A DOUBLE COMMUTANT RELATION IN THE CALKIN ALGEBRA ON THE BERGMAN SPACE

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Abstract. Let $\mathcal{T}$ be the Toeplitz algebra on the Bergman space $L^2_a(\mathbf{B}, dv)$ of the unit ball in $\mathbf{C}^n$. We show that the image of $\mathcal{T}$ in the Calkin algebra satisfies the double commutant relation: $\pi(\mathcal{T}) = \{\pi(\mathcal{T})\}''$. This is a surprising result, for it is the opposite of what happens in the Hardy-space case [16,17].

1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. Let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. Recall that the Bergman space $L^2_a(\mathbf{B}, dv)$ is just the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(\mathbf{B}, dv)$. Let $P : L^2(\mathbf{B}, dv) \to L^2_a(\mathbf{B}, dv)$ be the orthogonal projection. Each $f \in L^\infty(\mathbf{B}, dv)$ gives rise to the Toeplitz operator

$$T_f h = P(fh), \quad h \in L^2_a(\mathbf{B}, dv).$$

The *Toeplitz algebra* $\mathcal{T}$ on the Bergman space $L^2_a(\mathbf{B}, dv)$ is the C^*-algebra generated by the full collection of Toeplitz operators $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$.

We only consider separable Hilbert spaces in this paper. Recall that if $\mathcal{Z}$ is a collection of bounded operators on a Hilbert space $\mathcal{H}$, then its *essential commutant* is defined to be

$$\mathrm{EssCom}(\mathcal{Z}) = \{A \in \mathcal{B}(\mathcal{H}) : [A, T] \in \mathcal{K}(\mathcal{H}) \text{ for every } T \in \mathcal{Z}\},$$

where $\mathcal{K}(\mathcal{H})$ denotes the collection of compact operators on $\mathcal{H}$. Let $\mathcal{Q}$ denote the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and let

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}$$

be the quotient homomorphism. Then we obviously have $\pi(\mathrm{EssCom}(\mathcal{Z})) = \{\pi(\mathcal{Z})\}'$ for every subset $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$.

To motivate what we will do in this paper, let us mention the recent determination of the essential commutant of the Toeplitz algebra $\mathcal{T}$:

Theorem 1.1. [19] *The essential commutant of the Toeplitz algebra $\mathcal{T}$ equals*

$$\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\} + \mathcal{K}.$$

In the above, $\mathcal{K}$ denotes the collection of compact operators on $L^2_a(\mathbf{B}, dv)$, and

$$\mathrm{VO}_{\mathrm{bdd}} = \mathrm{VO} \cap L^\infty(\mathbf{B}, dv),$$

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where VO is the collection of functions of *vanishing oscillation* on $\mathbf{B}$, which were first introduced by Berger, Coburn and Zhu in [3]. These functions are defined in terms of the Bergman metric

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z, w \in \mathbf{B},$$

where φ_z is the Möbius transform of the ball $\mathbf{B}$ given on page 25 in [13]. Recall from [3] and [19] that VO denotes the collection of functions g on $\mathbf{B}$ satisfying the following two conditions: (1) g is continuous on $\mathbf{B}$; (2) the limit

$$\lim_{|z|\uparrow 1} \sup_{\beta(z,w)\le 1} |g(z) - g(w)| = 0$$

holds.

As was explained in [19], Theorem 1.1 is the Bergman-space analogue of the equality

$$\text{EssCom}(\mathcal{T}^{\text{Hardy}}) = \{T_f^{\text{Hardy}} : f \in \text{VMO} \cap L^\infty\} + \mathcal{K}^{\text{Hardy}}, \tag{1.1}$$

which was proved by Davidson in [4]. Here, $\mathcal{T}^{\text{Hardy}}$, T_f^{Hardy} and $\mathcal{K}^{\text{Hardy}}$ respectively denote the Toeplitz algebra, Toeplitz operator and the collection of compact operators on the *Hardy space* H^2. Recall that (1.1) has inspired several Hardy-space generalizations [5,6,7,8]. In comparison, the Bergman-space case requires a different approach, which perhaps partially explains the chronological gap between [4] and [19].

But given Theorem 1.1, one cannot help but wonder, what is the essential commutant of $\{T_g : g \in \text{VO}_{\text{bdd}}\}$? More than curiosity, this is the logical next step in the investigation. The purpose of this paper is to report the answer to this natural question:

Theorem 1.2. *The essential commutant of* $\{T_g : g \in \text{VO}_{\text{bdd}}\}$ *equals the Toeplitz algebra* $\mathcal{T}$.

While Theorem 1.1 is a direct analogue of (1.1), Theorem 1.2 comes as something of a surprise, for it is the opposite of what happens on the Hardy space. It is known that the essential commutant of $\{T_f^{\text{Hardy}} : f \in \text{VMO} \cap L^\infty\}$ is strictly larger than $\mathcal{T}^{\text{Hardy}}$ [16,17].

These results are better understood in the context of the double commutant relation in the Calkin algebra. There are two classes of unital C^*-subalgebras $\mathcal{A}$ of the Calkin algebra $\mathcal{Q}$ that are known to satisfy the double commutant relation $\mathcal{A} = \mathcal{A}''$:

(1) When $\mathcal{A}$ is separable. This was proved by Voiculescu in [14].

(2) When $\mathcal{A}$ is the image of a von Neumann algebra. This follows from the works of Johnson, Parrott [10] and Popa [12].

On the flip side, there are plenty of unital C^*-subalgebras of the Calkin algebra $\mathcal{Q}$ that do not satisfy the double commutant relation [2,10,15,16,17], among which $\pi(\mathcal{T}^{\text{Hardy}})$ is a notable example. In contrast, the following is an immediate consequence of Theorem 1.2:

Corollary 1.3. *The image of* $\mathcal{T}$ *in the Calkin algebra satisfies the double commutant relation.*

To conclude the Introduction, let us explain the basic ideas for proving Theorem 1.2. First of all, our proof revolves around the scalar quantity

$$\mathrm{diff}(f) = \sup\{|f(z) - f(w)| : \beta(z,w) \le 1\}.$$

Our first realization is that every $X \in \mathrm{EssCom}(\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\})$ satisfies the following "ϵ-δ" condition: given any $\epsilon > 0$, there is a $\delta = \delta(X, \epsilon) > 0$ such that

$$\|[X, T_f]\| \le \epsilon$$

for f satisfying the condition $\mathrm{diff}(f) \le \delta$ and some other additional technical restrictions. We will prove this in Section 3.

Our next realization is that it is possible to construct an approximate partition of the unity on $\mathbf{B}$ where the "diff" for the partition functions is arbitrarily small. More specifically, we need to construct, for each sufficiently large $m \in \mathbf{N}$, a family of partition functions $\{f_\omega : \omega \in I_m\}$ which itself admits a partition

$$I_m = I_m^{(1)} \cup \cdots \cup I_m^{(N)} \tag{1.2}$$

such that for each $\mu \in \{1, \dots, N\}$, the set $I_m^{(\mu)}$ has the following two properties. First, $f_\omega f_{\omega'} = 0$ for all $\omega \ne \omega'$ in $I_m^{(\mu)}$. Second,

$$\mathrm{diff}\left(\sum_{\omega \in I} f_\omega\right) = o(1) \quad \text{for every} \quad I \subset I_m^{(\mu)}, \tag{1.3}$$

where the $o(1)$ is relative to the growth of m. The key requirement here is that the N in (1.2) must be a constant independent of m. Because of this requirement, we cannot construct $\{f_\omega : \omega \in I_m\}$ based on coverings of $\mathbf{B}$ by balls with respect to the Bergman metric. Instead, we must use radial-spherical decompositions of $\mathbf{B}$, which are technically more demanding. This construction is the main content in Section 4.

This construction provides the functions $f^{(\mu)} = \sum_{\omega \in I_m^{(\mu)}} f_\omega$ and $F^{(\mu)} = \sum_{\omega \in I_m^{(\mu)}} f_\omega^2$, $\mu \in \{1, \dots, N\}$. Since N is fixed, given any $X \in \mathrm{EssCom}(\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\})$, it suffices to consider each individual $XT_{F^{(\mu)}}$. Note that $F^{(\mu)} = (f^{(\mu)})^2$. Thus

$$XT_{F^{(\mu)}} = XT_{(f^{(\mu)})^2} = T_{f^{(\mu)}} X T_{f^{(\mu)}} + [X, T_{f^{(\mu)}}] T_{f^{(\mu)}} + X H^*_{f^{(\mu)}} H_{f^{(\mu)}},$$

where $H_{f^{(\mu)}}$ is the Hankel operator with symbol $f^{(\mu)}$. Since $\mathrm{diff}(f^{(\mu)})$ is small, so are $\|[X, T_{f^{(\mu)}}]\|$ and $\|H_{f^{(\mu)}}\|$. In other words, $XT_{F^{(\mu)}}$ is a small perturbation of $T_{f^{(\mu)}} X T_{f^{(\mu)}}$. Then note that $T_{f^{(\mu)}} X T_{f^{(\mu)}} = D + W$, where

$$D = \sum_{\omega \in I_m^{(\mu)}} T_{f_\omega} X T_{f_\omega} \quad \text{and} \quad W = \sum_{\substack{\omega, \omega' \in I_m^{(\mu)} \\ \omega \ne \omega'}} T_{f_\omega} X T_{f_{\omega'}}.$$

One can think of D as the "diagonal terms" in $T_{f(\mu)}XT_{f(\mu)}$ and W as the "off-diagonal terms". As it turns, the "diagonal terms" form an operator in the Toeplitz algebra, $D \in \mathcal{T}$, which will be proved in Section 2.

A key issue in the proof is the handling of the "off-diagonal terms". We will show in Section 5 that $\|W\|$ can be dominated by a quantity of the form

$$\|[X, T_F]\| + \|X\|\|H_F\| + \|[X, T_G]\| + \|X\|\|H_G\| \tag{1.4}$$

with

$$F = \sum_{\omega \in I} f_\omega \quad \text{and} \quad G = \sum_{\omega \in J} f_\omega,$$

where I and J are two disjoint, finite subsets of $I_m^{(\mu)}$. By (1.3), diff(F) and diff(G) are small, consequently so is (1.4). In other words, $\|W\|$ is small. Adding up the operators of small norms mentioned above, this argument shows that $XT_{F(\mu)} = D + Z$, where $D \in \mathcal{T}$ and Z has a small norm, which is the essential part of the proof of Theorem 1.2.

The rest of the paper consists of the technical details of the argument outlined above.

2. Operators in the Toeplitz algebra

We begin by recalling certain standard fixtures associated with the Bergman space. First of all, the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{(n+1)/2}}{(1 - \langle \zeta, z \rangle)^{n+1}}, \quad z, \zeta \in \mathbf{B},$$

gives us the normalized reproducing kernel for $L_a^2(\mathbf{B}, dv)$. By [19,Corollary 4.4], we have

$$|k_z(\zeta)| \le (2e^{\beta(z,z')})^{n+1} |k_{z'}(\zeta)| \tag{2.1}$$

for all $z, z', \zeta \in \mathbf{B}$. Recall that for each $u \in \mathbf{B}$, the formula

$$(U_u f)(\zeta) = k_u(\zeta) f(\varphi_u(\zeta)),$$

$\zeta \in \mathbf{B}$ and $f \in L^2(\mathbf{B}, dv)$, defines a unitary operator on $L^2(\mathbf{B}, dv)$. Moreover, each U_u maps the Bergman space $L_a^2(\mathbf{B}, dv)$ to itself.

Definition 2.1. (1) For $z \in \mathbf{B}$ and $r > 0$, denote $D(z, r) = \{\zeta \in \mathbf{B} : \beta(z, \zeta) < r\}$.
(2) Let $a > 0$. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z, a) \cap D(w, a) = \emptyset$ for all distinct elements z, w in Γ.
(3) A subset Γ of $\mathbf{B}$ is simply said to be separated if it is a-separated for some $a > 0$.

Lemma 2.2. *Given any $0 < r < \infty$, there is a constant $C_{2.2}(r)$ which depends only on r and the complex dimension n such that $\|\chi_{D(u,r)} k_z\| \le C_{2.2}(r) |\langle k_z, k_u \rangle|$ for all $u, z \in \mathbf{B}$.*

Proof. By the properties of the Möbius transform φ_u (see [13,Theorem 2.2.2]) and the Möbius invariance of the Bergman metric β, we have $\chi_{D(u,r)} \circ \varphi_u = \chi_{\varphi_u(D(u,r))} = \chi_{D(0,r)}$ for all $u \in \mathbf{B}$ and $r > 0$. Therefore, for each $z \in \mathbf{B}$,

$$\|\chi_{D(u,r)} k_z\| = \|U_u(\chi_{D(u,r)} k_z)\| = \|\chi_{D(0,r)} k_{\varphi_u(z)}\|. \tag{2.2}$$

Given an $r \in (0, \infty)$, set $\rho = (e^{2r} - 1)/(e^{2r} + 1)$. Then $D(0,r) = \{\zeta \in \mathbf{B} : |\zeta| < \rho\}$, and consequently

$$\|\chi_{D(0,r)} k_{\varphi_u(z)}\|^2 = \int_{|\zeta|<\rho} |k_{\varphi_u(z)}(\zeta)|^2 dv(\zeta) \le \frac{(1 - |\varphi_u(z)|^2)^{n+1}}{(1-\rho)^{2n+2}} = \frac{|\langle k_z, k_u\rangle|^2}{(1-\rho)^{2n+2}},$$

where for the second $=$ we again cite [13,Theorem 2.2.2]. Combining this with (2.2), we see that the constant $C_{2.2}(r) = (1-\rho)^{-n-1}$ will do for the lemma. $\square$

Let $d\lambda$ denote the standard Möbius-invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

In this paper, we will again need the notion of *weakly localized* operators on the Bergman space, which was first introduced by Isralowitz, Mitkovski and Wick in [9], and which played an essential role in [18,19].

Definition 2.3. [9,18] Let a positive number $(n-1)/(n+1) < s < 1$ be given.
(a) A bounded operator B on the Bergman space $L_a^2(\mathbf{B}, dv)$ is said to be s-weakly localized if it satisfies the conditions

$$\sup_{z\in\mathbf{B}} \int |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z\in\mathbf{B}} \int |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle Bk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0 \quad \text{and}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle B^*k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let $\mathcal{A}_s$ denote the collection of s-weakly localized operators defined as above.
(c) Let $C^*(\mathcal{A}_s)$ denote the C^*-algebra generated by $\mathcal{A}_s$.

Theorem 2.4. [18,Theorem 1.3] *For every $(n-1)/(n+1) < s < 1$ we have $C^*(\mathcal{A}_s) = \mathcal{T}$.*

Lemma 2.5. [18,Lemma 2.2] *Let Γ be a separated set in $\mathbf{B}$. For every $0 < r < \infty$, there is a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $i \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta(u,v) > r$.*

Lemma 2.6. *Let T be a bounded operator on the Bergman space $L_a^2(\mathbf{B}, dv)$. Suppose that there is a separated set Γ in $\mathbf{B}$ and a constant $0 < C < \infty$ such that*

$$|\langle Tk_z, k_w\rangle| \le C \sum_{u\in\Gamma} |\langle k_z, k_u\rangle\langle k_u, k_w\rangle| \tag{2.3}$$

for all $z, w \in \mathbf{B}$. Then $T \in \mathcal{A}_s$ for every $(n-1)/(n+1) < s < 1$.

Proof. Rewriting (2.3), for all $z, w \in \mathbf{B}$ we have

$$|\langle Tk_z, k_w\rangle| \le C(1-|z|^2)^{\frac{n+1}{2}}(1-|w|^2)^{\frac{n+1}{2}} \sum_{u\in\Gamma} |k_u(z)||k_u(w)|.$$

Since Γ is separated, there is an $a > 0$ such that $D(u, a) \cap D(v, a) = \emptyset$ for all $u \neq v$ in Γ. For each $u \in \Gamma$, it follows from (2.1) that

$$|k_u(z)||k_u(w)| \le (2e^a)^{2n+2}|k_{x_u}(z)||k_{x_u}(w)| \quad \text{for every} \quad x_u \in D(u, a).$$

Thus if we set $C_1 = C(2e^a)^{2n+2}$, then the above gives us

$$|\langle Tk_z, k_w\rangle| \le C_1 \sum_{u\in\Gamma} |\langle k_z, k_{x_u}\rangle\langle k_{x_u}, k_w\rangle| \tag{2.4}$$

for all $z, w \in \mathbf{B}$, where $x_u \in D(u, a)$ for every $u \in \Gamma$.

Let $(n-1)/(n+1) < s < 1$ be given. To prove that $T \in \mathcal{A}_s$, we need to verify that

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0, \tag{2.5}$$

which works in much the same way as the proofs of [9,Proposition 2.2] and [19,Proposition 4.5]. Indeed for any $z \in \mathbf{B}$ and $r > 0$, it follows from (2.4) that

$$\begin{aligned}
&\int_{\mathbf{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \\
&\quad \le \int_{\beta(z,w)\ge r} C_1 \sum_{u\in\Gamma} \int_{D(u,a)} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \frac{d\lambda(x)}{\lambda(D(u,a))} \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \\
&\quad \le \frac{C_1}{\lambda(D(0,a))} \int \int_{\beta(z,w)\ge r} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) d\lambda(x).
\end{aligned}$$

Once we have this inequality, writing the last integral in the same form of $I_1 + I_2$ as on page 5203 in [19], (2.5) is proved by the argument given there.

Since $|\langle T^*k_z, k_w\rangle| = |\langle k_z, Tk_w\rangle| = |\langle Tk_w, k_z\rangle|$, (2.3) also holds with T^* in place of T. Hence (2.5) also holds with T^* in place of T. This completes the verification of the membership $T \in \mathcal{A}_s$. □

Definition 2.7. Let X be a bounded operator on the Bergman space $L_a^2(\mathbf{B}, dv)$. Then $\mathrm{LOC}(X)$ denotes the collection of operators of the form

$$T = \sum_{u \in \Gamma} T_{f_u} X T_{f_u}, \tag{2.6}$$

where Γ is any separated set in $\mathbf{B}$ and $\{f_u : u \in \Gamma\}$ are continuous functions on $\mathbf{B}$ satisfying the following two conditions:
(1) There is an $0 < r < \infty$ such that for every $u \in \Gamma$, $f_u = 0$ on $\mathbf{B} \backslash D(u, r)$.
(2) The inequality $0 \leq f_u \leq 1$ holds on $\mathbf{B}$ for every $u \in \Gamma$.

Note that the T given by (2.6) a bounded operator on $L_a^2(\mathbf{B}, dv)$. This is because, by conditions (1), (2) above and Lemma 2.5, the operator

$$T' = \sum_{u \in \Gamma} M_{f_u} X P M_{f_u}$$

is obviously bounded on $L^2(\mathbf{B}, dv)$. Since T is the compression of T' to the subspace $L_a^2(\mathbf{B}, dv)$, the boundedness of T follows. This argument also shows that the sum in (2.6) converges in the strong operator topology on $L_a^2(\mathbf{B}, dv)$. We think of the operators in $\mathrm{LOC}(X)$ as *localized* versions of X, hence the notation.

Below is the main goal of this section:

Proposition 2.8. *For every bounded operator X on $L_a^2(\mathbf{B}, dv)$, we have* $\mathrm{LOC}(X) \subset \mathcal{T}$.

Proof. Let Γ be a separated set in $\mathbf{B}$, and let $\{f_u : u \in \Gamma\}$ be continuous functions on $\mathbf{B}$ satisfying the two conditions in Definition 2.7. For $z, w \in \mathbf{B}$, we have

$$|\langle T_{f_u} X T_{f_u} k_z, k_w \rangle| = |\langle X T_{f_u} k_z, T_{f_u} k_w \rangle| \leq \|X\| \|T_{f_u} k_z\| \|T_{f_u} k_w\| \leq \|X\| \|f_u k_z\| \|f_u k_w\|.$$

By the two conditions in Definition 2.7, we can apply Lemma 2.2 in the above to obtain

$$|\langle T_{f_u} X T_{f_u} k_z, k_w \rangle| \leq C_{2.2}^2(r) \|X\| |\langle k_z, k_u \rangle| |\langle k_u, k_w \rangle|,$$

$u \in \Gamma$. Thus, if T is given by (2.6), then

$$|\langle T k_z, k_w \rangle| \leq C_{2.2}^2(r) \|X\| \sum_{u \in \Gamma} |\langle k_z, k_u \rangle \langle k_u, k_w \rangle|$$

for all $z, w \in \mathbf{B}$. Since Γ is separated, by Lemma 2.6 we have $T \in \mathcal{A}_s$ for every $(n-1)/(n+1) < s < 1$. By Theorem 2.4, this means $T \in \mathcal{T}$ as promised. $\square$

3. An epsilon-delta condition

For any continuous function f on $\mathbf{B}$, we define

$$\mathrm{diff}(f) = \sup\{|f(z) - f(w)| : \beta(z, w) \leq 1\}.$$

This turns out to be the most crucial scalar quantity for the proof of Theorem 1.2.

Lemma 3.1. *Let $f_1, \dots, f_k \dots$ be a sequence of continuous functions on* $\mathbf{B}$ *satisfying the following four conditions*:

(1) *There is a* $0 < C < \infty$ *such that* $\|f_k\|_\infty \le C$ *for every* $k \in \mathbf{N}$.

(2) *For every* $k \in \mathbf{N}$, *there exist* $a_k < b_k$ *in* $(0, 1)$ *such that* $f_k = 0$ *on* $\{z \in \mathbf{B} : |z| \le a_k\} \cup \{z \in \mathbf{B} : b_k \le |z| < 1\}$.

(3) $\lim_{k\to\infty} a_k = 1$.

(4) $\lim_{k\to\infty} \mathrm{diff}(f_k) = 0$.

Then there is an infinite subset I *of* $\mathbf{N}$ *such that* $f_J \in \mathrm{VO}_{\mathrm{bdd}}$ *for every* $J \subset I$, *where*

$$f_J = \sum_{k\in J} f_k.$$

Proof. It is elementary that

$$\beta(z, w) \ge \frac{1}{2}\left|\log \frac{(1 + |w|)(1 - |z|)}{(1 - |w|)(1 + |z|)}\right| \tag{3.1}$$

for $z, w \in \mathbf{B}$. By (3), we can inductively pick a sequence of natural numbers $k(1) < k(2) < \cdots < k(j) < \cdots$ such that

$$\frac{1}{2}\log \frac{(1 + a_{k(j+1)})(1 - b_{k(j)})}{(1 - a_{k(j+1)})(1 + b_{k(j)})} \ge 2 \tag{3.2}$$

for every $j \in \mathbf{N}$. Let $I = \{k(1), k(2), \dots, k(j), \dots\}$.

For each $k \in \mathbf{N}$, define $R_k = \{z \in \mathbf{B} : a_k \le |z| \le b_k\}$. Then (2) says that $f_k = 0$ on $\mathbf{B}\backslash R_k$. It follows from (3.1) and (3.2) that

$$\text{if } z \in R_{k(j)} \text{ and } w \in R_{k(j')} \text{ for } j \ne j' \text{ in } \mathbf{N}, \text{ then } \beta(z, w) \ge 2. \tag{3.3}$$

This immediately implies that if $J \subset I$, then f_J is continuous on $\mathbf{B}$. Moreover, since $R_{k(j)} \cap R_{k(j')} = \emptyset$ whenever $j \ne j'$, it follows from (1) and (2) that $\|f_J\|_\infty \le C$ for every $J \subset I$. That is, such an f_J is bounded on $\mathbf{B}$.

Let $j_0 \in \mathbf{N}$, and let $z, w \in \mathbf{B}$ satisfy the conditions $|z| \ge a_{k(j_0)}$ and $\beta(z, w) \le 1$. Then it follows from (3.3) that there is at most one $j \in \mathbf{N}$ such that $f_{k(j)}(z) - f_{k(j)}(w) \ne 0$. Furthermore, by (3.3), if such a j exist, then it must satisfy the condition $j \ge j_0$. Thus for $z, w \in \mathbf{B}$ satisfying the conditions $|z| \ge a_{k(j_0)}$ and $\beta(z, w) \le 1$, we have

$$|f_J(z) - f_J(w)| \le \sup\{\mathrm{diff}(f_{k(j)}) : j \ge j_0\}$$

for every $J \subset I$. Applying conditions (3) and (4), this completes the verification of the membership $f_J \in \mathrm{VO}_{\mathrm{bdd}}$ for $J \subset I$. $\square$

Definition 3.2. (a) For each $0 < t < 1$, the symbol $\Lambda(t)$ denotes the collection of *continuous* functions g on $\mathbf{B}$ satisfying the following three conditions:

(1) $0 \le g(z) \le 1$ for every $z \in \mathbf{B}$.
(2) $g(z) = 1$ whenever $|z| \le t$.
(3) There is a $t' = t'(g) \in (t, 1)$ such that $g(z) = 0$ whenever $t' \le |z| < 1$.
(b) Let $0 < t < 1$ and $\delta > 0$. Then $\Lambda(t;\delta)$ denotes the collection of functions $g \in \Lambda(t)$ satisfying the additional condition $\mathrm{diff}(g) \le \delta$.

Lemma 3.3. *For all $t \in (0, 1)$ and $\delta > 0$, we have $\Lambda(t;\delta) \neq \emptyset$.*

Proof. It follows from the triangle inequality that $|\beta(z, 0)-\beta(w, 0)| \le \beta(z, w)$ for all $z, w \in \mathbf{B}$. Using this fact, the promised function $g \in \Lambda(t;\delta)$ can be easily constructed in the form $g(z) = \psi(\beta(z, 0))$, where ψ is an appropriate Lipschitz function on $[0,\infty)$ with a small Lipschitz constant. We omit the elementary details. $\square$

Lemma 3.4. *Given any pair of $f \in L^\infty(\mathbf{B}, dv)$ and $h \in L^2_a(\mathbf{B}, dv)$, we have*

$$\lim_{t\uparrow 1} \sup\{\|T_{fg}h - T_f h\| : g \in \Lambda(t)\} = 0. \tag{3.4}$$

Proof. By conditions (1) and (2) in Definition 3.2(a), for every $0 < t < 1$ we have

$$\|T_{fg}h - T_f h\|^2 \le \|fgh - fh\|^2 \le \|f\|^2_\infty \int_{t\le|z|<1} |h(z)|^2 dv(z)$$

for all $g \in \Lambda(t)$, $f \in L^\infty(\mathbf{B}, dv)$ and $h \in L^2_a(\mathbf{B}, dv)$. This obviously implies (3.4). $\square$

For a bounded operator A on a Hilbert space $\mathcal{H}$, denote

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \text{ is any compact operator on } \mathcal{H}\},$$

which is the essential norm of A.

Lemma 3.5. [11,Lemma 2.1] *Let $\{B_i\}$ be a sequence of compact operators on a Hilbert space $\mathcal{H}$ satisfying the following conditions*:
(a) *Both sequences $\{B_i\}$ and $\{B^*_i\}$ converge to 0 in the strong operator topology.*
(b) *The limit $\lim_{i\to\infty} \|B_i\|$ exists.*
Then there exist natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$ such that the sum

$$\sum_{m=1}^{\infty} B_{i(m)} = \lim_{N\to\infty} \sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology and we have

$$\left\| \sum_{m=1}^{\infty} B_{i(m)} \right\|_{\mathcal{Q}} = \lim_{i\to\infty} \|B_i\|.$$

Definition 3.6. For $0 < t < 1$ and $\delta > 0$, the symbol $\Phi(t;\delta)$ denotes the collection of *continuous* functions f on $\mathbf{B}$ satisfying the following three conditions:

(1) $0 \le f(z) \le 1$ for every $z \in \mathbf{B}$.
(2) $f(z) = 0$ whenever $|z| \le t$.
(3) $\mathrm{diff}(f) \le \delta$.

The main goal of this section is to show that every operator in $\mathrm{EssCom}(\{T_g : g \in \mathrm{VO_{bdd}}\})$ satisfies the following "ϵ-δ" condition:

Proposition 3.7. *Let X be an operator in the essential commutant of $\{T_g : g \in \mathrm{VO_{bdd}}\}$. Then for every $\epsilon > 0$, there is a $\delta = \delta(X, \epsilon) > 0$ such that*

$$\lim_{t \uparrow 1} \sup\{\|[X, T_f]\| : f \in \Phi(t; \delta)\} \le \epsilon.$$

Proof. Let $X \in \mathrm{EssCom}(\{T_g : g \in \mathrm{VO_{bdd}}\})$ and $\epsilon > 0$ be given. Suppose that no such $\delta > 0$ existed as promised above. We will show that this leads to a contradiction.

First of all, the non-existence of such $\delta > 0$ means that for every $k \in \mathbf{N}$, there is an $f_k \in \Phi(1 - (1/k); 1/k)$ such that $\|[X, T_{f_k}]\| > \epsilon$. Thus for every $k \in \mathbf{N}$, there are unit vectors $h_k, \psi_k \in L_a^2(\mathbf{B}, dv)$ such that

$$|\langle [X, T_{f_k}] h_k, \psi_k \rangle| > \epsilon.$$

Applying Lemma 3.4, we see that for every $k \in \mathbf{N}$, there is a $1 - (1/k) < t_k < 1$ such that

$$|\langle [X, T_{f_k g}] h_k, \psi_k \rangle| > \epsilon \quad \text{for every} \quad g \in \Lambda(t_k).$$

Lemma 3.3 tells us that $\Lambda(t_k; 1/k)$ is not empty. This allows us to pick a $g_k \in \Lambda(t_k; 1/k)$. Define $q_k = f_k g_k$, $k \in \mathbf{N}$. Then the above gives us

$$|\langle [X, T_{q_k}] h_k, \psi_k \rangle| > \epsilon \quad \text{for every} \quad k \in \mathbf{N}.$$

Since h_k and ψ_k are unit vectors, this means

$$\|[X, T_{q_k}]\| > \epsilon \quad \text{for every} \quad k \in \mathbf{N}. \tag{3.5}$$

Next, we examine the properties of q_k. First of all, the properties that $0 \le f_k \le 1$ and $0 \le g_k \le 1$ imply that $0 \le q_k \le 1$ on $\mathbf{B}$. Furthermore, these properties also imply that

$$|q_k(z) - q_k(w)| \le |f_k(z) - f_k(w)| + |g_k(z) - g_k(w)|$$

for all $z, w \in \mathbf{B}$. It follows that for every $k \in \mathbf{N}$, we have

$$\mathrm{diff}(q_k) \le \mathrm{diff}(f_k) + \mathrm{diff}(g_k) \le (1/k) + (1/k) = 2/k.$$

Recall from Definition 3.6 that the membership $g_k \in \Lambda(t_k; 1/k)$ means that there is a $t_k < t_k' < 1$ such that $g_k(z) = 0$ whenever $t_k' \le |z| < 1$. Therefore, for each $k \in \mathbf{N}$, we have $q_k(z) = 0$ if either $t_k' \le |z| < 1$ or $|z| \le 1 - (1/k)$. In conclusion, the sequence of continuous functions $q_1, q_2, \ldots, q_k, \ldots$ satisfy all four conditions in Lemma 3.1.

Thus by Lemma 3.1, there is an infinite subset I of $\mathbf{N}$ such that for every $J \subset I$, we have $q_J \in \mathrm{VO}_{\mathrm{bdd}}$, where

$$q_J = \sum_{k \in J} q_k.$$

Since $\|q_k\|_\infty \le 1$, we have $\|[X, T_{q_k}]\| \le 2\|X\|$ for every k. Since I is an infinite set, it contains a sequence $k_1 < k_2 < \cdots < k_i < \cdots$ of natural numbers such that the limit

$$d = \lim_{i \to \infty} \|[X, T_{q_{k_i}}]\|$$

exists. Obviously, (3.5) implies $d \ge \epsilon$. Define $B_i = [X, T_{q_{k_i}}]$ for every $i \in \mathbf{N}$. By the preceding paragraph, we have $q_{k_i}(z) = 0$ whenever $t'_{k_i} \le |z| < 1$. It is well known that this implies that the Toeplitz operator $T_{q_{k_i}}$ is compact. Thus each B_i is a compact operator, $i \in \mathbf{N}$. Moreover, by the properties that $q_{k_i} = 0$ on the set $\{z \in \mathbf{B} : |z| \le 1 - (1/k_i)\}$ and $0 \le q_{k_i} \le 1$ on $\mathbf{B}$, we have the strong convergence $T_{q_{k_i}} \to 0$ as $i \to \infty$. Therefore we also have the strong convergence $B_i \to 0$ and $B_i^* \to 0$ as $i \to \infty$. That is, we have shown that the sequence $\{B_i\}$ satisfies the conditions in Lemma 3.5. By that lemma, there is a sequence of natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$ such that the limit

$$B = \lim_{N \to \infty} \sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology with $\|B\|_{\mathcal{Q}} = d \ge \epsilon > 0$. That is, B is not compact.

Define $E = \{k_{i(1)}, k_{i(2)}, \ldots, k_{i(m)}, \ldots\}$. Then obviously $E \subset \{k_1, k_2, \ldots, k_i, \ldots\} \subset I$. Therefore we have $q_E \in \mathrm{VO}_{\mathrm{bdd}}$. Since q_E is a bounded function on $\mathbf{B}$ and since every $q_{k_{i(m)}}$ is non-negative, by the dominated convergence theorem, we have the convergence

$$T_{q_E} = \lim_{N \to \infty} T_{q_{k_{i(1)}} + \cdots + q_{k_{i(N)}}} = \lim_{N \to \infty} \sum_{m=1}^{N} T_{q_{k_{i(m)}}}$$

in the strong operator topology. Thus

$$B = \lim_{N \to \infty} \sum_{m=1}^{N} B_{i(m)} = \lim_{N \to \infty} \left[X, \sum_{m=1}^{N} T_{q_{k_{i(m)}}} \right] = [X, T_{q_E}].$$

Since $q_E \in \mathrm{VO}_{\mathrm{bdd}}$ and B is not compact, this contradicts the assumption that X is in the essential commutant of $\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\}$. This completes the proof. □

Lemma 3.8. *Let $h_1, \ldots, h_k \ldots$ be a sequence of continuous functions on $\mathbf{B}$, and denote $U_k = \{z \in \mathbf{B} : h_k(z) \ne 0\}$, $k \in \mathbf{N}$. Suppose that this sequence has the property that there is an $a > 1$ such that $\inf\{\beta(z, w) : z \in U_j, w \in U_k\} \ge a$ for every pair of $j \ne k$ in $\mathbf{N}$. Then the function $h = \sum_{k=1}^{\infty} h_k$ has the property that $\mathrm{diff}(h) \le \sup_{k \in \mathbf{N}} \mathrm{diff}(h_k)$.*

Proof. Observe that, under the assumption, for any pair of $z, w \in \mathbf{B}$ satisfying the condition $\beta(z, w) \le 1$, the cardinality of the set $\{k \in \mathbf{N} : h_k(z) - h_k(w) \ne 0\}$ is at most 1. □

4. Radial-spherical decomposition

To prove Theorem 1.2, we need to partition the unit ball $\mathbf{B}$ by functions with small "diff". At the same time, the supports of these functions must not have excess overlap. Fortunately, we can satisfy these two competing requirements by decomposing the ball in both the radial and the spherical directions. But unfortunately, as always, any explicit radial-spherical decomposition of the ball involves complicated notation and messy details.

Let S denote $\{\xi \in \mathbf{C}^n : |\xi| = 1\}$, the unit sphere in $\mathbf{C}^n$. Recall that the formula

$$d(u,\xi) = |1 - \langle u,\xi\rangle|^{1/2}, \qquad u,\xi \in S,$$

defines a metric on S [13,page 66]. For any pair of $u \in S$ and $r > 0$, we write

$$B(u,r) = \{\xi \in S : d(u,\xi) < r\}.$$

Let σ be the standard spherical measure on S with the usual normalization $\sigma(S) = 1$. There is a constant $A_0 \in (2^{-n},\infty)$ such that

$$\min\{2^{-n},\pi^{-1}\}r^{2n} \le \sigma(B(u,r)) \le A_0 r^{2n} \tag{4.1}$$

for all $u \in S$ and $0 < r \le \sqrt{2}$ [13,Proposition 5.1.4].

With regard to the radial direction of $\mathbf{B}$, we set

$$\rho_k = 1 - 2^{-2k}$$

for every $k \in \mathbf{Z}_+$. For each pair of natural numbers $m \ge 6$ and $j \in \mathbf{N}$, let us denote

$$\alpha_{m,j} = m(1 - \rho_{jm}^2)^{1/2} = m \cdot 2^{-jm} \cdot (2 - 2^{-2jm})^{1/2}. \tag{4.2}$$

Note that $8\alpha_{m,j} \le \sqrt{2}$ for all $m \ge 6$ and $j \in \mathbf{N}$. For each pair of $m \ge 6$ and $j \in \mathbf{N}$, let $E_{m,j}$ be a subset of S that is *maximal* with respect to the property

$$B(u,\alpha_{m,j}/2) \cap B(v,\alpha_{m,j}/2) = \emptyset \quad \text{for all} \;\; u \ne v \;\; \text{in} \;\; E_{m,j}. \tag{4.3}$$

It follows from the maximality of $E_{m,j}$ that

$$\bigcup_{u \in E_{m,j}} B(u,\alpha_{m,j}) = S. \tag{4.4}$$

For each triple of $m \ge 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we define

$$\begin{aligned} A_{m,j,u} &= \{r\xi : \xi \in B(u,\alpha_{m,j}), r \in [\rho_{(j+2)m},\rho_{(j+3)m}]\} \quad \text{and} \\ B_{m,j,u} &= \{r\xi : \xi \in B(u,3\alpha_{m,j}), r \in [\rho_{jm},\rho_{(j+5)m}]\}. \end{aligned} \tag{4.5}$$

Then it follows from (4.4) that

$$\bigcup_{j=1}^{\infty} \bigcup_{u \in E_{m,j}} A_{m,j,u} = \{z \in \mathbf{B} : \rho_{3m} \le |z| < 1\}. \tag{4.6}$$

By (4.1) and (4.3), there is a natural number N_0 such that for every triple of $m \ge 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we have

$$\operatorname{card}\{v \in E_{m,j} : d(u,v) < 7\alpha_{m,j}\} \le N_0. \tag{4.7}$$

By a standard maximality argument, each $E_{m,j}$ admits a partition

$$E_{m,j} = E^{(1)}_{m,j} \cup \cdots \cup E^{(N_0)}_{m,j}$$

such that for every $\nu \in \{1, \dots, N_0\}$, we have $d(u,v) \ge 7\alpha_{m,j}$ for all $u \ne v$ in $E^{(\nu)}_{m,j}$. This number N_0 and the above partition will be fixed for the rest of the paper.

Lemma 4.1. [20,Lemma 2.4] *Suppose that* $0 < \rho < 1$ *and let* $z, w \in \mathbf{B}$. *If* $\rho < |z| < 1$ *and* $\rho < |w| < 1$, *then* $\beta((\rho/|z|)z, (\rho/|w|)w) \le \beta(z,w)$.

Lemma 4.2. (a) *Let* $m \ge 6$, $j \in \mathbf{N}$ *and* $\nu \in \{1, \dots, N_0\}$. *If* $u, v \in E^{(\nu)}_{m,j}$ *and* $u \ne v$, *then we have* $\beta(z,w) > 2$ *for all* $z \in B_{m,j,u}$ *and* $w \in B_{m,j,v}$.
(b) *Let* $m \ge 6$. *If* $u \in E_{m,j}$, $v \in E_{m,k}$ *and* $k \ge j+6$, *then we have* $\beta(z,w) > 3$ *for all* $z \in B_{m,j,u}$ *and* $w \in B_{m,k,v}$.
(c) *Let* $m \ge 6$, $j \in \mathbf{N}$ *and* $u \in E_{m,j}$. *Then* $\beta(z,w) \ge 2\log m$ *for all* $z \in \mathbf{B} \backslash B_{m,j,u}$ *and* $w \in A_{m,j,u}$.

Proof. (a) Consider any $z \in B_{m,j,u}$ and $w \in B_{m,j,v}$, where $u, v \in E^{(\nu)}_{m,j}$ and $u \ne v$. Then $z = |z|\xi$ and $w = |w|\eta$, where $\xi \in B(u, 3\alpha_{m,j})$ and $\eta \in B(v, 3\alpha_{m,j})$. Since $d(u,v) \ge 7\alpha_{m,j}$, we have $d(\xi,\eta) \ge \alpha_{m,j}$. Set $z' = \rho_{jm}\xi$ and $w' = \rho_{jm}\eta$. By [13,Theorem 2.2.2], we have

$$1 - |\varphi_{z'}(w')|^2 = \frac{(1-\rho_{jm}^2)^2}{|1-\rho_{jm}^2\langle\xi,\eta\rangle|^2} \le 4\frac{(1-\rho_{jm}^2)^2}{|1-\langle\xi,\eta\rangle|^2} = 4\left(\frac{1-\rho_{jm}^2}{d^2(\xi,\eta)}\right)^2 \le 4\left(\frac{1-\rho_{jm}^2}{\alpha_{m,j}^2}\right)^2.$$

Recalling (4.2), we obtain $1 - |\varphi_{z'}(w')|^2 \le 4m^{-4}$. Thus

$$\beta(z',w') \ge \frac{1}{2}\log\frac{1}{1-|\varphi_{z'}(w')|^2} \ge \frac{1}{2}\log\frac{m^4}{4} \ge \log(3^2\cdot 2) > 2\log 3 > 2.$$

Since $|z| \ge \rho_{jm}$ and $|w| \ge \rho_{jm}$, by Lemma 4.1 we have $\beta(z,w) \ge \beta(z',w') > 2$.

(b) Let $z \in B_{m,j,u}$ and $w \in B_{m,k,v}$, where $u \in E_{m,j}$, $v \in E_{m,k}$ and $k \ge j+6$. Then it follows from (4.5) and (3.1) that

$$\beta(z,w) \ge \frac{1}{2}\log\frac{1-|z|}{1-|w|} \ge \frac{1}{2}\log\frac{1-\rho_{(j+5)m}}{1-\rho_{km}} = \frac{1}{2}\log\frac{2^{-2(j+5)m}}{2^{-2km}} = (k-j-5)m\log 2.$$

Since $k-j-5 \geq 1$, $m \geq 6$ and $2\log 2 > 1$, we have $\beta(z,w) > 3$ as promised.

(c) Given $z \in \mathbf{B}\backslash B_{m,j,u}$ and $w \in A_{m,j,u}$, we have $z = |z|\xi$ and $w = |w|\eta$ with $\xi \in S$ and $\eta \in B(u, \alpha_{m,j})$. We consider three cases, according to the value of $|z|$. First, suppose that $\rho_{jm} \leq |z| \leq \rho_{(j+5)m}$. By (4.5), we have $\xi \notin B(u, 3\alpha_{m,j})$, and consequently $d(\xi,\eta) \geq 2\alpha_{m,j}$. Define $z' = \rho_{jm}\xi$ and $w' = \rho_{jm}\eta$. Then

$$1-|\varphi_{z'}(w')|^2 = \frac{(1-\rho_{jm}^2)^2}{|1-\rho_{jm}^2\langle\xi,\eta\rangle|^2} \leq 4\frac{(1-\rho_{jm}^2)^2}{|1-\langle\xi,\eta\rangle|^2} = 4\left(\frac{1-\rho_{jm}^2}{d^2(\xi,\eta)}\right)^2 \leq 4\left(\frac{1-\rho_{jm}^2}{4\alpha_{m,j}^2}\right)^2.$$

By (4.2), this means $1-|\varphi_{z'}(w')|^2 \leq (4m^4)^{-1}$. Thus

$$\beta(z',w') \geq \frac{1}{2}\log\frac{1}{1-|\varphi_{z'}(w')|^2} \geq \frac{1}{2}\log(4m^4) \geq 2\log m.$$

Applying Lemma 4.1, we obtain $\beta(z,w) \geq \beta(z',w') \geq 2\log m$.

Now consider the case where $|z| < \rho_{jm}$. Since $|w| \geq \rho_{(j+2)m}$, from (3.1) we obtain

$$\beta(z,w) \geq \frac{1}{2}\log\frac{1-|z|}{1-|w|} \geq \frac{1}{2}\log\frac{1-\rho_{jm}}{1-\rho_{(j+2)m}} = \frac{1}{2}\log\frac{2^{-2jm}}{2^{-2(j+2)m}} = m\log 4 > m.$$

Similarly, in the case $|z| > \rho_{(j+5)m}$, since $|w| \leq \rho_{(j+3)m}$, we have

$$\beta(z,w) \geq \frac{1}{2}\log\frac{1-|w|}{1-|z|} \geq \frac{1}{2}\log\frac{1-\rho_{(j+3)m}}{1-\rho_{(j+5)m}} = \frac{1}{2}\log\frac{2^{-2(j+3)m}}{2^{-2(j+5)m}} = m\log 4 > m.$$

To complete the proof, note that for $m \geq 6$, we always have $m \geq 2\log m$. □

Lemma 4.3. *For each triple of $m \geq 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, define*

$$z_{m,j,u} = \rho_{jm}u. \tag{4.8}$$

Then we have $B_{m,j,u} \subset D(z_{m,j,u}, R_m)$, where $R_m = 2 + 5m + \log\left(1 + 2^{10m} \times 18m^2\right)$.

Proof. Let $w \in B_{m,j,u}$. By (4.5), we have $w = r\eta$, where $\eta \in B(u, 3\alpha_{m,j})$ and $\rho_{jm} \leq r \leq \rho_{(j+5)m}$. Define $w' = ru$. Then $\beta(z_{m,j,u}, w) \leq \beta(z_{m,j,u}, w') + \beta(w', w)$. We estimate the two terms $\beta(z_{m,j,u}, w')$ and $\beta(w', w)$ separately.

First of all,

$$\beta(z_{m,j,u}, w') = \frac{1}{2}\log\frac{(1+r)(1-\rho_{jm})}{(1-r)(1+\rho_{jm})} \leq \frac{1}{2}\log 2 + \frac{1}{2}\log\frac{1-\rho_{jm}}{1-\rho_{(j+5)m}} \leq 1 + \frac{1}{2}\log\frac{2^{-2jm}}{2^{-2(j+5)m}}.$$

Thus $\beta(z_{m,j,u}, w') \leq 1 + 5m\log 2 < 1 + 5m$. On the other hand, by [13,Theorem 2.2.2],

$$1-|\varphi_{w'}(w)|^2 = \frac{(1-r^2)^2}{|1-r^2\langle u,\eta\rangle|^2} \geq \frac{(1-r^2)^2}{(1-r^2+|1-\langle u,\eta\rangle|)^2} \geq \frac{(1-\rho_{(j+5)m}^2)^2}{(1-\rho_{(j+5)m}^2+|1-\langle u,\eta\rangle|)^2}.$$

Hence

$$\begin{aligned}\beta(w',w) &\le \frac{1}{2}\log\frac{4}{1-|\varphi_{w'}(w)|^2}\\ &< 1+\log\left(1+\frac{|1-\langle u,\eta\rangle|}{1-\rho_{(j+5)m}}\right) = 1+\log\left(1+2^{10m}\frac{d^2(u,\eta)}{1-\rho_{jm}}\right).\end{aligned}$$

Since $\eta \in B(u, 3\alpha_{m,j})$, we have $d^2(u,\eta) \le 9\alpha_{m,j}^2$. Therefore

$$\beta(w',w) < 1+\log\left(1+2^{10m}\frac{9\alpha_{m,j}^2}{1-\rho_{jm}}\right) \le 1+\log\left(1+2^{10m}\frac{18\alpha_{m,j}^2}{1-\rho_{jm}^2}\right).$$

Recalling (4.2), we obtain $\beta(w',w) < 1+\log\left(1+2^{10m}\cdot 18m^2\right)$. Combining this with the fact that $\beta(z_{m,j,u},w') < 1+5m$, we have $\beta(z_{m,j,u},w) < R_m$. This completes the proof. $\square$

For any $z \in \mathbf{B}$ and any non-empty subset E of $\mathbf{B}$, we denote

$$\beta(z,E) = \inf\{\beta(z,\zeta) : \zeta \in E\},$$

which is the Bergman distance between z and E. For all $z, w \in \mathbf{B}$, we have

$$|\beta(z,E) - \beta(w,E)| \le \beta(z,w). \tag{4.9}$$

This is because, for any $\zeta \in E$, it follows from the triangle inequality that $\beta(z,E) - \beta(w,\zeta) \le \beta(z,w)$. Taking any sequence $\{\zeta_k\}$ in E such that $\beta(w,\zeta_k) \to \beta(w,E)$ as $k \to \infty$, we find that $\beta(z,E) - \beta(w,E) \le \beta(z,w)$. Similarly, we also have $\beta(w,E) - \beta(z,E) \le \beta(z,w)$. Therefore (4.9) holds.

For every $m \ge 6$, define the function

$$\tilde{f}_m(x) = \begin{cases} 1-(\log m)^{-1}x & \text{for} \quad 0 \le x \le \log m \\ \\ 0 & \text{for} \quad \log m < x < \infty \end{cases}. \tag{4.10}$$

Obviously, this function satisfies the Lipschitz condition $|\tilde{f}_m(x) - \tilde{f}_m(y)| \le (\log m)^{-1}|x-y|$ for all $x, y \in [0,\infty)$. Given any triple of $m \ge 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we now define

$$f_{m,j,u}(z) = \tilde{f}_m(\beta(z, A_{m,j,u})) \quad \text{for} \quad z \in \mathbf{B}. \tag{4.11}$$

Lemma 4.4. *For every triple of $m \ge 6$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, the function $f_{m,j,u}$ defined above has the following five properties*:
(a) *The inequality $0 \le f_{m,j,u} \le 1$ holds on $\mathbf{B}$.*
(b) *$f_{m,j,u} = 1$ on the set $A_{m,j,u}$.*
(c) *$f_{m,j,u}$ is continuous on $\mathbf{B}$.*
(d) *The set $\{z \in \mathbf{B} : f_{m,j,u}(z) \ne 0\}$ is contained in $B_{m,j,u}$.*

(e) *We have* $\mathrm{diff}(f_{m,j,u}) \le (\log m)^{-1}$.

Proof. (a) and (b) follow directly from the definitions of $\tilde{f}_m$ and $f_{m,j,u}$. (c) follows from the continuity of $\tilde{f}_m$ and (4.9). For (d), note that if $z \in \mathbf{B}\backslash B_{m,j,u}$, then Lemma 4.2(c) gives us $\beta(z, A_{m,j,u}) \ge 2\log m$. By (4.10), we have $\tilde{f}_m(\beta(z, A_{m,j,u})) = 0$.

To verify (e), let $z, w \in \mathbf{B}$ be given, and suppose that $\beta(z,w) \le 1$. By the Lipschitz condition for $\tilde{f}_m$ and (4.9), we have

$$\begin{aligned}|f_{m,j,u}(z) - f_{m,j,u}(w)| &= |\tilde{f}_m(\beta(z, A_{m,j,u})) - \tilde{f}_m(\beta(w, A_{m,j,u}))| \\ &\le \frac{1}{\log m}|\beta(z, A_{m,j,u}) - \beta(w, A_{m,j,u})| \le \frac{\beta(z,w)}{\log m} \le \frac{1}{\log m}.\end{aligned}$$

This completes the proof. □

The triple subscript in $f_{m,j,u}$, while necessary for our construction, is obviously quite cumbersome as a notation. Let us try to alleviate this problem by introducing:

Definition 4.5. Let $m \ge 6$ be given. (a) For each pair of $\kappa \in \{1,2,3,4,5,6\}$ and $\nu \in \{1,\dots,N_0\}$, where N_0 is the integer that appears in (4.7), let $I_m^{(\nu,\kappa)}$ denote the collection of all triples $m, 6j+\kappa, u$ satisfying the conditions $j \in \mathbf{Z}_+$ and $u \in E^{(\nu)}_{m,6j+\kappa}$.
(b) For $\kappa \in \{1,2,3,4,5,6\}$, $\nu \in \{1,\dots,N_0\}$ and $J \in \mathbf{N}$, let $I^{(\nu,\kappa)}_{m,J}$ denote the collection of all triples $m, 6j+\kappa, u$ satisfying the conditions $0 \le j \le J$ and $u \in E^{(\nu)}_{m,6j+\kappa}$.
(c) Denote $I_m = \cup_{\kappa=1}^6 \cup_{\nu=1}^{N_0} I_m^{(\nu,\kappa)}$.
(d) For any subset I of I_m, denote $f_I = \sum_{\omega\in I} f_\omega$ and $F_I = \sum_{\omega \in I} f_\omega^2$.

Lemma 4.6. *Let $m \ge 6$, $\kappa \in \{1,2,3,4,5,6\}$ and $\nu \in \{1,\dots,N_0\}$. Then for every subset I of $I_m^{(\nu,\kappa)}$, we have $f_I \in \Phi(\rho_m; (\log m)^{-1})$.*

Proof. Let $I \subset I_m^{(\nu,\kappa)}$. Recall from Lemma 4.4 that for each $\omega \in I$, we have $\{z \in \mathbf{B} : f_\omega(z) \ne 0\} \subset B_\omega$ and $\mathrm{diff}(f_\omega) \le (\log m)^{-1}$. By Lemma 4.2 (a) and (b), for every pair of $\omega \ne \omega'$ in I, the Bergman distance between B_ω and $B_{\omega'}$ is at least 2. Therefore Lemma 3.8 tells us that $\mathrm{diff}(f_I) \le \sup_{\omega\in I}\mathrm{diff}(f_\omega) \le (\log m)^{-1}$. Lemma 4.4 also provides that for each ω, f_ω is continuous on $\mathbf{B}$ and satisfies the condition $0 \le f_\omega \le 1$. Hence the fact that the Bergman distance between B_ω and $B_{\omega'}$ is at least 2 for $\omega \ne \omega'$ in I also ensures that f_I is continuous on $\mathbf{B}$ and that $0 \le f_I \le 1$. If $|z| < \rho_{\kappa m}$, then $z \notin B_\omega$ for every $\omega \in I$. Thus by Lemma 4.4(d), if $|z| < \rho_{\kappa m}$, then $f_I(z) = 0$. By continuity, we also have $f_I(z) = 0$ when $|z| \le \rho_{\kappa m}$. Since $\kappa \ge 1$, recalling Definition 3.6, this completes the verification of the membership $f_I \in \Phi(\rho_m; (\log m)^{-1})$. □

Lemma 4.7. *Let $m \ge 6$, $\kappa \in \{1,2,3,4,5,6\}$ and $\nu \in \{1,\dots,N_0\}$, and let I be any subset of $I_m^{(\nu,\kappa)}$. Then for every bounded operator X on $L^2_a(\mathbf{B}, dv)$, we have*

$$\sum_{\omega\in I} T_{f_\omega} X T_{f_\omega} \in \mathrm{LOC}(X). \tag{4.12}$$

Proof. Given any $I \subset I_m^{(\nu,\kappa)}$, consider the set $\Gamma = \{z_\omega : \omega \in I\}$, where z_ω was defined by (4.8). By (4.5), we have $z_\omega \in B_\omega$. Thus it follows from Lemma 4.2 that Γ is a separated set in $\mathbf{B}$. Lemma 4.3 tells us that for each $\omega \in I$, we have $B_\omega \subset D(z_\omega, R_m)$. By Lemma 4.4(d), we have $f_\omega = 0$ on $\mathbf{B} \backslash D(z_\omega, R_m)$. Recalling Definition 2.7, (4.12) follows. □

As usual, for each $f \in L^\infty(\mathbf{B}, dv)$, we define the Hankel operator

$$H_f h = (1 - P)(fh), \quad h \in L^2_a(\mathbf{B}, dv).$$

Lemma 4.8. *There is a constant $0 < C_{4.8} < \infty$ such that $\|H_f\| \leq C_{4.8}\text{diff}(f)$ for every bounded continuous function f on $\mathbf{B}$.*

Proof. Recall that for $f \in L^2(\mathbf{B}, dv)$, the formula $\|f\|_{\text{BMO}} = \sup_{z \in \mathbf{B}} \|(f - \langle fk_z, k_z\rangle)k_z\|$ defines its BMO norm. It is well known that there is a constant C_1 such that $\|H_f\| \leq C_1\|f\|_{\text{BMO}}$ for every $f \in L^\infty(\mathbf{B}, dv)$ [1,Theorem 22]. Thus it suffices to produce a constant C_2 such that $\|f\|_{\text{BMO}} \leq C_2\text{diff}(f)$ for every bounded continuous function f on $\mathbf{B}$.

To find such a C_2, note that for $j \in \mathbf{Z}_+$, $1 - 2^{-j} \leq t \leq 1 - 2^{-j-1}$ and $\xi \in S$, we have

$$\beta((1 - 2^{-j})\xi, t\xi) = \frac{1}{2}\log\frac{(1 + t)2^{-j}}{(1 - t)(2 - 2^{-j})} \leq \frac{1}{2}\log 4 < 1. \tag{4.13}$$

Define $Q_j = \{w \in \mathbf{B} : 1 - 2^{-j} \leq |w| < 1 - 2^{-j-1}\}$, $j \in \mathbf{Z}_+$. By (4.13) and an obvious telescoping sum, we see that if f is a bounded continuous function on $\mathbf{B}$, then

$$|f(w) - f(0)| \leq (j + 1)\text{diff}(f) \quad \text{for every} \quad w \in Q_j,$$

$j \in \mathbf{Z}_+$. Set $C_2 = \{\sum_{j=0}^\infty (j + 1)^2 v(Q_j)\}^{1/2}$, which is obviously finite. We have

$$\|f - f(0)\|^2 = \int |f(w) - f(0)|^2 dv(w) \leq \sum_{j=0}^\infty (j + 1)^2 v(Q_j)(\text{diff}(f))^2 = C_2^2(\text{diff}(f))^2.$$

For each $z \in \mathbf{B}$, denote $f_z = f \circ \varphi_z$. Then it follows from the above that

$$\|(f - \langle fk_z, k_z\rangle)k_z\| \leq \|(f - f(z))k_z\| = \|f_z - f_z(0)\| \leq C_2\text{diff}(f_z) = C_2\text{diff}(f),$$

where the second = is due to the Möbius invariance of β. This completes the proof. □

5. Proof of Theorem 1.2

To prove Theorem 1.2, we need to fully exploit the properties of Toeplitz operators:

Lemma 5.1. *Let $\{f_1, \ldots, f_\ell\}$ be a finite set of functions in $L^\infty(\mathbf{B}, dv)$ with the property that $f_j f_k = 0$ for all $j \neq k$ in $\{1, \ldots, \ell\}$. Let A be any bounded operator on the Bergman space $L^2_a(\mathbf{B}, dv)$. Then there exist complex numbers $\{\gamma_1, \ldots, \gamma_\ell\}$ with $|\gamma_k| = 1$ for every $k \in \{1, \ldots, \ell\}$ and a subset E of $\{1, \ldots, \ell\}$ such that if we define*

$$F = \sum_{k \in E} f_k, \quad G = \sum_{k \in \{1,\ldots,\ell\}\backslash E} f_k, \quad F' = \sum_{k \in E} \gamma_k f_k \quad \textit{and} \quad G' = \sum_{k \in \{1,\ldots,\ell\}\backslash E} \gamma_k f_k,$$

then

$$\left\| \sum_{j\neq k} T_{f_j} A T_{f_k} \right\| \leq 4(\|T_{F'} A T_G\| + \|T_{G'} A T_F\|).$$

Proof. It suffices to consider the case $\ell \geq 2$. Denote $U_k = \{z \in \mathbf{B} : f_k(z) \neq 0\}$ for each $k \in \{1, \dots, \ell\}$. Then $U_j \cap U_k = \emptyset$ for all $j \neq k$ in $\{1, \dots, \ell\}$ by our assumption. Write

$$Z = \sum_{j\neq k} T_{f_j} A T_{f_k} \quad \text{and} \quad Z_\theta = \sum_{j\neq k} e^{i(j-k)\theta} T_{f_j} A T_{f_k}, \quad \theta \in \mathbf{R}.$$

Then obviously we have

$$Z = \frac{1}{2\pi} \int_0^{2\pi} (Z - Z_\theta) d\theta.$$

This shows that there is a $\theta^* \in [0, 2\pi]$ such that $\|Z\| \leq \|Z - Z_{\theta^*}\|$.

Write $\gamma_k = e^{ik\theta^*}$ for every $k \in \{1, \dots, \ell\}$. Define the operators

$$B = \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} M_{f_j} A P M_{f_k} \quad \text{and} \quad B' = \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} \gamma_j \bar{\gamma}_k M_{f_j} A P M_{f_k}$$

on $L^2(\mathbf{B}, dv)$. Also, define

$$\psi = \sum_{k=1}^{\ell} \gamma_k \chi_{U_k}.$$

Using the properties that $U_j \cap U_k = \emptyset$ for $j \neq k$ and that $f_k = 0$ on $\mathbf{B} \backslash U_k$, we have

$$B - B' = B - M_\psi B M_{\bar{\psi}} = M_\psi (M_{\bar{\psi}} B - B M_{\bar{\psi}}).$$

For each $k \in \{1, \dots, \ell\}$, let us write $\gamma_k = c_k + i d_k$, where $c_k, d_k \in [-1, 1]$. Define

$$p = \sum_{k=1}^{\ell} c_k \chi_{U_k} \quad \text{and} \quad q = \sum_{k=1}^{\ell} d_k \chi_{U_k}.$$

Then the above gives us $B - B' = M_\psi X - i M_\psi Y$, where

$$X = M_p B - B M_p \quad \text{and} \quad Y = M_q B - B M_q.$$

Since $\gamma_k \bar{\gamma}_k = 1$ for every $k \in \{1, \dots, \ell\}$, $Z - Z_{\theta^*}$ is the compression of $B - B'$ to the subspace $L_a^2(\mathbf{B}, dv)$. Hence $\|Z - Z_{\theta^*}\| = \|P(B - B')P\|$. Consequently, we have either $\|Z\| \leq \|Z - Z_{\theta^*}\| \leq 2\|P M_\psi X P\|$ or $\|Z\| \leq \|Z - Z_{\theta^*}\| \leq 2\|P M_\psi Y P\|$.

In the case $\|Z\| \leq 2\|P M_\psi X P\|$, consider $c_1, \dots, c_\ell$, which are real numbers in $[-1, 1]$. There is a permutation $\tau(1), \dots, \tau(\ell)$ of the integers $1, \dots, \ell$ such that

$$c_{\tau(j)} \geq c_{\tau(j-1)} \quad \text{for every} \quad j \in \{2, \dots, \ell\}.$$

For each $j \in \{1, \ldots, \ell\}$, define the subset $E_j = \{\tau(k) : j \le k \le \ell\}$ of $\{1, \ldots, \ell\}$. Then

$$p = \sum_{k=1}^{\ell} c_{\tau(k)} \chi_{U_{\tau(k)}} = c_{\tau(1)} \sum_{\mu \in E_1} \chi_{U_\mu} + \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) \sum_{\mu \in E_j} \chi_{U_\mu}.$$

Since $\chi_{U_j} f_k = 0$ when $j \neq k$ and $\chi_{U_k} f_k = f_k$, we have

$$\sum_{k=1}^{\ell} c_k f_k = p \sum_{k=1}^{\ell} f_k = c_{\tau(1)} g_1 + \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) g_j, \quad \text{where} \quad g_j = \sum_{\mu \in E_j} f_\mu$$

for every $1 \le j \le \ell$. Note that $E_1 = \{1, \ldots, \ell\}$. Thus

$$\begin{aligned} X = M_p B - B M_p &= \sum_{j=1}^{\ell} c_j M_{f_j} A P M_{g_1} - M_{g_1} A P \sum_{j=1}^{\ell} c_j M_{f_j} \\ &= \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) (M_{g_j} A P M_{g_1} - M_{g_1} A P M_{g_j}) \\ &= \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) (M_{g_j} A P M_{h_j} - M_{h_j} A P M_{g_j}), \end{aligned}$$

where

$$h_j = \sum_{\mu \in \{1, \ldots, \ell\} \setminus E_j} f_\mu,$$

$2 \le j \le \ell$. Since $(c_{\tau(2)} - c_{\tau(1)}) + \cdots + (c_{\tau(\ell)} - c_{\tau(\ell-1)}) = c_{\tau(\ell)} - c_{\tau(1)} \le 2$, we have

$$\begin{aligned} \|P M_\psi X P\| &\le \sum_{j=2}^{\ell} (c_{\tau(j)} - c_{\tau(j-1)}) \|P M_\psi (M_{g_j} A P M_{h_j} - M_{h_j} A P M_{g_j}) P\| \\ &\le 2 \max_{2 \le j \le \ell} (\|T_{\psi g_j} A T_{h_j}\| + \|T_{\psi h_j} A T_{g_j}\|). \end{aligned}$$

That is, there is a $j_0 \in \{2, \ldots, \ell\}$ such that

$$\|P M_\psi X P\| \le 2(\|T_{\psi g_{j_0}} A T_{h_{j_0}}\| + \|T_{\psi h_{j_0}} A T_{g_{j_0}}\|).$$

If we simply let $E = E_{j_0}$, then $g_{j_0} = F$, $\psi g_{j_0} = F'$, $h_{j_0} = G$ and $\psi h_{j_0} = G'$. This proves the lemma in the case $\|Z\| \le 2\|P M_\psi X P\|$.

In the case $\|Z\| \le 2\|P M_\psi Y P\|$, we just apply the argument in the preceding paragraph with $d_1, \ldots, d_\ell$ in place of $c_1, \ldots, c_\ell$. This completes the proof of the lemma. □

Proof of Theorem 1.2. Since we know that $\mathrm{EssCom}(\{T_g : g \in \mathrm{VO_{bdd}}\}) \supset \mathcal{T}$, we only need to prove that $\mathrm{EssCom}(\{T_g : g \in \mathrm{VO_{bdd}}\}) \subset \mathcal{T}$.

Let $X \in \mathrm{EssCom}(\{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\})$. To show that $X \in \mathcal{T}$, pick any $\epsilon > 0$. It suffices to show that X admits a decomposition $X = Y + Z$ with $Y \in \mathcal{T}$ and

$$\|Z\| \le 6N_0\{C_{4.8}\|X\| + 2 + 16(2 + C_{4.8}\|X\|)\}\epsilon, \tag{5.1}$$

where $C_{4.8}$ and N_0 are the constants that appear in Lemma 4.8 and (4.7) respectively.

First of all, by Proposition 3.7, there exist a $\delta > 0$ and a $0 < t^* < 1$ such that

$$\|[X, T_f]\| \le 2\epsilon \quad \text{for every} \quad f \in \Phi(t^*; \delta). \tag{5.2}$$

With δ and t^* so fixed, we pick an integer $m \ge 6$ satisfying the conditions

$$(\log m)^{-1} \le \min\{\epsilon, \delta\} \quad \text{and} \quad \rho_m \ge t^*. \tag{5.3}$$

With m so fixed, let us consider the function F_{I_m} given in Definition 4.5(d). Since

$$F_{I_m} = \sum_{\kappa=1}^{6} \sum_{\nu=1}^{N_0} F_{I_m^{(\nu,\kappa)}} \tag{5.4}$$

and since by Lemma 4.6 each $F_{I_m^{(\nu,\kappa)}}$ satisfies the inequality $0 \le F_{I_m^{(\nu,\kappa)}} \le 1$ on $\mathbf{B}$, we have $0 \le F_{I_m} \le 6N_0$ on $\mathbf{B}$. By Lemma 4.4(b) and (4.6), we have $F_{I_m}(z) \ge 1$ whenever $\rho_{3m} \le |z| < 1$. Define $\Delta_m = \{z \in \mathbf{B} : |z| < \rho_{3m}\}$. Thus we have shown that the function

$$h = \chi_{\Delta_m} + F_{I_m} \tag{5.5}$$

satisfies the inequality $1 \le h \le 6N_0 + 1$ on $\mathbf{B}$. This guarantees that the positive Toeplitz operator T_h is both bounded and invertible on $L^2_a(\mathbf{B}, dv)$. Moreover, $\|T_h^{-1}\| \le 1$. Since $T_h \in \mathcal{T}$ and $\mathcal{T}$ is a C^*-algebra, we have $T_h^{-1} \in \mathcal{T}$.

By (5.5) and (5.4), we have the decomposition

$$X = XT_hT_h^{-1} = X_0 + \sum_{\kappa=1}^{6} \sum_{\nu=1}^{N_0} X_{\nu,\kappa}, \tag{5.6}$$

where

$$X_0 = XT_{\chi_{\Delta_m}}T_h^{-1} \quad \text{and} \quad X_{\nu,\kappa} = XT_{F_{I_m^{(\nu,\kappa)}}}T_h^{-1}$$

for $1 \le \kappa \le 6$ and $1 \le \nu \le N_0$. It is well known that $\mathcal{T} \supset \mathcal{K}$. Since $\Delta_m = \{z \in \mathbf{B} : |z| < \rho_{3m}\}$, the Toeplitz operator $T_{\chi_{\Delta_m}}$ is compact. Hence $X_0 \in \mathcal{K} \subset \mathcal{T}$.

Next, consider each $X_{\nu,\kappa}$. It is a consequence of Lemma 4.4(d) and Lemma 4.2 that $F_{I_m^{(\nu,\kappa)}} = f^2_{I_m^{(\nu,\kappa)}}$. (Again, we refer the reader to Definition 4.5(d).) Therefore

$$T_{F_{I_m^{(\nu,\kappa)}}} = T_{f^2_{I_m^{(\nu,\kappa)}}} = T^2_{f_{I_m^{(\nu,\kappa)}}} + H^*_{f_{I_m^{(\nu,\kappa)}}} H_{f_{I_m^{(\nu,\kappa)}}}.$$

Accordingly, we have

(5.7) $X_{\nu,\kappa} = X^{(1)}_{\nu,\kappa} + Z^{(1)}_{\nu,\kappa}$, where $X^{(1)}_{\nu,\kappa} = XT^2_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1}$ and $Z^{(1)}_{\nu,\kappa} = XH^*_{f_{I_m^{(\nu,\kappa)}}} H_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1}$.

We have $\|H^*_{f_{I_m^{(\nu,\kappa)}}} H_{f_{I_m^{(\nu,\kappa)}}}\| \le \|H_{f_{I_m^{(\nu,\kappa)}}}\| \le C_{4.8}\mathrm{diff}(f_{I_m^{(\nu,\kappa)}})$ by Lemma 4.8. By Lemma 4.6 and (5.3), we have $\mathrm{diff}(f_{I_m^{(\nu,\kappa)}}) \le (\log m)^{-1} \le \epsilon$. Hence

$$\|Z^{(1)}_{\nu,\kappa}\| \le C_{4.8}\|X\|\epsilon. \tag{5.8}$$

We further decompose $X^{(1)}_{\nu,\kappa}$: we have

$$X^{(1)}_{\nu,\kappa} = X^{(2)}_{\nu,\kappa} + Z^{(2)}_{\nu,\kappa},$$

where

$$X^{(2)}_{\nu,\kappa} = T_{f_{I_m^{(\nu,\kappa)}}} XT_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1} \quad \text{and} \quad Z^{(2)}_{\nu,\kappa} = [X, T_{f_{I_m^{(\nu,\kappa)}}}]T_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1}.$$

Recall from Lemma 4.6 that $f_{I_m^{(\nu,\kappa)}} \in \Phi(\rho_m; (\log m)^{-1})$. Therefore it follows from (5.3) and (5.2) that

$$\|Z^{(2)}_{\nu,\kappa}\| \le \|[X, T_{f_{I_m^{(\nu,\kappa)}}}]\| \le 2\epsilon. \tag{5.9}$$

Then note that

$$X^{(2)}_{\nu,\kappa} = Y_{\nu,\kappa} + Z^{(3)}_{\nu,\kappa},$$

where

$$Y_{\nu,\kappa} = \sum_{\omega\in I_m^{(\nu,\kappa)}} T_{f_\omega} XT_{f_\omega} T_h^{-1} \quad \text{and} \quad Z^{(3)}_{\nu,\kappa} = \sum_{\substack{\omega,\omega'\in I_m^{(\nu,\kappa)} \\ \omega\neq\omega'}} T_{f_\omega} XT_{f_{\omega'}} T_h^{-1}. \tag{5.10}$$

Since $T_h^{-1} \in \mathcal{T}$, it follows from Lemma 4.7 and Proposition 2.8 that $Y_{\nu,\kappa} \in \mathcal{T}$.

To estimate $\|Z^{(3)}_{\nu,\kappa}\|$, first observe that we have the strong convergence

$$\sum_{\substack{\omega,\omega'\in I_{m,J}^{(\nu,\kappa)} \\ \omega\neq\omega'}} M_{f_\omega} XPM_{f_{\omega'}} \to \sum_{\substack{\omega,\omega'\in I_m^{(\nu,\kappa)} \\ \omega\neq\omega'}} M_{f_\omega} XPM_{f_{\omega'}} \quad \text{as} \quad J \to \infty$$

on $L^2(\mathbf{B}, dv)$, where $I_{m,J}^{(\nu,\kappa)}$ was given by Definition 4.5(b). Compressing this strong convergence to the subspace $L^2_a(\mathbf{B}, dv)$, we see that there is a $J \in \mathbf{N}$ such that

$$\|Z^{(3)}_{\nu,\kappa}\| \le 2\|Z^{(4)}_{\nu,\kappa}\|, \quad \text{where} \quad Z^{(4)}_{\nu,\kappa} = \sum_{\substack{\omega,\omega'\in I_{m,J}^{(\nu,\kappa)} \\ \omega\neq\omega'}} T_{f_\omega} XT_{f_{\omega'}}. \tag{5.11}$$

(Note that the fact that $\|T_h^{-1}\| \le 1$ is also involved here.) The significance of $Z^{(4)}_{\nu,\kappa}$ is that $I^{(\nu,\kappa)}_{m,J}$ is a finite set. Since $f_\omega f_{\omega'} = 0$ for $\omega \neq \omega'$ in $I^{(\nu,\kappa)}_{m,J}$, by Lemma 5.1, there are complex numbers $\{\gamma_\omega : \omega \in I^{(\nu,\kappa)}_{m,J}\}$ of modulus 1 and a subset I of $I^{(\nu,\kappa)}_{m,J}$ such that if we set

$$F = \sum_{\omega\in I} f_\omega, \quad G = \sum_{\omega\in I^{(\nu,\kappa)}_{m,J}\setminus I} f_\omega, \quad F' = \sum_{\omega\in I}\gamma_\omega f_\omega \quad \text{and} \quad G' = \sum_{\omega\in I^{(\nu,\kappa)}_{m,J}\setminus I} \gamma_\omega f_\omega,$$

then

$$\|Z^{(4)}_{\nu,\kappa}\| \le 4(\|T_{F'}XT_G\| + \|T_{G'}XT_F\|). \tag{5.12}$$

Note that $T_{G'}XT_F = T_{G'}[X, T_F] + T_{G'}T_F X$. By Lemma 4.6, we have $F \in \Phi(\rho_m; (\log m)^{-1})$. Hence it follows from (5.3) and (5.2) that

$$\|T_{G'}[X, T_F]\| \le \|[X, T_F]\| \le 2\epsilon. \tag{5.13}$$

We know that $B_\omega \cap B_{\omega'} = \emptyset$ for all $\omega \neq \omega'$ in $I^{(\nu,\kappa)}_m$. Therefore $G'F = 0$ on $\mathbf{B}$, and consequently $T_{G'}T_F = -H^*_{\overline{G'}}H_F$. Thus by Lemmas 4.8 and 4.6 and by (5.3), we have

$$\|T_{G'}T_F X\| \le \|H_F\|\|X\| \le C_{4.8}\mathrm{diff}(F)\|X\| \le C_{4.8}(\log m)^{-1}\|X\| \le C_{4.8}\|X\|\epsilon.$$

Combining this with (5.13), we see that $\|T_{G'}XT_F\| \le (2 + C_{4.8}\|X\|)\epsilon$. The same argument also shows that $\|T_{F'}XT_G\| \le (2 + C_{4.8}\|X\|)\epsilon$. Substituting these in (5.12) and recalling (5.11), we have

$$\|Z^{(3)}_{\nu,\kappa}\| \le 16(2 + C_{4.8}\|X\|)\epsilon. \tag{5.14}$$

Recapping the above, for each pair of $1 \le \kappa \le 6$ and $1 \le \nu \le N_0$ we obtain the decomposition

$$X_{\nu,\kappa} = Y_{\nu,\kappa} + Z^{(1)}_{\nu,\kappa} + Z^{(2)}_{\nu,\kappa} + Z^{(3)}_{\nu,\kappa},$$

where $Y_{\nu,\kappa} \in \mathcal{T}$ and $Z^{(1)}_{\nu,\kappa}$, $Z^{(2)}_{\nu,\kappa}$ and $Z^{(3)}_{\nu,\kappa}$ satisfy estimates (5.8), (5.9) and (5.14) respectively. Combining this with (5.6), we obtain the decomposition $X = Y + Z$, where

$$Y = X_0 + \sum_{\kappa=1}^{6}\sum_{\nu=1}^{N_0} Y_{\nu,\kappa} \quad \text{and} \quad Z = \sum_{\kappa=1}^{6}\sum_{\nu=1}^{N_0}(Z^{(1)}_{\nu,\kappa} + Z^{(2)}_{\nu,\kappa} + Z^{(3)}_{\nu,\kappa}). \tag{5.15}$$

Now, (5.1) follows from (5.8), (5.9) and (5.14), and we have shown that $Y \in \mathcal{T}$. This completes the proof. $\square$

Remark 5.2. Note that, other than its boundedness, the only property of X that we used in the above proof is that it satisfies the "ϵ-δ" condition in Proposition 3.7. Thus the above proof actually shows that for any bounded operator X on $L^2_a(\mathbf{B}, dv)$, if it satisfies the "ϵ-δ" condition in Proposition 3.7, then it belongs to $\mathcal{T}$. In other words, the "ϵ-δ" condition in Proposition 3.7 characterizes the membership $X \in \mathcal{T}$.

Remark 5.3. The proof given above has broader implications than just Theorem 1.2. As an example of such implications, we present a compactness criterion for operators in $\mathcal{T}$.

Proposition 5.4. *Let $X \in \mathcal{T}$. Then X is compact if and only if* $\mathrm{LOC}(X) \subset \mathcal{K}$.

Proof. Let $X \in \mathcal{T}$ and suppose that $\mathrm{LOC}(X) \subset \mathcal{K}$. As we showed above, for every $\epsilon > 0$, X admits a decomposition $X = Y + Z$, where Y and Z are given by (5.15), with X_0 known to be compact. Recalling (5.10), the assumption $\mathrm{LOC}(X) \subset \mathcal{K}$ implies that every $Y_{\nu,\kappa}$ is compact. Thus Y is compact. Since Z satisfies (5.1), this shows that X is compact.

Conversely, suppose that X is compact. Let Γ be any separated set in $\mathbf{B}$ and let $\{f_u : u \in \Gamma\}$ be any family of functions satisfying the conditions in Definition 2.7. Using Lemma 2.5, from the compactness of X we deduce that the operator

$$\sum_{u\in\Gamma} M_{f_u} X P M_{f_u}$$

is compact on $L^2(\mathbf{B}, dv)$. Compressing the above to the subspace $L^2_a(\mathbf{B}, dv)$, we see that every operator in $\mathrm{LOC}(X)$ is compact. □

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