SCHATTEN CLASS COMPOSITION OPERATORS

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Abstract. Let C_φ be a composition operator on the Bergman space A^2 of the unit disc. A well-known problem asks whether the condition $\int_D \big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\big)^p d\lambda(z) < \infty$ is equivalent to the membership of C_φ in the Schatten class $\mathcal{C}_p$, $1 < p < \infty$. This was settled in the negative for the case $2 < p < \infty$ in [3]. When $2 < p < \infty$, this condition is not sufficient for $C_\varphi \in \mathcal{C}_p$. In this paper we take up the case $1 < p < 2$. We show that when $1 < p < 2$, this condition is not necessary for $C_\varphi \in \mathcal{C}_p$.

1. Introduction

Let D be the unit disk in the complex plane $\mathbf{C}$ and $H(D)$ be the class of functions analytic in D. Let dA be the area measure on D normalized in such a way that $A(D) = 1$. We write $d\lambda$ for the Möbius-invariant measure on D, i.e., $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$.

Recall that the Bergman space A^2 is defined by

$$A^2 = \{f : f \in H(D),\ \|f\|_{A^2}^2 = \int_D |f(z)|^2 dA(z) < \infty\}.$$

The Hardy space H^2 is the Hilbert space of analytic functions f on D such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Given an analytic function $\varphi : D \to D$, we have the composition operator C_φ on A^2 or H^2 defined by the formula $C_\varphi(f) = f \circ \varphi$. Recall that such a C_φ is always bounded [5].

Let H be a separable Hilbert space. For any $1 \le p < \infty$, the Schatten p-class $\mathcal{C}_p$ consists of bounded linear operators T on H satisfying the condition $\|T\|_p < \infty$, where the p-norm is defined by the formula

$$\|T\|_p = \{\text{tr}\,(|T|^p)\}^{1/p} = \{\text{tr}\,((T^*T)^{p/2})\}^{1/p}.$$

The membership of composition operator C_φ in the Schatten class $\mathcal{C}_p$ has been a constant source of fascination for operator theorists. In the case of the Bergman space A^2, Luecking and Zhu showed that $C_\varphi \in \mathcal{C}_p$ if and only if the function $z \mapsto \{\log(1/|z|)\}^{-2} N_{\varphi,2}(z)$ belongs to $L^{p/2}(D, d\lambda)$, where $N_{\varphi,2}$ is a counting function associated with φ [2].

On the other hand, it would be more desirable to obtain a criterion for the membership $C_\varphi \in \mathcal{C}_p$ in which φ appears in a more explicit way. One such approach involves the *Berezin transform*. Let k_z be the normalized reproducing kernel for A^2. Recall that for $T \in \mathcal{B}(A^2)$, the Berezin transform of T is the function

$$z \mapsto \langle Tk_z, k_z\rangle$$

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on D. It is known (see [6]) that $C_\varphi \in \mathcal{C}_p$ if and only if

$$\int_D \langle C_\varphi^* C_\varphi k_z, k_z\rangle^{p/2} d\lambda(z) < \infty. \tag{1.1}$$

In view of this, one naturally considers the condition

$$\int_D \langle C_\varphi C_\varphi^* k_z, k_z\rangle^{p/2} d\lambda(z) < \infty. \tag{1.2}$$

Compared with (1.1), (1.2) appears more desirable because, by an easy calculation,

$$\langle C_\varphi C_\varphi^* k_z, k_z\rangle = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^2,$$

which involves φ in a very direct way. Thus the following problem arose:

Problem 1.1. [1] [2] [5] Let $\varphi : D \to D$ be an analytic function. Is it true that for $1 < p < \infty$, the composition operator $C_\varphi : A^2 \to A^2$ is in the Schatten class $\mathcal{C}_p$ if and only if

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) < \infty? \tag{1.3}$$

It is trivial that $C_\varphi \in \mathcal{C}_2$ if and only if

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^2 d\lambda(z) < \infty.$$

In [3], it was shown that when $2 < p < \infty$, (1.3) is not sufficient for the membership $C_\varphi \in \mathcal{C}_p$. That is, for each $2 < p < \infty$, there is an analytic $\varphi : D \to D$ such that

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) < \infty$$

and $C_\varphi \notin \mathcal{C}_p$. In this paper we settle the remaining case, the case $1 < p < 2$. We will show that when $1 < p < 2$, (1.3) is not necessary for the membership $C_\varphi \in \mathcal{C}_p$. Here is our main result.

Theorem 1.1. *For each $1 < p < 2$, there exists an analytic function $\varphi : D \to D$ such that the composition operator $C_\varphi : A^2 \to A^2$ belongs to the Schatten class $\mathcal{C}_p$, and yet*

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) = \infty. \tag{1.4}$$

Together with the result in [3], Theorem 1.1 completes the contrast between (1.1) and (1.2). From the view point of operator theory, it is truly amazing that there is such a sharp contrast.

It will be interesting to consider what happens in the case of the Hardy space. Let k_z^{Har} denote the normalized reproducing kernel for the Hardy space H^2. By an easy calculation,

$$\langle C_\varphi C_\varphi^* k_z^{\mathrm{Har}}, k_z^{\mathrm{Har}}\rangle = \frac{1-|z|^2}{1-|\varphi(z)|^2}.$$

Thus the Hardy-space equivalent of condition (1.3) is

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p/2} d\lambda(z) < \infty. \tag{1.5}$$

Recently, Yang and Yuan showed for each $2 < p < \infty$, there is an analytic $\varphi : D \to D$ such that

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{p/2} d\lambda(z) < \infty$$

and such that the composition operator $C_\varphi : H^2 \to H^2$ does not belong to $\mathcal{C}_p(H^2)$ [4]. This settles the entire Hardy-space case. This is because (1.5) holds only if $p > 2$. For every $p \le 2$, we have

$$\int_D (1-|z|^2)^{p/2} d\lambda(z) = \infty.$$

The remainder of the paper consists of the proof of Theorem 1.1.

2. The proof of Theorem 1.1

The proof of Theorem 1.1 begins with a construction adapted from [3]. For $n = 1, 2, \ldots$, define

$$T_n = \big(2^{-(n+1)},\ 2^{-n}\big] \quad \text{and} \quad S_n = \big((4/3)2^{-(n+1)},\ (5/3)2^{-(n+1)}\big].$$

That is, S_n is the middle third of T_n. Denote $t_n = (4/3)2^{-(n+1)}$, the left end-point of S_n, $n \in \mathbf{N}$.

Let $1 < p < 2$ be given. We choose an ϵ such that

$$0 < \epsilon < 1/p$$

and such that $p\epsilon$ is a rational number. Thus $p^{-1} > (p-1)\epsilon$, and $\lim_{k\to\infty} 2^{-(p^{-1}-(p-1)\epsilon)k} = 0$. We can choose a strictly increasing sequence $k(1) < \ldots < k(n) < \ldots$ of positive integers such that

$$2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot 2^{p\epsilon k(n)} = 2^{-(p^{-1}-(p-1)\epsilon)k(n)+1} \le (1/3)2^{-(n+1)} = |S_n| \tag{2.1}$$

for every n and such that every $p\epsilon k(n)$ is an integer. Note the difference between the choice of $k(n)$ in this paper and the choice in [3].

For integers $n \ge 1$ and $1 \le j \le 2^{p\epsilon k(n)}$, define the intervals

$$J_{n,j} = (a_{n,j}, c_{n,j}) = \big(t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot (j-1),\ t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot j\big),$$

$$I_{n,j} = (a_{n,j}, b_{n,j}) = \big(t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot (j-1),\ t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot (2j-1)\big).$$

It is easy to check that $I_{n,j}$ is the left half of $J_{n,j}$ and the $J_{n,j}$'s are pairwise disjoint. (2.1) ensures that

$$\bigcup_{j=1}^{2^{p\epsilon k(n)}} J_{n,j} \subset S_n.$$

We denote the length of the interval $I_{n,j}$ by ρ_n. That is,

$$\rho_n = |I_{n,j}| = b_{n,j} - a_{n,j} = 2^{-(p^{-1}+\epsilon)k(n)}.$$

We now define a measurable function u on the unit circle $\mathbf{T} = \{w \in \mathbf{C} : |w| = 1\}$ as follows:

$$u(e^{it}) = 2^{-k(n)} \quad \text{if} \quad t \in \bigcup_{j=1}^{2^{p\epsilon k(n)}} I_{n,j},\ n \ge 1,$$

$$u(e^{it}) = 1 \quad \text{if} \quad t \in (-\pi, \pi] \setminus \Big\{ \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{p\epsilon k(n)}} I_{n,j} \Big\}.$$

The harmonic extension of u to D will be denoted by the same symbol. Finally, define

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})dt,$$

$$\varphi(z) = \exp(-h(z)), \quad z \in D. \tag{2.2}$$

Obviously, $\text{Re}\{h(z)\} = u(z) > 0$, and consequently

$$|\varphi(z)| = e^{-\text{Re}\{h(z)\}} = e^{-u(z)} < 1$$

for every $z \in D$. This implies $\varphi(D) \subset D$.

For $z \in D$ and $e^{it} \in \mathbf{T}$, let $P(z, e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$ be the Poisson kernel. It was shown in [3, p. 2508] that if $1/2 \le r < 1$ and $|\theta - t| \le 5$, then there exist constants $0 < \alpha < \beta < \infty$ such that

$$\frac{\alpha(1-r)}{(1-r)^2 + (\theta - t)^2} \le \frac{1}{2\pi} P(re^{i\theta}, e^{it}) \le \frac{\beta(1-r)}{(1-r)^2 + (\theta - t)^2}. \tag{2.3}$$

For any $n \in \mathbf{N}$ and $1 \le j \le 2^{p\epsilon k(n)}$, define

$$G_{n,j} = \{re^{i\theta} : \theta \in I_{n,j},\ 0 < 1 - r \le \rho_n\}. \tag{2.4}$$

Given such a pair of n, j, we have

$$G_{n,j} = \bigcup_{\nu=0}^{k(n)} G_{n,j}^{\nu},$$

where

$$G_{n,j}^0 = \{re^{i\theta} : \theta \in I_{n,j},\ 0 < 1 - r \le \rho_n \cdot 2^{-k(n)}\},$$

$$G_{n,j}^{\nu} = \{re^{i\theta} : \theta \in I_{n,j},\ \rho_n \cdot 2^{-k(n)} \cdot 2^{\nu-1} < 1 - r \le \rho_n \cdot 2^{-k(n)} \cdot 2^{\nu}\}$$

for $1 \le \nu \le k(n)$. By [3, (2.6) and (2.7)], there is a constant $0 < c < 1$ independent of n, j such that

$$u(z) \ge c2^{-k(n)+\nu} \quad \text{if } z \in G_{n,j}^{\nu},\ 0 \le \nu \le k(n). \tag{2.5}$$

Recalling [3, (2.10)], we have

$$A(G_{n,j}^{\nu}) \le \rho_n^2 \cdot 2^{-k(n)} \cdot 2^{\nu}, \quad 0 \le \nu \le k(n). \tag{2.6}$$

The following two lemmas are quoted from [3, Lemma 7] and [3, Lemma 5], respectively.

Lemma 2.1. *There is a $c_1 > 0$ such that*

$$u(z) \ge c_1 \quad \textit{for every} \quad z \in D \Big\backslash \Big\{ \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{p\epsilon k(n)}} G_{n,j} \Big\}$$

where $G_{n,j}$ is defined by (2.4).

Lemma 2.2. *For any $n \ge 1$ and $1 \le j \le 2^{p\epsilon k(n)}$, let $B_{n,j}$ be the middle third of $I_{n,j}$. That is, $B_{n,j} = (3^{-1}(b_{n,j} + 2a_{n,j}),\ 3^{-1}(2b_{n,j} + a_{n,j}))$, where $a_{n,j} < b_{n,j}$ are the end-points of $I_{n,j}$. Furthermore, for such n and j, define*

$$E_{n,j} = \{re^{it} : t \in B_{n,j},\ 0 < 1 - r \le \rho_n \cdot 2^{-k(n)}\}.$$

Then $\sup_{z \in E_{n,j}} u(z) \le (1 + 6\beta)2^{-k(n)}$, where β is the constant that appears in (2.3).

We need one more lemma:

Lemma 2.3. *There is a $c_3 > 0$ such that*

$$\int_{E_{n,j}} \frac{(1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} dA(z) \ge c_3 2^{-p\epsilon k(n)}$$

for all $n \ge 1$ and $1 \le j \le 2^{p\epsilon k(n)}$.

Proof. Denote $\varphi_{n,j} = \inf\{|\varphi(z)| : z \in E_{n,j}\}$, $n \ge 1$ and $1 \le j \le 2^{p\epsilon k(n)}$. Then $\varphi_{n,j} \ge e^{-C2^{-k(n)}}$ by Lemma 2.2. Writing $\sigma = \sup_{0<x\le C}(1-e^{-x})/x$, we have

$$\frac{1}{1-|\varphi(z)|} \ge \frac{1}{1-\varphi_{n,j}} \ge \frac{1}{\sigma C 2^{-k(n)}} = \frac{2^{k(n)}}{\sigma C} \quad \text{for } z \in E_{n,j}.$$

Let $c_2 = 2^{-2}(\sigma C)^{-p}$. Then

$$\begin{aligned}\int_{E_{n,j}} \frac{(1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} dA(z) &\ge c_2 2^{pk(n)} \int_{E_{n,j}} (1-|z|)^{p-2} dA(z) \\ &\ge c_3 2^{pk(n)} \cdot (\rho_n 2^{-k(n)})^{p-1} \cdot \rho_n = c_3 2^{-p\epsilon k(n)}.\end{aligned}$$

This completes the proof. ☐

Proof of Theorem 1.1: We must show that the analytic function $\varphi : D \to D$ defined by (2.2) has the property that $C_\varphi \in \mathcal{C}_p$ and satisfies (1.4). Let us first verify $C_\varphi \in \mathcal{C}_p$.

To show that $C_\varphi \in \mathcal{C}_p$, we need the following inequality: For any $0 < \rho < 1$ and $0 < x < 1$, using Hölder's inequality with conjugate exponents $1/\rho$ and $1/(1-\rho)$, we have

$$\begin{aligned}\sum_{l=0}^{\infty}(l+1)^\rho x^l &= \sum_{l=0}^{\infty}(l+1)^\rho \cdot x^{l\rho} \cdot x^{l(1-\rho)} \\ &\le \Big\{\sum_{l=0}^{\infty}\big((l+1)^\rho x^{l\rho}\big)^{1/\rho}\Big\}^\rho \cdot \Big\{\sum_{l=0}^{\infty}\big(x^{l(1-\rho)}\big)^{1/(1-\rho)}\Big\}^{1-\rho} \\ &= \Big(\frac{1}{(1-x)^2}\Big)^\rho \Big(\frac{1}{1-x}\Big)^{1-\rho} \\ &= \frac{1}{(1-x)^{\rho+1}}. \end{aligned} \tag{2.7}$$

Let $e_l(z) = (l+1)^{1/2} z^l, l = 0, 1, 2, \dots$. Recall that $\{e_l : l \ge 0\}$ is the standard orthonormal basis for the Bergman space A^2. Because $1 < p < 2$ and $\|e_l\| = 1$, it follows that

$$\begin{aligned}\langle (C_\varphi^* C_\varphi)^{p/2} e_l, e_l\rangle \le \{\langle C_\varphi^* C_\varphi e_l, e_l\rangle\}^{p/2} &= \|C_\varphi e_l\|_{A^2}^p = (l+1)^{p/2}\|\varphi^l\|_{A^2}^p \\ &= (l+1)^{p/2}\Big\{\int_D |\varphi(z)|^{2l} dA(z)\Big\}^{p/2}.\end{aligned}$$

Let

$$G = \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{2^{p\epsilon k(n)}} G_{n,j},$$

where $G_{n,j}$ is given by (2.4). For $z \in D\backslash G$, Lemma 2.1 implies that

$$|\varphi(z)| = e^{-\mathrm{Re}(h(z))} = e^{-u(z)} \le e^{-c_1}. \tag{2.8}$$

We have

$$\begin{aligned}
\mathrm{tr}\,((C_\varphi{}^* C_\varphi)^{p/2}) &= \sum_{l=0}^{\infty} \langle (C_\varphi^* C_\varphi)^{p/2} e_l, e_l\rangle \le \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \int_D |\varphi(z)|^{2l} dA(z) \Big\}^{p/2} \\
&= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \int_{D\backslash G} |\varphi(z)|^{2l} dA(z) + \int_G |\varphi(z)|^{2l} dA(z) \Big\}^{p/2} \\
&\le I + J,
\end{aligned}$$

where

$$\begin{aligned}
I &= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \int_{D\backslash G} |\varphi(z)|^{2l} dA(z) \Big\}^{p/2} \quad \text{and} \\
J &= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \int_{G} |\varphi(z)|^{2l} dA(z) \Big\}^{p/2}.
\end{aligned}$$

Applying (2.8), we obtain

$$\begin{aligned}
I &= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \int_{D\backslash G} |\varphi(z)|^{2l} dA(z) \Big\}^{p/2} \le \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ (e^{-c_1})^{2l} \int_{D\backslash G} dA(z) \Big\}^{p/2} \\
&\le \sum_{l=0}^{\infty} (l+1)^{p/2} \big\{ (e^{-c_1})^{2l} \big\}^{p/2} = \sum_{l=0}^{\infty} (l+1)^{p/2} (e^{-pc_1})^l \le \frac{1}{(1-e^{-pc_1})^{(p/2)+1}},
\end{aligned}$$

where the last $\le$ follows from the condition $p/2 < 1$ and (2.7).

Next we show that $J < \infty$. Note that

$$\Big(\sum_n a_n \Big)^s \le \sum_n a_n^s$$

if $s \le 1$ and $a_n \ge 0$. Applying (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}
J &= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \sum_{n=1}^{\infty} \sum_{j=1}^{2^{p\epsilon k(n)}} \sum_{\nu=0}^{k(n)} \int_{G_{n,j}^{\nu}} |\varphi(z)|^{2l} dA(z) \Big\}^{p/2} \\
&\le \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \sum_{n=1}^{\infty} \sum_{j=1}^{2^{p\epsilon k(n)}} \sum_{\nu=0}^{k(n)} (e^{-c2^{-k(n)+\nu}})^{2l} \rho_n^2 2^{-k(n)+\nu} \Big\}^{p/2} \\
&= \sum_{l=0}^{\infty} (l+1)^{p/2} \Big\{ \sum_{n=1}^{\infty} 2^{p\epsilon k(n)} \sum_{\nu=0}^{k(n)} (e^{-c2^{-k(n)+\nu}})^{2l} \rho_n^2 2^{-k(n)+\nu} \Big\}^{p/2} \\
&\le \sum_{l=0}^{\infty} (l+1)^{p/2} \sum_{n=1}^{\infty} \sum_{\nu=0}^{k(n)} e^{-C_1 l 2^{-k(n)+\nu}} \cdot \rho_n^p \cdot 2^{(p^2/2)\epsilon k(n)} \cdot (2^{-k(n)+\nu})^{p/2} \\
&= \sum_{n=1}^{\infty} \sum_{\nu=0}^{k(n)} \Big(\sum_{l=0}^{\infty} (l+1)^{p/2} e^{-C_1 l 2^{-k(n)+\nu}} \Big) \rho_n^p \cdot 2^{(p^2/2)\epsilon k(n)} \cdot (2^{-k(n)+\nu})^{p/2}
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty}\sum_{\nu=0}^{k(n)} \frac{\rho_n^p \cdot 2^{(p^2/2)\epsilon k(n)} \cdot (2^{-k(n)+\nu})^{p/2}}{(1-e^{-C_1 2^{-k(n)+\nu}})^{(p/2)+1}}.$$

Let $\delta = \inf_{0<x\leq C_1} x^{-1}(1-e^{-x})$. Continuing with the above, we obtain

$$\begin{aligned} J &\leq \sum_{n=1}^{\infty}\sum_{\nu=0}^{k(n)} \frac{\rho_n^p \cdot 2^{(p^2/2)\epsilon k(n)} \cdot (2^{-k(n)+\nu})^{p/2}}{(\delta C_1 2^{-k(n)+\nu})^{(p/2)+1}} \\ &= \frac{1}{(\delta C_1)^{(p/2)+1}} \sum_{n=1}^{\infty}\sum_{\nu=0}^{k(n)} \frac{\rho_n^p \cdot 2^{(p^2/2)\epsilon k(n)}}{2^{-k(n)+\nu}} \\ &\leq \frac{2}{(\delta C_1)^{(p/2)+1}} \sum_{n=1}^{\infty} \frac{2^{-(1+p\epsilon)k(n)} \cdot 2^{(p^2/2)\epsilon k(n)}}{2^{-k(n)}} \\ &= \frac{2}{(\delta C_1)^{(p/2)+1}} \sum_{n=1}^{\infty} 2^{-(1-(p/2))p\epsilon k(n)} < \infty, \end{aligned}$$

where the last step again uses the condition $p/2 < 1$. Therefore

$$\mathrm{tr}\,((C_\varphi{}^* C_\varphi)^{p/2}) \leq I + J < \infty.$$

This implies that $C_\varphi \in \mathcal{C}_p$.

It remains to verify that

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) = \infty.$$

Obviously,

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) \geq \sum_{n=1}^{\infty}\sum_{j=1}^{2^{p\epsilon k(n)}} \int_{E_{n,j}} \frac{(1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} dA(z).$$

Applying Lemma 2.3, we have

$$\int_D \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) \geq \sum_{n=1}^{\infty}\sum_{j=1}^{2^{p\epsilon k(n)}} c_3 2^{-p\epsilon k(n)} = c_3 \sum_{n=1}^{\infty} 2^{p\epsilon k(n)} \cdot 2^{-p\epsilon k(n)} = \infty.$$

This completes the proof.

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