DEFECT OPERATORS ASSOCIATED WITH SUBMODULES OF THE HARDY MODULE

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Abstract. Let $H^2(S)$ be the Hardy space on the unit sphere S in $\mathbf{C}^n$, $n \geq 2$. Then $H^2(S)$ is a natural Hilbert module over the ball algebra $A(\mathbf{B})$. Let M_{z_1}, ..., M_{z_n} be the module operators corresponding to the multiplication by the coordinated functions. Each submodule $\mathcal{M} \subset H^2(S)$ gives rise to the module operators $Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}$, $j = 1$, ..., n, on $\mathcal{M}$. In this paper we establish the following commonly believed, but never previously proven result: whenever $\mathcal{M} \neq \{0\}$, the sum of the commutators

$$[Z^*_{\mathcal{M},1}, Z_{\mathcal{M},1}] + ... + [Z^*_{\mathcal{M},n}, Z_{\mathcal{M},n}]$$

does not belong to the Schatten class $\mathcal{C}_n$.

1. Introduction

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Throughout the paper, we assume that the complex dimension n is greater than or equal to 2. The open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$ will be denoted by $\mathbf{B}$. We write $A(\mathbf{B})$ for the ball algebra. That is, $A(\mathbf{B})$ consists of functions which are analytic on $\mathbf{B}$ and continuous on the closed ball $\{z \in \mathbf{C}^n : |z| \leq 1\}$.

Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. We take the usual normalization $\sigma(S) = 1$. As usual, let $H^2(S)$ denote the Hardy space on the unit sphere S. That is, $H^2(S)$ is the closure in $L^2(S, d\sigma)$ of the polynomials in the coordinate variables $z_1, ..., z_n$. For each $i \in \{1, ..., n\}$, let Z_i be the operator of multiplication by the coordinate function z_i on $H^2(S)$.

The study of naturally arising operators on $H^2(S)$ has a long history. In recent years, an increasingly common approach in this study is to treat the Hardy space $H^2(S)$ as a Hilbert module over the algebra $A(\mathbf{B})$ [7], where the module operation is, of course, the natural multiplication of functions. In this context we will call $H^2(S)$ the *Hardy module* over $A(\mathbf{B})$. A great advantage of the framework of Hilbert modules is that it leads to many new and challenging questions.

A closed, linear subspace $\mathcal{M}$ of $H^2(S)$ is said to be a *submodule* of the Hardy module if it is invariant under the multiplication by the functions in $A(\mathbf{B})$. Each submodule $\mathcal{M}$ gives rise to the restricted operators

$$Z_{\mathcal{M},i} = Z_i|\mathcal{M}, \quad i = 1, ..., n.$$

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A natural question about the submodules is the Schatten class membership, or the lack thereof, of the commutators $[Z_{\mathcal{M},i}^*, Z_{\mathcal{M},j}]$. The purpose of this paper is to actually prove a "negative" result in this regard. Although this result is widely believed to be true, it has never been established in the literature. As we will see, its proof is not a trivial matter.

Recall that for each $1 \leq p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition $\|A\|_p < \infty$, where the p-norm is given by the formula $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$. In terms of the s-numbers $s_1(A)$, $s_2(A)$, ..., $s_k(A)$, ... of A (see [11,Section II.7]), we have

$$\|A\|_p = \left(\sum_{k=1}^{\infty}\{s_k(A)\}^p\right)^{1/p}.$$

Also recall that for each $k \in \mathbf{N}$,

$$s_k(A) = \inf\{\|A + K\| : \mathrm{rank}(K) \leq k-1\}.$$

It is well known that if $p > n$, then $[Z_i^*, Z_j] \in \mathcal{C}_p$ for all $i, j \in \{1, ..., n\}$. It is also well known that for each $i \in \{1, ..., n\}$, $[Z_i^*, Z_i] \notin \mathcal{C}_n$. See, for example, [10]. This leads to the natural question, what happens if we consider $Z_{\mathcal{M},1}$, ..., $Z_{\mathcal{M},n}$ instead of Z_1, ..., Z_n?

Given a submodule $\mathcal{M}$, let us denote

$$D_{\mathcal{M}} = \sum_{i=1}^{n}[Z_{\mathcal{M},i}^*, Z_{\mathcal{M},i}]. \tag{1.1}$$

The main result of the paper is an estimate for the distribution of the s-numbers of $D_{\mathcal{M}}$.

Theorem 1.1. *Let $\mathcal{M}$ be any submodule of the Hardy module $H^2(S)$. If $\mathcal{M} \neq \{0\}$, then there is a positive number $\epsilon = \epsilon(\mathcal{M}) > 0$ such that*

$$s_1(D_{\mathcal{M}}) + ... + s_k(D_{\mathcal{M}}) \geq \epsilon k^{(n-1)/n} \tag{1.2}$$

for every $k \in \mathbf{N}$. Consequently, $D_{\mathcal{M}}$ does not belong to the Schatten class $\mathcal{C}_n$ whenever $\mathcal{M} \neq \{0\}$.

In the above theorem, the conclusion $D_{\mathcal{M}} \notin \mathcal{C}_n$ follows from (1.2) immediately. This is because, if $1 < p < \infty$ and if $\{a_k\} \in \ell_+^p$, then $k^{-(p-1)/p}\sum_{j=1}^k a_j \to 0$ as $k \to \infty$.

An analogue of Theorem 1.1 also holds in the context of the Drury-Arveson space. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ which has the function $(1 - \langle w, z\rangle)^{-1}$ as its reproducing kernel [1,2,3]. One can alternately describe the Drury-Arveson space H_n^2 as the collection of analytic functions

$$f(z) = \sum_{\alpha \in \mathbf{Z}_+^n} c_\alpha z^\alpha$$

on $\mathbf{B}$ satisfying the condition

$$\sum_{\alpha\in\mathbf{Z}_+^n}|c_\alpha|^2\frac{\alpha!}{|\alpha|!}<\infty.$$

We will also write Z_1, ..., Z_n for the operators of multiplication by the coordinate functions z_1, ..., z_n on H_n^2. With the identification of each z_i with Z_i, H_n^2 is a *free* Hilbert module over the polynomial ring $\mathbf{C}[z_1,...,z_n]$ [1]. We will refer to H_n^2 as the *Drury-Arveson module*.

A closed, linear subspace $\mathcal{M}$ of H_n^2 is said to be a *submodule* of the Drury-Arveson module if it is invariant under the multiplication by the polynomials in $\mathbf{C}[z_1,...,z_n]$. Each submodule $\mathcal{M}$ of H_n^2 gives rise to the restricted operators

$$Z_{\mathcal{M},i}=Z_i|\mathcal{M},\qquad i=1,...,n.$$

Given a submodule $\mathcal{M}$ of the Drury-Arveson module H_n^2, we also define the operator $D_{\mathcal{M}}$ by (1.1). Note that for $\mathcal{M}\subset H_n^2$, $\mathcal{M}\neq\{0\}$, the operator $D_{\mathcal{M}}$ differs from

$$1-\sum_{i=1}^n Z_{\mathcal{M},i}Z_{\mathcal{M},i}^*. \tag{1.3}$$

In fact, (1.3) detects the "defect" of the row contraction $(Z_{\mathcal{M},1},...,Z_{\mathcal{M},n})$ and can even have finite rank for submodules of the Drury-Arveson module. But the story for $D_{\mathcal{M}}$ is quite different.

Theorem 1.2. *Let $\mathcal{M}$ be any submodule of the Drury-Arveson module H_n^2. If $\mathcal{M}\neq\{0\}$, then there is a positive number $\epsilon=\epsilon(\mathcal{M})>0$ such that*

$$s_1(D_{\mathcal{M}})+...+s_k(D_{\mathcal{M}})\geq\epsilon k^{(n-1)/n}$$

for every $k\in\mathbf{N}$. Consequently, $D_{\mathcal{M}}$ does not belong to the Schatten class $\mathcal{C}_n$ whenever $\mathcal{M}\neq\{0\}$.

Although Theorem 1.2 looks the same as Theorem 1.1, its proof is much easier than the proof of Theorem 1.1. Ultimately, this is due to the freeness of H_n^2 as a Hilbert module over $\mathbf{C}[z_1,...,z_n]$. Specifically, in the proof of Theorem 1.2 we use the fact that if $\mathcal{M}$ is a submodule of H_n^2 and if $\mathcal{M}\neq\{0\}$, then $\mathcal{M}$ contains a non-trivial *multiplier* of H_n^2 [2]. While this fact itself is not trivial, it leads to an easy proof of Theorem 1.2, as we will see.

By contrast, a multiplier for the Hardy space $H^2(S)$ is a function in $H^\infty(S)$. But if $\mathcal{M}$ is a submodule of the Hardy module, it is not known whether $\mathcal{M}\cap H^\infty(S)$ contains anything other than 0. This is the main difficulty in the proof of Theorem 1.1. In other words, in the proof of Theorem 1.1 we need a scheme to get around a certain unboundedness.

Notwithstanding the technical merit in the proof of Theorem 1.1, it is a fair question to ask, why should one care about these results? Or, put differently, what is the motivation for proving Theorems 1.1 and 1.2 in the first place? We are motivated by two considerations.

The first motivation is related to what is now commonly referred to as the Arveson conjecture. Simply stated, it is this:

Problem 1.3. [3] For a submodule $\mathcal{M}$ of H_n^2, do the commutators $[Z_{\mathcal{M},i}^*, Z_{\mathcal{M},j}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$?

In [5], Douglas proposed the analogous problem for the Bergman space. From there it does not take too much imagination for one to think about the case of the Hardy space $H^2(S)$, since all of these are reproducing-kernel Hilbert spaces. In all these versions of the problem, one conspicuous feature is the lower limit $p > n$ that one sets for the Schatten class. One might say that this lower limit is dictated by known examples. For instance, it is well known that $[Z_i^*, Z_i] \notin \mathcal{C}_n$ on $H^2(S)$ and H_n^2, and the same is also true on the Bergman space of the ball $\mathbf{B}$. In other words, examples show that the lower limit $p > n$ is necessary for *some* submodules. The first motivation for this investigation was to find out whether the lower limit $p > n$ is necessary for *every* submodule $\mathcal{M} \neq \{0\}$.

The second motivation is related to extensions of the C^*-algebra $C(S)$ by the compact operators. More specifically, this stems from a paper of Douglas and Voiculescu [8]. Suppose that $(T_1, ..., T_n)$ is an essentially commuting tuple of bounded operators on a separable Hilbert space $\mathcal{H}$. Furthermore, suppose that the tuple $(T_1, ..., T_n)$ generates an exact sequence

$$\{0\} \to \mathcal{K} \to \mathcal{T} \to C(S) \to \{0\},$$

where $\mathcal{K}$ is the collection of compact operators, $\mathcal{T}$ is the C^*-algebra generated by T_1, ..., T_n and $\mathcal{K}$, and the homomorphism $\mathcal{T} \to C(S)$ is an extension of the map $T_i \mapsto z_i$, $i = 1$, ..., n. Such an exact sequence, of course, represents an element $[\tau]$ in $\mathrm{Ext}(S)$ [4]. The class $[\tau]$ can be determined in the following way. There exists a $2^n \times 2^n$ matrix α whose entries are polynomials in $2n$ variables such that if we set

$$A = \alpha(T_1, T_1^*, ..., T_n, T_n^*), \tag{1.4}$$

then A is Fredholm and, under the identification $\mathrm{Ext}(S) \cong \mathbf{Z}$ [4], we have

$$[\tau] = \mathrm{index}(A).$$

See page 107 in [8]. Douglas and Voiculescu proved the following index formula, which is generally regarded as a precursor to what is now called non-commutative geometry.

Theorem 1.4. [8,Proposition 2] *Suppose that the operators T_1, ..., T_n satisfy the conditions*

$$[T_i, T_j] \in \mathcal{C}_n, \quad [T_i^*, T_j] \in \mathcal{C}_n \quad \textit{for all } \ 1 \le i, j \le n \tag{1.5}$$

and

$$1 - \sum_{i=1}^{n} T_i^* T_i \in \mathcal{C}_n. \tag{1.6}$$

Then for the operator A given by (1.4) *we have*

$$\mathrm{index}(A) = \mathrm{tr}[T_1, T_1^*, ..., T_n, T_n^*], \tag{1.7}$$

where $[T_1, T_1^*, ..., T_n, T_n^*]$ *is the antisymmetric sum of* T_1, T_1^*, ..., T_n, T_n^*.

When [8] was published, it was not known whether one can have $\text{index}(A) \neq 0$ for a tuple $(T_1, ..., T_n)$ satisfying conditions (1.5) and (1.6). Later, however, Gong showed that for any $m \in \mathbf{Z}$, there exists a tuple $(T_1, ..., T_n)$ satisfying (1.5) and (1.6) with

$$\text{index}(A) = m.$$

Indeed Gong showed that one can even replace Schatten class $\mathcal{C}_n$ with $\mathcal{C}_p$ for any $p > n - (1/2)$ [12,Theorems 2.2 and 2.4].

From the view point of function-theoretical operator theory, however, Gong's paper is not the end of the story. It does not tell us whether index formula (1.7) can *ever* be applied to *canonical* operator tuples associated with the sphere. Being canonical is, of course, not something that one can precisely define. But by any standard, tuples of the form

$$(T_1, ..., T_n) = (Z_{\mathcal{M},1}, ..., Z_{\mathcal{M},n}) \tag{1.8}$$

must be considered canonical in the setting of the sphere, while perturbations of the above by unspecified compact operators are certainly not. Thus one of the motivating questions for us was whether index formula (1.7) can ever be applied in the case of (1.8). In other words, do (1.8), (1.5) and (1.6) ever hold simultaneously? Although it is generally believed that the answer is always negative for submodules $\mathcal{M} \neq \{0\}$ of the Hardy, Drury-Arveson and Bergman modules, no such results have been proven until Theorems 1.1 and 1.2. But we do not know what happens in the case of the Bergman module.

Problem 1.5. Does the analogue of Theorems 1.1 and 1.2 hold true for submodules of the Bergman module?

Having explained the motivation for this investigation, let us turn to the techniques involved in the proof of Theorem 1.1. This paper benefits greatly from our recent work on Hankel operators on the sphere [9]. More specifically, the proof of Theorem 1.1 uses many ideas in the proof of [9,Theorem 1.6]. To prove (1.2), for each $k \in \mathbf{N}$ we need to construct an operator F with

$$\text{rank}(F) \approx k$$

and

$$\|F\| \leq C, \tag{1.9}$$

where C is independent of k, such that $\text{tr}(D_{\mathcal{M}}F)$ is on the order of $k^{(n-1)/n}$. Of the three requirements, (1.9) turns out to be the biggest obstacle, which we over come by using the Schur multiplier m_z introduced in [9,Section 3] and certain techniques in [14].

The rest of the paper is organized as follows. Section 2 contains some of the estimates needed in the proof of Theorem 1.1. In Section 3 we combine these estimates and others to complete the proof of Theorem 1.1. In Section 4 we give a proof of Theorem 1.2. Thanks to Arveson's paper [2], the proof of Theorem 1.2 is just a rather straightforward estimate using the standard orthonormal basis for H_n^2. We include Theorem 1.2 in the paper partly because its proof provides such a sharp contrast to the proof of Theorem 1.1. Finally, in Section 5 we explain the meaning of inequality (1.2) in the language of norm ideals [11].

2. Preliminaries and Estimates

Recall that the normalized reproducing kernel for $H^2(S)$ is given by the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{n/2}}{(1-\langle \zeta, z\rangle)^n}, \qquad |\zeta| \le 1 \quad \text{and} \quad |z| < 1.$$

Lemma 2.1. *If $f \in L^2(S, d\sigma)$, then*

$$\lim_{r\uparrow 1} \int \|(f - f(\xi))k_{r\xi}\|^2 d\sigma(\xi) = 0. \tag{2.1}$$

Proof. Let $f \in L^2(S, d\sigma)$ and $0 < r < 1$. Then

$$\begin{aligned}\int \|fk_{r\xi}\|^2 d\sigma(\xi) &= \int \left(\int |f(\zeta)|^2 |k_{r\xi}(\zeta)|^2 d\sigma(\zeta)\right) d\sigma(\xi) \\ &= \int |f(\zeta)|^2 \left(\int \frac{(1-r^2)^n}{|1 - r\langle \zeta, \xi\rangle|^{2n}} d\sigma(\xi)\right) d\sigma(\zeta) = \int |f(\zeta)|^2 d\sigma(\zeta) = \|f\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int \|(f - f(\xi))k_{r\xi}\|^2 d\sigma(\xi) &= \int (\|fk_{r\xi}\|^2 - 2\mathrm{Re}\{\overline{f(\xi)}\langle fk_{r\xi}, k_{r\xi}\rangle\} + |f(\xi)|^2) d\sigma(\xi) \\ &= 2(\|f\|^2 - \mathrm{Re}\langle f_r, f\rangle),\end{aligned} \tag{2.2}$$

where

$$f_r(\xi) = \langle fk_{r\xi}, k_{r\xi}\rangle = \int f(\zeta) \frac{(1-r^2)^n}{|1 - r\langle \zeta, \xi\rangle|^{2n}} d\sigma(\zeta).$$

By the well-known properties of the Poisson integral on S (see, e.g., [13,Theorem 3.3.4(b)]),

$$\lim_{r\uparrow 1} \langle f_r, f\rangle = \langle f, f\rangle = \|f\|^2. \tag{2.3}$$

Thus (2.1) follows from (2.2) and (2.3). □

Let us denote

$$D = \sum_{i=1}^n [Z_i^*, Z_i], \tag{2.4}$$

and let us keep in mind that D acts on the entire Hardy space $H^2(S)$.

In conformity with [13], we write $\mathcal{U}$ for the collection of unitary transformations on $\mathbf{C}^n$. Each $U \in \mathcal{U}$ induces a unitary operator W_U on $H^2(S)$ by the formula

$$(W_U h)(\zeta) = h(U\zeta), \qquad h \in H^2(S).$$

It is straightforward to verify that

$$[D, W_U] = 0 \quad \text{for every} \ \ U \in \mathcal{U}. \tag{2.5}$$

For each $z \in \mathbf{B}$, we define the Schur multiplier

$$m_z(\zeta) = \frac{1 - |z|}{1 - \langle \zeta, z\rangle} \tag{2.6}$$

as in [9]. Furthermore, for each $z \in \mathbf{B}$ we define the function

$$u_z(\zeta) = \frac{(1 - |z|^2)^{(n/2)+n+1}}{(1 - \langle \zeta, z\rangle)^{2n+1}}. \tag{2.7}$$

In the proof of our next lemma, the assumption $n \geq 2$ enters the proof of Theorem 1.1 in a dramatic fashion.

Lemma 2.2. *There exists a constant $c_{2.2} > 0$ which depends only on the complex dimension n such that*

$$\langle Du_z, u_z\rangle \geq c_{2.2}(1 - |z|^2)$$

for every $z \in \mathbf{B}$.

Proof. Given any $z \in \mathbf{B}$, write $\rho = |z|$ and define the vector $\hat{z} = (\rho, 0, ..., 0)$ in $\mathbf{B}$. There exists a $V \in \mathcal{U}$ such that $V^* z = \hat{z}$. Thus, by (2.5), we have

$$\langle Du_z, u_z\rangle = \langle DW_V u_z, W_V u_z\rangle = \langle Du_{V^*z}, u_{V^*z}\rangle = \langle Du_{\hat{z}}, u_{\hat{z}}\rangle.$$

Since $[Z_i^*, Z_i] \geq 0$ for each $i \in \{1, ..., n\}$, the above implies

$$\langle Du_z, u_z\rangle \geq \langle [Z_2^*, Z_2]u_{\hat{z}}, u_{\hat{z}}\rangle. \tag{2.8}$$

Let us write ζ_i for the i-th component of ζ. Since $\hat{z} = (\rho, 0, ..., 0)$, we have

$$u_{\hat{z}}(\zeta) = \frac{(1 - \rho^2)^{(n/2)+n+1}}{(1 - \rho\zeta_1)^{2n+1}}.$$

Since $u_{\hat{z}}$ only depends on ζ_1, it follows that $Z_2^* u_{\hat{z}} = 0$. Thus from (2.8) we obtain

$$\begin{aligned} \langle Du_z, u_z\rangle &\geq \langle [Z_2^*, Z_2]u_{\hat{z}}, u_{\hat{z}}\rangle = \langle Z_2^* Z_2 u_{\hat{z}}, u_{\hat{z}}\rangle \\ &= \|Z_2 u_{\hat{z}}\|^2 = (1 - \rho^2)^{3n+2} \int \frac{|\zeta_2|^2}{|1 - \rho\zeta_1|^{4n+2}} d\sigma(\zeta). \end{aligned} \tag{2.9}$$

To estimate the above integral, recall that, for each $m \in \mathbf{N}$, the expansion

$$\frac{1}{(1-v)^{m+1}} = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!m!} v^k$$

holds on $\{v \in \mathbf{C} : |v| < 1\}$. Since $\zeta_1^j \zeta_2 \perp \zeta_1^k \zeta_2$ for each pair of $j \neq k$ in $\mathbf{Z}_+$, we have

$$\begin{aligned}\int \frac{|\zeta_2|^2}{|1-\rho\zeta_1|^{4n+2}} d\sigma(\zeta) &= \sum_{k=0}^{\infty} \left(\frac{(k+2n)!}{k!(2n)!}\right)^2 \rho^{2k} \int |\zeta_1^k \zeta_2|^2 d\sigma(\zeta) \\ &= \sum_{k=0}^{\infty} \left(\frac{(k+2n)!}{k!(2n)!}\right)^2 \rho^{2k} \frac{k!1!(n-1)!}{(n-1+k+1)!},\end{aligned}$$

where the second $=$ follows from [13,Proposition 1.4.9]. Hence

$$\begin{aligned}\int \frac{|\zeta_2|^2}{|1-\rho\zeta_1|^{4n+2}} d\sigma(\zeta) &= \frac{(n-1)!}{(2n)!} \sum_{k=0}^{\infty} \frac{(k+2n)!}{k!(2n)!} \rho^{2k} \prod_{j=1}^{n} (k+n+j) \\ &\geq \frac{(n-1)!}{(2n)!2^n} \sum_{k=0}^{\infty} \frac{(k+3n)!}{k!(3n)!} \rho^{2k} = \frac{(n-1)!}{(2n)!2^n} \cdot \frac{1}{(1-\rho^2)^{3n+1}}.\end{aligned}$$

Substituting this in (2.9) and recalling the fact that $\rho = |z|$, the lemma follows. $\square$

It is elementary that if c is a complex number with $|c| \leq 1$ and if $0 < t < 1$, then

$$2|1-tc| \geq |1-c|. \tag{2.10}$$

It is well known that the formula

$$d(x,y) = |1-\langle x,y\rangle|^{1/2}, \qquad x,y \in S, \tag{2.11}$$

defines a metric on S [13,page 66]. Throughout the paper, we denote

$$B(x,r) = \{\zeta \in S : |1-\langle \zeta,x\rangle|^{1/2} < r\}$$

for $x \in S$ and $r > 0$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \leq \sigma(B(x,r)) \leq A_0 r^{2n} \tag{2.12}$$

for all $x \in S$ and $0 < r \leq \sqrt{2}$ [13,Proposition 5.1.4]. Note that the upper bound above also holds for $r \geq \sqrt{2}$.

Lemma 2.3. (i) If $x, y \in S$, $x \neq y$, $0 < \rho < 1$, and $\zeta \in S$, then

$$|m_{\rho x}(\zeta) m_{\rho y}(\zeta)| \leq \frac{8(1-\rho)}{|1-\langle x,y\rangle|}. \tag{2.13}$$

(ii) *For all* $x, \zeta \in S$ *and* $0 < \rho < 1$, *we have* $|\zeta - x||m_{\rho x}(\zeta)| \le 2(1-\rho)^{1/2}$.

Proof. (i) Given $x \ne y$ in S, write $t = d(x, y)$. Let $0 < \rho < 1$ and $\zeta \in S$ also be given. Since $B(x, t/2) \cap B(y, t/2) = \emptyset$, we have either $\zeta \notin B(x, t/2)$ or $\zeta \notin B(y, t/2)$. If $\zeta \notin B(x, t/2)$, then

$$|m_{\rho x}(\zeta)| = \frac{1-\rho}{|1-\rho\langle\zeta, x\rangle|} \le \frac{2(1-\rho)}{|1-\langle\zeta, x\rangle|} = \frac{2(1-\rho)}{(d(\zeta, x))^2} \le \frac{2(1-\rho)}{(t/2)^2} = \frac{8(1-\rho)}{|1-\langle x, y\rangle|}.$$

Since $|m_{\rho y}(\zeta)| \le 1$, (2.13) holds in this case. Similarly, if $\zeta \notin B(y, t/2)$, then $|m_{\rho y}(\zeta)| \le 8(1-\rho)|1-\langle x, y\rangle|^{-1}$ and $|m_{\rho x}(\zeta)| \le 1$. Hence (2.13) also holds in the latter case.

(ii) Let $x, \zeta \in S$. Then $|\zeta - x|^2 = 2 - 2\mathrm{Re}\langle\zeta, x\rangle$. Thus for each $0 < \rho < 1$ we have

$$\begin{aligned} |\zeta - x||m_{\rho x}(\zeta)| &\le \sqrt{2|1-\langle\zeta, x\rangle|}\frac{1-\rho}{|1-\rho\langle\zeta, x\rangle|} \\ &= \left\{\frac{|1-\langle\zeta, x\rangle|}{|1-\rho\langle\zeta, x\rangle|}\right\}^{1/2} \cdot \left\{\frac{1-\rho}{|1-\rho\langle\zeta, x\rangle|}\right\}^{1/2} \cdot \sqrt{2} \cdot \sqrt{1-\rho}. \end{aligned}$$

Obviously, the second $\{\cdots\}^{1/2}$ above is at most 1, while the first $\{\cdots\}^{1/2}$ does not exceed $\sqrt{2}$ according to (2.10). This completes the proof. □

Next we recall the following counting lemma:

Lemma 2.4. [14,Lemma 4.1] *Let* X *be a set and let* E *be a subset of* $X \times X$. *Suppose that* m *is a natural number such that*

$$\mathrm{card}\{y \in X : (x, y) \in E\} \le m \quad \textit{and} \quad \mathrm{card}\{y \in X : (y, x) \in E\} \le m$$

for every $x \in X$. *Then there exist pairwise disjoint subsets* $E_1, E_2, \ldots, E_{2m}$ *of* E *such that*

$$E = E_1 \cup E_2 \cup \ldots \cup E_{2m}$$

and such that for each $1 \le j \le 2m$, *the conditions* $(x, y), (x', y') \in E_j$ *and* $(x, y) \ne (x', y')$ *imply both* $x \ne x'$ *and* $y \ne y'$.

For each $z \in \mathbf{B}$, define the function

$$v_z(\zeta) = \left(\frac{1-|z|^2}{1-\langle\zeta, z\rangle}\right)^{n+1}. \tag{2.14}$$

The following is the key estimate in the proof of Theorem 1.1.

Lemma 2.5. *There is a constant* $0 < C_{2.5} < \infty$ *which depends only on the complex dimension* n *such that the following estimate holds*: *Let* $0 < t < 1$ *and suppose that* $\{y_j : j \in J\}$ *is a subset of* S *satisfying the condition*

$$B(y_i, t) \cap B(y_j, t) = \emptyset \quad \textit{if} \;\; i \ne j. \tag{2.15}$$

For each $j \in J$, let

$$z_j = (1 - t^2)^{1/2} y_j.$$

Let $\{f_j : j \in J\}$ be a set of functions in $L^2(S, d\sigma)$ satisfying the condition $\|f_j\| \le 1$, $j \in J$. Then the norm of the operator

$$F = \sum_{j \in J} (v_{z_j} f_j) \otimes (v_{z_j} f_j)$$

satisfies the inequality $\|F\| \le C_{2.5}$.

Proof. Let $\{\eta_j : j \in J\}$ be an orthonormal set and define the operator

$$T = \sum_{j \in J} (v_{z_j} f_j) \otimes \eta_j.$$

Then $F = TT^*$. Since $\|TT^*\| = \|T^*T\|$, it suffices to estimate $\|T^*T\|$. We have

$$T^*T = \sum_{i,j \in J} \langle v_{z_j} f_j, v_{z_i} f_i \rangle \eta_i \otimes \eta_j = B + \sum_{k=0}^{\infty} Y_k, \tag{2.16}$$

where

$$B = \sum_{j \in J} \|v_{z_j} f_j\|^2 \eta_j \otimes \eta_j$$

and

$$Y_k = \sum_{2^k t \le d(y_i, y_j) < 2^{k+1} t} \langle v_{z_j} f_j, v_{z_i} f_i \rangle \eta_i \otimes \eta_j,$$

$k \in \mathbf{Z}_+$. By (2.14), $\|v_z\|_\infty \le (1 + |z|)^{n+1} \le 2^{n+1}$ for each $z \in \mathbf{B}$. Since $\|f_j\| \le 1$, we have $\|v_{z_j} f_j\| \le 2^{n+1}$, $j \in J$. Since $\{\eta_j : j \in J\}$ is an orthonormal set, we conclude that

$$\|B\| \le 4^{n+1}. \tag{2.17}$$

Next we estimate $\|Y_k\|$.

For each $k \in \mathbf{Z}_+$, define

$$E^{(k)} = \{(i, j) \in J \times J : 2^k t \le d(y_i, y_j) < 2^{k+1} t\}.$$

Using the conditions $\|f_j\| \le 1$, $\|f_i\| \le 1$ and (2.14), we have

$$|\langle v_{z_j} f_j, v_{z_i} f_i \rangle| \le \int |v_{z_j} v_{z_i}| |f_j f_i| d\sigma \le \|v_{z_j} v_{z_i}\|_\infty \le 4^{n+1} \|m_{z_j} m_{z_i}\|_\infty^{n+1}.$$

For each $(i, j) \in E^{(k)}$, it follows from Lemma 2.3(i) and the condition $d(y_i, y_j) \ge 2^k t$ that

$$\|m_{z_j} m_{z_i}\|_\infty \le \frac{8(1 - (1 - t^2)^{1/2})}{(2^k t)^2} \le \frac{8t^2}{2^{2k} t^2} = \frac{8}{2^{2k}}.$$

Hence

$$|\langle v_{z_j} f_j, v_{z_i} f_i\rangle| \le \frac{(32)^{n+1}}{2^{2k(n+1)}} \quad \text{if } \ (i,j) \in E^{(k)}. \tag{2.18}$$

For each $i \in J$, if $d(y_i, y_j) < 2^{k+1}t$, then $B(y_j, t) \subset B(y_i, 2^{k+2}t)$. By (2.15) and the fact that $\sigma(B(x,t)) = \sigma(B(y,t))$ for all $x, y \in S$, for each $i \in J$ we have

$$\text{card}\{j \in J : d(y_i, y_j) < 2^{k+1}t\} \le \frac{\sigma(B(y_i, 2^{k+2}t))}{\sigma(B(y_i, t))} \le \frac{A_0(2^{k+2}t)^{2n}}{2^{-n}t^{2n}} = C_1 2^{2nk},$$

where A_0 is the constant that appears in (2.12) and $C_1 = 2^{5n}A_0$. Set

$$\ell(k) = \min\{\ell \in \mathbf{N} : \ell \ge C_1 2^{2nk}\}. \tag{2.19}$$

According to Lemma 2.4, we can decompose $E^{(k)}$ as the union of pairwise disjoint subsets

$$E_1^{(k)}, ..., E_{2\ell(k)}^{(k)}$$

such that for each $\nu \in \{1, ..., 2\ell(k)\}$, if $(i,j), (i',j') \in E_\nu^{(k)}$ and if $(i,j) \ne (i',j')$, then we have both $i \ne i'$ and $j \ne j'$. This decomposition of $E^{(k)}$ allows us to write

$$Y_k = Y_{k,1} + ... + Y_{k,2\ell(k)}, \tag{2.20}$$

where

$$Y_{k,\nu} = \sum_{(i,j)\in E_\nu^{(k)}} \langle v_{z_j} f_j, v_{z_i} f_i\rangle \eta_i \otimes \eta_j,$$

$1 \le \nu \le 2\ell(k)$. The property of $E_\nu^{(k)}$ simply means that the projection onto the first component, $(i,j) \mapsto i$, is injective on $E_\nu^{(k)}$. Similarly, the projection onto the second component, $(i,j) \mapsto j$, is also injective on each $E_\nu^{(k)}$. Combining these injectivities with the fact that $\{\eta_j : j \in J\}$ is an orthonormal set and with (2.18), we obtain

$$\|Y_{k,\nu}\| \le \frac{(32)^{n+1}}{2^{2k(n+1)}}$$

for each $\nu \in \{1, ..., 2\ell(k)\}$. Recalling (2.20) and (2.19), we now have

$$\|Y_k\| \le \frac{(32)^{n+1}}{2^{2k(n+1)}} \cdot 2\ell(k) \le \frac{(32)^{n+1}}{2^{2k(n+1)}} \cdot 2(C_1+1)2^{2nk} = \frac{2(32)^{n+1}(C_1+1)}{2^{2k}}.$$

Combining this estimate with (2.16) and (2.17), we see that if we set

$$C_{2.5} = 4^{n+1} + 2(32)^{n+1}(C_1+1)\sum_{k=0}^{\infty} \frac{1}{2^{2k}},$$

then $\|F\| \le C_{2.5}$. $\square$

3. Proof of Theorem 1.1

Let a submodule $\mathcal{M}$ of the Hardy module $H^2(S)$ be given. Let $P_{\mathcal{M}}$ be the orthogonal projection from $H^2(S)$ onto $\mathcal{M}$. Let $i \in \{1, ..., n\}$ and $h \in \mathcal{M}$. Then $Z^*_{\mathcal{M},i}h = P_{\mathcal{M}} Z_i^* h$ and $Z_{\mathcal{M},i}h = Z_i h$, which leads to

$$\langle [Z^*_{\mathcal{M},i}, Z_{\mathcal{M},i}]h, h\rangle = \|Z_{\mathcal{M},i}h\|^2 - \|Z^*_{\mathcal{M},i}h\|^2 \ge \|Z_i h\|^2 - \|Z_i^* h\|^2 = \langle [Z_i^*, Z_i]h, h\rangle.$$

Therefore

$$\langle D_{\mathcal{M}}h, h\rangle \ge \langle Dh, h\rangle \quad \text{for every} \quad h \in \mathcal{M}. \tag{3.1}$$

Our proof of Theorem 1.1 is based on the realization that, with enough work and further exploitation of the invariance of $\mathcal{M}$ under the multiplication by functions in $A(\mathbf{B})$, (1.2) can be deduced from (3.1). Here is how we proceed.

Suppose that $\mathcal{M} \neq \{0\}$. We pick an arbitrary $\psi \in \mathcal{M}$ with $\|\psi\| \neq 0$. Since $0 < \|\psi\| < \infty$, there are positive numbers $0 < a < b < \infty$ such that if we set

$$G = \{\xi \in S : a \le |\psi(\xi)| \le b\},$$

then $\sigma(G) > 0$. Next we set

$$c_{3.2} = \frac{a c_{2.2}^{1/2}}{\sqrt{n}2^{n+3}}, \tag{3.2}$$

where $c_{2.2}$ is the constant that appears in Lemma 2.2. For each $0 < r < 1$, define the set

$$G_r = \{\xi \in G : \|(\psi - \psi(\xi))k_{r\xi}\| \le c_{3.2}\}.$$

Obviously,

$$\sigma(G \backslash G_r) \le c_{3.2}^{-2} \int \|(\psi - \psi(\xi))k_{r\xi}\|^2 d\sigma(\xi).$$

Thus it follows from Lemma 2.1 that there exists a $0 < \rho < 1$ such that

$$\sigma(G_r) \ge (1/2)\sigma(G) \quad \text{for every} \quad \rho \le r < 1. \tag{3.4}$$

With this ρ we define

$$\delta = (1 - \rho^2)^{1/2}. \tag{3.5}$$

Now suppose a $0 < t < \delta$ is given. We set $r(t) = (1 - t^2)^{1/2}$. The relation between δ and ρ ensures $\rho < r(t) < 1$. By (3.4), this gives us

$$\sigma(G_{r(t)}) \ge (1/2)\sigma(G). \tag{3.6}$$

There is a subset $\{x_j : j \in J\}$ of $G_{r(t)}$ which is *maximal* with respect to the property

$$B(x_i, t) \cap B(x_j, t) = \emptyset \qquad \text{if} \;\; i \neq j. \tag{3.7}$$

The maximality of $\{x_j : j \in J\}$ implies $\cup_{j\in J} B(x_j, 2t) \supset G_{r(t)}$. Combining this with (2.12) and (3.6), we see that there exist a constant $0 < c_1 < \infty$ which are determined by n and $\sigma(G)$, and a constant $0 < C_2 < \infty$ which depends on n only, such that

$$c_1 t^{-2n} \leq \mathrm{card}(J) \leq C_2 t^{-2n}. \tag{3.8}$$

For each $j \in J$, define $z_j = (1 - t^2)^{1/2} x_j$. Then define the operator

$$F_t = \sum_{j\in J} (u_{z_j}\psi) \otimes (u_{z_j}\psi),$$

where u_{z_j} is defined by (2.7). Since $x_j \in G_{r(t)}$ and $z_j = r(t)x_j$, the definition of $G_{r(t)}$ ensures $\|(\psi - \psi(x_j))k_{z_j}\| \leq c_{3.2}$ for each $j \in J$. Since $G_{r(t)} \subset G$, we have $|\psi(x_j)| \leq b$, $j \in J$. Combining these two inequalities, we find that

$$\|\psi k_{z_j}\| \leq c_{3.2} + b \qquad \text{for each} \;\; j \in J. \tag{3.9}$$

By (2.7) and (2.14), $u_{z_j}\psi = v_{z_j}\psi k_{z_j}$. Thus it follows from (3.9) and Lemma 2.5 that

$$\|F_t\| \leq C_{2.5}(c_{3.2} + b)^2. \tag{3.10}$$

Since $\psi \in \mathcal{M}$ and since $\mathcal{M}$ is a submodule, we have $u_{z_j}\psi \in \mathcal{M}$ for each $j \in J$. Hence F_t is an operator on the Hilbert space $\mathcal{M}$. Next we estimate $\mathrm{tr}(D_{\mathcal{M}} F_t)$.

Applying (3.1), we have

$$\mathrm{tr}(D_{\mathcal{M}} F_t) = \sum_{j\in J} \langle D_{\mathcal{M}} u_{z_j}\psi, u_{z_j}\psi\rangle \geq \sum_{j\in J} \langle D u_{z_j}\psi, u_{z_j}\psi\rangle = \sum_{j\in J} \|D^{1/2} u_{z_j}\psi\|^2. \tag{3.11}$$

We need to estimate $\|D^{1/2}u_{z_j}\psi\|^2$ for each $j \in J$. Obviously,

$$\begin{aligned}
\|D^{1/2}u_{z_j}\psi\| &\geq |\psi(x_j)|\|D^{1/2}u_{z_j}\| - \|D^{1/2}(\psi - \psi(x_j))u_{z_j}\| \\
&\geq a\|D^{1/2}u_{z_j}\| - \|D^{1/2}(\psi - \psi(x_j))u_{z_j}\| \\
&\geq a\{c_{2.2}(1 - |z_j|^2)\}^{1/2} - \|D^{1/2}(\psi - \psi(x_j))u_{z_j}\| \\
&= ac_{2.2}^{1/2} t - \|D^{1/2}(\psi - \psi(x_j))u_{z_j}\|, \qquad (3.12)
\end{aligned}$$

where the third $\geq$ follows from Lemma 2.2.

For each $\nu \in \{1, ..., n\}$, write $(x_j)_\nu$ for the ν-th component of x_j. Also, for each $g \in A(\mathbf{B})$, let M_g denote the operator of multiplication by g on $H^2(S)$ as usual. Then

$$\begin{aligned}
\|D^{1/2}(\psi - \psi(x_j))u_{z_j}\|^2 &= \langle D(\psi - \psi(x_j))u_{z_j}, (\psi - \psi(x_j))u_{z_j}\rangle \\
&= \sum_{\nu=1}^{n} \langle [Z_\nu^*, Z_\nu](\psi - \psi(x_j))u_{z_j}, (\psi - \psi(x_j))u_{z_j}\rangle \\
&= \sum_{\nu=1}^{n} \langle [(Z_\nu - (x_j)_\nu)^*, Z_\nu - (x_j)_\nu](\psi - \psi(x_j))u_{z_j}, (\psi - \psi(x_j))u_{z_j}\rangle \\
&\le \sum_{\nu=1}^{n} \|(Z_\nu - (x_j)_\nu)(\psi - \psi(x_j))u_{z_j}\|^2 \\
&= \sum_{\nu=1}^{n} \|(Z_\nu - (x_j)_\nu)M_{v_{z_j}}(\psi - \psi(x_j))k_{z_j}\|^2, \\
&\le \sum_{\nu=1}^{n} \|(Z_\nu - (x_j)_\nu)M_{v_{z_j}}\|^2 \|(\psi - \psi(x_j))k_{z_j}\|^2. \qquad (3.13)
\end{aligned}$$

By (2.14) and (2.6), $\|(Z_\nu - (x_j)_\nu)M_{v_{z_j}}\| \le 2^{n+1}\|(Z_\nu - (x_j)_\nu)M_{m_{z_j}}\|$. On the other hand, Lemma 2.3(ii) tells us that $\|(Z_\nu - (x_j)_\nu)M_{m_{z_j}}\| \le 2(1 - r(t))^{1/2} \le 2t$. Thus

$$\|(Z_\nu - (x_j)_\nu)M_{v_{z_j}}\|^2 \le 4^{n+2}t^2.$$

As we mentioned previously, $\|(\psi - \psi(x_j))k_{z_j}\| \le c_{3.2}$. Hence from (3.13) we obtain

$$\|D^{1/2}(\psi - \psi(x_j))u_{z_j}\|^2 \le nc_{3.2}^2 4^{n+2}t^2.$$

Taking square-root on both sides and then bringing the result into (3.12), we find that

$$\|D^{1/2}u_{z_j}\psi\| \ge \{ac_{2.2}^{1/2} - \sqrt{n}c_{3.2}2^{n+2}\}t = (1/2)ac_{2.2}^{1/2}t, \qquad (3.14)$$

where the = follows from (3.2).

Combining (3.11), (3.14) and the lower bound in (3.8), we have

$$\mathrm{tr}(D_{\mathcal{M}}F_t) \ge \frac{1}{4}a^2c_{2.2}t^2 \cdot \mathrm{card}(J) \ge \frac{1}{4}a^2c_{2.2}t^2 \cdot c_1t^{-2n} = c_3t^{2-2n}, \qquad (3.15)$$

where we set $c_3 = (1/4)a^2c_1c_{2.2}$. Let $\|.\|_1$ denote the norm of the trace class. Recall that s-numbers obey the relation $s_k(AB) \le s_k(A)\|B\|$ [11,page 61]. Hence

$$\mathrm{tr}(D_{\mathcal{M}}F_t) \le \|D_{\mathcal{M}}F_t\|_1 = \sum_{\ell=1}^{\mathrm{rank}(F_t)} s_\ell(D_{\mathcal{M}}F_t) \le \sum_{\ell=1}^{\mathrm{rank}(F_t)} s_\ell(D_{\mathcal{M}})\|F_t\|. \qquad (3.16)$$

Obviously, the upper bound in (3.8) implies $\mathrm{rank}(F_t) \le C_2 t^{-2n}$. Combine this with (3.16), (3.10) and (3.15), we obtain

$$C_{2.5}(c_{3.2}+b)^2 \sum_{1\le \ell \le C_2 t^{-2n}} s_\ell(D_{\mathcal{M}}) \ge \mathrm{tr}(D_{\mathcal{M}} F_t) \ge c_3 t^{2-2n}.$$

Thus we have shown that, if we set $c_4 = \{C_{2.5}(c_{3.2}+b)^2\}^{-1} c_3$, then the inequality

$$\sum_{1\le \ell \le C_2 t^{-2n}} s_\ell(D_{\mathcal{M}}) \ge c_4 t^{2-2n}$$

holds for every $0 < t < \delta$.

Let K be the smallest natural number greater than $C_2\delta^{-2n}$. If $k \ge K$, then there is a $0 < t_k < \delta$ such that $C_2 t_k^{-2n} = k$. Hence

$$s_1(D_{\mathcal{M}}) + \dots + s_k(D_{\mathcal{M}}) = \sum_{1\le \ell \le C_2 t_k^{-2n}} s_\ell(D_{\mathcal{M}}) \ge c_4 t_k^{2-2n} = c_4 C_2^{-(n-1)/n} k^{(n-1)/n}$$

for each $k \ge K$. Thus if we set $\epsilon = K^{-1} c_4 C_2^{-(n-1)/n}$, then (1.2) holds for every $k \in \mathbf{N}$. This completes the proof of Theorem 1.1.

4. Submodules of the Drury-Arveson Module

We still assume, of course, $n \ge 2$. Let $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ be the standard orthonormal basis for the Drury-Arveson space H_n^2. That is, for each $\alpha \in \mathbf{Z}_+^n$,

$$e_\alpha(\zeta) = \left(\frac{|\alpha|!}{\alpha!}\right)^{1/2} \zeta^\alpha.$$

Recall that on H_n^2, we also write $Z_1, \dots, Z_n$ for the operators by the coordinate functions $\zeta_1, \dots, \zeta_n$. And, just as in Section 2, on H_n^2 we also have the operator

$$D = \sum_{i=1}^n [Z_i^*, Z_i].$$

Lemma 4.1. *For each*

$$f = \sum_{\alpha \in \mathbf{Z}_+^n} b_\alpha e_\alpha \in H_n^2$$

we have

$$\langle Df, f\rangle = |b_0|^2 + (n-1) \sum_{\alpha \in \mathbf{Z}_+^n} \frac{|b_\alpha|^2}{|\alpha|+1}. \tag{4.1}$$

Proof. This follows from the last line on page 191 in [1]. □

Proof of Theorem 1.2. Let a submodule $\mathcal{M}$ of the Drury-Arveson module H_n^2 be given. Arveson showed that there exists a sequence $\{\varphi_1, ..., \varphi_k, ...\}$ contained in $\mathcal{M}$ such that each φ_k is a *multiplier* of H_n^2 and such that the operator

$$M_{\varphi_1} M_{\varphi_1}^* + ... + M_{\varphi_k} M_{\varphi_k}^* + ...$$

is the orthogonal projection from H_n^2 onto $\mathcal{M}$ [2,page 191]. Now suppose that $\mathcal{M} \neq \{0\}$. Then this result of Arveson tells us that there is a $\varphi \in \mathcal{M}$, $\varphi \neq 0$, which is a multiplier of H_n^2. That is, M_φ is a bounded operator on H_n^2 [1,Proposition 2.2].

For each $\ell \in \mathbf{N}$, denote

$$A_\ell = \{(\alpha_1, ..., \alpha_n) \in \mathbf{Z}_+^n : \ell < \alpha_i \leq 2\ell,\ 1 \leq i \leq n\}.$$

With the φ obtained above, we define the operator

$$F_\ell = \sum_{\alpha \in A_\ell} (\varphi e_\alpha) \otimes (\varphi e_\alpha),$$

$\ell \in \mathbf{N}$. Obviously, $F_\ell = M_\varphi Q_\ell M_\varphi^*$, where $Q_\ell = \sum_{\alpha \in A_\ell} e_\alpha \otimes e_\alpha$, which is an orthogonal projection. Therefore

$$\|F_\ell\| \leq \|M_\varphi\|^2 \tag{4.2}$$

for every $\ell \in \mathbf{N}$. Now suppose that

$$\varphi = \sum_{\beta \in \mathbf{Z}_+^n} c_\beta e_\beta.$$

Then for each $\alpha \in \mathbf{Z}_+^n$ we have

$$\varphi e_\alpha = \sum_{\beta \in \mathbf{Z}_+^n} c_\beta u(\alpha, \beta) e_{\alpha+\beta}, \tag{4.3}$$

where

$$u(\alpha, \beta) = \left(\frac{|\beta|!}{\beta!} \cdot \frac{|\alpha|!}{\alpha!} \cdot \frac{(\alpha+\beta)!}{|\alpha+\beta|!} \right)^{1/2}.$$

For each $\alpha = (\alpha_1, ..., \alpha_n) \in A_\ell$, it follows from the condition $\ell < \alpha_i \leq 2\ell$ for each $i \in \{1, ..., n\}$ that

$$\frac{|\alpha|!}{\alpha!} \cdot \frac{(\alpha+\beta)!}{|\alpha+\beta|!} = \frac{(\alpha+\beta)!/\alpha!}{|\alpha+\beta|!/|\alpha|!} \geq \left(\frac{\ell}{2n\ell + |\beta|} \right)^{|\beta|}.$$

Therefore

$$u(\alpha,\beta) \geq \left(\frac{|\beta|!}{\beta!}\right)^{1/2} \cdot \left(\frac{\ell}{2n\ell+|\beta|}\right)^{|\beta|/2} \tag{4.4}$$

for every $\alpha \in A_\ell$.

By the argument in the first paragraph of Section 3, in the present case we also have

$$\langle D_{\mathcal{M}}h, h\rangle \geq \langle Dh, h\rangle \quad \text{for every} \quad h \in \mathcal{M}.$$

Since $\varphi \in \mathcal{M}$, we have $\varphi e_\alpha \in \mathcal{M}$ for each $\alpha \in \mathbf{Z}_+^n$. Combining these facts with (4.3) and (4.1), we have

$$\operatorname{tr}(D_{\mathcal{M}}F_\ell) = \sum_{\alpha\in A_\ell} \langle D_{\mathcal{M}}\varphi e_\alpha, \varphi e_\alpha\rangle \geq \sum_{\alpha\in A_\ell} \langle D\varphi e_\alpha, \varphi e_\alpha\rangle = (n-1)\sum_{\alpha\in A_\ell}\sum_{\beta\in\mathbf{Z}_+^n} \frac{|c_\beta|^2 u^2(\alpha,\beta)}{|\alpha+\beta|+1}.$$

Since $\varphi \neq 0$, there is a $\beta_0 \in \mathbf{Z}_+^n$ such that $c_{\beta_0} \neq 0$. The above gives us

$$\operatorname{tr}(D_{\mathcal{M}}F_\ell) \geq (n-1)|c_{\beta_0}|^2 \sum_{\alpha\in A_\ell} \frac{u^2(\alpha,\beta_0)}{|\alpha+\beta_0|+1} \tag{4.5}$$

for every $\ell \in \mathbf{N}$. Now suppose that $\ell > |\beta_0|$. Then it follows from (4.4) that $u^2(\alpha,\beta_0) \geq (|\beta_0|!/\beta_0!)\cdot(1/3n)^{|\beta_0|}$ when $\alpha \in A_\ell$. Also, if $\ell > |\beta_0|$ and $\alpha \in A_\ell$, then $|\alpha+\beta_0|+1 = |\alpha|+|\beta_0|+1 \leq 2n\ell+\ell \leq 3n\ell$. Substituting these inequalities in (4.5), we find that

$$\operatorname{tr}(D_{\mathcal{M}}F_\ell) \geq \frac{(n-1)|c_{\beta_0}|^2|\beta_0|!}{(3n)^{|\beta_0|}\beta_0!} \cdot \frac{1}{3n\ell} \cdot \operatorname{card}(A_\ell) = \delta_1 \ell^{n-1} \tag{4.6}$$

for each $\ell > |\beta_0|$, where $\delta_1 = (n-1)|c_{\beta_0}|^2|\beta_0|!(3n)^{-|\beta_0|-1}(\beta_0!)^{-1}$. Note that $\operatorname{rank}(F_\ell) \leq \operatorname{card}(A_\ell) = \ell^n$. Therefore

$$\operatorname{tr}(D_{\mathcal{M}}F_\ell) \leq \|D_{\mathcal{M}}F_\ell\|_1 = \sum_{j=1}^{\operatorname{rank}(F_\ell)} s_j(D_{\mathcal{M}}F_\ell) \leq \sum_{j=1}^{\ell^n} s_j(D_{\mathcal{M}})\|F_\ell\| \leq \|M_\varphi\|^2 \sum_{j=1}^{\ell^n} s_j(D_{\mathcal{M}}),$$

where the last $\leq$ follows from (4.2). Combining this with (4.6), we obtain

$$\sum_{j=1}^{\ell^n} s_j(D_{\mathcal{M}}) \geq \delta \ell^{n-1} \tag{4.7}$$

for each $\ell > |\beta_0|$, where $\delta = \delta_1/\|M_\varphi\|^2$.

Now set $K = (|\beta_0| + 1)^n$. Given any $k \geq K$, there is an $\ell > |\beta_0|$ such that $\ell^n \leq k < (\ell+1)^n$. Applying (4.7), we have

$$\sum_{j=1}^{k} s_j(D_{\mathcal{M}}) \geq \sum_{j=1}^{\ell^n} s_j(D_{\mathcal{M}}) \geq \delta \ell^{n-1} \geq \delta \left(\frac{\ell}{\ell+1} \right)^{n-1} k^{(n-1)/n} \geq \frac{\delta}{2^{n-1}} k^{(n-1)/n}.$$

From this inequality Theorem 1.2 follows. $\square$

5. Norm Ideals

We will now explain the meaning of Theorems 1.1 and 1.2 in the language of *norm ideals*. Recall that, for each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k \geq 1} \frac{s_1(A) + s_2(A) + ... + s_k(A)}{1^{-1/p} + 2^{-1/p} + ... + k^{-1/p}} \tag{5.1}$$

defines a *symmetric norm* for operators [11,Section III.14]. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is an ideal in $\mathcal{B}(\mathcal{H})$. It is well known that $\mathcal{C}_p^+ \supset \mathcal{C}_p$ and that $\mathcal{C}_p^+ \neq \mathcal{C}_p$. Indeed $\mathcal{C}_p^+$ and $\mathcal{C}_p$ are just some of the most commonly known examples of norm ideals. More generally, we recall the following construction.

Let $\hat{c}$ be the linear space of sequences $\{a_j\}_{j \in \mathbf{N}}$, where $a_j \in \mathbf{R}$ and for each sequence $a_j \neq 0$ only for a finite number of j's. A *symmetric gauge function* [11,page 71] is a map $\Phi : \hat{c} \to [0, \infty)$ which has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, ..., 0, ...\}) = 1$.

(c) $\Phi(\{a_j\}_{j \in \mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j \in \mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

Each symmetric gauge function Φ gives rise to the symmetric norm

$$\|A\|_\Phi = \sup_{k \geq 1} \Phi(\{s_1(A), ..., s_k(A), 0, ..., 0, ...\})$$

for operators. On any Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\}$$

is a norm ideal [11,page 68]. This term refers to the following properties of $\mathcal{C}_\Phi$:

- For any B, $C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \leq \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \leq \|A\|_\Phi$, and the equality holds when $\text{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|.\|_\Phi$.

As an example, let us mention the symmetric gauge function $\Phi_p^+ : \hat{c} \to [0,\infty)$, $1 \le p < \infty$, defined as follows. For each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, let

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{k\ge1} \frac{|a_{\pi(1)}| + |a_{\pi(2)}| + ... + |a_{\pi(k)}|}{1^{-1/p} + 2^{-1/p} + ... + k^{-1/p}},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \ge |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$, which exists because $a_j = 0$ for all but a finite number of j's.

Obviously, $\mathcal{C}_p^+$ is none other than the norm ideal $\mathcal{C}_{\Phi_p^+}$.

Theorems 1.1 and 1.2 set a size requirement on norm ideals of the form $\mathcal{C}_\Phi$ to which a nonzero $D_\mathcal{M}$ considered in this paper can belong:

Proposition 5.1. *Let Φ be a symmetric gauge function. If there is a submodule $\mathcal{M}$ of either $H^2(S)$ or H_n^2 such that $0 < \|D_\mathcal{M}\|_\Phi < \infty$, then $\mathcal{C}_\Phi \supset \mathcal{C}_n^+$.*

Proof. The assumption $\|D_\mathcal{M}\|_\Phi > 0$ implies $\mathcal{M} \ne \{0\}$. Hence, by Theorem 1.1 or Theorem 1.2, there is an $\epsilon = \epsilon(\mathcal{M}) > 0$ such that

$$s_1(D_\mathcal{M}) + ... + s_k(D_\mathcal{M}) \ge \epsilon k^{(n-1)/n} \tag{5.2}$$

for every $k \in \mathbf{N}$. Let $A \in \mathcal{C}_n^+$. Then by (5.1) we have

$$s_1(A) + ... + s_k(A) \le \|A\|_n^+(1^{-1/n} + ... + k^{-1/n}) \le 3\|A\|_n^+ k^{(n-1)/n}, \tag{5.3}$$

$k \in \mathbf{N}$. The combination of (5.2) and (5.3) gives us

$$s_1(A) + ... + s_k(A) \le (3\|A\|_n^+/\epsilon)\{s_1(D_\mathcal{M}) + ... + s_k(D_\mathcal{M})\},$$

$k \in \mathbf{N}$. By [11,Lemma III.3.1], this implies

$$\|A\|_\Phi \le (3\|A\|_n^+/\epsilon)\|D_\mathcal{M}\|_\Phi.$$

Since $\|D_\mathcal{M}\|_\Phi < \infty$, we have $\|A\|_\Phi < \infty$ for each $A \in \mathcal{C}_n^+$. That is, $\mathcal{C}_\Phi \supset \mathcal{C}_n^+$. □

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