FUGLEDE COMMUTATIONS MODULO LORENTZ IDEALS

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Abstract. We examine Fuglede commutation properties, particularly those in the context of Lorentz ideals, from a new perspective. We also show that Fuglede commutation property fails for a number of ideals.

1. Introduction

In this paper, all Hilbert spaces are assumed to be separable, and all operators are assumed to be bounded.

The famous theorem of Fuglede [11] tells us that if N is a normal operator, then for any operator X the condition $[N, X] = 0$ implies $[N^*, X] = 0$. In [22], Weiss proved the remarkable identity

$$\|[N, X]\|_2 = \|[N^*, X]\|_2, \tag{1.1}$$

where N is any normal operator and $\|\cdot\|_2$ is the Hilbert-Schmidt norm. In particular, for a normal operator N, if $[N, X]$ is a Hilbert-Schmidt operator, then so is $[N^*, X]$. This is called the Fuglede commutation property modulo the Hilbert-Schmidt class $\mathcal{C}_2$. Weiss's identity was later generalized to an inequality for Schatten p-norms for $1 < p < \infty$.

Recall that for any $1 \le p < \infty$, the Schatten p-norm of an operator A is defined by the formula $\|A\|_p = \{\text{tr}((A^*A)^{p/2})\}^{1/p}$. On any Hilbert space $\mathcal{H}$, the Schatten p-class is defined to be the collection of operators $\mathcal{C}_p = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p < \infty\}$.

For each $1 < p < \infty$, there is a constant $0 < C_p < \infty$ such that

$$\|[N^*, X]\|_p \le C_p\|[N, X]\|_p \tag{1.2}$$

whenever N is a normal operator. This was proved by Abdessemed and Davies for the case $2 < p < \infty$ in [1] and by Shulman for the case $1 < p < 2$ in [19]. In particular, Fuglede commutation property holds modulo each Schatten class $\mathcal{C}_p$, $1 < p < \infty$. That is, if N is a normal operator and $1 < p < \infty$, then $[N, X] \in \mathcal{C}_p$ if and only if $[N^*, X] \in \mathcal{C}_p$.

The analogue of (1.2) also holds for Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$, $1 < p < \infty$, which are defined as follows.

Let $\mathcal{H}$ be a Hilbert space. For any given $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{j\ge1} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

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defines a norm for operators on $\mathcal{H}$. Here and in what follows, we write $s_1(A)$, $s_2(A)$, $\dots$, $s_j(A)$, $\dots$ for the s-numbers [12] of the operator A. It is well known that the collection of operators

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

form a norm ideal, for which we cite [12] as our primary reference.

For each $1 \le p < \infty$, the formula

$$\|A\|_p^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j^{(p-1)/p}}$$

also defines a norm for operators on $\mathcal{H}$. Denote

$$\mathcal{C}_p^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^- < \infty\},$$

which is also a norm ideal of operators on $\mathcal{H}$ [12].

In recent decades, Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$ have gained prominence due to the study of non-commutative geometry [3] and other advances in operator theory and operator algebras (see, e.g., [21,5,24,8,9,14,10]). The ideal $\mathcal{C}_1^+$ commands special interest in that it is the domain of every Dixmier trace [6,3,18].

It follows from the results [17, Corollary 3.8] and [17, Theorem 4.5] of Kissin and Shulman that for each $1 < p < \infty$, there are constants $0 < B_p < \infty$ and $0 < D_p < \infty$ such that if N is a normal operator on a Hilbert space $\mathcal{H}$ and if X is any operator on $\mathcal{H}$, then

$$\|[N^*, X]\|_p^+ \le B_p \|[N, X]\|_p^+ \quad \text{and} \tag{1.3}$$

$$\|[N^*, X]\|_p^- \le D_p \|[N, X]\|_p^-. \tag{1.4}$$

These two inequalities imply the following commutation properties: if N is a normal operator and if $1 < p < \infty$, then $[N, X] \in \mathcal{C}_p^+$ if and only if $[N^*, X] \in \mathcal{C}_p^+$; similarly, $[N, X] \in \mathcal{C}_p^-$ if and only if $[N^*, X] \in \mathcal{C}_p^-$.

In this paper, we will take another look at inequalities (1.3) and (1.4) from a different perspective, one that circumvents some of the general Banach-space techniques in previous investigations. Our approach is based on a contraction J, defined on the Hilbert-Schmidt class $\mathcal{C}_2$ of a particular kind of Hilbert spaces. Let us introduce this operator.

Let ν be a compactly supported regular Borel measure on $\mathbf{C}$. For each $n \in \mathbf{N}$, $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ is the Hilbert space of $\mathbf{C}^n$-valued functions that are square-integrable with respect to $d\nu$. Suppose that K is a Hilbert-Schmidt operator on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$. Then there is an $n \times n$ matrix-valued Borel function $G(z, w)$ such that

$$\iint \mathrm{tr}\{G^*(z, w)G(z, w)\} d\nu(z) d\nu(w) < \infty$$

and such that

$$(Kf)(z) = \int G(z,w)f(w)d\nu(w)$$

for every $f \in L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$. We will refer to this G as the *integral kernel* of K. There is a Hilbert-Schmidt operator K' on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ whose integral kernel equals

$$G'(z,w) = \begin{cases} \frac{\bar{z}-\bar{w}}{z-w} G(z,w) & \text{if} \quad z \neq w \\ \\ 0 & \text{if} \quad z = w \end{cases}.$$

For such a pair of Hilbert-Schmidt operators K and K', we define

$$J(K) = K'. \tag{1.5}$$

Obviously, J is a contraction on the $\mathcal{C}_2$ of $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$.

Theorem 1.1. *Given any $1 < p < \infty$, there are constants $0 < C_p^+ < \infty$ and $0 < C_p^- < \infty$ such that for every compactly supported regular Borel measure ν on $\mathbf{C}$, every $n \in \mathbf{N}$ and every Hilbert-Schmidt operator K on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, we have*

$$\|J(K)\|_p^+ \leq C_p^+ \|K\|_p^+ \quad \textit{and} \quad \|J(K)\|_p^- \leq C_p^- \|K\|_p^-.$$

The relevance of Theorem 1.1 to Fuglede commutation properties is best explained in terms of general *symmetrically normed ideals*, which we introduce next.

Following [12], let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j \in \mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A symmetric gauge function (also called *symmetric norming function*) is a map

$$\Phi : \hat{c} \to [0, \infty)$$

that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \dots, 0, \dots\}) = 1$.

(c) $\Phi(\{a_j\}_{j \in \mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j \in \mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [12, page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j \geq 1} \Phi(\{s_1(A), \dots, s_j(A), 0, \dots, 0, \dots\})$$

for operators. On any Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{1.6}$$

is a symmetrically normed ideal [12, page 68].

Let us recall some familiar examples. For each $1 \le p < \infty$, the formula $\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = (\sum_{j=1}^{\infty} |a_j|^p)^{1/p}$ defines a symmetric gauge function on $\hat{c}$, and the corresponding ideal $\mathcal{C}_{\Phi_p}$ defined by (1.6) is just the Schatten class $\mathcal{C}_p$. For each $1 \le p < \infty$, we define the symmetric gauge functions Φ_p^+ and Φ_p^- by the formulas

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\ge 1} \frac{|a_{\pi(1)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + \cdots + j^{-1/p}} \quad \text{and} \quad \Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^{\infty} \frac{|a_{\pi(j)}|}{j^{(p-1)/p}},$$

$\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \ge |a_{\pi(2)}| \ge \cdots \ge |a_{\pi(j)}| \ge \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. Then the ideals $\mathcal{C}_{\Phi_p^+}$ and $\mathcal{C}_{\Phi_p^-}$ defined by (1.6) using Φ_p^+ and Φ_p^- are none other than the Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$ introduced earlier.

Theorem 1.2. *Let Φ be a symmetric gauge function. Suppose that there is a constant $0 < \Lambda = \Lambda(\Phi) < \infty$ such that for every compactly supported regular Borel measure ν on $\mathbf{C}$, every $n \in \mathbf{N}$ and every Hilbert-Schmidt operator K on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, we have*

$$\|J(K)\|_\Phi \le \Lambda \|K\|_\Phi. \tag{1.7}$$

Then for every normal operator N on a Hilbert space $\mathcal{H}$ and every $X \in \mathcal{B}(\mathcal{H})$, we have

$$\|[N^*, X]\|_\Phi \le \Lambda \|[N, X]\|_\Phi. \tag{1.8}$$

We emphasize that it is the same constant Λ that appears in both (1.7) and (1.8).

Since $\mathcal{C}_p^+ = \mathcal{C}_{\Phi_p^+}$ and $\mathcal{C}_p^- = \mathcal{C}_{\Phi_p^-}$, Theorem 1.2 tells us that for each $1 < p < \infty$, inequalities (1.3) and (1.4) respectively hold for the constants $B_p = C_p^+$ and $D_p = C_p^-$, where C_p^+ and C_p^- are provided by Theorem 1.1.

Next, we switch gears and consider the other direction. In [23], Weiss asked whether or not Fuglede commutation property holds modulo the trace class $\mathcal{C}_1$. A negative answer to this question was given in [17]. More precisely, Kissin and Shulman showed that there exist a compact normal operator N and a compact operator X such that $[N, X] \in \mathcal{C}_1$ while $[N^*, X] \notin \mathcal{C}_1$ [17, Corollary 5.9]. Using a general technique, we will show that this means that Fuglede commutation property also fails modulo the ideal $\mathcal{C}_1^+$:

Theorem 1.3. *There exist a normal operator N and a compact operator X such that $[N, X] \in \mathcal{C}_1^+$ while $[N^*, X] \notin \mathcal{C}_1^+$.*

Perhaps it is not coincidental that $\mathcal{C}_1$ and $\mathcal{C}_1^+$ are norm ideals modulo which Fuglede commutation property fails: both ideals carry some kind of trace. We have the ordinary trace on $\mathcal{C}_1$, and we have the Dixmier trace on $\mathcal{C}_1^+$. In each case, the failure of the Fuglede commutation property can be proved by using the particular trace.

Then there is the matter of the Macaev ideal $\mathcal{C}_\infty^-$. Recall that $\mathcal{C}_\infty^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\infty^- < \infty\}$, where

$$\|A\|_\infty^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j}.$$

It is well known that $\mathcal{C}_\infty^-$ is the pre-dual of $\mathcal{C}_1^+$ [12].

Given what we know so far, an obvious question becomes unavoidable: does Fuglede commutation property hold modulo the Macaev ideal $\mathcal{C}_\infty^-$? In other words, for a normal operator N, does the membership $[N, X] \in \mathcal{C}_\infty^-$ imply $[N^*, X] \in \mathcal{C}_\infty^-$? The answer is negative:

Theorem 1.4. *There exist a normal operator N and a compact operator X such that $[N, X] \in \mathcal{C}_\infty^-$ while $[N^*, X] \notin \mathcal{C}_\infty^-$.*

Even though $\mathcal{C}_\infty^-$ does not carry any kind of trace, Theorem 1.4 tells us that modulo $\mathcal{C}_\infty^-$ Fuglede commutation property still fails. Moreover, compared with $\mathcal{C}_1$ and $\mathcal{C}_1^+$, the Macaev ideal $\mathcal{C}_\infty^-$ is at the other end of the scale. That is, $\mathcal{C}_\infty^-$ is a large ideal. In fact, $\mathcal{C}_\infty^-$ is not much smaller than $\mathcal{K}$, the ideal of compact operators. Fuglede's original theorem [11] implies that if N is a normal operator, then $[N, X] \in \mathcal{K}$ if and only if $[N^*, X] \in \mathcal{K}$. In this connection, Theorem 1.4 provides a sharp contrast.

Note that [17, Corollary 5.9], Theorem 1.3 and Theorem 1.4 all deal with "endpoint" cases of one kind or another. This may give the reader the impression that it is rare for Fuglede commutation property to fail. But we can easily generalize the proof of Theorem 1.4 to produce failed Fuglede commutation properties on a wholesale basis. In other words, with very little additional effort, the proof of Theorem 1.4 can be generalized to cover a class of ideals. First, let us introduce these ideals.

Let $\alpha = \{\alpha_j\}$ be a non-increasing sequence of positive numbers starting with $\alpha_1 = 1$. We assume that the sequence α is *binormalizing* [12, page 141], i.e.,

$$\sum_{j=1}^{\infty} \alpha_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \alpha_j = 0.$$

On any Hilbert space $\mathcal{H}$, such a sequence α gives rise to the operator ideal

$$\mathcal{C}_\alpha = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\alpha < \infty\},$$

where the norm $\|\cdot\|_\alpha$ is defined by the formula

$$\|A\|_\alpha = \sum_{j=1}^{\infty} \alpha_j s_j(A).$$

See [12, Section III.15]. We assume that the sequence α satisfies the additional condition that there is a constant $0 < C = C(\alpha) < \infty$ such that

$$\sum_{j=1}^{n^2} \alpha_j \leq C \sum_{j=1}^{n} \alpha_j \quad \text{for every} \quad n \in \mathbf{N}. \tag{1.9}$$

Obviously, the sequence $\{j^{-1}\}$ is binormalizing and satisfies (1.9), and the corresponding ideal $\mathcal{C}_{\{j^{-1}\}}$ is just the Macaev ideal $\mathcal{C}_\infty^-$. For each $0 < t \le 1$, the sequence

$$\left\{\frac{1}{j(1+\log j)^t}\right\}$$

is also binormalizing, and it is easy to verify that it satisfies (1.9). Thus there are plenty of such α. We have the following generalization of Theorem 1.4:

Theorem 1.5. *Let $\alpha = \{\alpha_j\}$ be any binormalizing sequence that satisfies condition* (1.9). *Then there exist a normal operator N and a compact operator X such that $[N, X] \in \mathcal{C}_\alpha$ while $[N^*, X] \notin \mathcal{C}_\alpha$.*

On any Hilbert space $\mathcal{H}$, a binormalizing sequence $\alpha = \{\alpha_j\}$ also gives rise to the operator ideal

$$\mathcal{C}_\alpha^\dagger = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\alpha^\dagger < \infty\},$$

where the norm $\|\cdot\|_\alpha^\dagger$ is defined by the formula

$$\|A\|_\alpha^\dagger = \sup_{k\ge1} \frac{s_1(A) + \cdots + s_k(A)}{\alpha_1 + \cdots + \alpha_k}$$

[12, Theorem III.14.1]. In fact, [12, Theorem III.15.2] tells us that $\mathcal{C}_\alpha^\dagger$ is the dual of the ideal $\mathcal{C}_\alpha$ defined earlier. For example, for the sequence $\{j^{-1}\}$, we have $\mathcal{C}_{\{j^{-1}\}}^\dagger = \mathcal{C}_1^+$, which is the dual of the Macaev ideal $\mathcal{C}_\infty^- = \mathcal{C}_{\{j^{-1}\}}$. For this class of ideals, we have

Theorem 1.6. *Let $\alpha = \{\alpha_j\}$ be any binormalizing sequence that satisfies condition* (1.9). *Then there exist a normal operator N and a compact operator X such that $[N, X] \in \mathcal{C}_\alpha^\dagger$ while $[N^*, X] \notin \mathcal{C}_\alpha^\dagger$.*

Let us now briefly describe the organization of the paper.

Our main idea for the proof of Theorem 1.2 is a particular representation for general normal operators. This representation is not taught in the usual textbooks, but it is particularly convenient for approximations that arise in connection with Fuglede commutations. We give this representation in Section 2.

The representation in Section 2 leads to a particular kind of "integral kernel" for $[N, X]$ when $[N, X] \in \mathcal{C}_2$. Using a well-known result of Voiculescu [21], we show in Section 3 that if N is the operator of multiplication by the coordinate z on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ and if $[N, X] \in \mathcal{C}_2$, then $J([N, X]) = [N^*, X]$.

In Section 4 we recall the projections P_θ on $\mathcal{C}_2$, $0 \le \theta \le 2\pi$, which were introduced in [1]. The $\|\cdot\|_p$-bound for P_θ proved in [1] is a key ingredient in the proof of Theorem 1.1.

Then, by unconventional interpolation techniques, in Section 5 we show that P_θ satisfies similar bounds with respect to the norms $\|\cdot\|_p^+$ and $\|\cdot\|_p^-$, $1 < p < \infty$. Theorem 1.1 is then proved by using the integral formula that represents J in terms of P_θ.

With the preparations in Sections 2 and 3, and with additional technical steps, the proof of Theorem 1.2 is completed in Section 6.

Finally, we prove Theorems 1.3-1.6 in Section 7.

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2. A representation of normal operators

The proof of Theorem 1.2 depends on the particular representation of normal operators given below. Even though the spectral theory of normal operators appears in every textbook on the subject, this particular representation does not. Therefore it will be beneficial to go through its details. We remind the reader that we only consider separable Hilbert spaces in this paper.

Let ν be any compactly supported regular Borel measure on $\mathbf{C}$. We write N_ν for the operator of multiplication by the coordinate function z on the function space $L^2(\mathbf{C}, d\nu)$. It is well known that if ν_1 and ν_2 are mutually absolutely continuous, then N_{ν_1} and N_{ν_2} are unitarily equivalent [4, Section IX.3].

As usual, for any Borel set Δ in $\mathbf{C}$, $L^2(\Delta, d\nu)$ denotes the subspace $\{f \in L^2(\mathbf{C}, d\nu) : f = 0 \text{ on } \mathbf{C}\backslash\Delta\}$ of $L^2(\mathbf{C}, d\nu)$. Furthermore, we will write $N_{\nu,\Delta}$ for the restriction of N_ν to the reducing subspace $L^2(\Delta, d\nu)$.

Let N be a normal operator on a Hilbert space $\mathcal{H}$. Then it is well known that there exists a countable collection of compactly supported regular Borel measures $\{\nu_i : i \in I\}$ on $\mathbf{C}$ such that N is unitarily equivalent to the orthogonal sum $\oplus_{i\in I} N_{\nu_i}$ on $\oplus_{i\in I} L^2(\mathbf{C}, d\nu_i)$. Since the collection $\{\nu_i : i \in I\}$ is countable, there is a compactly supported regular Borel measure ν on $\mathbf{C}$ such that very ν_i, $i \in I$, is absolutely continuous wit respect to ν. In the case where N has a pure point spectrum, we can choose each ν_i to be a single point masses, and therefore ν consists purely of point masses.

Thus if N is a normal operator on a Hilbert space $\mathcal{H}$, then there exist a compactly supported regular Borel measure ν on $\mathbf{C}$ and a countable collection of Borel sets $\{\Delta_i : i \in I\}$ in $\mathbf{C}$ such that N is unitarily equivalent to

$$\bigoplus_{i\in I} N_{\nu,\Delta_i}.$$

For a compactly supported regular Borel measure ν on $\mathbf{C}$, let $L^2(\mathbf{C}, d\nu) \otimes \ell^2$ denote the collection of ℓ^2-valued Borel functions f on $\mathbf{C}$ such that

$$\int_{\mathbf{C}} \|f(z)\|^2 d\nu(z) < \infty.$$

We define the operator $\tilde{N}_\nu$ on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$ by the formula

$$(\tilde{N}_\nu f)(z) = z f(z), \quad f \in L^2(\mathbf{C}, d\nu) \otimes \ell^2.$$

Summarizing the above, below is the particular representation of normal operators that we need in this paper:

Proposition 2.1. *Let N be a normal operator on a Hilbert space $\mathcal{H}$. Then there exist a compactly supported regular Borel measure ν on $\mathbf{C}$ and a reducing subspace $\mathcal{R}$ of the operator $\tilde{N}_\nu$ defined above such that N is unitarily equivalent to the restriction of $\tilde{N}_\nu$ to $\mathcal{R}$. In the case where N has a pure point spectrum, we can require that ν consist purely of point masses.*

If $E : L^2(\mathbf{C}, d\nu) \otimes \ell^2 \to \mathcal{R}$ is the orthogonal projection, then the restriction of $\tilde{N}_\nu$ to $\mathcal{R}$ is naturally identified with the operator $E\tilde{N}_\nu E = \tilde{N}_\nu E$ on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$. Thus we see that for the proof of Theorem 1.2, it suffices to consider pairs of the form $\tilde{N}_\nu$ and $\tilde{X}$, where $\tilde{X}$ acts on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$. But the situation can be further simplified by natural approximations in the space ℓ^2.

For this paper, ℓ^2 means $\ell^2(\mathbf{N})$. That is, any sequence in ℓ^2 is indexed by the natural numbers $\mathbf{N}$. Furthermore, for each $n \in \mathbf{N}$, we identify $\mathbf{C}^n$ with the subspace $\{(z_1, \dots, z_n, 0 \dots, 0, \dots) : z_j \in \mathbf{C}, 1 \le j \le n\}$ of ℓ^2. In this spirit, for each $n \in \mathbf{N}$, we write $N_\nu^{(n)}$ for the restriction of $\tilde{N}_\nu$ to the subspace $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ of $L^2(\mathbf{C}, d\nu) \otimes \ell^2$.

The estimates that matter will be done on the spaces $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, $n \in \mathbf{N}$. The results will then be extended to operators on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$ by the obvious approximation. This approach allows us to take advantage of the fact that Hilbert-Schmidt operators on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ are represented by integral kernels.

3. A relation between integral kernels

Recall that the contraction J is defined by (1.5). Its connection to Fuglede commutations is made through the following proposition, which is in essence a further development of Weiss's identity (1.1).

Proposition 3.1. *Let Y be an operator on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ such that $[N_\nu^{(n)}, Y] \in \mathcal{C}_2$. Then*

$$J([N_\nu^{(n)}, Y]) = [(N_\nu^{(n)})^*, Y].$$

Proof. By a well-known result of Voiculescu, there exists a sequence of finite-rank orthogonal projections $\{F_k\}$ on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ such that

$$\lim_{k \to \infty} F_k = 1 \tag{3.2}$$

in the strong operator topology and

$$\lim_{k \to \infty} \|[N_\nu^{(n)}, F_k]\|_2 = 0. \tag{3.3}$$

See [21, Corollary 2.6]. Note that each $F_k Y F_k$ is a finite-rank operator, and therefore has an integral kernel. For each k, let $H_k(z, w)$, $G_k(z, w)$ and $G'_k(z, w)$ respectively denote the integral kernels of $F_k Y F_k$, $[N_\nu^{(n)}, F_k Y F_k]$ and $[(N_\nu^{(n)})^*, F_k Y F_k]$. Then, of course,

$$G_k(z, w) = (z - w) H_k(z, w) \quad \text{and} \quad G'_k(z, w) = (\bar{z} - \bar{w}) H_k(z, w).$$

From these two relations we deduce

$$G_k'(z,w) = \begin{cases} \frac{\bar{z}-\bar{w}}{z-w} G_k(z,w) & \text{if} \quad z \neq w \\ \\ 0 & \text{if} \quad z = w \end{cases}.$$

That is,

$$J([N_\nu^{(n)}, F_k Y F_k]) = [(N_\nu^{(n)})^*, F_k Y F_k] \tag{3.4}$$

for every k. On the other hand, for each k we have

$$[N_\nu^{(n)}, F_k Y F_k] = [N_\nu^{(n)}, F_k] Y F_k + F_k [N_\nu^{(n)}, Y] F_k + F_k Y [N_\nu^{(n)}, F_k] \quad \text{and}$$
$$[(N_\nu^{(n)})^*, F_k Y F_k] = [(N_\nu^{(n)})^*, F_k] Y F_k + F_k [(N_\nu^{(n)})^*, Y] F_k + F_k Y [(N_\nu^{(n)})^*, F_k].$$

By Weiss's theorem, the condition $[N_\nu^{(n)}, Y] \in \mathcal{C}_2$ implies $[(N_\nu^{(n)})^*, Y] \in \mathcal{C}_2$. Thus from these two identities and (3.2), (3.3) it follows that

$$\lim_{k\to\infty} \|[N_\nu^{(n)}, F_k Y F_k] - [N_\nu^{(n)}, Y]\|_2 = 0 \quad \text{and}$$
$$\lim_{k\to\infty} \|[(N_\nu^{(n)})^*, F_k Y F_k] - [(N_\nu^{(n)})^*, Y]\|_2 = 0.$$

Since J is a contraction on $\mathcal{C}_2$, combining these limits with (3.4), we see that $J([N_\nu^{(n)}, Y]) = [(N_\nu^{(n)})^*, Y]$. This completes the proof. □

4. Boundedness on Schatten classes

The material in this section is a variation of the results in [1, Section 2].

Let ν be a compactly supported regular Borel measure on $\mathbf{C}$, and consider any $n \in \mathbf{N}$. In this section, by the symbol $\mathcal{C}_p$ we mean the collection of Schatten p-class operators on the particular Hilbert space $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$. Thus if $K \in \mathcal{C}_2$, then K has an integral kernel, as defined in Section 3.

For each $0 \leq \theta \leq 2\pi$, we define the subset

$$A_\theta = \{(z,w) : z \neq w \text{ and } 0 \leq \arg(z-w) < \theta\}$$

of $\mathbf{C} \times \mathbf{C}$. If $K \in \mathcal{C}_2$ has integral kernel $G(z,w)$, we define $P_\theta(K)$ to be the operator on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ which has

$$\chi_{A_\theta}(z,w) G(z,w)$$

as its integral kernel. Obviously, P_θ is a contraction on $\mathcal{C}_2$. If we view $\mathcal{C}_2$ as a Hilbert space with the inner product

$$\langle K, L \rangle = \operatorname{tr}(KL^*),$$

$K, L \in \mathcal{C}_2$, then P_θ is an orthogonal projection on $\mathcal{C}_2$.

Lemma 4.1. *Given any $1 < p < \infty$, there is a constant $0 < C_p < \infty$ which depends only on p such that*

$$\|P_\theta(K)\|_p \le C_p \|K\|_p$$

for every $0 \le \theta \le 2\pi$ and every $K \in \mathcal{C}_2 \cap \mathcal{C}_p$.

Proof. If $n = 1$, then this is just a specialized version of [1, Lemma 2.5]. But a careful checking of the estimates in [1, Section 2] finds that the same proof works for all $n \in \mathbf{N}$, with the bounding constant C_p independent of n. □

5. Boundedness of P_θ on Lorentz ideals

Next we prove the analogue of Lemma 4.1 for $\mathcal{C}_2 \cap \mathcal{C}_p^+$, $1 < p < \infty$. This requires unconventional techniques of interpolation.

Given a symmetric gauge $\Phi : \hat{c} \to [0, \infty)$, as defined in Section 1, we need to extend its domain beyond $\hat{c}$ in the following way. Suppose that $\{b_j\}_{j \in \mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., the set $\{j \in \mathbf{N} : b_j \neq 0\}$ is not necessarily finite. Then we define

$$\Phi(\{b_j\}_{j \in \mathbf{N}}) = \sup_{k \ge 1} \Phi(\{b_1, \dots, b_k, 0, \dots, 0, \dots\}).$$

For any sequence $a = \{a_1, \dots, a_j, \dots\}$ of non-negative numbers and $s > 0$, we denote

$$N(a; s) = \mathrm{card}\{j \in \mathbf{N} : a_j > s\}.$$

It is well known that for $s > 0$ and $1 < p < \infty$, we have

$$N(a; s) \le (\Phi_p(a)/s)^p, \tag{5.1}$$

where Φ_p is the symmetric gauge function for the Schatten class $\mathcal{C}_p$.

Lemma 5.1. [8, Lemma 5.6] *Suppose that $1 < p < \infty$. Let $\alpha = \{\alpha_1, \dots, \alpha_k, \dots\}$ be a non-increasing sequence of non-negative numbers. Define*

$$F_p(\alpha) = \sup_{k \ge 1} k^{1/p} \alpha_k.$$

Then

$$\frac{p-1}{p} F_p(\alpha) \le \Phi_p^+(\alpha) \le F_p(\alpha).$$

Lemma 5.2. *Let $a = \{a_1, \dots, a_k, \dots\}$ be a sequence of non-negative numbers. Let $1 < p < \infty$, $0 < M < \infty$ and $0 < \tau < \infty$. If the inequality*

$$N(a; s) \le M(\tau/s)^p \tag{5.2}$$

holds for every $s > 0$, then $\Phi_p^+(a) \le 2M^{1/p}\tau$.

Proof. There is a bijection $\pi : \mathbf{N} \to \mathbf{N}$ such that $a_{\pi(i)} \geq a_{\pi(i+1)}$ for every $i \in \mathbf{N}$. Consider any $i \in \mathbf{N}$ such that $a_{\pi(i)} \neq 0$. Set $s = a_{\pi(i)}/2$. Then by (5.2),

$$i \leq N(a; s) \leq M(\tau/s)^p = M(2\tau)^p a_{\pi(i)}^{-p}.$$

Solving this, we find that

$$a_{\pi(i)} \leq 2M^{1/p}\tau i^{-1/p}$$

if $a_{\pi(i)} \neq 0$. This inequality, of course, also holds in the case $a_{\pi(i)} = 0$. Obviously, this inequality implies $\Phi_p^+(a) \leq 2M^{1/p}\tau$. $\square$

If A is any operator, we denote

$$N(A; s) = \mathrm{card}\{j \in \mathbf{N} : s_j(A) > s\}$$

for $s > 0$. It is known that for operators A, B and $s > 0$, we have

$$N(A + B; s) \leq N(A; s/2) + N(B; s/2)$$

[7, Section 7]. After the above preparation, we now turn to the P_θ defined in Section 4.

Proposition 5.3. *Given any* $1 < p < \infty$, *there is a constant* $0 < B(p) < \infty$ *which depends only on* p *such that*

$$\|P_\theta(K)\|_p^+ \leq B(p)\|K\|_p^+$$

for every $0 \leq \theta \leq 2\pi$ *and every* $K \in \mathcal{C}_2 \cap \mathcal{C}_p^+$ *on* $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, $n \in \mathbf{N}$.

Proof. For the given $1 < p < \infty$, we pick $1 < r' < r < \infty$ such that $r' < p < r$. Given a $K \in \mathcal{C}_2 \cap \mathcal{C}_p^+$, denote

$$R = \frac{p}{p-1}\|K\|_p^+.$$

Then it follows from Lemma 5.1 that

$$s_j(K) \leq R/j^{1/p} \quad \text{for every} \quad j \in \mathbf{N}. \tag{5.3}$$

There are orthonormal sets $\{x_j : j \in \mathbf{N}\}$ and $\{y_j : j \in \mathbf{N}\}$ such that

$$K = \sum_{j=1}^{\infty} s_j(K) x_j \otimes y_j.$$

For each $s > 0$, we define

$$L_s = \sum_{1 \leq j < (R/s)^p} s_j(K) x_j \otimes y_j \quad \text{and} \quad M_s = \sum_{j \geq (R/s)^p} s_j(K) x_j \otimes y_j.$$

Applying (5.1), Lemma 4.1 and (5.3), and using the fact that $r'/p < 1$, we have

$$N(P_\theta(L_s); s) \le s^{-r'} \|P_\theta(L_s)\|_{r'}^{r'} \le C_{r'}^{r'} s^{-r'} \|L_s\|_{r'}^{r'} = C_{r'}^{r'} s^{-r'} \sum_{1\le j<(R/s)^p} (s_j(K))^{r'}$$

$$\le C_{r'}^{r'} s^{-r'} \sum_{1\le j<(R/s)^p} (R/j^{1/p})^{r'} \le C_1 s^{-r'} R^{r'} \{(R/s)^p\}^{1-(r'/p)} = C_1 (R/s)^p. \tag{5.4}$$

The membership $K \in \mathcal{C}_2 \cap \mathcal{C}_p^+$ obviously implies $M_s \in \mathcal{C}_2 \cap \mathcal{C}_p^+$. Applying (5.1), Lemma 4.1 and (5.3), and using the fact that $r/p > 1$, we also have

$$N(P_\theta(M_s); s) \le s^{-r} \|P_\theta(M_s)\|_r^r \le C_r^r s^{-r} \|M_s\|_r^r = C_r^r s^{-r} \sum_{j\ge(R/s)^p} (s_j(K))^r$$

$$\le C_r^r s^{-r} \sum_{j\ge(R/s)^p} (R/j^{1/p})^r \le C_2 s^{-r} R^r \{(R/s)^p\}^{1-(r/p)} = C_2 (R/s)^p. \tag{5.5}$$

Since $K = L_s + M_s$, from (5.4) and (5.5) we obtain

$$N(P_\theta(K); 2s) \le N(P_\theta(L_s); s) + N(P_\theta(M_s); s) \le C_3 (R/s)^p$$

for every $s > 0$, where $C_3 = C_1 + C_2$. A simple rescaling gives us the inequality

$$N(P_\theta(K); s) \le 2^p C_3 (R/s)^p$$

for every $s > 0$. By Lemma 5.2, this means $\|P_\theta(K)\|_p^+ \le 4C_3^{1/p} R$. Since $R = \{p/(p-1)\}\|K\|_p^+$, this completes the proof. □

Proof of Theorem 1.1. (1) We first deduce the constants C_p^+, $1 < p < \infty$, from Proposition 5.3 by using the argument in the proof of [1, Lemma 2.6]. Specifically, we have

$$\frac{\bar{z} - \bar{w}}{z - w} = \exp(-2i \arg(z - w)) = 1 + 2i \int_0^{2\pi} e^{-2i\theta} \chi_{A_\theta}(z, w) d\theta \quad \text{when} \quad z \ne w, \tag{5.6}$$

where

$$A_\theta = \{(u, v) : u \ne v \text{ and } 0 \le \arg(u - v) < \theta\}.$$

By (3.1) and the definition of P_θ in Section 4, (5.6) translates to the operator identity

$$J(K) = P_{2\pi}(K) + 2i \int_0^{2\pi} e^{-2i\theta} P_\theta(K) d\theta. \tag{5.7}$$

To prove the bound $\|J(K)\|_p^+ \le C_p^+ \|K\|_p^+$, we may assume $K \in \mathcal{C}_p^+$, for otherwise the desired bound trivially holds. For $K \in \mathcal{C}_2 \cap \mathcal{C}_p^+$, it follows from (5.7) and Proposition 5.3 that

$$\|J(K)\|_p^+ \le \|P_{2\pi}(K)\|_p^+ + 2 \int_0^{2\pi} \|P_\theta(K)\|_p^+ d\theta \le (1 + 4\pi) B(p) \|K\|_p^+.$$

Thus we can take $(1+4\pi)B(p)$ to be the desired constant C_p^+.

(2) To find the constants C_p^-, we fix a $1 < p < \infty$ and denote $q = p/(p-1)$. It is well known that $\mathcal{C}_q^+$ is the dual of $\mathcal{C}_p^-$. Thus for any operator A, we have

$$\|A\|_p^- = \sup\{|\mathrm{tr}(AF)| : \|F\|_q^+ \le 1 \text{ and } \mathrm{rank}(F) < \infty\}. \tag{5.8}$$

See [12, pages 148, 149 and 125]. For any $K_1, K_2 \in \mathcal{C}_2$ on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, it is obvious that $\mathrm{tr}(J(K_1)K_2) = \mathrm{tr}(K_1 J(K_2))$. Thus for any $K \in \mathcal{C}_2$ on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ and any finite-rank operator F on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ with $\|F\|_q^+ \le 1$, it follows from (1) that

$$|\mathrm{tr}(J(K)F)| = |\mathrm{tr}(KJ(F))| \le \|K\|_p^- \|J(F)\|_q^+ \le \|K\|_p^- C_q^+ \|F\|_q^+ \le C_q^+ \|K\|_p^-.$$

By (5.8), this implies $\|J(K)\|_p^- \le C_q^+ \|K\|_p^-$. That is, we can take the constant C_q^+ provided in (1) to be the desired constant C_p^-. This completes the proof. □

6. Proof of Theorem 1.2

Because of the necessary approximations, there are a few technical steps before we can get to the proof of Theorem 1.2. For the rest of the section, we assume that Φ is a symmetric gauge function for which (1.7) holds. In particular, Λ is the constant in (1.7).

Proposition 6.1. *Let ν be any compactly supported regular Borel measure on $\mathbf{C}$ which consists purely of point masses. Then for every $n \in \mathbf{N}$ and every operator Y on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, we have*

$$\|[(N_\nu^{(n)})^*, Y]\|_\Phi \le \Lambda \|[N_\nu^{(n)}, Y]\|_\Phi. \tag{6.1}$$

Proof. (1) First, suppose that ν has only a finite number of point masses. In this case, $\dim(L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n) < \infty$, which makes the condition $K \in \mathcal{C}_2$ superfluous. Thus (1.7) can be applied to all operators on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$.

Given any operator Y on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$, by Proposition 3.1, we have

$$J([N_\nu^{(n)}, Y]) = [(N_\nu^{(n)})^*, Y].$$

Thus, in this case, (6.1) follows from this identity and (1.7).

(2) Now we consider a general ν that consists purely of point masses. For such a ν, there exist finite sets $F_1, \dots, F_k, \dots$ in $\mathbf{C}$ such that $\nu(\mathbf{C} \backslash F_k) \to 0$ as $k \to \infty$. For each k, let E_k be the operator of multiplication by the function χ_{F_k} on $L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$. Then $E_k \to 1$ in the strong operator topology as $k \to \infty$. Obviously, $E_k(L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n)$ is none other than the subspace $L^2(F_k, d\nu) \otimes \mathbf{C}^n$. The restriction of ν to each F_k consists of a finite number of point masses. Therefore by case (1), we have

$$\|[(N_\nu^{(n)})^*, E_k Y E_k]\|_\Phi \le \Lambda \|[N_\nu^{(n)}, E_k Y E_k]\|_\Phi = \Lambda \|E_k [N_\nu^{(n)}, Y] E_k\|_\Phi.$$

Since $E_k \to 1$ strongly as $k \to \infty$, we also have

$$\|[(N_\nu^{(n)})^*, Y]\|_\Phi \le \limsup_{k\to\infty} \|[(N_\nu^{(n)})^*, E_kYE_k]\|_\Phi.$$

See [12, Theorem III.5.1]. Obviously, (6.1) follows from these inequalities. □

Proposition 6.2. *Suppose that ν is a compactly supported regular Borel measure on $\mathbf{C}$ which consists purely of point masses. Then the inequality*

$$\|[\tilde{N}_\nu^*, Y]\|_\Phi \le \Lambda\|[\tilde{N}_\nu, Y]\|_\Phi \tag{6.2}$$

hold for every operator Y on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$

Proof. For each $n \in \mathbf{N}$, let $E^{(n)} : L^2(\mathbf{C}, d\nu) \otimes \ell^2 \to L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ be the orthogonal projection. Then $E^{(n)}$ commutes with $\tilde{N}_\nu$ and

$$E^{(n)}\tilde{N}_\nu E^{(n)} = \tilde{N}_\nu E^{(n)} = N_\nu^{(n)}.$$

Therefore it follows from Proposition 6.1 that for every operator Y on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$,

$$\|[\tilde{N}_\nu^*, E^{(n)}YE^{(n)}]\|_\Phi \le \Lambda\|[\tilde{N}_\nu, E^{(n)}YE^{(n)}]\|_\Phi = \Lambda\|E^{(n)}[\tilde{N}_\nu, Y]E^{(n)}\|_\Phi.$$

Obviously, $E^{(n)}$ strongly converges to 1 as $n \to \infty$. Thus by a limit argument similar to the one at the end of the proof of Proposition 6.1, the above inequality implies (6.2). □

Proposition 6.3. *Let N be a normal operator on a Hilbert space $\mathcal{H}$. Suppose that N has a pure point spectrum. Then for every operator X on $\mathcal{H}$, we have*

$$\|[N^*, X]\|_\Phi \le \Lambda\|[N, X]\|_\Phi. \tag{6.3}$$

Proof. Since N has a pure point spectrum, by Proposition 2.1, there is a compactly supported regular Borel measure ν which consists purely of point masses and a reducing subspace $\mathcal{R}$ for $\tilde{N}_\nu$ such that N is unitarily equivalent to the restriction of $\tilde{N}_\nu$ to $\mathcal{R}$. That is, there is a unitary operator $U : \mathcal{H} \to \mathcal{R}$ such that $UNU^* = \tilde{N}_\nu|\mathcal{R}$. Define $Y = UXU^*$, which is an operator on $\mathcal{R}$. We then extend Y to an operator on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$ in such a way that $Y = 0$ on $\mathcal{R}^\perp$. Then

$$[\tilde{N}_\nu, Y] = U[N, X]U^* \oplus 0 \quad \text{and} \quad [\tilde{N}_\nu^*, Y] = U[N^*, X]U^* \oplus 0. \tag{6.4}$$

Since ν consists purely of point masses, by Proposition 6.2, we have

$$\|[\tilde{N}_\nu^*, Y]\|_\Phi \le \Lambda\|[\tilde{N}_\nu, Y]\|_\Phi. \tag{6.5}$$

If T is any operator on $\mathcal{H}$, then $\|UTU^* \oplus 0\|_\Phi = \|T\|_\Phi$. Thus (6.3) follows from (6.4) and (6.5). □

Proof of Theorem 1.2. Consider are the following two possibilities.

(1) Suppose that $\mathcal{C}_\Phi \not\subset \mathcal{C}_2^-$. Let $\epsilon > 0$. In this case, by a well-known theorem of Bercovici and Voiculescu [2], there exist a $K_\epsilon \in \mathcal{C}_\Phi$ with $\|K_\epsilon\|_\Phi \leq \epsilon$ and a diagonal operator N_ϵ, i.e., a normal operator with a pure point spectrum, such that $N = N_\epsilon + K_\epsilon$. For any operator X on $\mathcal{H}$, Proposition 6.3 tells us that

$$\|[N_\epsilon^*, X]\|_\Phi \leq \Lambda \|[N_\epsilon, X]\|_\Phi.$$

Therefore

$$\begin{aligned}
\|[N^*, X]\|_\Phi &\leq \|[N_\epsilon^*, X]\|_\Phi + \|[N^*, X] - [N_\epsilon^*, X]\|_\Phi \leq \Lambda\|[N_\epsilon, X]\|_\Phi + 2\|K_\epsilon^*\|_\Phi \|X\| \\
&\leq \Lambda\|[N, X]\|_\Phi + \Lambda\|[N_\epsilon, X] - [N, X]\|_\Phi + 2\|K_\epsilon^*\|_\Phi\|X\| \\
&\leq \Lambda\|[N, X]\|_\Phi + 2\Lambda\|K_\epsilon\|_2\|X\| + 2\|K_\epsilon^*\|_\Phi\|X\| \\
&\leq \Lambda\|[N, X]\|_\Phi + (2\Lambda + 2)\|X\|\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this proves (1.8) in the case where $\mathcal{C}_\Phi \not\subset \mathcal{C}_2^-$.

(2) Suppose that $\mathcal{C}_\Phi \subset \mathcal{C}_2^-$. To prove (1.8), we only need to consider the case where $\|[N, X]\|_\Phi < \infty$. That is, we assume $[N, X] \in \mathcal{C}_\Phi$.

By Proposition 2.1, there is a compactly supported regular Borel measure ν and a reducing subspace $\mathcal{R}$ for $\tilde{N}_\nu$ such that N is unitarily equivalent to the restriction of $\tilde{N}_\nu$ to $\mathcal{R}$. That is, there is a unitary operator $U : \mathcal{H} \to \mathcal{R}$ such that $UNU^* = \tilde{N}_\nu\big|\mathcal{R}$. Define $Y = UXU^*$, which is an operator on $\mathcal{R}$. We then extend Y to an operator on $L^2(\mathbf{C}, d\nu) \otimes \ell^2$ in such a way that $Y = 0$ on $\mathcal{R}^\perp$. Then

$$[\tilde{N}_\nu, Y] = U[N, X]U^* \oplus 0 \quad \text{and} \quad [\tilde{N}_\nu^*, Y] = U[N^*, X]U^* \oplus 0.$$

As we explained in the proof of Proposition 6.3, (1.8) will follow if we can show that

$$\|[\tilde{N}_\nu^*, Y]\|_\Phi \leq \Lambda\|[\tilde{N}_\nu, Y]\|_\Phi. \tag{6.7}$$

By the condition $[N, X] \in \mathcal{C}_\Phi$, we have $[\tilde{N}_\nu, Y] \in \mathcal{C}_\Phi$.

To prove (6.7), let $E^{(n)} : L^2(\mathbf{C}, d\nu) \otimes \ell^2 \to L^2(\mathbf{C}, d\nu) \otimes \mathbf{C}^n$ be the orthogonal projection, $n \in \mathbf{N}$. Since $E^{(n)}$ strongly converges to 1 as $n \to \infty$, (6.7) will follow if we can show that

$$\|[\tilde{N}_\nu^*, E^{(n)}YE^{(n)}]\|_\Phi \leq \Lambda\|[\tilde{N}_\nu, E^{(n)}YE^{(n)}]\|_\Phi \quad (= \Lambda\|E^{(n)}[\tilde{N}_\nu, Y]E^{(n)}\|_\Phi)$$

for every $n \in \mathbf{N}$. Equivalently, it suffices to prove that

$$\|[(N_\nu^{(n)})^*, E^{(n)}YE^{(n)}]\|_\Phi \leq \Lambda\|[N_\nu^{(n)}, E^{(n)}YE^{(n)}]\|_\Phi \tag{6.8}$$

for every $n \in \mathbf{N}$. Since $[\tilde{N}_\nu, Y] \in \mathcal{C}_\Phi$, for each $n \in \mathbf{N}$ we have

$$[N_\nu^{(n)}, E^{(n)}YE^{(n)}] = E^{(n)}[\tilde{N}_\nu, Y]E^{(n)} \in \mathcal{C}_\Phi.$$

By the assumption $\mathcal{C}_\Phi \subset \mathcal{C}_2^-$ and the inclusion $\mathcal{C}_2^- \subset \mathcal{C}_2$, each $[N_\nu^{(n)}, E^{(n)}YE^{(n)}]$ is a Hilbert-Schmidt operator. Thus (6.8) follows from Propositions 3.1 and (1.7). This completes the proof of Theorem 1.2. □

7. Proofs of Theorems 1.3-1.6

We begin with the following example due to Shulman and Turowska:

Example 7.1. [20, Example 8.5] Consider the Hilbert space $L^2(D, dA)$, where $D = \{z \in \mathbf{C} : |z| < 1\}$, the unit disc in $\mathbf{C}$, and dA is the area measure on $\mathbf{C}$. Let M be the normal operator on $L^2(D, dA)$ defined by the formula

$$(Mf)(z) = zf(z), \tag{7.1}$$

$f \in L^2(D, dA)$. Define the operator

$$(Yf)(z) = \int_D \frac{f(w)}{z-w} dA(w), \tag{7.2}$$

$f \in L^2(D, dA)$. Then Y is in the Schatten p-class for every $p > 2$. (In fact, this Y is known to be in the Lorentz ideal $\mathcal{C}_2^+$ [5]; also see [24].) It is obvious that $[M, Y]$ is the rank-one operator $1 \otimes 1$ on $L^2(D, dA)$. Therefore $\mathrm{tr}[M, Y] \neq 0$. This nonzero trace is an obstruction to the membership of $[M^*, Y]$ in $\mathcal{C}_1$ [22, page 15]. To see this, write $A = 2^{-1}(M + M^*)$ and $B = (2i)^{-1}(M - M^*)$. If it were true that $[M^*, Y] \in \mathcal{C}_1$, then we would have $[A, Y] \in \mathcal{C}_1$ and $[B, Y] \in \mathcal{C}_1$. Since Y is compact and A, B are self-adjoint, by a well-know result of Helton and Howe [13, Lemma 1.3], we would have $\mathrm{tr}[A, Y] = 0$ and $\mathrm{tr}[B, Y] = 0$. This contradicts the fact that $\mathrm{tr}[M, Y] \neq 0$. Hence $[M^*, Y] \notin \mathcal{C}_1$. □

For the proof of Theorem 1.3, we will use the following general fact:

Lemma 7.2. *Let A be an operator on a Hilbert space $\mathcal{H}$. On the space*

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \oplus \cdots,$$

define the operator

$$B = A \oplus \frac{1}{2}A \oplus \frac{1}{3}A \oplus \cdots \oplus \frac{1}{k}A \oplus \cdots.$$

Then $B \in \mathcal{C}_1^+$ if and only if $A \in \mathcal{C}_1$.

Proof. Obviously, B is compact if and only if A is compact. Therefore, to prove the lemma, it suffices to consider the case where A is a compact operator. Thus there are orthonormal sets $\{x_j : j \in \mathbf{N}\}$ and $\{y_j : j \in \mathbf{N}\}$ in $\mathcal{H}$ such that

$$A = \sum_{j=1}^{\infty} s_j(A) x_j \otimes y_j.$$

If $A \in \mathcal{C}_1$, then $\sum_{j=1}^{\infty} s_j(A) = \|A\|_1 < \infty$, and we can rewrite B in the form

$$B = \sum_{j=1}^{\infty} s_j(A) B_j,$$

where

$$B_j = (x_j \otimes y_j) \oplus \frac{1}{2}(x_j \otimes y_j) \oplus \frac{1}{3}(x_j \otimes y_j) \oplus \cdots \oplus \frac{1}{k}(x_j \otimes y_j) \oplus \cdots$$

for each $j \in \mathbf{N}$. Obviously, $\|B_j\|_1^+ = 1$ for every $j \in \mathbf{N}$. Hence the condition $A \in \mathcal{C}_1$ implies $B \in \mathcal{C}_1^+$.

Now suppose that $B \in \mathcal{C}_1^+$. Then we have $|B| \in \mathcal{C}_1^+$. Moreover,

$$|B| = |A| \oplus \frac{1}{2}|A| \oplus \frac{1}{3}|A| \oplus \cdots \oplus \frac{1}{k}|A| \oplus \cdots \tag{7.3}$$

and

$$|A| = \sum_{j=1}^{\infty} s_j(A) y_j \otimes y_j. \tag{7.4}$$

For each $n \in \mathbf{N}$, define the operator

$$C_n = \sum_{j=1}^{n} s_j(A)\{(y_j \otimes y_j) \oplus \frac{1}{2}(y_j \otimes y_j) \oplus \frac{1}{3}(y_j \otimes y_j) \oplus \cdots \oplus \frac{1}{k}(y_j \otimes y_j) \oplus \cdots\}.$$

From (7.3) and (7.4) we see that $|B| \geq C_n$ for every $n \in \mathbf{N}$. Let Tr_ω be any Dixmier trace [6,3,18]. It is obvious that

$$\mathrm{Tr}_\omega\left((y_j \otimes y_j) \oplus \frac{1}{2}(y_j \otimes y_j) \oplus \frac{1}{3}(y_j \otimes y_j) \oplus \cdots \oplus \frac{1}{k}(y_j \otimes y_j) \oplus \cdots\right) = 1$$

for every $j \in \mathbf{N}$. Combining this with the fact that $|B| \geq C_n$, we have

$$\mathrm{Tr}_\omega(|B|) \geq \mathrm{Tr}_\omega(C_n) = \sum_{j=1}^{n} s_j(A).$$

Since this holds for every $n \in \mathbf{N}$, it follows that $\|A\|_1 \leq \mathrm{Tr}_\omega(|B|) \leq \||B|\|_1^+ = \|B\|_1^+ < \infty$. This completes the proof. □

Proof of Theorem 1.3. Let M and Y be given by (7.1) and (7.2) respectively. We define the operators

$$N = M \oplus M \oplus M \oplus \cdots \oplus M \oplus \cdots \quad \text{and}$$
$$X = Y \oplus \frac{1}{2}Y \oplus \frac{1}{3}Y \oplus \cdots \oplus \frac{1}{k}Y \oplus \cdots .$$

Then obviously N is normal and X is compact. Moreover,

$$[N, X] = [M, Y] \oplus \frac{1}{2}[M, Y] \oplus \frac{1}{3}[M, Y] \oplus \cdots \oplus \frac{1}{k}[M, Y] \oplus \cdots \quad \text{and}$$
$$[N^*, X] = [M^*, Y] \oplus \frac{1}{2}[M^*, Y] \oplus \frac{1}{3}[M^*, Y] \oplus \cdots \oplus \frac{1}{k}[M^*, Y] \oplus \cdots .$$

Since $[M, Y] \in \mathcal{C}_1$ while $[M^*, Y] \notin \mathcal{C}_1$, it follows from Lemma 7.2 that

$$[N, X] \in \mathcal{C}_1^+ \quad \text{while} \quad [N^*, X] \notin \mathcal{C}_1^+.$$

This proves Theorem 1.3. □

It was shown in [15] that there does not exist any constant $0 < C < \infty$ such that the operator-norm inequality

$$\|[N^*, X]\| \le C\|[N, X]\|$$

holds whenever N is a normal operator. Also see [16] for further results. In this regard, the operator M defined by (7.1) also serves as a nice example:

Lemma 7.3. *Let M be the normal operator defined by* (7.1). *For each $n \in \mathbf{N}$, there is a compact operator X_n on $L^2(D, dA)$ such that*

$$\|[M, X_n]\| \le 2, \quad \operatorname{rank}([M, X_n]) < \infty \quad \textit{and} \quad \|[M^*, X_n]\| \ge n/\pi. \tag{7.5}$$

Proof. Denote $K = [M^*, Y]$, where Y is defined by (7.2). Since Y is compact, so is K. Therefore there are orthonormal sets $\{f_j : j \in \mathbf{N}\}$ and $\{g_j : j \in \mathbf{N}\}$ such that

$$K = \sum_{j=1}^{\infty} s_j(K) f_j \otimes g_j.$$

Since $K \notin \mathcal{C}_1$, we have $\sum_{j=1}^{\infty} s_j(K) = \infty$.

Let an $n \in \mathbf{N}$ be given. Then there is an $m_n \in \mathbf{N}$ such that

$$\sum_{j=1}^{m_n} s_j(K) \ge 2n.$$

With this m_n, we define

$$L_n = \sum_{j=1}^{m_n} g_j \otimes f_j.$$

Obviously,

$$\operatorname{tr}(KL_n) = \sum_{j=1}^{m_n} s_j(K) \ge 2n.$$

Since $L^\infty(D, dA)$ is dense in $L^2(D, dA)$, for each j, there are sequences $\{\varphi_j^{(k)}\}_{k\in\mathbf{N}}$ and $\{\psi_j^{(k)}\}_{k\in\mathbf{N}}$ in $L^\infty(D, dA)$ such that

$$\lim_{k\to\infty} \|f_j - \varphi_j^{(k)}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|g_j - \psi_j^{(k)}\| = 0. \tag{7.6}$$

For each $k \in \mathbf{N}$, define

$$L_n^{(k)} = \sum_{j=1}^{m_n} \psi_j^{(k)} \otimes \varphi_j^{(k)}.$$

Then (7.6) implies that $\|L_n - L_n^{(k)}\|_1 \to 0$ as $k \to \infty$. Consequently,

$$\lim_{k\to\infty} \mathrm{tr}(KL_n^{(k)}) = \mathrm{tr}(KL_n) = \sum_{j=1}^{m_n} s_j(K) \geq 2n. \tag{7.7}$$

Since $\{f_j : j \in \mathbf{N}\}$ and $\{g_j : j \in \mathbf{N}\}$ are orthonormal sets, we have $\|L_n\| = 1$. Thus the fact that $\|L_n - L_n^{(k)}\|_1 \to 0$ also implies that $\|L_n^{(k)}\| \to 1$ as $k \to \infty$. Combining this limit with (7.7), we see that there is a $k_n \in \mathbf{N}$ such that

$$|\mathrm{tr}(KL_n^{(k_n)})| \geq n \quad \text{and} \quad \|L_n^{(k_n)}\| \leq 2. \tag{7.8}$$

As usual, if $\varphi \in L^\infty(D, dA)$, we write M_φ for the operator of multiplication by the function φ on $L^\infty(D, dA)$. We now define the operator

$$X_n = \sum_{j=1}^{m_n} M^*_{\varphi_j^{(k_n)}} Y M_{\psi_j^{(k_n)}}. \tag{7.9}$$

Let us verify that X_n has all the desired properties. First of all, since Y is a compact operator and since the sequences $\{\varphi_j^{(k)}\}_{k\in\mathbf{N}}$ and $\{\psi_j^{(k)}\}_{k\in\mathbf{N}}$ are in $L^\infty(D, dA)$, X_n is a compact operator. Then from the fact $[M, Y] = 1 \otimes 1$ we obtain

$$[M, X_n] = \sum_{j=1}^{m_n} M^*_{\varphi_j^{(k_n)}} [M, Y] M_{\psi_j^{(k_n)}} = \sum_{j=1}^{m_n} \bar{\varphi}_j^{(k_n)} \otimes \bar{\psi}_j^{(k_n)}. \tag{7.10}$$

Therefore $\mathrm{rank}([M, X_n]) \leq m_n < \infty$. For every $f \in L^2(D, dA)$, we have

$$[M, X_n]f = \overline{(L_n^{(k_n)})^* \bar{f}}.$$

Note that $\|\bar{f}\| = \|f\|$ for every $f \in L^2(D, dA)$. Thus it follows from (7.8) that

$$\|[M, X_n]\| \leq 2. \tag{7.11}$$

On the other hand, since $[M^*, Y] = K$, we have

$$\langle [M^*, X_n]1, 1\rangle = \sum_{j=1}^{m_n} \langle M^*_{\varphi_j^{(k_n)}} K M_{\psi_j^{(k_n)}} 1, 1\rangle = \sum_{j=1}^{m_n} \langle K\psi_j^{(k_n)}, \varphi_j^{(k_n)}\rangle = \mathrm{tr}(KL_n^{(k_n)}). \tag{7.12}$$

Combining this with (7.8) and with the fact that $\|1\|^2 = \pi$, we see that $\|[M^*, X_n]\| \geq n/\pi$. This verifies (7.5) and completes the proof. □

Proof of Theorem 1.4. Let M be the operator defined by (7.1). For each $n \in \mathbf{N}$, let X_n be the compact operator provided by Lemma 7.3. Recall from (7.10) that for every $n \in \mathbf{N}$, we have $\mathrm{rank}([M, X_n]) \leq m_n$.

For each $n \in \mathbf{N}$, we pick a natural number $r(n) \geq 3 + m_n$ such that

$$\log r(n) \geq \|X_n\|. \tag{7.13}$$

If B is any operator and $k \in \mathbf{N}$, we denote

$$B^{[k]} = \overbrace{B \oplus \cdots \oplus B}^{k \text{ copies}}.$$

For $k \in \mathbf{N}$ and any operators A, B, it is obvious that

$$[A^{[k]}, B^{[k]}] = [A, B]^{[k]}.$$

Thus

$$\mathrm{rank}([M^{[r(n)]}, X_n^{[r(n)]}]) \leq r(n)m_n \quad \text{and} \quad \|[M^{[r(n)]}, X_n^{[r(n)]}]\| \leq 2$$

for each $n \in \mathbf{N}$, where the second $\leq$ follows from (7.5). Consequently,

$$\|[M^{[r(n)]}, X_n^{[r(n)]}]\|_\infty^- \leq \sum_{j=1}^{r(n)m_n} \frac{2}{j} \leq 2(1 + \log\{r(n)m_n\}) \leq 2 + 4\log r(n). \tag{7.14}$$

By (7.5), we have

$$s_1([M^*, X_n]) = \|[M^*, X_n]\| \geq n/\pi,$$

$n \in \mathbf{N}$. Hence

$$s_j([(M^*)^{[r(n)]}, X_n^{[r(n)]}]) \geq n/\pi \quad \text{for every} \quad 1 \leq j \leq r(n).$$

Consequently,

$$\|[(M^*)^{[r(n)]}, X_n^{[r(n)]}]\|_\infty^- \geq \frac{n}{\pi} \sum_{j=1}^{r(n)} \frac{1}{j} \geq \frac{n}{\pi} \log r(n). \tag{7.15}$$

We now define

$$N = \bigoplus_{n=1}^{\infty} M^{[r(n^3)]} \quad \text{and} \quad X = \bigoplus_{n=1}^{\infty} \frac{1}{n^2 \log r(n^3)} X_{n^3}^{[r(n^3)]}.$$

Let us verify that this pair of operators has the properties promised in Theorem 1.4.

First of all, N is obviously a normal operator. By (7.13), we have $\|X_{n^3}^{[r(n^3)]}\|/\log r(n^3) \leq 1$ for every $n \in \mathbf{N}$. By this norm bound and the fact that each X_n is compact, X is a compact operator. Applying (7.14), we have

$$\begin{aligned}\|[N, X]\|_\infty^- &= \left\| \bigoplus_{n=1}^{\infty} \frac{1}{n^2 \log r(n^3)} [M^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}] \right\|_\infty^- \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^2 \log r(n^3)} \|[M^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}]\|_\infty^- \leq \sum_{n=1}^{\infty} \frac{2 + 4\log r(n^3)}{n^2 \log r(n^3)} < \infty.\end{aligned}$$

That is, $[N, X] \in \mathcal{C}_\infty^-$. On the other hand, for every $k \in \mathbf{N}$ we have

$$\begin{aligned}\|[N^*, X]\|_\infty^- &= \left\| \bigoplus_{n=1}^{\infty} \frac{1}{n^2 \log r(n^3)} [(M^*)^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}] \right\|_\infty^- \\ &\geq \frac{1}{k^2 \log r(k^3)} \|[(M^*)^{[r(k^3)]}, X_{k^3}^{[r(k^3)]}]\|_\infty^- \geq \frac{(k^3/\pi) \log r(k^3)}{k^2 \log r(k^3)} = \frac{k}{\pi},\end{aligned}$$

where the second $\geq$ follows from (7.15). Since this holds for every $k \in \mathbf{N}$, it follows that $[N^*, X] \notin \mathcal{C}_\infty^-$. This completes the proof. $\square$

Proof of Theorem 1.5. As in the proof of Theorem 1.4, let M be the normal operator defined by (7.1). And again, for each $n \in \mathbf{N}$, let X_n be the compact operator provided by Lemma 7.3. Recall that $\text{rank}([M, X_n]) \leq m_n$.

Given a binormalizing sequence $\alpha = \{\alpha_j\}$ satisfying (1.9), we denote

$$\sigma(n) = \sum_{j=1}^{n} \alpha_j, \quad n \in \mathbf{N}.$$

Thus (1.9) translates to

$$\sigma(n^2) \leq C\sigma(n) \quad \text{for every} \quad n \in \mathbf{N}. \tag{7.16}$$

Since α is binormalizing, we have $\sigma(n) \to \infty$ as $n \to \infty$. This enables us to pick, for each $n \in \mathbf{N}$, a natural number $r(n) > m_n$ such that

$$\sigma(r(n)) \geq \|X_n\|. \tag{7.17}$$

As in the proof of Theorem 1.4, we have

$$\mathrm{rank}([M^{[r(n)]}, X_n^{[r(n)]}]) \le r(n)m_n \quad \text{and} \quad \|[M^{[r(n)]}, X_n^{[r(n)]}]\| \le 2,$$

$n \in \mathbf{N}$. Combining these facts with (7.16), we have

$$\|[M^{[r(n)]}, X_n^{[r(n)]}]\|_\alpha \le 2\sigma(r(n)m_n) \le 2\sigma((r(n))^2) \le 2C\sigma(r(n)). \tag{7.18}$$

As we explained in the proof of Theorem 1.4,

$$s_j([(M^*)^{[r(n)]}, X_n^{[r(n)]}]) \ge n/\pi \quad \text{for every} \;\; 1 \le j \le r(n).$$

Consequently,

$$\|[(M^*)^{[r(n)]}, X_n^{[r(n)]}]\|_\alpha \ge (n/\pi)\sigma(r(n)). \tag{7.19}$$

Now define

$$N = \bigoplus_{n=1}^{\infty} M^{[r(n^3)]} \quad \text{and} \quad X = \bigoplus_{n=1}^{\infty} \frac{1}{n^2\sigma(r(n^3))} X_{n^3}^{[r(n^3)]}.$$

Let us verify that this pair of operators has the promised properties.

Again, N is obviously a normal operator. By (7.17), we have $\|X_{n^3}^{[r(n^3)]}\|/\sigma(r(n^3)) \le 1$ for every $n \in \mathbf{N}$. Since X_n is compact for every $n \in \mathbf{N}$, X is a compact operator. Applying (7.18), we have

$$\begin{aligned}\|[N, X]\|_\alpha &= \left\| \bigoplus_{n=1}^{\infty} \frac{1}{n^2\sigma(r(n^3))} [M^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}] \right\|_\alpha \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^2\sigma(r(n^3))} \|[M^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}]\|_\alpha \le \sum_{n=1}^{\infty} \frac{2C\sigma(r(n^3))}{n^2\sigma(r(n^3))} < \infty.\end{aligned}$$

That is, $[N, X] \in \mathcal{C}_\alpha$. On the other hand, for every $k \in \mathbf{N}$ we have

$$\begin{aligned}\|[N^*, X]\|_\alpha &= \left\| \bigoplus_{n=1}^{\infty} \frac{1}{n^2\sigma(r(n^3))} [(M^*)^{[r(n^3)]}, X_{n^3}^{[r(n^3)]}] \right\|_\alpha \\ &\ge \frac{1}{k^2\sigma(r(k^3))} \|[(M^*)^{[r(k^3)]}, X_{k^3}^{[r(k^3)]}]\|_\alpha \ge \frac{(k^3/\pi)\sigma(r(k^3))}{k^2\sigma(r(k^3))} = \frac{k}{\pi},\end{aligned}$$

where the second $\ge$ follows from (7.19). Since this holds for every $k \in \mathbf{N}$, it follows that $[N^*, X] \notin \mathcal{C}_\alpha$. This completes the proof. □

Proof of Theorem 1.6. Let M and Y be given by (7.1) and (7.2) respectively. Define

$$\begin{aligned}&N = M \oplus M \oplus \cdots \oplus M \oplus \cdots \quad \text{and} \\ &X = \alpha_1 Y \oplus \alpha_2 Y \oplus \cdots \oplus \alpha_j Y \oplus \cdots .\end{aligned}$$

Let us verify that these operators have the desired properties. As before, N is obviously normal. Since Y is compact and since $\alpha_j \to 0$ as $j \to \infty$, X is compact. Since $[M, Y] = 1 \otimes 1$ on $L^2(D, dA)$, we have

$$[N, X] = \alpha_1(1 \otimes 1) \oplus \alpha_2(1 \otimes 1) \oplus \cdots \oplus \alpha_j(1 \otimes 1) \oplus \cdots ,$$

which is obviously in the ideal $\mathcal{C}_\alpha^\dagger$.

To show that $[N^*, X] \notin \mathcal{C}_\alpha^\dagger$, we denote

$$\sigma(n) = \sum_{j=1}^{n} \alpha_j, \quad n \in \mathbf{N},$$

as in the proof of Theorem 1.5. Since the sequence $\alpha = \{\alpha_j\}$ satisfies (1.9), inequality (7.16) again holds. Writing $K = [M^*, Y]$ as in the proof of Lemma 7.3, we have

$$[N^*, X] = \alpha_1 K \oplus \alpha_2 K \oplus \cdots \oplus \alpha_j K \oplus \cdots .$$

Thus for every $n \in \mathbf{N}$,

$$\sum_{k=1}^{n^2} s_k([N^*, X]) \geq \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} s_i(K) = \sigma(n) \sum_{i=1}^{n} s_i(K) \geq \frac{\sigma(n^2)}{C} \sum_{i=1}^{n} s_i(K),$$

where the last $\geq$ follows from (7.16). By the definition of $\| \cdot \|_\alpha^\dagger$, we now have

$$\|[N^*, X]\|_\alpha^\dagger \geq \frac{1}{\sigma(n^2)} \sum_{k=1}^{n^2} s_k([N^*, X]) \geq \frac{1}{C} \sum_{i=1}^{n} s_i(K)$$

for every $n \in \mathbf{N}$. Since $K \notin \mathcal{C}_1$, it follows that $\|[N^*, X]\|_\alpha^\dagger = \infty$, i.e., $[N^*, X] \notin \mathcal{C}_\alpha^\dagger$. This completes the proof. □

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