each $j \in \{1, \dots, n\}$, the maps $z \mapsto (D - z)^{-1} u_j$ and $z \mapsto (D^* - \bar{z})^{-1} v_j$ are continuous on s_0 with respect to the norm topology. But these maps are obviously continuous on $s_1 \cup s_2 \cup s_3$. Hence for each $j \in \{1, \dots, n\}$, the maps $z \mapsto (D - z)^{-1} u_j$ and $z \mapsto (D^* - \bar{z})^{-1} v_j$ are norm continuous on the entire contour Γ. Consequently, Lemma 3.3 tells us that the map $z \mapsto K(z)$ is continuous on Γ with respect to the operator norm. This allows us to define

$$K = \frac{1}{2\pi i} \int_\Gamma K(z)dz. \tag{5.3}$$

Since $\mathrm{rank}(K(z)) \le n$ for every $z \in \Gamma$ and since the numerical function $\|K(z)\|$ is bounded on Γ, Lemma 4.2 tells us that this K is a compact operator. We will show that this is the K promised in (5.2).

Let $\mathcal{L}$ denote the collection of (finite) linear combinations of the vectors $\{e_k\}$. Let $\mathcal{W}$ be the collection of vectors w in $\mathcal{H}$ satisfying the following two conditions:

- For each $z \in s_0$, w belongs to the domain of $(D - z)^{-1}$.

- The map $z \mapsto (D-z)^{-1}w$ from s_0 into $\mathcal{H}$ is continuous with respect to the norm topology.

By (α), we have $\mathcal{W} \supset \mathcal{L}$. Thus $\mathcal{W}$ is dense in $\mathcal{H}$. We now define

$$R(z)w = (D-z)^{-1}w - K(z)w \quad \text{for} \quad z \in \Gamma \quad \text{and} \quad w \in \mathcal{W}.$$

Then Lemma 3.4 tells us that

$$(T-z)R(z)w = w \quad \text{for} \quad z \in \Gamma \quad \text{and} \quad w \in \mathcal{W}. \tag{5.4}$$

This will be crucial later on.

For each $w \in \mathcal{W}$, since the map $z \mapsto (D-z)^{-1}w$ is continuous on s_0, it is continuous on the entire contour Γ. Thus $(D-z)^{-1}w$ can be integrated over Γ. We claim that

$$\frac{1}{2\pi i}\int_\Gamma (D-z)^{-1}w dz = -Pw \quad \text{for every} \quad w \in \mathcal{W}, \tag{5.5}$$

where P is the orthogonal projection given by (5.1). To prove this, take any $w \in \mathcal{W}$. If $h = \sum_{k=1}^\infty c_k e_k \in \mathcal{L}$, i.e., if $c_k = 0$ for all but a finite number of k's, then

$$\begin{aligned}\left\langle \frac{1}{2\pi i}\int_\Gamma (D-z)^{-1}w dz, h\right\rangle &= \frac{1}{2\pi i}\int_\Gamma \langle (D-z)^{-1}w, h\rangle dz = \frac{1}{2\pi i}\int_\Gamma \langle w, (D^*-\bar{z})^{-1}h\rangle dz \\ &= \sum_{k=1}^\infty \frac{1}{2\pi i}\int_\Gamma \frac{\bar{c}_k\langle w, e_k\rangle}{\lambda_k - z} dz.\end{aligned}$$

Evaluating the contour integrals in the above sum, we find that

$$\left\langle \frac{1}{2\pi i}\int_\Gamma (D-z)^{-1}w dz, h\right\rangle = -\sum_{\mathrm{Re}(\lambda_k)<x_0} \bar{c}_k\langle w, e_k\rangle = -\langle Pw, h\rangle.$$

Since $\mathcal{L}$ is dense in $\mathcal{H}$ and since the map $z \mapsto (D-z)^{-1}w$ is norm continuous on Γ, this proves (5.5). Combining (5.5) and (5.3), we have

$$\frac{1}{2\pi i}\int_\Gamma R(z)w dz = -(P+K)w \quad \text{for every} \quad w \in \mathcal{W}. \tag{5.6}$$

Let $S \in \{T\}'$ be given. To show that $S \in \{P+K\}'$, we first show that $S\mathcal{W} \subset \mathcal{W}$. Indeed for each $w \in \mathcal{W}$, we apply (5.4) to obtain

$$Sw = S(T-z)R(z)w = (T-z)SR(z)w = (D-z)SR(z)w + \sum_{j=1}^n \langle SR(z)w, v_j\rangle u_j$$

for every $z \in s_0$. Thus by (β), if $z \in s_0$, then Sw is in the domain of $(D-z)^{-1}$, and

$$(D-z)^{-1}Sw = SR(z)w + \sum_{j=1}^{n} \langle SR(z)w, v_j\rangle (D-z)^{-1}u_j.$$

Since the maps $z \mapsto R(z)w$ and $z \mapsto (D-z)^{-1}u_j$, $1 \le j \le n$, are norm continuous on s_0, so is the map $z \mapsto (D-z)^{-1}Sw$. This proves the assertion that $S\mathcal{W} \subset \mathcal{W}$.

Using (5.4) again, for each pair of $w \in \mathcal{W}$ and $z \in \Gamma$ we have

$$(T-z)R(z)Sw = Sw \quad \text{and} \quad (T-z)SR(z)w = S(T-z)R(z)w = Sw.$$

Since T has no eigenvalues, these two identities imply that

$$R(z)Sw = SR(z)w \quad \text{for all} \;\; w \in \mathcal{W} \;\; \text{and} \;\; z \in \Gamma.$$

Combining this with (5.6), for each $w \in \mathcal{W}$ we have

$$\begin{aligned} -(P+K)Sw &= \frac{1}{2\pi i}\int_\Gamma R(z)Swdz = \frac{1}{2\pi i}\int_\Gamma SR(z)wdz \\ &= \frac{1}{2\pi i}S\int_\Gamma R(z)wdz = -S(P+K)w. \end{aligned}$$

Since $\mathcal{W}$ is dense in $\mathcal{H}$, we conclude that S commutes with $P+K$. This proves (5.2) and completes the proof of the theorem. □

References

1. J. Conway, *A course in functional analysis*, 2nd edition, Graduate Texts in Mathematics, **96**, Springer-Verlag, New York, 1990.
2. C. Foias, I. Jung, E. Ko and C. Pearcy, *On rank-one perturbations of normal operators*, J. Funct. Anal. **253** (2007), 628-646.
3. E. Hille, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications **31**, American Mathematical Society, New York, 1948.
4. V. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator*, Funct. Anal. and Appl. **7** (1973), 213-214.
5. C. Pearcy, *Topics in Operator Theory*, CBMS Reg. Conf. Ser. Math., **36**, Amer. Math. Soc., Providence, 1978.
6. H. Radjav and P. Rosenthal, *Invariant subspaces*, 2nd edition, Dover Publications, Mineola, NY, 2003.

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