CORRIGENDUM TO "MULTIPLIERS AND ESSENTIAL NORM ON THE DRURY-ARVESON SPACE"

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Let H^2_n be the Drury-Arveson space of analytic functions on the unit ball $\mathbf{B}$ in $\mathbf{C}^n$, and let $\mathcal{M}$ denote the collection of multipliers of H^2_n. This corrigendum concerns the following lemma in our original article "Multipliers and essential norm on the Drury-Arveson space", Proc. Amer. Math. Soc. **139** (2011), 2497-2504:

Lemma 3.1. *Let $f \in \mathcal{M}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then $1/f$ is also a multiplier of H^2_n.*

This lemma is, of course, an immediate consequence of the recently proved corona theorem for $\mathcal{M}$ [7]. But in our original article, we also asserted that Lemma 3.1 also follows from Theorem 2 in Chen's paper [6], which claims that for $g \in H^2_n$, $\|g\|^2$ is equivalent to

$$\|g\|^2_{\#} = |g(0)|^2 + \iint \frac{|g(z) - g(w)|^2}{|1 - \langle z, w\rangle|^{2n+1}} dv(z)dv(w).$$

It has since been discovered that this theorem of Chen's is false. We would like to thank Dechao Zheng for informing us of this fact, who in turn attributes his source to Carl Sundberg. Once one knows this, the fact that $\|g\|^2$ is not equivalent to $\|g\|^2_{\#}$ can be directly shown. For example, if one tries the special case where the complex dimension n equals 2, one finds that

$$\iint \frac{|z_1 - w_1|^2}{|1 - \langle z, w\rangle|^5} dv(z)dv(w) = \infty,$$

where z_1 and w_1 denote the first component of z and w respectively.

Thus question arises as to whether there is a proof of Lemma 3.1 that does not invoke the corona theorem of Costea, Sawyer and Wick [7]. One might call Lemma 3.1 the "one-function corona theorem" for $\mathcal{M}$. We learned this term from Dechao Zheng, and we also learned that there is considerable interest in finding an elementary proof of the one-function corona theorem, a proof that does not involve hard analysis in the style of [7].

In this somewhat expanded version of corrigendum, we will give an elementary proof of Lemma 3.1. The virtue of our proof is that it involves only soft analysis, and very little of it indeed. What makes this proof particularly worth reporting is the fact that it is

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based on the very essence of the theory of Drury-Arveson space, namely the von Neumann inequality for commuting row contractions.

The proof begins with some elementary facts. For each $f \in \mathcal{M}$, we write $\|f\|_{\mathcal{M}}$ for its multiplier norm. That is,

$$\|f\|_{\mathcal{M}} = \sup\{\|fg\| : g \in H_n^2, \|g\| \leq 1\}.$$

For each analytic function h on $\mathbf{B}$ and each $0 \leq r < 1$, we define the function

$$h_r(z) = h(rz), \quad z \in \mathbf{B}.$$

Lemma 1. *Let $f \in \mathcal{M}$. Then for each $0 \leq r < 1$, we have $f_r \in \mathcal{M}$ and $\|f_r\|_{\mathcal{M}} \leq \|f\|_{\mathcal{M}}$.*

Proof. Let $\mathbf{T}^n$ denote the n-dimensional torus $\{(\tau_1, \ldots, \tau_n) : |\tau_j| = 1, 1 \leq j \leq n\}$. Let dm_n be the Lebesgue measure on $\mathbf{T}^n$ with the normalization $m_n(\mathbf{T}^n) = 1$. For each $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{T}^n$, define the unitary transformation U_τ on $\mathbf{C}^n$ by the formula

$$U_\tau(z_1, \ldots, z_n) = (\tau_1 z_1, \ldots, \tau_n z_n).$$

Let $f \in \mathcal{M}$. Then we obviously have $\|f\|_{\mathcal{M}} = \|f \circ U_\tau\|_{\mathcal{M}}$, $\tau \in \mathbf{T}^n$. For each $0 \leq r < 1$, define the function

$$P_r(\tau_1, \ldots, \tau_n) = \prod_{j=1}^{n} \frac{1 - r^2}{|1 - r\bar{\tau}_j|^2}$$

on $\mathbf{T}^n$. By the well-known properties of the Poisson kernel, we have

$$M_{f_r} = \int M_{f \circ U_\tau} P_r(\tau) dm_n(\tau).$$

Since the integral of P_r on $\mathbf{T}^n$ equals 1 and $P_r \geq 0$, the lemma follows. □

For each real number $-n \leq t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)} \tag{1}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Obviously, we have $\mathcal{H}^{(-n)} = H_n^2$. Moreover, $\mathcal{H}^{(-1)}$ is the Hardy space $H^2(S)$, and $\mathcal{H}^{(0)}$ is the Bergman space on the unit ball.

Lemma 2. *If $f \in \mathcal{M}$, then f is also a multiplier for every $\mathcal{H}^{(t)}$, $-n < t < \infty$. Moreover, for each $-n < t < \infty$ we have*

$$\|fg\|_t \le \|f\|_{\mathcal{M}} \|g\|_t \quad \textit{whenever} \;\; g \in \mathcal{H}^{(t)}.$$

Proof. Let N be the number operator introduced by Arveson [2]. That is, $Nz^\alpha = |\alpha| z^\alpha$, $\alpha \in \mathbf{Z}_+^n$. For each $-n < t < \infty$, let $M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)}$ denote the operators of multiplication by the coordinate functions on $\mathcal{H}^{(t)}$. Using (1), it is straightforward to verify that

$$M_{z_1}^{(t)} M_{z_1}^{(t)*} + \cdots + M_{z_n}^{(t)} M_{z_n}^{(t)*} = N(n + t + N)^{-1}.$$

Thus for each $-n < t < \infty$, the commuting tuple $(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})$ is a row contraction. Consequently, the von Neumann inequality

$$\|p(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})\| \le \|p\|_{\mathcal{M}}$$

holds for every polynomial p. See [2,8]. That is, for each polynomial p, we have

$$\|pg\|_t \le \|p\|_{\mathcal{M}} \|g\|_t \quad \text{whenever} \;\; g \in \mathcal{H}^{(t)}. \tag{2}$$

Thus our task is to show that (2) still holds if we replace p by $f \in \mathcal{M}$. For this we use the power series expansion

$$\frac{1}{(1-u)^n} = \sum_{j=0}^{\infty} c_j u^j, \quad \text{where} \;\; c_j = \frac{(j+n-1)!}{j!(n-1)!},$$

which holds when $|u| < 1$. Let any $f \in \mathcal{M}$ be given. By the Cauchy integral formula

$$f(z) = \int \frac{f(\xi)}{(1 - \langle z, \xi \rangle)^n} d\sigma(\xi),$$

where $d\sigma$ is the spherical measure on $S = \{\xi \in \mathbf{C}^n : |\xi| = 1\}$, for each $0 \le r < 1$ we have

$$f_r = \sum_{j=0}^{\infty} c_j r^j \psi_j, \quad \text{where} \;\; \psi_j(z) = \int f(\xi) \langle z, \xi \rangle^j d\sigma(\xi).$$

For each $\xi \in S$, the operator of multiplication by the function $\langle z, \xi \rangle$ is obviously a contraction on $\mathcal{H}^{(t)}$ as well as on H_n^2. Therefore $\|\psi_j\|_{\mathcal{M}} \le \int |f| d\sigma \le \|f\|_\infty$ and, similarly, $\|M_{\psi_j}^{(t)}\| \le \|f\|_\infty$ for each $j \ge 0$. Thus the operators

$$\sum_{j=0}^{k} c_j r^j M_{\psi_j}^{(t)}, \quad k = 0, 1, 2, \dots,$$

form a Cauchy sequence with respect the operator norm on $\mathcal{H}^{(t)}$. Hence

$$\|M_{f_r}^{(t)}\| = \lim_{k\to\infty}\left\|\sum_{j=0}^{k} c_j r^j M_{\psi_j}^{(t)}\right\| \le \liminf_{k\to\infty}\left\|\sum_{j=0}^{k} c_j r^j \psi_j\right\|_{\mathcal{M}},$$

where the $\le$ follows from (2). Since the operators $\sum_{j=0}^{k} c_j r^j M_{\psi_j}$, $k = 0, 1, 2, \ldots$, converge to M_{f_r} on H_n^2 with respect to the operator norm, we have

$$\|M_{f_r}^{(t)}\| \le \lim_{k\to\infty}\left\|\sum_{j=0}^{k} c_j r^j \psi_j\right\|_{\mathcal{M}} = \|f_r\|_{\mathcal{M}} \le \|f\|_{\mathcal{M}},$$

where the second $\le$ follows from Lemma 1. Now, if $g \in \mathcal{H}^{(t)}$, then $f_r g_r = (fg)_r$. Hence

$$\|(fg)_r\|_t = \|M_{f_r}^{(t)} g_r\|_t \le \|f\|_{\mathcal{M}}\|g_r\|_t \le \|f\|_{\mathcal{M}}\|g\|_t.$$

Since this holds for every $0 \le r < 1$, we have $\|fg\|_t \le \|f\|_{\mathcal{M}}\|g\|_t$ as promised. □

Write $R = z_1\partial_1 + \cdots + z_n\partial_n$, the radial derivative in n variables. We now pick a fixed nature number m such that $2m - n \ge 0$. For each integer $0 \le k \le m$, we define the norm $\|\cdot\|_{*,k}$ by the formula

$$\|g\|_{*,k}^2 = |g(0)|^2 + \int |(R^k g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z).$$

For each $\alpha \in \mathbf{Z}_+^n$, we have $Rz^\alpha = |\alpha| z^\alpha$ and

$$\int |z^\alpha|^2 (1 - |z|^2)^{2m-n} dv(z) = \frac{n!(2m-n)!\alpha!}{(|\alpha| + 2m)!}.$$

Comparing this with (1), for each $0 \le k \le m$ there are $0 < c_k \le C_k < \infty$ such that

$$c_k\|g\|_{*,k} \le \|g\|_{2m-n-2k} \le C_k\|g\|_{*,k} \quad \text{for every} \quad g \in \mathcal{H}^{(2m-n-2k)}. \tag{3}$$

In particular, $\|\cdot\|_{*,m}$ is equivalent to the norm on the Drury-Arveson space H_n^2.

Proposition 3. *Suppose that $Y_1, \ldots, Y_K$ are operators satisfying the following conditions:*
(a) *For each $1 \le j \le K$, either $Y_j = R$ or $Y_j = M_{f_j}$ for some $f_j \in \mathcal{M}$.*
(b) *$Y_1 = R$.*
(c) *$\mathrm{card}\{j : Y_j = R, 1 \le j \le K\} = m$.*
Then there is a constant $C = C(Y_1, \ldots, Y_K)$ such that

$$\int |(Y_1 \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \le C\|g\|^2 \tag{4}$$

for every $g \in H_n^2$, where $\|\cdot\|$ denotes the norm on H_n^2.

Proof. We use an induction on K. If $K = m$, then, of course, (4) holds by virtue of (3) in the case $k = m$. Now suppose that $\ell \geq m$ and that the proposition has been proved for all $m \leq K \leq \ell$. Consider the case $K = \ell + 1$. Let B denote the collection of j's in $\{1, \ldots, K\}$ such that $Y_j = M_{f_j}$ for some $f_j \in \mathcal{M}$. Then $B \neq \emptyset$. Let j_0 be the smallest integer in B. If $B = \{j : j_0 \leq j \leq K\}$, then

$$Y_1 \cdots Y_K = R^m M_{f_{j_0} \cdots f_K},$$

and (4) again follows from (3) and the assumption that $f_{j_0}, \ldots, f_K$ are multipliers of H_n^2.

Let us suppose that $B \neq \{j : j_0 \leq j \leq K\}$. Then the induction hypothesis gives us a C_1 such that

$$\int |(Y_1 \cdots Y_{j_0-1} Y_{j_0+1} \cdots Y_K h)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \leq C_1 \|h\|^2 \tag{5}$$

for every $h \in H_n^2$. Now, given a $g \in H_n^2$, write

$$\tilde{g} = Y_{j_0+1} \cdots Y_K g.$$

Then

$$Y_1 \cdots Y_K g = R^{j_0-1}(f_{j_0} \tilde{g}).$$

Since $B \neq \{j : j_0 \leq j \leq K\}$, there is at least one $j \in \{j_0 + 1, \ldots, K\}$ such that $Y_j = R$. Thus $\tilde{g}(0) = 0$. Applying (3) and Lemma 2, we have

$$\begin{aligned}
\int |(Y_1 \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) &= \int |(R^{j_0-1}(f_{j_0} \tilde{g}))(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \\
&\leq c_{j_0-1}^{-2} \|f_{j_0} \tilde{g}\|_{2m-n-2j_0+2}^2 \leq c_{j_0-1}^{-2} \|f_{j_0}\|_{\mathcal{M}}^2 \|\tilde{g}\|_{2m-n-2j_0+2}^2 \\
&\leq (C_{j_0-1}/c_{j_0-1})^2 \|f_{j_0}\|_{\mathcal{M}}^2 \int |(R^{j_0-1}\tilde{g})(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \\
&= (C_{j_0-1}/c_{j_0-1})^2 \|f_{j_0}\|_{\mathcal{M}}^2 \int |(Y_1 \cdots Y_{j_0-1} Y_{j_0+1} \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z).
\end{aligned}$$

Combining this with (5), the induction on K is complete. □

With the above preparation, we now have

An elementary proof of Lemma 3.1. Let f be given as in the lemma. By (3), it suffices to show that there is a $0 < C < \infty$ such that

$$\int |(R^m M_{f^{-1}} g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \leq C \|g\|^2$$

for every $g \in H_n^2$. By Proposition 3 and the assumption $|f| \geq c > 0$ on $\mathbf{B}$, this inequality will follow if we can prove the following assertion: the operator $R^m M_{f^{-1}}$ is the sum of a finite number terms of the form

$$a M_{f^{-\nu}} Y_1 \cdots Y_K,$$

where $a \in \mathbf{R}$, $\nu \in \mathbf{N}$, and the operators $Y_1, \dots, Y_K$ satisfy the conditions

(a) for each $1 \le j \le K$, either $Y_j = R$ or $Y_j = M_f$;

(b) $Y_1 = R$;

(c) $\mathrm{card}\{j : Y_j = R, 1 \le j \le K\} = m$.

To prove this, note that for each natural number k, we have the commutation relation

$$RM_{f^{-k}} = (k+1)M_{f^{-k}}R - kM_{f^{-k-1}}RM_f.$$

Obviously, the above assertion about $R^m M_{f^{-1}}$ follows from this identity and an easy induction on m. □

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