ON THE ESSENTIAL COMMUTANT OF THE TOEPLITZ ALGEBRA ON THE BERGMAN SPACE

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Abstract. Let $\mathcal{T}$ be the C^*-algebra generated by the Toeplitz operators $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ on the Bergman space of the unit ball. We show that the essential commutant of $\mathcal{T}$ equals $\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\} + \mathcal{K}$, where $\mathrm{VO}_{\mathrm{bdd}}$ is the collection of bounded functions of vanishing oscillation on $\mathbf{B}$ and $\mathcal{K}$ denotes the collection of compact operators on $L^2_a(\mathbf{B}, dv)$.

1. Introduction

Suppose that $\mathcal{Z}$ is a collection of bounded operators on a Hilbert space $\mathcal{H}$. Recall that the *essential commutant* of $\mathcal{Z}$ is defined to be

$$\mathrm{EssCom}(\mathcal{Z}) = \{A \in \mathcal{B}(\mathcal{H}) : [A, T] \text{ is compact for every } T \in \mathcal{Z}\}.$$

Obviously, $\mathrm{EssCom}(\mathcal{Z})$ is always a norm-closed unital operator algebra that contains $\mathcal{K}(\mathcal{H})$, the collection of compact operators on $\mathcal{H}$. If $\mathcal{Z}$ is closed under the *-operation, then $\mathrm{EssCom}(\mathcal{Z})$ is a C^*-algebra.

The story about essential commutant began with the classic papers [11,13], where Johnson, Parrott and Popa characterized the essential commutant of every von Neumann algebra. Ever since, essential-commutant problems have always attracted attention. The purpose of this paper is to determine the essential commutant of the Toeplitz algebra on the *Bergman space* of the unit ball. Before stating our result, let us first explain the historical background of this problem.

Recall that the essential-commutant problem for the Toeplitz algebra on the *Hardy space* was solved long ago. To avoid confusion with the notation that we will use later, let us write $\mathcal{T}^{\mathrm{Hardy}}$ for the Toeplitz algebra on the Hardy space H^2. Also, write T_f^{Hardy} for Toeplitz operators on H^2. In [2], Davidson showed that

$$\mathrm{EssCom}(\mathcal{T}^{\mathrm{Hardy}}) = \{T_f^{\mathrm{Hardy}} : f \in \mathrm{QC}\} + \mathcal{K}^{\mathrm{Hardy}}, \tag{1.1}$$

where $\mathrm{QC} = \mathrm{VMO} \cap L^\infty$ and $\mathcal{K}^{\mathrm{Hardy}}$ is the collection of compact operators on the Hardy space. Later, this result was generalized in [4,9] to the setting of the Hardy space $H^2(S)$ of the unit sphere in $\mathbf{C}^n$. (For the latest developments along this line, see [3,5].) Moreover, we now even know that the essential commutant of $\{T_f^{\mathrm{Hardy}} : f \in \mathrm{QC}\}$ is strictly larger than $\mathcal{T}^{\mathrm{Hardy}}$ [15,17]. In other words, the image of $\mathcal{T}^{\mathrm{Hardy}}$ in the Calkin algebra does not satisfy the double-commutant relation.

Keywords: Bergman space, Toeplitz algebra, essential commutant.

In view of these Hardy-space results, it may appear surprising that in the decades since [2], no progress has been made on the corresponding essential-commutant problems on the Bergman space. This paper will fundamentally change the situation by proving the Bergman-space analogue of (1.1). At the same time, the material of the paper helps explain why it took so long for progress to be made on the Bergman space: the Bergman-space case deals with a different kind of structure and requires ideas and techniques that were developed only in the last few years.

Let us turn to the technical details of the paper. Let $\mathbf{B}$ denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. Let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. The Bergman space $L^2_a(\mathbf{B}, dv)$ is the subspace

$$\{h \in L^2(\mathbf{B}, dv) : h \text{ is analytic on } \mathbf{B}\}$$

of $L^2(\mathbf{B}, dv)$. Write P for the orthogonal projection from $L^2(\mathbf{B}, dv)$ onto $L^2_a(\mathbf{B}, dv)$. For each $f \in L^\infty(\mathbf{B}, dv)$, we have the Toeplitz operator T_f defined by the formula

$$T_f h = P(fh), \quad h \in L^2_a(\mathbf{B}, dv).$$

The *Toeplitz algebra* $\mathcal{T}$ on the Bergman space $L^2_a(\mathbf{B}, dv)$ is the C^*-algebra generated by the collection of Toeplitz operators

$$\{T_f : f \in L^\infty(\mathbf{B}, dv)\}.$$

In a recent paper [19], the structure of the Toeplitz algebra $\mathcal{T}$ was explored in some depth. It is the knowledge gained there that enables us to determine $\mathrm{EssCom}(\mathcal{T})$ in this paper.

The natural description of $\mathrm{EssCom}(\mathcal{T})$ involves functions of *vanishing oscillation* on $\mathbf{B}$, which were first introduced by Berger, Coburn and Zhu in [1]. These functions are defined in terms of the *Bergman metric* on $\mathbf{B}$. For each $z \in \mathbf{B}\backslash\{0\}$, we have the Möbius transform φ_z given by the formula

$$\varphi_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle}\left\{z - \frac{\langle \zeta, z\rangle}{|z|^2}z - (1 - |z|^2)^{1/2}\left(\zeta - \frac{\langle \zeta, z\rangle}{|z|^2}z\right)\right\}, \quad \zeta \in \mathbf{B},$$

[14, page 25]. In the case $z = 0$, we define $\varphi_0(\zeta) = -\zeta$. Then the formula

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B},$$

gives us the Bergman metric on $\mathbf{B}$. Recall that a function g on $\mathbf{B}$ is said to have vanishing oscillation if it satisfies the following two conditions: (1) g is continuous on $\mathbf{B}$; (2) the limit

$$\lim_{|z|\uparrow 1} \sup_{\beta(z,w)\le 1} |g(z) - g(w)| = 0$$

holds. We will write VO for the collection of functions of vanishing oscillation on $\mathbf{B}$. Moreover, we write

$$\mathrm{VO}_{\mathrm{bdd}} = \mathrm{VO} \cap L^\infty(\mathbf{B}, dv).$$

In other words, $\mathrm{VO_{bdd}}$ denotes the collection of functions of vanishing oscillation that are also bounded (hence the subscript "bdd") on $\mathbf{B}$.

Let us write $\mathcal{K}$ for the collection of compact operators on $L^2_a(\mathbf{B}, dv)$. It is well known that $\mathcal{T} \supset \mathcal{K}$. The following is the main result of the paper:

Theorem 1.1. *The essential commutant of the Toeplitz algebra $\mathcal{T}$ equals*

$$\{T_g : g \in \mathrm{VO_{bdd}}\} + \mathcal{K}.$$

It was already known in [1] that if $g \in \mathrm{VO_{bdd}}$, then the operators $PM_g(1-P)$ and $(1-P)M_gP$ are compact on $L^2(\mathbf{B}, dv)$. Therefore it follows that

$$\mathrm{EssCom}(\mathcal{T}) \supset \{T_g : g \in \mathrm{VO_{bdd}}\} + \mathcal{K}. \tag{1.2}$$

Our task for this paper is to prove the inclusion

$$\mathrm{EssCom}(\mathcal{T}) \subset \{T_g : g \in \mathrm{VO_{bdd}}\} + \mathcal{K}, \tag{1.3}$$

which will take quite a few steps. We conclude the Introduction by giving an outline of the proof of (1.3), which also serves to explain the organization of the paper.

The proof of (1.3) involves a "reverse bound" for certain matrix norms. While the bound itself is elementary, we will prove it in Section 2 as our first step.

An ingredient that is essential to the proof is the modified kernel $\psi_{z,i}$, $z \in \mathbf{B}$ and $i \in \mathbf{Z}_+$. Modified kernels were previously used in various spaces [7,8,18]. We recall these functions and other relevant material in Section 3. The section ends with Proposition 3.7, which is a step in the proof of (1.3). This proposition allows us to "harvest" a specific piece of a non-compact operator for analysis: if A is a non-compact operator on $L^2_a(\mathbf{B}, dv)$, then there is an operator of the form

$$F = \sum_{z\in\Gamma} \psi_{z,i} \otimes e_z, \tag{1.4}$$

where Γ is a set that is separated with respect to the Bergman metric and $\{e_z : z \in \Gamma\}$ is an orthonormal set, such that AF is not compact.

Our proof requires a special class of operators to test the membership $A \in \mathrm{EssCom}(\mathcal{T})$. In fact, our test operators have the form

$$T = \sum_{z\in\Gamma} c_z \psi_{z,i} \otimes \psi_{\gamma(z),i},$$

where the set Γ is separated, the map $\gamma : \Gamma \to \mathbf{B}$ satisfies the condition $\beta(z, \gamma(z)) \le C$ for every $z \in \Gamma$, and the set of coefficients $\{c_z : z \in \Gamma\}$ is bounded. But to use such a T as a test operator, we must know that $T \in \mathcal{T}$. In Section 4, we show that such a T is *weakly localized*. Therefore by a result from [19], we have $T \in \mathcal{T}$. Then, using the fact that

$T \in \mathcal{T}$, we prove Lemma 4.7, which provides conditions for excluding an operator from $\text{EssCom}(\mathcal{T})$, another necessary step in the proof of Theorem 1.1.

Using the modified kernel $\psi_{z,i}$, in Section 5 we introduced the modified Berezin transforms $\mathcal{B}_i(X)$ of any operator X, $i \in \mathbf{Z}_+$. When $i = 0$, $\mathcal{B}_0(X)$ is just the usual Berezin transform of X. But our proof uses $\mathcal{B}_i(X)$ for an $i \geq 8n+1$, which necessitates the introduction of the modified Berezin transforms. Using the fact that $T \in \mathcal{T}$ from Section 4, we show that if $X \in \text{EssCom}(\mathcal{T})$, then $\mathcal{B}_i(X) \in \text{VO}_{\text{bdd}}$ for every $i \in \mathbf{Z}_+$, which is also a step in the proof of Theorem 1.1.

Section 6 contains some specific estimates required in the proof, which involve the condition $i \geq 8n+1$.

With all these preparations, in Section 7 we prove Theorem 1.1; more specifically, we prove inclusion (1.3). To do that, fix an $i \geq 8n+1$. Let $X \in \text{EssCom}(\mathcal{T})$ be given. Since we know that $\mathcal{B}_i(X) \in \text{VO}_{\text{bdd}}$, it suffices to show that $X - T_{\mathcal{B}_i(X)}$ is compact. If $X - T_{\mathcal{B}_i(X)}$ were not compact, then there would be an F of the form (1.4) such that $(X - T_{\mathcal{B}_i(X)})F$ is not compact. Then, by a lengthy deduction process that involves the bounds provided in Sections 2 and 6, we show that the non-compactness of $(X - T_{\mathcal{B}_i(X)})F$ implies that

$$A = (X - T_{\mathcal{B}_i(X)})^*(X - T_{\mathcal{B}_i(X)})$$

satisfies the conditions in Lemma 4.7. By that lemma, we would have to conclude that $A \notin \text{EssCom}(\mathcal{T})$, which is obviously a contradiction.

2. A reverse matrix bound

For each $k \in \mathbf{N}$, let M_k denote the collection of $k \times k$ matrices. Each $A \in M_k$ is naturally identified with the corresponding operator on (the column Hilbert space) $\mathbf{C}^k$. We write $\mathcal{D}_k$ for the collection of $k \times k$ diagonal matrices. Let DP_k denote the collection of $k \times k$ diagonal matrices whose diagonal entries are either 1 or 0. That is, each $E \in \text{DP}_k$ is a diagonal matrix that is also a projection. For every $A \in M_k$, $k \in \mathbf{N}$, we define

$$C_k(A) = \max\{\|[A, E]\| : E \in \text{DP}_k\}.$$

Lemma 2.1. *For $D \in \mathcal{D}_k$ and $A \in M_k$, $k \in \mathbf{N}$, we have $\|[A, D]\| \leq 4\|D\|C_k(A)$.*

Proof. It suffices to show that $\|[A, D]\| \leq 2\|D\|C_k(A)$ for $D \in \mathcal{D}_k$ with real entries. Given such a D, we list its diagonal entries in the ascending order as

$$d_1 \leq \cdots \leq d_k.$$

Then there is a permutation $\sigma(1), \ldots, \sigma(k)$ of $1, \ldots, k$ such that for every $i \in \{1, \ldots, k\}$, d_i is in the intersection of the $\sigma(i)$-th row and the $\sigma(i)$-th column of D. Note that $d_1 \geq -\|D\|$ and $d_k \leq \|D\|$. For each $i \in \{1, \ldots, k\}$, let E_i be the $k \times k$ diagonal matrix whose entry in the intersection of the $\sigma(j)$-th row and the $\sigma(j)$-th column equals 1 for every $i \leq j \leq k$, and whose other entries are all 0. Then E_1 is the $k \times k$ identity matrix, and we have

$$D = d_1E_1 + (d_2 - d_1)E_2 + \cdots + (d_k - d_{k-1})E_k.$$

Accordingly, for any $A \in M_k$, we have

$$\begin{aligned}\|[A,D]\| &\le |d_1|\|[A,E_1]\| + (d_2-d_1)\|[A,E_2]\| + \cdots + (d_k-d_{k-1})\|[A,E_k]\| \\ &\le (d_2-d_1)C_k(A) + \cdots + (d_k-d_{k-1})C_k(A) = (d_k-d_1)C_k(A) \le 2\|D\|C_k(A)\end{aligned}$$

as promised. □

For each $k \in \mathbf{N}$, let VD_k be the collection of $A \in M_k$ whose diagonal entries are all zero. That is, VD stands for *vanishing diagonal*.

Lemma 2.2. *If $A \in \mathrm{VD}_k$, $k \in \mathbf{N}$, then $\|A\| \le \sup\{\|[A,D]\| : D \in \mathcal{D}_k, \|D\| \le 1\}$.*

Proof. Let $k \in \mathbf{N}$ be given. For each $\theta \in \mathbf{R}$, let V_θ be the $k \times k$ diagonal matrix whose diagonal entries are, in the natural order, $e^{i\theta}, e^{i2\theta}, \dots, e^{ik\theta}$. Let $A \in \mathrm{VD}_k$. Since the diagonal entries of A are all zero, elementary calculation shows that

$$\int_0^{2\pi} V_\theta^* A V_\theta d\theta = 0.$$

Hence

$$A = \frac{1}{2\pi}\int_0^{2\pi}(A - V_\theta^* A V_\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}[A, V_\theta^*]V_\theta d\theta.$$

Thus for each $A \in \mathrm{VD}_k$, there is a $\theta(A) \in [0, 2\pi]$ such that $\|[A, V_{\theta(A)}^*]V_{\theta(A)}\| \ge \|A\|$. Since $\|[A, V_{\theta(A)}^*]\| = \|[A, V_{\theta(A)}^*]V_{\theta(A)}\|$, the lemma follows. □

The following bound is a step in the proof of (1.3):

Proposition 2.3. *If $A \in \mathrm{VD}_k$, $k \in \mathbf{N}$, then $\|A\| \le 4C_k(A)$.*

Proof. The conclusion follows immediately from Lemmas 2.2 and 2.1. □

3. Modified kernel and separated sets

As usual, we write $H^\infty(\mathbf{B})$ for the collection of bounded analytic functions on $\mathbf{B}$. Also, we write $\|h\|_\infty = \sup_{\zeta \in \mathbf{B}} |h(\zeta)|$ for $h \in H^\infty(\mathbf{B})$. Naturally, we consider $H^\infty(\mathbf{B})$ as a subset of the Bergman space $L^2_a(\mathbf{B}, dv)$.

Recall that the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle \zeta, z\rangle)^{n+1}}, \quad z, \zeta \in \mathbf{B},$$

gives us the normalized reproducing kernel for $L^2_a(\mathbf{B}, dv)$. For each integer $i \ge 0$, we define the modified kernel function

$$\psi_{z,i}(\zeta) = \frac{(1-|z|^2)^{\{(n+1)/2\}+i}}{(1-\langle \zeta, z\rangle)^{n+1+i}}, \quad z, \zeta \in \mathbf{B}.$$

If we introduce the multiplier

$$m_z(\zeta) = \frac{1 - |z|^2}{1 - \langle \zeta, z \rangle}$$

for each $z \in \mathbf{B}$, then we have the relation $\psi_{z,i} = m_z^i k_z$. As we have seen previously [7,8,18], this modification gives $\psi_{z,i}$ a faster "decaying rate" than k_z, which is what makes the estimate in Lemma 6.4 possible.

Obviously, $\|m_z\|_\infty \le 1 + |z| < 2$ for every $z \in \mathbf{B}$. Therefore for every $i \in \mathbf{Z}_+$ we have $\|\psi_{z,i}\| \le 2^i$. On the other hand, $\langle \psi_{z,i}, k_z \rangle = 1$. Hence the inequality

$$1 \le \|\psi_{z,i}\| \le 2^i \tag{3.1}$$

holds for all $i \in \mathbf{Z}_+$ and $z \in \mathbf{B}$.

Definition 3.1. (1) For $z \in \mathbf{B}$ and $r > 0$, denote $D(z, r) = \{\zeta \in \mathbf{B} : \beta(z, \zeta) < r\}$.
(2) Let $a > 0$. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z, a) \cap D(w, a) = \emptyset$ for all distinct elements z, w in Γ.
(3) A subset Γ of $\mathbf{B}$ is simply said to be separated if it is a-separated for some $a > 0$.

Lemma 3.2. [19,Lemma 2.2] *Let Γ be a separated set in $\mathbf{B}$.*
(a) *For each $0 < R < \infty$, there is a natural number $N = N(\Gamma, R)$ such that* $\mathrm{card}\{v \in \Gamma : \beta(u, v) \le R\} \le N$ *for every $u \in \Gamma$.*
(b) *For every pair of $z \in \mathbf{B}$ and $\rho > 0$, there is a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $i \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta(\varphi_u(z), \varphi_v(z)) > \rho$.*

Recall that for each $z \in \mathbf{B}$, the formula

$$(U_z h)(\zeta) = k_z(\zeta) h(\varphi_z(\zeta)), \quad \zeta \in \mathbf{B} \ \text{ and } \ h \in L_a^2(\mathbf{B}, dv),$$

defines a unitary operator.

Lemma 3.3. [19,Lemma 2.6] *Given any separated set Γ in $\mathbf{B}$, there exists a constant $0 < B(\Gamma) < \infty$ such that the following estimate holds: Let $\{h_u : u \in \Gamma\}$ be functions in $H^\infty(\mathbf{B})$ such that $\sup_{u \in \Gamma} \|h_u\|_\infty < \infty$, and let $\{e_u : u \in \Gamma\}$ be any orthonormal set. Then*

$$\left\| \sum_{u \in \Gamma} (U_u h_u) \otimes e_u \right\| \le B(\Gamma) \sup_{u \in \Gamma} \|h_u\|_\infty.$$

Suppose that Γ is a separated set in $\mathbf{B}$, $i \in \mathbf{Z}_+$ and $z \in \mathbf{B}$. For each such triple $\{\Gamma, z, i\}$, we define the operator

$$E_{\Gamma;z;i} = \sum_{u \in \Gamma} \psi_{\varphi_u(z),i} \otimes \psi_{\varphi_u(z),i}.$$

Corollary 3.4. *Let Γ be a separated set in $\mathbf{B}$ and let $i \in \mathbf{Z}_+$. Given any $R > 0$, there is a constant $C_{3.4} = C_{3.4}(R)$ such that the inequality $\|E_{\Gamma;z;i}\| \le C_{3.4}$ holds for every $z \in \mathbf{B}$ satisfying the condition $\beta(z, 0) \le R$.*

Proof. Let $u \in \Gamma$ and $z \in \mathbf{B}$. By [14,Theorem 2.2.2], we have $\psi_{\varphi_u(z),i} = U_u h_{u,z;i}$, where

$$h_{u,z;i} = \left(\frac{1 - \langle u, z \rangle}{|1 - \langle u, z \rangle|} \right)^{n+1} (m_{\varphi_u(z)} \circ \varphi_u)^i k_z.$$

Obviously, $\|h_{u,z;i}\|_\infty \le 2^i \|k_z\|_\infty \le C(1-|z|)^{-(n+1)/2}$, where $C = 2^{i+(1/2)(n+1)}$. Combining this with the fact that $|z| = (e^{2\beta(z,0)} - 1)/(e^{2\beta(z,0)} + 1)$, we obtain

$$\|h_{u,z;i}\|_\infty \le C(e^{2\beta(z,0)} + 1)^{(n+1)/2}.$$

Applying Lemma 3.3, we see that the constant $C_{3.4} = C_{3.4}(R) = B^2(\Gamma)C^2(e^{2R} + 1)^{n+1}$ suffices for the given $R > 0$. □

Let $d\lambda$ denote the standard Möbius-invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

Proposition 3.5. [18,Proposition 4.1] *For each integer $i \ge 0$, there exist scalars $0 < c \le C < \infty$ which are determined by i and n such that the self-adjoint operator*

$$R_i = \int \psi_{z,i} \otimes \psi_{z,i} d\lambda(z)$$

satisfies the operator inequality $cP \le R_i \le CP$ on the Hilbert space $L^2(\mathbf{B}, dv)$.

Lemma 3.6. [6,Lemma 4.2] *Let $\{X, \mathcal{M}, \mu\}$ be a (finite or infinite) measure space. Let H be a separable Hilbert space and let $\mathcal{K}(H)$ denote the collection of compact operators on H. Suppose that $F : X \to \mathcal{K}(H)$ is a weakly $\mathcal{M}$-measurable map. If*

$$\int_X \|F(x)\| d\mu(x) < \infty,$$

then

$$K = \int_X F(x) d\mu(x)$$

is a compact operator on the Hilbert space H.

Using the above facts, we can "decompose" non-compactness on the Bergman space:

Proposition 3.7. *Let A be a bounded, non-compact operator on $L^2_a(\mathbf{B}, dv)$. Then for every $i \in \mathbf{Z}_+$, there is a 1-separated set Γ in $\mathbf{B}$ such that the operator*

$$A \sum_{z \in \Gamma} \psi_{z,i} \otimes e_z$$

is not compact, where $\{e_z : z \in \Gamma\}$ is any orthonormal set.

Proof. Let $\mathcal{L}$ be a subset of $\mathbf{B}$ which is maximal with respect to the property that

$$D(u,1)\cap D(v,1)=\emptyset \quad \text{for all} \quad u\neq v \quad \text{in} \quad \mathcal{L}. \tag{3.2}$$

The maximality of $\mathcal{L}$ implies that

$$\bigcup_{u\in\mathcal{L}} D(u,2)=\mathbf{B}. \tag{3.3}$$

Now define the function

$$\Phi=\sum_{u\in\mathcal{L}}\chi_{D(u,2)}$$

on $\mathbf{B}$. By (3.2) and Lemma 3.2(a), there is a natural number $N\in\mathbf{N}$ such that

$$\operatorname{card}\{v\in\mathcal{L}: D(u,2)\cap D(v,2)\neq\emptyset\}\leq N$$

for every $u\in\mathcal{L}$. This and (3.3) together tell us that the inequality

$$1\leq\Phi\leq N \tag{3.4}$$

holds on the unit ball $\mathbf{B}$.

Given any integer $i\in\mathbf{Z}_+$, we define the operator

$$R_i'=\int\Phi(z)\psi_{z,i}\otimes\psi_{z,i}d\lambda(z)=\sum_{u\in\mathcal{L}}\int_{D(u,2)}\psi_{z,i}\otimes\psi_{z,i}d\lambda(z).$$

Then (3.4) implies that $R_i\leq R_i'\leq NR_i$. Applying Proposition 3.5, we see that the operator inequality $c\leq R_i'\leq NC$ holds on the Bergman space $L_a^2(\mathbf{B},dv)$. That is, R_i' is both bounded and invertible on $L_a^2(\mathbf{B},dv)$. By the Möbius invariance of both β and $d\lambda$, we have

$$R_i'=\sum_{u\in\mathcal{L}}\int_{D(0,2)}\psi_{\varphi_u(z),i}\otimes\psi_{\varphi_u(z),i}d\lambda(z)=\int_{D(0,2)}E_{\mathcal{L};z;i}d\lambda(z).$$

Let A be a bounded, non-compact operator on $L_a^2(\mathbf{B},dv)$. Since R_i' is invertible, the operator

$$AR_i'=\int_{D(0,2)}AE_{\mathcal{L};z;i}d\lambda(z)$$

is not compact. By Corollary 3.4, there is a finite bound for $\|E_{\mathcal{L};z;i}\|$, $z\in D(0,2)$. Thus Lemma 3.6 tells us that there is a $w\in D(0,2)$ such that $AE_{\mathcal{L};w;i}$ is not compact, i.e.,

$$A\sum_{u\in\mathcal{L}}\psi_{\varphi_u(w),i}\otimes\psi_{\varphi_u(w),i} \tag{3.5}$$

is not compact. Since $\mathcal{L}$ is 1-separated, Lemma 3.2(b) provides a partition $\mathcal{L}=\mathcal{L}_1\cup\cdots\cup\mathcal{L}_m$ such that for each $j\in\{1,\dots,m\}$, we have $\beta(\varphi_u(w),\varphi_v(w))>2$ for all $u\neq v$ in $\mathcal{L}_j$. That is, each set $\Gamma_j = \{\varphi_u(w) : u \in \mathcal{L}_j\}$ is also 1-separated, $j \in \{1, \dots, m\}$. The non-compactness of (3.5) implies that there is a $j_0 \in \{1, \dots, m\}$ such that the operator

$$A \sum_{z \in \Gamma_{j_0}} \psi_{z,i} \otimes \psi_{z,i} \tag{3.6}$$

is not compact. Finally, let $\{e_z : z \in \Gamma_{j_0}\}$ be any orthonormal set and define the operator

$$F = \sum_{z \in \Gamma_{j_0}} \psi_{z,i} \otimes e_z.$$

By Corollary 3.4, F is a bounded operator. Since (3.6) equals AFF^*, we conclude that AF is not compact. This completes the proof. $\square$

4. Membership criterion

To prove Theorem 1.1, we obviously need plenty of operators to test the membership $A \in \mathrm{EssCom}(\mathcal{T})$. In view of Proposition 3.7, it is easy to understand that the most suitable "test operators" are discrete sums constructed from the modified kernel $\psi_{z,i}$. But then a problem immediate arises: how do we know that these operators belong to $\mathcal{T}$?

It was first discovered in [20] that *localization* is a powerful tool for analyzing operators on reproducing-kernel Hilbert spaces. Recently, Isralowitz, Mitkovski and Wick further explored this idea in [10] by introducing the notion of *weakly localized operators* on the Bergman space. This in turn led to the author's work [19], which settles the membership problem mentioned in the preceding paragraph.

Definition 4.1. Let a positive number $(n-1)/(n+1) < s < 1$ be given.
(a) A bounded operator B on the Bergman space $L^2_a(\mathbf{B}, dv)$ is said to be s-weakly localized if it satisfies the conditions

$$\sup_{z \in \mathbf{B}} \int |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z \in \mathbf{B}} \int |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r\to\infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0 \quad \text{and}$$

$$\lim_{r\to\infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let $\mathcal{A}_s$ denote the collection of s-weakly localized operators defined as above.
(c) Let $C^*(\mathcal{A}_s)$ denote the C^*-algebra generated by $\mathcal{A}_s$.

Theorem 4.2. [19,Theorem 1.3] *For every* $(n-1)/(n+1) < s < 1$ *we have* $C^*(\mathcal{A}_s) = \mathcal{T}$.

Lemma 4.3. [19,Lemma 2.3] *For all $u, v, x, y \in \mathbf{B}$ we have*

$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_v(y)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_v(y)\rangle|} \le 2e^{\beta(x,0)+\beta(y,0)}\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}.$$

Corollary 4.4. *For every triple of $z, z', \zeta \in \mathbf{B}$ we have $|k_z(\zeta)| \le (2e^{\beta(z,z')})^{n+1}|k_{z'}(\zeta)|$.*

Proof. Given any triple of $z, z', \zeta \in \mathbf{B}$, we apply Lemma 4.3 to the case where $u = z'$, $v = \zeta$, $x = \varphi_{z'}(z)$ and $y = 0$, which gives us

$$\frac{(1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}}{|1-\langle z,\zeta\rangle|} \le 2e^{\beta(\varphi_{z'}(z),0)}\frac{(1-|z'|^2)^{1/2}(1-|\zeta|^2)^{1/2}}{|1-\langle z',\zeta\rangle|}.$$

Since $\beta(\varphi_{z'}(z),0) = \beta(z,z')$, this implies the conclusion of the corollary. □

We now present the "test operators" mentioned earlier.

Proposition 4.5. *Let Γ be a separated set in $\mathbf{B}$. Suppose that $\gamma : \Gamma \to \mathbf{B}$ is a map for which there is a $0 < C < \infty$ such that*

$$\beta(u,\gamma(u)) \le C \tag{4.1}$$

for every $u \in \Gamma$. Then for every $i \in \mathbf{Z}_+$ and every bounded set of complex coefficients $\{c_u : u \in \Gamma\}$, the operator

$$T = \sum_{u\in\Gamma} c_u \psi_{u,i} \otimes \psi_{\gamma(u),i} \tag{4.2}$$

belongs to the Toeplitz algebra $\mathcal{T}$.

Proof. We need the Forelli-Rudin estimates in [10]. Fix an $(n-1)/(n+1) < s < 1$ and set

$$A = \sup_{x\in\mathbf{B}} \int |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \quad \text{and}$$

$$B(R) = \sup_{x\in\mathbf{B}} \int_{\beta(w,x)\ge R} |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)$$

for $R > 0$. From [10,page 1558] we know that $A < \infty$ and $B(R) \to 0$ as $R \to \infty$. To show that $T \in \mathcal{T}$, by Theorem 4.2, it suffices to show that $T \in \mathcal{A}_s$.

Thus we need to verify that

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0. \tag{4.3}$$

To prove this, let us write $C_1 = \sup\{|c_u| : u \in \Gamma\}$, which is assumed to be finite. For every pair of $z, w \in \mathbf{B}$ we have

$$
\begin{aligned}
|\langle Tk_z, k_w\rangle| &\le C_1 \sum_{u\in\Gamma} |\langle k_z, \psi_{\gamma(u),i}\rangle\langle \psi_{u,i}, k_w\rangle| \\
&\le 2^{2i} C_1 (1-|z|^2)^{\frac{n+1}{2}} (1-|w|^2)^{\frac{n+1}{2}} \sum_{u\in\Gamma} |k_{\gamma(u)}(z) k_u(w)|. \qquad (4.4)
\end{aligned}
$$

By the assumption on Γ, there is an $a > 0$ such that $D(u, a) \cap D(v, a) = \emptyset$ for all $u \neq v$ in Γ. By Corollary 4.4, we have $|k_u(w)| \le (2e^a)^{n+1}|k_x(w)|$ for every $x \in D(u, a)$. Similarly, by (4.1) and Corollary 4.4, we have $|k_{\gamma(u)}(z)| \le (2e^{a+C})^{n+1}|k_x(z)|$ for every $x \in D(u, a)$. Substituting these in (4.4), we find that if we set $C_2 = 2^{2i}C_1(4e^{2a+C})^{n+1}$, then

$$
|\langle Tk_z, k_w\rangle| \le C_2 \sum_{u\in\Gamma} |\langle k_z, k_{x_u}\rangle\langle k_{x_u}, k_w\rangle|,
$$

where $x_u \in D(u, a)$ for every $u \in \Gamma$. Thus for any $z \in \mathbf{B}$ and $r > 0$, we have

$$
\begin{aligned}
&\int_{\mathbf{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \\
&\le \int_{\beta(z,w)\ge r} C_2 \sum_{u\in\Gamma} \int_{D(u,a)} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \frac{d\lambda(x)}{\lambda(D(u,a))} \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \\
&\le \frac{C_2}{\lambda(D(0,a))} \int \int_{\beta(z,w)\ge r} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)d\lambda(x).
\end{aligned}
$$

The rest of the proof resembles the proof of [10,Proposition 2.2]: Write the last integral in the form of $I_1 + I_2$, where

$$
I_1 = \int_{\beta(z,x)<r/2} \int_{\beta(z,w)\ge r} \quad \text{and} \quad I_2 = \int_{\beta(z,x)\ge r/2} \int_{\beta(z,w)\ge r}.
$$

If $\beta(z, x) < r/2$ and $\beta(z, w) \ge r$, then $\beta(x, w) \ge r/2$. Hence

$$
I_1 \le \int |\langle k_z, k_x\rangle| \left(\frac{1-|x|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} \int_{\beta(w,x)\ge r/2} |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)d\lambda(x).
$$

Since the inner integral is at most $B(r/2)$, we have $I_1 \le AB(r/2)$. On the other hand,

$$
I_2 \le \int_{\beta(z,x)\ge r/2} |\langle k_z, k_x\rangle| \left(\frac{1-|x|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} \int |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)d\lambda(x).
$$

Since the inner integral does not exceed A, we have $I_2 \le AB(r/2)$. Hence

$$\int_{\mathbf{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \le \frac{2C_2AB(r/2)}{\lambda(D(0,a))}$$

for all $z \in \mathbf{B}$ and $r > 0$, which proves (4.3). By the same argument, if we replace T by T^* in (4.3), the limit also holds. This completes the verification that $T \in \mathcal{A}_s$. □

For an operator A on a Hilbert space $\mathcal{H}$, we write $\|A\|_{\mathcal{Q}}$ for its *essential norm*, i.e.,

$$\|A\|_{\mathcal{Q}} = \inf\{\|A-K\| : K \text{ is any compact operator on } \mathcal{H}\}.$$

Next we use operators of the form (4.2) to test membership in EssCom($\mathcal{T}$). To do this, we also need a familiar lemma:

Lemma 4.6. [12,Lemma 2.1] *Let $\{B_i\}$ be a sequence of compact operators on a Hilbert space $\mathcal{H}$ satisfying the following conditions*:
(a) *Both sequences $\{B_i\}$ and $\{B_i^*\}$ converge to 0 in the strong operator topology.*
(b) *The limit $\lim_{i\to\infty}\|B_i\|$ exists.*
Then there exist natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$ such that the sum

$$\sum_{m=1}^{\infty} B_{i(m)} = \lim_{N\to\infty}\sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology and we have

$$\left\|\sum_{m=1}^{\infty} B_{i(m)}\right\|_{\mathcal{Q}} = \lim_{i\to\infty}\|B_i\|.$$

Lemma 4.7. *Let A be a bounded operator on $L_a^2(\mathbf{B}, dv)$. Suppose that there exist an $i \in \mathbf{Z}_+$, a separated set Γ in $\mathbf{B}$ and a $c > 0$ such that the following two conditions hold*:
(1) *There is a sequence $E_1, \ldots, E_j, \ldots$ of finite subsets of Γ such that*

$$\left\|\left[A, \sum_{z\in E_j}\psi_{z,i}\otimes\psi_{z,i}\right]\right\| \ge c \quad \textit{for every } \ j \ge 1.$$

(2) $\inf\{|z| : z \in E_j\} \to 1$ *as $j \to \infty$.*
Then A does not belong to the essential commutant of $\mathcal{T}$.

Proof. For convenience, for each subset E of Γ we denote

$$S_E = \sum_{z\in E}\psi_{z,i}\otimes\psi_{z,i}.$$

By Corollary 3.4, S_Γ is a bounded operator. Therefore $\|S_E\| \leq \|S_\Gamma\| < \infty$ for every $E \subset \Gamma$.

Since each E_j is a finite set and since (2) holds, passing to a subsequence if necessary, we may assume that $E_j \cap E_k = \emptyset$ for all $j \neq k$ in $\mathbf{N}$. Denote $\mathcal{E} = \cup_{j=1}^\infty E_j$. Then

$$\langle S_{\mathcal{E}} h, h\rangle = \sum_{j=1}^\infty \langle S_{E_j} h, h\rangle$$

for every $h \in L_a^2(\mathbf{B}, dv)$. Obviously, this implies that the sequence of operators $\{S_{E_j}\}$ converges to 0 weakly. But since $S_{E_j} \geq 0$ for every j, from this weak convergence we deduce that the operator sequence $\{S_{E_j}\}$ converges to 0 strongly. Define

$$B_j = [A, S_{E_j}]$$

for every $j \in \mathbf{N}$. Then we have the strong convergence $B_j \to 0$ and $B_j^* \to 0$ as $j \to \infty$, i.e., condition (a) in Lemma 4.6 is satisfied by these operators. Since $\|B_j\| \leq 2\|A\|\|S_\Gamma\|$ for every j, there is a subsequence $\{j_\nu\}$ of the natural numbers such that the limit

$$d = \lim_{\nu \to \infty} \|B_{j_\nu}\|$$

exists. That is, the subsequence $\{B_{j_\nu}\}$ satisfies both condition (a) and condition (b) in Lemma 4.6. By that lemma, there are $\nu(1) < \nu(2) < \cdots < \nu(m) < \cdots$ such that the sum

$$B = \sum_{m=1}^\infty B_{j_{\nu(m)}} = \lim_{N \to \infty} \sum_{m=1}^N B_{j_{\nu(m)}}$$

converges in the strong operator topology with $\|B\|_{\mathcal{Q}} = d$. By condition (1), $d \geq c > 0$. Thus B is not a compact operator.

For each $N \in \mathbf{N}$, define

$$T_N = \sum_{m=1}^N S_{E_{j_{\nu(m)}}}.$$

If we set $\mathcal{F} = \cup_{m=1}^\infty E_{j_{\nu(m)}}$, then we obviously have the weak convergence $T_N \to S_{\mathcal{F}}$ as $N \to \infty$. Thus, taking weak limit, we obtain

$$B = \lim_{N \to \infty} \sum_{m=1}^N [A, S_{E_{j_{\nu(m)}}}] = \lim_{N \to \infty} [A, T_N] = [A, S_{\mathcal{F}}].$$

This shows that the commutator $[A, S_{\mathcal{F}}]$ is not compact. Since Proposition 4.5 tells us that $S_{\mathcal{F}} \in \mathcal{T}$, we conclude that $A \notin \mathrm{EssCom}(\mathcal{T})$. $\square$

5. Modified Berezin transforms

We begin this section with some general elementary facts.

Lemma 5.1. *Let T be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for each unit vector $x \in \mathcal{H}$ we have*

$$\|[T, x \otimes x]\| = \|(T - \langle Tx, x\rangle)x\|.$$

Proof. Let $x \in \mathcal{H}$. By the self-adjointness of T, we have $\langle Tx, x\rangle \in \mathbf{R}$. Therefore

$$[T, x \otimes x] = \{(T - \langle Tx, x\rangle)x\} \otimes x - x \otimes \{(T - \langle Tx, x\rangle)x\} = h \otimes x - x \otimes h,$$

where we write $h = (T - \langle Tx, x\rangle)x$. In the case $\|x\| = 1$, we have $\langle h, x\rangle = 0$. Hence

$$\|[T, x \otimes x]\| = \|h \otimes x - x \otimes h\| = \|h\|\|x\| = \|h\| = \|(T - \langle Tx, x\rangle)x\|$$

for every unit vector $x \in \mathcal{H}$. $\square$

Lemma 5.2. *Let T be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for every pair of unit vectors $x, y \in \mathcal{H}$ we have*

$$|\langle Tx, x\rangle - \langle Ty, y\rangle| \le \|[T, x \otimes y]\| + \|[T, x \otimes x]\| + \|[T, y \otimes y]\|. \tag{5.1}$$

Proof. By the self-adjointness of T, we have

$$\begin{aligned}[T, x \otimes y] &= (Tx) \otimes y - x \otimes (Ty)\\ &= \{(T - \langle Tx, x\rangle)x\} \otimes y - x \otimes \{(T - \langle Ty, y\rangle)y\} + (\langle Tx, x\rangle - \langle Ty, y\rangle)x \otimes y.\end{aligned}$$

Since x and y are unit vectors, we have $\|x \otimes y\| = \|x\|\|y\| = 1$. Therefore

$$\begin{aligned}|\langle Tx, x\rangle - \langle Ty, y\rangle| &= \|(\langle Tx, x\rangle - \langle Ty, y\rangle)x \otimes y\|\\ &\le \|[T, x \otimes y]\| + \|\{(T - \langle Tx, x\rangle)x\} \otimes y\| + \|x \otimes \{(T - \langle Ty, y\rangle)y\}\|\\ &= \|[T, x \otimes y]\| + \|(T - \langle Tx, x\rangle)x\| + \|(T - \langle Ty, y\rangle)y\|.\end{aligned}$$

Applying Lemma 5.1 to the last two terms above, we obtain (5.1). $\square$

Lemma 5.3. *Let $\{z_j\}$ be a sequence in $\mathbf{B}$ such that*

$$\limsup_{j\to\infty} |z_j| = 1. \tag{5.2}$$

Then there is a sequence $j_1 < j_2 < \cdots < j_i < \cdots$ of natural numbers such that $|z_{j_i}| < |z_{j_{i+1}}|$ for every $i \in \mathbf{N}$ and such that the set $\{z_{j_i} : i \in \mathbf{N}\}$ is separated.

Proof. For $z \in \mathbf{B}$, we have $\beta(z, 0) = (1/2)\log\{(1 + |z|)/(1 - |z|)\}$. Therefore (5.2) implies

$$\limsup_{j\to\infty} \beta(z_j, 0) = \infty.$$

Using the triangle inequality for β, the conclusion of the lemma follows from an easy inductive selection of $j_1 < j_2 < \cdots < j_i < \cdots$. $\square$

Proposition 5.4. *Suppose that* $\{z_j\}$ *is a sequence in* $\mathbf{B}$ *such that*

$$\lim_{j\to\infty} |z_j| = 1. \tag{5.3}$$

Furthermore, suppose that $\{w_j\}$ *is a sequence in* $\mathbf{B}$ *for which there is a constant* $0 < C < \infty$ *such that*

$$\beta(z_j, w_j) \le C \tag{5.4}$$

for every $j \in \mathbf{N}$. *Then for every* $i \in \mathbf{Z}_+$ *and every* $X \in \mathrm{EssCom}(\mathcal{T})$ *we have*

$$\lim_{j\to\infty} \|[X, \psi_{z_j,i} \otimes \psi_{w_j,i}]\| = 0. \tag{5.5}$$

Proof. For the given $\{z_j\}$, $\{w_j\}$, i and X, suppose that (5.5) did not hold. Then, replacing $\{z_j\}$, $\{w_j\}$ by subsequences if necessary, we may assume that there is a $c > 0$ such that

$$\lim_{j\to\infty} \|[X, \psi_{z_j,i} \otimes \psi_{w_j,i}]\| = c. \tag{5.6}$$

We will show that this leads to a contradiction.

By (5.3) and Lemma 5.3, there is a sequence $j_1 < j_2 < \cdots < j_\nu < \cdots$ of natural numbers such that $|z_{j_\nu}| < |z_{j_{\nu+1}}|$ for every $\nu \in \mathbf{N}$ and such that the set $\{z_{j_\nu} : \nu \in \mathbf{N}\}$ is separated. For each $\nu \in \mathbf{N}$, we now define the operator

$$B_\nu = [X, \psi_{z_{j_\nu},i} \otimes \psi_{w_{j_\nu},i}],$$

whose rank is at most 2. Since $\beta(w_j, 0) = (1/2)\log\{(1 + |w_j|)/(1 - |w_j|)\}$, (5.3) and (5.4) together imply that $|w_j| \uparrow 1$ as $j \to \infty$. Thus both sequences of vectors $\{\psi_{z_j,i}\}$ and $\{\psi_{w_j,i}\}$ converge to 0 weakly in $L^2_a(\mathbf{B}, dv)$. Consequently we have the convergence

$$\lim_{\nu\to\infty} B_\nu = 0 \quad \text{and} \quad \lim_{\nu\to\infty} B^*_\nu = 0$$

in the strong operator topology. Thus by (5.6) and Lemma 4.6, there is a subsequence $\nu(1) < \nu(2) < \cdots < \nu(m) < \cdots$ of natural numbers such that the sum

$$B = \sum_{m=1}^{\infty} B_{\nu(m)}$$

converges strongly with $\|B\|_{\mathcal{Q}} = c > 0$. Thus B is not compact. Now define the operator

$$A = \sum_{m=1}^{\infty} \psi_{z_{j_{\nu(m)}},i} \otimes \psi_{w_{j_{\nu(m)}},i}.$$

Since the set $\{z_{j_\nu} : \nu \in \mathbf{N}\}$ is separated and since (5.4) holds, by Proposition 4.5 we have $A \in \mathcal{T}$. Since $X \in \mathrm{EssCom}(\mathcal{T})$, the commutator $[X, A]$ is compact. On the other hand, we clearly have $[X, A] = B$, which, according to Lemma 4.6, is a non-compact operator. This is the contradiction promised earlier. □

Next we introduce modified Berezin transforms. To do this, we first need to normalize $\psi_{z,i}$. For all $i \in \mathbf{Z}_+$ and $z \in \mathbf{B}$, we define

$$\tilde{\psi}_{z,i} = \frac{\psi_{z,i}}{\|\psi_{z,i}\|}.$$

Keep in mind that $1 \le \|\psi_{z,i}\| \le 2^i$ (see (3.1)). Suppose that A is a bounded operator on $L^2_a(\mathbf{B}, dv)$. Then for each $i \in \mathbf{Z}_+$ we define the function

$$\mathcal{B}_i(A)(z) = \langle A\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle, \quad z \in \mathbf{B},$$

on the unit ball. Of course, $\mathcal{B}_0(A)$ is just the usual Berezin transform (also called Berezin *symbol*) of A. For each $i > 0$, we consider $\mathcal{B}_i(A)$ as a modified Berezin transform of A.

Proposition 5.5. *If $X \in \mathrm{EssCom}(\mathcal{T})$, then $\mathcal{B}_i(X) \in \mathrm{VO}_{\mathrm{bdd}}$ for every $i \in \mathbf{Z}_+$.*

Proof. Let $i \in \mathbf{Z}_+$ be given. Since $\mathcal{T}$ is closed under the *-operation, so is $\mathrm{EssCom}(\mathcal{T})$. Hence it suffices to consider a self-adjoint $X \in \mathrm{EssCom}(\mathcal{T})$. Obviously, $\mathcal{B}_i(X)$ is both bounded and continuous on $\mathbf{B}$. If it were true that $\mathcal{B}_i(X) \notin \mathrm{VO}$, then there would be a $c > 0$ and sequences $\{z_j\}$, $\{w_j\}$ in $\mathbf{B}$ with

$$\lim_{j\to\infty} |z_j| = 1 \tag{5.7}$$

such that for every $j \in \mathbf{N}$, we have $\beta(z_j, w_j) \le 1$ and

$$|\langle X\tilde{\psi}_{z_j,i}, \tilde{\psi}_{z_j,i}\rangle - \langle X\tilde{\psi}_{w_j,i}, \tilde{\psi}_{w_j,i}\rangle| = |\mathcal{B}_i(X)(z_j) - \mathcal{B}_i(X)(w_j)| \ge c. \tag{5.8}$$

But on the other hand, it follows from Lemma 5.2 that

$$\begin{aligned} &|\langle X\tilde{\psi}_{z_j,i}, \tilde{\psi}_{z_j,i}\rangle - \langle X\tilde{\psi}_{w_j,i}, \tilde{\psi}_{w_j,i}\rangle| \\ &\qquad \le \|[X, \tilde{\psi}_{z_j,i} \otimes \tilde{\psi}_{w_j,i}]\| + \|[X, \tilde{\psi}_{z_j,i} \otimes \tilde{\psi}_{z_j,i}]\| + \|[X, \tilde{\psi}_{w_j,i} \otimes \tilde{\psi}_{w_j,i}]\|. \end{aligned} \tag{5.9}$$

By (5.7) and the condition $\beta(z_j, w_j) \le 1$, $j \in \mathbf{N}$, we can apply Proposition 5.4 to obtain

$$\lim_{j\to\infty} \|[X, \psi_{z_j,i} \otimes \psi_{w_j,i}]\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|[X, \psi_{z_j,i} \otimes \psi_{z_j,i}]\| = 0. \tag{5.10}$$

Obviously, conditions (5.7) and $\beta(z_j, w_j) \le 1$, $j \in \mathbf{N}$, also imply $\lim_{j\to\infty} |w_j| = 1$. Thus Proposition 5.4 also provides that

$$\lim_{j\to\infty} \|[X, \psi_{w_j,i} \otimes \psi_{w_j,i}]\| = 0. \tag{5.11}$$

By (3.1), we have $\|[X, \tilde{\psi}_{z,i} \otimes \tilde{\psi}_{w,i}]\| \le \|[X, \psi_{z,i} \otimes \psi_{w,i}]\|$ for all $z, w \in \mathbf{B}$. Thus (5.9), (5.10) and (5.11) together contradict (5.8). $\square$

Lemma 5.6. [1,Theorem 11] *If $g \in \mathrm{VO}_{\mathrm{bdd}}$, then*

$$\lim_{|z|\uparrow 1} \|(g - g(z))k_z\| = 0.$$

Proposition 5.7. *If $X \in \mathrm{EssCom}(\mathcal{T})$, then for every $i \in \mathbf{Z}_+$ we have*

$$\lim_{|z|\uparrow 1} \|(X - T_{\mathcal{B}_i(X)})\psi_{z,i}\| = 0.$$

Proof. Again, it suffices to consider any self-adjoint $X \in \mathrm{EssCom}(\mathcal{T})$. Let $i \in \mathbf{Z}_+$ be given. Then from Lemma 5.1, Proposition 5.4 and (3.1) we deduce that

$$\lim_{|z|\uparrow 1} \|(X - \mathcal{B}_i(X)(z))\psi_{z,i}\| \le 2^i \lim_{|z|\uparrow 1} \|(X - \mathcal{B}_i(X)(z))\tilde{\psi}_{z,i}\| \le 2^i \lim_{|z|\uparrow 1} \|[X, \psi_{z,i} \otimes \psi_{z,i}]\| = 0.$$

Therefore the proposition will follow if we can show that

$$\lim_{|z|\uparrow 1} \|(T_{\mathcal{B}_i(X)} - \mathcal{B}_i(X)(z))\psi_{z,i}\| = 0. \tag{5.12}$$

But

$$\begin{aligned} \|(T_{\mathcal{B}_i(X)} - \mathcal{B}_i(X)(z))\psi_{z,i}\| &\le \|(\mathcal{B}_i(X) - \mathcal{B}_i(X)(z))\psi_{z,i}\| = \|(\mathcal{B}_i(X) - \mathcal{B}_i(X)(z))m_z^i k_z\| \\ &\le 2^i \|(\mathcal{B}_i(X) - \mathcal{B}_i(X)(z))k_z\|. \end{aligned} \tag{5.13}$$

Proposition 5.5 tells us that $\mathcal{B}_i(X) \in \mathrm{VO}_{\mathrm{bdd}}$, which enables us to apply Lemma 5.6 in the case $g = \mathcal{B}_i(X)$. Thus (5.12) follows from (5.13) and Lemma 5.6. $\square$

6. Some quantitative estimates

Here we present a number of estimates that will be needed in the proof of Theorem 1.1. First of all, we need a more precise version of Lemma 3.2(a).

Lemma 6.1. *There is a constant $C_{6.1}$ such that if Γ is any 1-separated set in $\mathbf{B}$ and if $1 \le R < \infty$, then for every $u \in \Gamma$ we have*

$$\mathrm{card}\{v \in \Gamma : \beta(u, v) \le R\} \le C_{6.1} e^{2nR}. \tag{6.1}$$

Proof. Since Γ is 1-separated, for every pair of $x \ne y$ in $\{v \in \Gamma : \beta(u, v) \le R\}$, we have $D(x, 1) \cap D(y, 1) = \emptyset$. Also, if $\beta(u, v) \le R$, then $D(v, 1) \subset D(u, R + 1)$. By the Möbius invariance of both β and $d\lambda$, we have $\lambda(D(z, t)) = \lambda(D(0, t))$ for all $z \in \mathbf{B}$ and $t > 0$. Hence every $u \in \Gamma$ we have

$$\mathrm{card}\{v \in \Gamma : \beta(u, v) \le R\} \le \frac{\lambda(D(u, R + 1))}{\lambda(D(0, 1))} = \frac{\lambda(D(0, R + 1))}{\lambda(D(0, 1))}. \tag{6.2}$$

On the other hand, $\beta(0,z) = (1/2)\log\{(1+|z|)/(1-|z|)\}$, $z \in \mathbf{B}$. Therefore

$$D(0,R+1) = \{z \in \mathbf{B} : |z| < \rho\}, \quad \text{where } \ \rho = (e^{2R+2}-1)/(e^{2R+2}+1).$$

By the radial-spherical decomposition of the volume measure dv, we have

$$\lambda(D(0,R+1)) = \int_{|z|<\rho} \frac{dv(z)}{(1-|z|^2)^{n+1}} = \int_0^\rho \frac{2nr^{2n-1}dr}{(1-r^2)^{n+1}} \le \int_0^\rho \frac{2ndr}{(1-r)^{n+1}} \le \frac{2}{(1-\rho)^n}.$$

Obviously, $1-\rho \ge e^{-2R-2}$. Therefore $\lambda(D(0,R+1)) \le 2e^{2n}e^{2nR}$. Substituting this in (6.2), we see that (6.1) holds for the constant $C_{6.1} = 2e^{2n}/\lambda(D(0,1))$. □

Lemma 6.2. [18,Lemma 4.2] *Given any integer $i \ge 1$, there is a constant $C_{6.2}$ such that*

$$|\langle \psi_{z,i}, \psi_{w,i}\rangle| \le C_{6.2}e^{-i\beta(z,w)} \tag{6.3}$$

for all $z, w \in \mathbf{B}$.

Remark. Even though [18] was published only four years ago, by what we know now, (6.3) is a rather crude estimate. Using techniques employed in analogous situations on the Hardy space [7,Proposition 3.1] and the Drury-Arveson space [8,Lemma 5.1], it can be shown that

$$|\langle \psi_{z,i}, \psi_{w,i}\rangle| \le Ce^{-(n+1+i)\beta(z,w)}$$

for $z, w \in \mathbf{B}$. But since for the proof of Theorem 1.1 we can pick as large an i as we please, (6.3) suffices for our purpose, and we will not try to improve it in this paper.

Lemma 6.3. [16,Lemma 4.1] *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\operatorname{card}\{y \in X : (x,y) \in E\} \le m \quad \text{and} \quad \operatorname{card}\{y \in X : (y,x) \in E\} \le m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup ... \cup E_{2m}$$

and such that for each $1 \le j \le 2m$, the conditions $(x,y),(x',y') \in E_j$ and $(x,y) \ne (x',y')$ imply both $x \ne x'$ and $y \ne y'$.

Lemma 6.4. *Let an integer $i \ge 8n+1$ be given. Then there is a constant $C_{6.4}$ such that the following estimate holds: Let Γ be a 1-separated set in $\mathbf{B}$ and let $\{e_u : u \in \Gamma\}$ be any orthonormal set. Let $1 \le R < \infty$. Then*

$$\left\| \sum_{(u,v)\in F} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v \right\| \le C_{6.4}e^{-(4n+1)R}$$

for every $F \subset \{(u,v) \in \Gamma \times \Gamma : \beta(u,v) \ge R\}$.

Proof. We partition such an F in the form

$$F = E^{(1)} \cup E^{(2)} \cup \cdots \cup E^{(k)} \cdots, \quad \text{where}$$
$$E^{(k)} = \{(u,v) \in F : kR \leq \beta(u,v) < (k+1)R\}, \quad k \in \mathbf{N}.$$

Accordingly,

$$\sum_{(u,v)\in F} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v = T^{(1)} + T^{(2)} + \cdots + T^{(k)} + \cdots, \quad \text{where} \tag{6.4}$$
$$T^{(k)} = \sum_{(u,v)\in E^{(k)}} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v, \quad k \in \mathbf{N}.$$

By Lemma 6.1, for each $u \in \Gamma$ we have

$$\text{card}\{v \in \Gamma : (u,v) \in E^{(k)}\} \leq C_{6.1} e^{2n(k+1)R} \quad \text{and}$$
$$\text{card}\{v \in \Gamma : (v,u) \in E^{(k)}\} \leq C_{6.1} e^{2n(k+1)R}.$$

Thus, by Lemma 6.3, each $E^{(k)}$ admits a partition

$$E^{(k)} = E_1^{(k)} \cup \cdots \cup E_{2m_k}^{(k)} \quad \text{with} \;\; m_k \leq C_{6.1} e^{2n(k+1)R}$$

such that for every $j \in \{1, \ldots, 2m_k\}$, the conditions $(u,v), (u',v') \in E_j^{(k)}$ and $(u,v) \neq (u',v')$ imply both $u \neq u'$ and $v \neq v'$.

Accordingly, we decompose each $T^{(k)}$ in the form

$$T^{(k)} = T_1^{(k)} + \cdots + T_{2m_k}^{(k)}, \quad \text{where} \tag{6.5}$$
$$T_j^{(k)} = \sum_{(u,v)\in E_j^{(k)}} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v, \quad j \in \{1, \ldots, 2m_k\}.$$

The above-mentioned property of $E_j^{(k)}$ means that both projections $(u,v) \mapsto u$ and $(u,v) \mapsto v$ are injective on $E_j^{(k)}$. Therefore

$$\|T_j^{(k)}\| = \sup_{(u,v)\in E_j^{(k)}} |\langle \psi_{v,i}, \psi_{u,i}\rangle|.$$

Applying Lemma 6.2, this gives us $\|T_j^{(k)}\| \leq C_{6.2} e^{-ikR}$. By (6.5), we now have

$$\|T^{(k)}\| \leq 2m_k C_{6.2} e^{-ikR} \leq 2C_{6.1}C_{6.2} e^{2n(k+1)R} e^{-ikR} = C_1 e^{-\{ik-2n(k+1)\}R},$$

where $C_1 = 2C_{6.1}C_{6.2}$. Since $i \geq 8n+1$ and $k \geq 1$, we have

$$ik - 2n(k+1) \geq (8n+1)k - 2n \cdot 2k = (4n+1)k.$$

Hence $\|T^{(k)}\| \le C_1 e^{-(4n+1)kR}$. Combining this with (6.4), we obtain

$$\left\| \sum_{(u,v)\in F} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v \right\| \le \sum_{k=1}^{\infty} \|T^{(k)}\| \le C_1 \sum_{k=1}^{\infty} e^{-(4n+1)kR}.$$

Recall that we assume $R \ge 1$. Thus, factoring out $e^{-(4n+1)R}$ on the right, we see that the lemma holds for the constant $C_{6.4} = C_1 \sum_{k=1}^{\infty} e^{-(4n+1)(k-1)}$. □

Lemma 6.5. *Given any $i \ge 8n+1$, there is a positive number $2 \le R(i) < \infty$ such that the following holds true for every $R \ge R(i)$: Let Γ be a subset of $\mathbf{B}$ with the property that $\beta(u,v) \ge R$ for $u \ne v$ in Γ, and let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then the operator*

$$\Psi = \sum_{u,v\in\Gamma} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v$$

satisfies the condition $\|\Psi x\| \ge (1/2)\|x\|$ for every vector x of the form

$$x = \sum_{u\in\Gamma} c_u e_u, \quad \sum_{u\in\Gamma} |c_u|^2 < \infty. \tag{6.6}$$

Proof. Given any $i \ge 8n+1$, let $2 \le R(i) < \infty$ be such that $C_{6.4} e^{-(4n+1)R(i)} \le 1/2$, where $C_{6.4}$ is the constant provided by Lemma 6.4. Let $R \ge R(i)$, and suppose that Γ has the property that $\beta(u,v) \ge R$ for $u \ne v$ in Γ. We have $\Psi = D + Y$, where

$$D = \sum_{u\in\Gamma} \|\psi_{u,i}\|^2 e_u \otimes e_u \quad \text{and} \quad Y = \sum_{\substack{u,v\in\Gamma \\ u\ne v}} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v.$$

By (3.1), we have $\|Dx\| \ge \|x\|$ for every vector x of the form (6.6). By the property of Γ, we can apply Lemma 6.4 to obtain $\|Y\| \le C_{6.4} e^{-(4n+1)R} \le C_{6.4} e^{-(4n+1)R(i)} \le 1/2$. Clearly, the conclusion of the lemma follows from these two inequalities. □

7. Proof of Theorem 1.1

As we have already mentioned, (1.2) is known and we only need to prove (1.3). To do this, we first fix an integer $i \ge 8n+1$. Let X be any operator in $\mathrm{EssCom}(\mathcal{T})$. Then Proposition 5.5 tells us that $\mathcal{B}_i(X) \in \mathrm{VO}_{\mathrm{bdd}}$. Thus $T_{\mathcal{B}_i(X)} \in \mathrm{EssCom}(\mathcal{T})$ by (1.2). To prove (1.3), it suffices to show that the operator $X - T_{\mathcal{B}_i(X)}$ is compact. Assume the contrary, i.e., $X - T_{\mathcal{B}_i(X)}$ is not compact. We will show that this non-compactness leads to the conclusion that the operator

$$A = (X - T_{\mathcal{B}_i(X)})^*(X - T_{\mathcal{B}_i(X)}) \tag{7.1}$$

does not belong to $\mathrm{EssCom}(\mathcal{T})$, which is a contradiction.

Since $X - T_{\mathcal{B}_i(X)}$ is assumed not to be compact, Proposition 3.7 provides a 1-separated set Γ in $\mathbf{B}$ such that the operator

$$Y = (X - T_{\mathcal{B}_i(X)}) \sum_{u \in \Gamma} \psi_{u,i} \otimes e_u \tag{7.2}$$

is also not compact, where $\{e_u : u \in \Gamma\}$ is an orthonormal set, which will be fixed for the rest of the proof. Our next step is to fix certain constants.

First of all, the non-compactness of Y means that

$$\|Y\|_{\mathcal{Q}} = d > 0. \tag{7.3}$$

Since $\psi_{u,i} = m_u^i k_u = U_u m_u^i \circ \varphi_u$ and since $\|m_u\|_\infty \le 2$, $u \in \Gamma$, by Lemma 3.3 we have

$$\left\| \sum_{u \in G} \psi_{u,i} \otimes e_u \right\| \le 2^i B(\Gamma) \quad \text{for every} \;\; G \subset \Gamma. \tag{7.4}$$

Let $R(i) \ge 2$ be the positive number provided by Lemma 6.5 for the selected integer i. We then pick a positive number $R > R(i)$ such that

$$4\|X\|^2 C_{6.4} e^{-R} \le \frac{d^2}{2^{4i+6} B^4(\Gamma) C_{6.1}^2}, \tag{7.5}$$

where $C_{6.1}$ and $C_{6.4}$ are the constants provided by Lemmas 6.1 and 6.4 respectively. By Lemma 6.1, there is a natural number $N \le C_{6.1} e^{2nR}$ such that

$$\mathrm{card}\{v \in \Gamma : \beta(u,v) \le R\} \le N$$

for every $u \in \Gamma$. By a standard maximality argument, there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$ such that for every $\nu \in \{1, \dots, N\}$,

$$\text{the conditions } u, v \in \Gamma_\nu \text{ and } u \ne v \text{ imply } \beta(u,v) > R. \tag{7.6}$$

For each $\nu \in \{1, \dots, N\}$, define

$$Y_\nu = (X - T_{\mathcal{B}_i(X)}) \sum_{u \in \Gamma_\nu} \psi_{u,i} \otimes e_u.$$

By (7.2) and (7.3), there is a $\mu \in \{1, \dots, N\}$ such that $\|Y_\mu\|_{\mathcal{Q}} \ge d/N$.

By Lemma 4.7, to obtain the promised contradiction $A \notin \mathrm{EssCom}(\mathcal{T})$, it suffices to produce, for each $j \in \mathbf{N}$, a finite subset $E_j \subset \Gamma_\mu \cap \{z \in \mathbf{B} : |z| \ge 1 - (1/j)\}$ such that

$$\left\| \left[A, \sum_{u \in E_j} \psi_{u,i} \otimes \psi_{u,i} \right] \right\| \ge \frac{d^2}{2^{4i+6} B^4(\Gamma) C_{6.1}^2 e^{4nR}}. \tag{7.7}$$

Let $j \in \mathbf{N}$ be given. To find the E_j described above, we set $G_j = \Gamma_\mu \cap \{z \in \mathbf{B} : |z| \geq 1 - (1/j)\}$. Note that $\Gamma_\mu \backslash G_j$ is a finite set. Thus if we define

$$Z_j = (X - T_{\mathcal{B}_i(X)}) \sum_{u \in G_j} \psi_{u,i} \otimes e_u,$$

then $Y_\mu - Z_j$ is a finite-rank operator, and consequently $\|Z_j\|_{\mathcal{Q}} = \|Y_\mu\|_{\mathcal{Q}} \geq d/N$. Hence

$$\|Z_j^* Z_j\|_{\mathcal{Q}} = \|Z_j\|_{\mathcal{Q}}^2 \geq (d/N)^2 \geq \frac{d^2}{C_{6.1}^2 e^{4nR}}.$$

Obviously, we have $Z_j^* Z_j = D + W$, where

$$D = \sum_{u \in G_j} \|(X - T_{\mathcal{B}_i(X)})\psi_{u,i}\|^2 e_u \otimes e_u \quad \text{and} \quad W = \sum_{\substack{u,v \in G_j \\ u \neq v}} \langle A\psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v.$$

Proposition 5.7 implies that D is a compact operator. Hence

$$\|W\| \geq \|W\|_{\mathcal{Q}} = \|Z_j^* Z_j\|_{\mathcal{Q}} \geq \frac{d^2}{C_{6.1}^2 e^{4nR}}.$$

For each $k \in \mathbf{N}$, define the orthogonal projection

$$F_k = \sum_{\substack{u \in G_j \\ |u| \leq 1-(1/k)}} e_u \otimes e_u.$$

Then obviously we have the strong convergence $F_k W F_k \to W$ as $k \to \infty$. Therefore there is a $k(j) \in \mathbf{N}$ such that if we set $G_j' = \{u \in G_j : |u| \leq 1 - (1/k(j))\}$ and

$$W' = \sum_{\substack{u,v \in G_j' \\ u \neq v}} \langle A\psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v, \tag{7.8}$$

then

$$\|W'\| \geq (1/2)\|W\| \geq \frac{d^2}{2C_{6.1}^2 e^{4nR}}. \tag{7.9}$$

Obviously, G_j' is a finite set and the diagonal of W' vanishes.

We now apply Proposition 2.3 to the finite-rank operator W'. By that proposition, there is an $E_j \subset G_j'$ such that the orthogonal projection

$$Q = \sum_{u \in E_j} e_u \otimes e_u$$

has the property $4\|[W', Q]\| \geq \|W'\|$. If we define

$$J = \sum_{u \in G'_j \setminus E_j} e_u \otimes e_u,$$

then $[W', Q] = JW'Q - QW'J$. Since W' is self-adjoint, this gives us $\|[W', Q]\| = \|JW'Q\|$. Combining these facts with (7.9), we obtain

$$\|JW'Q\| \geq \frac{d^2}{8C_{6.1}^2 e^{4nR}}.$$

On the other hand, since $\{G'_j \setminus E_j\} \cap E_j = \emptyset$, from (7.8) we see that

$$JW'Q = \sum_{u \in G'_j \setminus E_j} \sum_{v \in E_j} \langle A\psi_{v,i}, \psi_{u,i} \rangle e_u \otimes e_v = S^*AT,$$

where

$$S = \sum_{u \in G'_j \setminus E_j} \psi_{u,i} \otimes e_u \quad \text{and} \quad T = \sum_{u \in E_j} \psi_{u,i} \otimes e_u.$$

By the finite dimensionalities involved here, there are unit vectors $\xi \in \text{span}\{e_u : u \in E_j\}$ and $\eta \in \text{span}\{e_u : u \in G'_j \setminus E_j\}$ such that $|\langle S^*AT\xi, \eta \rangle| = \|S^*AT\|$. Hence

$$|\langle S^*AT\xi, \eta \rangle| \geq \frac{d^2}{8C_{6.1}^2 e^{4nR}}.$$

Since $R > R(i)$ and $E_j \subset G'_j \subset \Gamma_\mu$, by (7.6) and Lemma 6.5, we have $\|T^*Tx\| \geq (1/2)\|x\|$ for every $x \in \text{span}\{e_u : u \in E_j\}$. This implies that T^*T is surjective on $\text{span}\{e_u : u \in E_j\}$. Hence there is an $x_0 \in \text{span}\{e_u : u \in E_j\}$ with $\|x_0\| \leq 2$ such that $\xi = T^*Tx_0$. Similarly, there is a $y_0 \in \text{span}\{e_u : u \in G'_j \setminus E_j\}$ with $\|y_0\| \leq 2$ such that $\eta = S^*Sy_0$. Therefore

$$|\langle S^*SS^*ATT^*Tx_0, y_0 \rangle| = |\langle S^*ATT^*Tx_0, S^*Sy_0 \rangle| = |\langle S^*AT\xi, \eta \rangle| \geq \frac{d^2}{8C_{6.1}^2 e^{4nR}}.$$

Since $\|x_0\| \leq 2$ and $\|y_0\| \leq 2$, this implies

$$\|S^*SS^*ATT^*T\| \geq \frac{d^2}{32C_{6.1}^2 e^{4nR}}.$$

By (7.4), we have $\|T\| \leq 2^i B(\Gamma)$ and $\|S\| \leq 2^i B(\Gamma)$. Hence

$$\|SS^*ATT^*\| \geq \frac{d^2}{2^{2i+5} B^2(\Gamma) C_{6.1}^2 e^{4nR}}. \tag{7.10}$$

On the other hand,

$$
\begin{aligned}
\|SS^*ATT^*\| &\le \|SS^*[A,TT^*]\| + \|SS^*TT^*A\| \\
&\le \|SS^*\|\|[A,TT^*]\| + \|S\|\|S^*T\|\|T^*\|\|A\| \\
&\le 2^{2i}B^2(\Gamma)\|[A,TT^*]\| + 2^{2i}B^2(\Gamma)\|A\|\|S^*T\|. \qquad (7.11)
\end{aligned}
$$

Recalling (7.1), we clearly have $\|A\| \le 4\|X\|^2$. Thus from (7.10) and (7.11) we deduce

$$
\|[A,TT^*]\| + 4\|X\|^2\|S^*T\| \ge \frac{d^2}{2^{4i+5}B^4(\Gamma)C_{6.1}^2 e^{4nR}}. \qquad (7.12)
$$

To estimate $\|S^*T\|$, note that

$$
S^*T = \sum_{u\in G_j'\backslash E_j} \sum_{v\in E_j} \langle \psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v.
$$

Obviously, $\{G_j'\backslash E_j\} \times E_j \subset \{(u,v) \in \Gamma_\mu \times \Gamma_\mu : u \ne v\}$. By (7.6), we can apply Lemma 6.4 to obtain $\|S^*T\| \le C_{6.4}e^{-(4n+1)R}$. Substituting this in (7.12), we have

$$
\|[A,TT^*]\| + 4\|X\|^2 C_{6.4}e^{-(4n+1)R} \ge \frac{d^2}{2^{4i+5}B^4(\Gamma)C_{6.1}^2 e^{4nR}}.
$$

We now apply condition (7.5) in the above and then simplify. The result of this is

$$
\|[A,TT^*]\| \ge \frac{d^2}{2^{4i+6}B^4(\Gamma)C_{6.1}^2 e^{4nR}}.
$$

Since

$$
TT^* = \sum_{u\in E_j} \psi_{u,i} \otimes \psi_{u,i},
$$

this proves (7.7) and completes the proof of Theorem 1.1.

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