COMMUTATORS AND LOCALIZATION ON THE DRURY-ARVESON SPACE

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Abstract. Let f be a multiplier for the Drury-Arveson space H_n^2 of the unit ball, and let ζ_1, ..., ζ_n denote the coordinate functions. We show that for each $1 \leq i \leq n$, the commutator $[M_f^*, M_{\zeta_i}]$ belongs to the Schatten class $\mathcal{C}_p$, $p > 2n$. This leads to a localization result for multipliers.

1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z : |z| < 1\}$ in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is assumed to be greater than or equal to 2. A multivariable analogue of the classical Hardy space of the unit circle is the Drury-Arveson space H_n^2 on $\mathbf{B}$ [3, 9]. Because of its close relation to a number of topics in operator theory, among which we mention the von Neumann inequality for commuting row contractions, H_n^2 has been the subject of intense study of late [2-7,10,12,13].

The space H_n^2 is a reproducing kernel Hilbert space with the kernel

$$K(z, w) = \frac{1}{1 - \langle z, w\rangle}, \quad z, w \in \mathbf{B},$$

which is a multivariable generalization of the one-variable Szegö kernel. An orthonormal basis of H_n^2 is given by $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$, where

$$e_\alpha(\zeta) = \sqrt{\frac{|\alpha|!}{\alpha!}}\zeta^\alpha.$$

In this paper we use the standard multi-index notation: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$,

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}.$$

For functions $f, g \in H_n^2$ with Taylor expansions

$$f(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} c_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} d_\alpha \zeta^\alpha,$$

the inner product is given by

$$\langle f, g\rangle = \sum_{\alpha\in\mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}.$$

Throughout the paper, we let $M_{\zeta_1}, \ldots, M_{\zeta_n}$ denote the operators of multiplication by the coordinate functions $\zeta_1, \ldots, \zeta_n$ on H_n^2. With the identification of each ζ_i with each M_{ζ_i}, H_n^2 is often called the Drury-Arveson module over the polynomial ring $\mathbf{C}[\zeta_1, \ldots, \zeta_n]$.

A holomorphic function f on $\mathbf{B}$ is called a *multiplier* for the space H_n^2 if $fH_n^2 \subset H_n^2$. If f is a multiplier, then the multiplication operator M_f defined by $M_f(g) = fg$ is necessarily bounded on

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H^2_n [3], and the multiplier norm of f is defined to be the operator norm of M_f. In [3], Arveson showed that, when $n \geq 2$, the collection of multipliers of H^2_n is strictly smaller than H^∞. On H^2_n, multipliers can be used to express orthogonal projections. Suppose that $\mathcal{E}$ is a submodule of the Drury-Arveson module, i.e., $\mathcal{E}$ is a closed linear subspace of H^2_n which is invariant under $M_{\zeta_1}, \dots, M_{\zeta_n}$. Then there exist multipliers $\{f_1, \dots, f_k, \dots\}$ of H^2_n such that the operator

$$M_{f_1}M^*_{f_1} + \cdots + M_{f_k}M^*_{f_k} + \cdots$$

is the orthogonal projection from H^2_n onto $\mathcal{E}$ (see page 191 in [4]).

Among the recent results related to multipliers, we would like to mention the following developments. Interpolation problems for multipliers and model theory related to the Drury-Arveson space also have been intensely studied over the past decade or so [5, 6, 10, 12, 13]. Recently, Arcozzi, Rochberg and Sawyer gave a characterization of the multipliers in terms of Carleson measures for H^2_n [2]. In another study, Costea, Sawyer and Wick [7] proved a corona theorem for the Drury-Arveson space multipliers.

Since H^2_n is a natural analogue of the Hardy space, it is natural to take a list of Hardy-space results and try to determine which ones have analogues on H^2_n and which ones do not. Commutators are certainly very high on any such list. One prominent part of the theory of the Hardy space is the Toeplitz operators on it. Since there is no L^2 associated with H^2_n, the only analogue of Toeplitz operators on H^2_n are the multipliers. In this paper we are interested in the commutators of the form $[M^*_f, M_{\zeta_i}]$, where f is a multiplier for the Drury-Arveson space. Since the story about the commutators of the form $[M^*_f, M_{\zeta_i}]$ is well known on the Hardy space, one would certainly like to know the analogous story on H^2_n.

Recall that for each $1 \leq p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition $\|A\|_p < \infty$, where the p-norm is given by the formula

$$\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}.$$

Arveson showed in his seminal paper [3] that commutators of the form $[M^*_{\zeta_j}, M_{\zeta_i}]$ all belong to $\mathcal{C}_p$, $p > n$. As the logical next step, one certainly expects a Schatten class result for commutators on H^2_n involving multipliers other than the simplest coordinate functions. The following is the main result of the paper:

Theorem 1.1. *Let f be a multiplier for the Drury-Arveson space H^2_n. For each $1 \leq i \leq n$, the commutator $[M^*_f, M_{\zeta_i}]$ belongs to the Schatten class $\mathcal{C}_p$, $p > 2n$. Moreover, for each $2n < p < \infty$, there is a constant C which depends only on p and n such that*

$$\|[M^*_f, M_{\zeta_i}]\|_p \leq C\|M_f\|$$

for every multiplier f of H^2_n and every $1 \leq i \leq n$.

This Schatten-class result has C^*-algebraic implications.

Throughout the paper, we denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$ by S.

Let $\mathcal{T}_n$ be the C^*-algebra generated by $M_{\zeta_1}, \cdots, M_{\zeta_n}$ on H^2_n. Recall that $\mathcal{T}_n$ was introduced by Arveson in [3]. In more ways than one, $\mathcal{T}_n$ is the analogue of the C^*-algebra generated by Toeplitz operators with *continuous* symbols. Indeed Arveson showed that there is an exact sequence

$$\{0\} \to \mathcal{K} \to \mathcal{T}_n \xrightarrow{\tau} C(S) \to \{0\}, \tag{1.1}$$

where $\mathcal{K}$ is the collection of compact operators on H_n^2. But there is another natural C^*-algebra on H_n^2 which is also related to "Toeplitz operators", where the symbols are not necessarily continuous. We define

$$\mathcal{TM}_n = \text{the } C^*\text{-algebra generated by } \{M_f : fH_n^2 \subset H_n^2\}.$$

Theorem 1.1 tells us that $\mathcal{T}_n$ is contained in the essential center of $\mathcal{TM}_n$, in analogy with the classic situation on the Hardy space of the unit sphere S. This opens the door for us to use the classic localization technique [8] to analyze multipliers.

Recall that the *essential norm* of a bounded operator A on a Hilbert space $\mathcal{H}$ is

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \text{ is compact on } \mathcal{H}\}.$$

Alternately, $\|A\|_{\mathcal{Q}} = \|\pi(A)\|$, where π denotes the quotient map from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

To state our localization result, we need to introduce a class of Schur multipliers. For each $z \in \mathbf{B}$, let

$$s_z(\zeta) = \frac{1 - |z|}{1 - \langle \zeta, z\rangle}. \tag{1.2}$$

The reason we call s_z a Schur multiplier is that the norm of the operator M_{s_z} on H_n^2 is 1, as we will see in Section 2. Using Theorem 1.1, we will prove

Theorem 1.2. *Let $A \in \mathcal{TM}_n$. Then for each $\xi \in S$, the limit*

$$\lim_{r\uparrow 1} \|AM_{s_{r\xi}}\| \tag{1.3}$$

exists. Moreover, we have

$$\|A\|_{\mathcal{Q}} = \sup_{\xi\in S} \lim_{r\uparrow 1} \|AM_{s_{r\xi}}\|.$$

The C^*-algebraic meaning of the "localized limit" (1.3) will be explained in Section 6. Alternately, we can state Theorem 1.2 in a version which may be better suited for applications:

Theorem 1.3. *For each $A \in \mathcal{TM}_n$, we have*

$$\|A\|_{\mathcal{Q}} = \lim_{r\uparrow 1} \sup_{r\le |z|<1} \|AM_{s_z}\|.$$

The rest of the paper is organized as follows. Section 2 begins with an orthogonal decomposition of H_n^2. This decomposition allows us to obtain the subnormality of certain multipliers. We then use this decomposition to make a number of norm estimates. In Section 3 we derive a "quasi-resolution" of the identity operator of H_n^2, which plays the key role in the proof of Theorem 1.1. In Section 4 we estimate the Schatten p-norm and the operator norm of certain finite-rank operators which arise from the "quasi-resolution". With this preparation, the proof of Theorem 1.1 is completed in Section 5. Section 6 deals with localization and proves Theorems 1.2 and 1.3.

In terms of techniques, the reader will notice that this paper is quite different from previous works on the Drury-Arveson space. This is highlighted by the fact that the unit sphere S and the spherical measure $d\sigma$ play a prominent role in our estimates. Many of the techniques we use in this paper are inspired by our earlier work on Hankel operators [11]. The best example to illustrate this is the idea of using "quasi-resolution" of the identity operator. This interchangeability of techniques serves to show that there is indeed much in common between the Hardy space and the Drury-Arveson space. This view was one of the motivating factors which started this investigation.

2. Estimates for Certain Multipliers

First of all, let us introduce the subset $\mathcal{B} = \{(0, \beta_2, \dots, \beta_n) : \beta_2, \dots, \beta_n \in \mathbf{Z}_+\}$ of $\mathbf{Z}_+^n$. As we indicated in Section 1, we denote the components of ζ by $\zeta_1, \dots, \zeta_n$. For each $\beta \in \mathcal{B}$, define the closed linear subspace

$$H_\beta = \overline{\operatorname{span}\{\zeta_1^k \zeta^\beta : k \geq 0\}}$$

of H_n^2. Then we have the orthogonal decomposition

$$H_n^2 = \bigoplus_{\beta \in \mathcal{B}} H_\beta.$$

For each $\beta \in \mathcal{B}$, we have an orthonormal basis $\{e_{k,\beta} : k \geq 0\}$ for H_β, where

$$e_{k,\beta}(\zeta) = \sqrt{\frac{(k + |\beta|)!}{k!\beta!}} \zeta_1^k \zeta^\beta. \tag{2.1}$$

It is well known that $H_0 = H_1^2$, the Hardy space associated with the unit circle T. For our proofs, we need to identify each H_β, $\beta \neq 0$, as a weighted Bergman space on the unit disc.

Denote $D = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc in the complex plane. Let dA be the area measure on D with the normalization $A(D) = 1$. For each integer $m \geq 0$, let

$$B^{(m)} = L_a^2(D, (1 - |z|^2)^m dA(z)), \tag{2.2}$$

the usual weighted Bergman space of weight m. It is well known that the standard orthonormal basis for $B^{(m)}$ is $\{e_k^{(m)} : k \in \mathbf{Z}_+\}$, where

$$e_k^{(m)}(z) = \sqrt{\frac{(k + m + 1)!}{k!m!}} z^k. \tag{2.3}$$

For each $\beta \in \mathcal{B} \backslash \{0\}$, define the unitary operator $W_\beta : H_\beta \to B^{(|\beta|-1)}$ by the formula

$$W_\beta e_{k,\beta} = e_k^{(|\beta|-1)}, \quad k \in \mathbf{Z}_+. \tag{2.4}$$

Using (2.1) and (2.3), it is straightforward to verify that the weighted shift $M_{\zeta_1}|H_\beta$ is unitarily equivalent to M_z on $B^{(|\beta|-1)}$. More precisely, if $\beta \in \mathcal{B} \backslash \{0\}$, then

$$W_\beta M_{\zeta_1} h_\beta = M_z W_\beta h_\beta \quad \text{for every} \quad h_\beta \in H_\beta. \tag{2.5}$$

The operator $M_{\zeta_1}|H_0$ is, of course, the unilateral shift.

Lemma 2.1. *For each individual $i \in \{1, \dots, n\}$, the multiplication operator M_{ζ_i} is subnormal on H_n^2. Moreover, each M_{ζ_i} has a normal extension of norm* 1.

Proof. This is actually a known fact. See [1]. But this fact also follows from (2.5) for M_{ζ_1}. By the obvious symmetry, the entire lemma follows from (2.5). □

For each $z \in \mathbf{B}$, define the multiplier

$$m_z(\zeta) = \frac{1-|z|^2}{1-\langle \zeta, z\rangle}. \tag{2.6}$$

Obviously, m_z is just a minor modification of the Schur multiplier s_z defined in (1.2). For many purposes, it is easier to work with m_z than s_z, as we will see. The proof of Theorem 1.1 involves the subnormality of $M_{m_z^k}$ and an estimate for $\|M_{m_z m_w}\|$.

Let $\mathcal{U}$ denote the collection of unitary transformations on $\mathbf{C}^n$. It is obvious that if f is a multiplier for H_n^2 and if $U \in \mathcal{U}$, then the function $f \circ U$ is also a multiplier for H_n^2. Moreover, the multiplication operators

$$M_f \quad \text{and} \quad M_{f\circ U}$$

are unitarily equivalent on H_n^2. This fact will be used several times.

Corollary 2.2. *For all $k \in \mathbf{Z}_+$ and $z \in \mathbf{B}$, the operator $M_{m_z^k}$ is subnormal on H_n^2.*

Proof. Given a $z \in \mathbf{B}$, pick a $U \in \mathcal{U}$ such that

$$U^* z = (|z|, 0, \ldots, 0).$$

Then for each $k \in \mathbf{Z}_+$ we have

$$m_z^k(U\zeta) = m_{U^*z}^k(\zeta) = \left(\frac{1-|z|^2}{1-|z|\zeta_1}\right)^k.$$

By Lemma 2.1 and the above-mentioned unitary equivalence, $M_{m_z^k}$ has a normal extension. □

The following lemma provides a key estimate:

Lemma 2.3. *If $0 < s < 1$, then the norm of the operator of multiplication by the function*

$$\frac{\zeta_2}{1-s\zeta_1}$$

on H_n^2 does not exceed

$$\frac{2}{\sqrt{1-s}}.$$

Proof. Consider an arbitrary $h_\beta \in H_\beta$, where $\beta = (0, \beta_2, \ldots, \beta_n)$. Then

$$h_\beta(\zeta) = \sum_{k=0}^\infty c_k \zeta_1^k \zeta^\beta.$$

First we assume that $\beta \neq 0$. By (2.4), we have

$$(W_\beta h_\beta)(z) = \sqrt{\frac{\beta!}{(|\beta|-1)!}} \sum_{k=0}^\infty c_k z^k, \quad z \in D,$$

which is a vector in $B^{(|\beta|-1)}$. Denote $e_2 = (0, 1, 0, \ldots, 0)$. Since $\zeta_2\zeta^\beta = \zeta^{\beta+e_2}$, we have

$$(W_{\beta+e_2}\zeta_2 h_\beta)(z) = \sqrt{\frac{(\beta+e_2)!}{|\beta|!}} \sum_{k=0}^\infty c_k z^k, \quad z \in D,$$

which is a vector in $B^{(|\beta|)}$. Now suppose that

$$h_\beta(\zeta) = (1 - s\zeta_1)^{-1} f_\beta(\zeta),$$

where

$$f_\beta(\zeta) = \sum_{k=0}^{\infty} a_k \zeta_1^k \zeta^\beta.$$

For $z \in D$ and $0 < s < 1$, we have $|1 - sz| \geq 1 - |z|$ and $|1 - sz| \geq 1 - s$. Thus the above yields

$$\begin{aligned}
\|\zeta_2(1 - s\zeta_1)^{-1} f_\beta\|^2_{H^2_n} &= \|\zeta_2 h_\beta\|^2_{H^2_n} = \|W_{\beta+e_2}\zeta_2 h_\beta\|^2_{B^{(|\beta|)}} \\
&= \frac{(\beta+e_2)!}{|\beta|!} \int_D \left| \sum_{k=0}^{\infty} c_k z^k \right|^2 (1 - |z|^2)^{|\beta|} dA(z) \\
&= \frac{(\beta+e_2)!}{|\beta|!} \int_D \left| \frac{1}{1 - sz} \sum_{k=0}^{\infty} a_k z^k \right|^2 (1 - |z|^2)^{|\beta|} dA(z) \\
&\leq \frac{2}{1-s} \cdot \frac{(\beta+e_2)!}{|\beta|!} \int_D \left| \sum_{k=0}^{\infty} a_k z^k \right|^2 (1 - |z|^2)^{|\beta|-1} dA(z) \\
&= \frac{2}{1-s} \cdot \frac{\beta_2 + 1}{|\beta|} \cdot \frac{\beta!}{(|\beta|-1)!} \int_D \left| \sum_{k=0}^{\infty} a_k z^k \right|^2 (1 - |z|^2)^{|\beta|-1} dA(z) \\
&= \frac{2}{1-s} \cdot \frac{\beta_2 + 1}{|\beta|} \|W_\beta f_\beta\|^2_{B^{(|\beta|-1)}} \\
&= \frac{2}{1-s} \cdot \frac{\beta_2 + 1}{|\beta|} \|f_\beta\|^2_{H^2_n} \leq \frac{4}{1-s} \|f_\beta\|^2_{H^2_n}.
\end{aligned}$$

Thus we have shown that for $\beta \neq 0$, the norm of the restriction of the operator of multiplication by $\zeta_2(1 - s\zeta_1)^{-1}$ to H_β does not exceed $2(1 - s)^{-1/2}$. Next we consider the case where $\beta = 0$.

We know that $H_0 = H^2_1$, the Hardy space on the unit circle T. Let $h \in H_0$. Then

$$h(\zeta) = \sum_{k=0}^{\infty} c_k \zeta_1^k.$$

We have

$$(W_{e_2}\zeta_2 h)(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in D,$$

which is a vector in the unweighted Bergman space $B^{(0)}$. Now suppose

$$h(\zeta) = (1 - s\zeta_1)^{-1} f(\zeta)$$

for some

$$f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta_1^k.$$

Using the polar decomposition of dA, we see that

$$\begin{aligned}\|\zeta_2(1-s\zeta_1)^{-1}f\|^2_{H^2_n} = \|W_{e_2}\zeta_2 h\|^2_{B^{(0)}} &= \int_D \left|\sum_{k=0}^\infty c_k z^k\right|^2 dA(z) = \int_D \left|\frac{1}{1-sz}\sum_{k=0}^\infty a_k z^k\right|^2 dA(z)\\ &= 2\int_0^1 r\int_T \left|\frac{1}{1-sr\tau}\sum_{k=0}^\infty a_k(r\tau)^k\right|^2 dm(\tau)dr\\ &\le 2\int_0^1 \frac{1}{(1-sr)^2}dr\sum_{k=0}^\infty |a_k|^2 = 2\int_0^1 \frac{1}{(1-sr)^2}dr\|f\|^2_{H^2_n}\\ &= \frac{2}{1-s}\|f\|^2_{H^2_n}.\end{aligned}$$

Thus we have shown that the norm of the restriction of the operator of multiplication by $\zeta_2(1-s\zeta_1)^{-1}$ to H_0 does not exceed $\sqrt{2}(1-s)^{-1/2}$.

Obviously, if $f_\beta \in H_\beta$, $f_{\beta'} \in H_{\beta'}$ and $\beta \ne \beta'$, then

$$\frac{\zeta_2}{1-s\zeta_1}f_\beta \perp \frac{\zeta_2}{1-s\zeta_1}f_{\beta'}.$$

Thus it follows from the above two paragraphs that the norm of $M_{\zeta_2/(1-s\zeta_1)}$ on the entire H^2_n does not exceed $2(1-s)^{-1/2}$. This completes the proof. □

The proof of Theorem 1.1 involves Möbius transform. For each $z \in \mathbf{B}\backslash\{0\}$, let

$$\varphi_z(\zeta) = \frac{1}{1-\langle\zeta,z\rangle}\left\{z - \frac{\langle\zeta,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(\zeta - \frac{\langle\zeta,z\rangle}{|z|^2}z\right)\right\}. \tag{2.7}$$

Then φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \text{id}$. Recall that

$$k_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{1-\langle\zeta,z\rangle}, \qquad z,\zeta \in \mathbf{B}, \tag{2.8}$$

is the normalized reproducing kernel for H^2_n. Define the operator U_z by the formula

$$(U_z f)(\zeta) = f(\varphi_z(\zeta))k_z(\zeta), \qquad f \in H^2_n, \tag{2.9}$$

for each $z \in \mathbf{B}\backslash\{0\}$. Using Theorem 2.2.2 in [14], it is straightforward to verify that

$$\langle U_z k_x, U_z k_y\rangle = \frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{1-\langle y,x\rangle} = \langle k_x, k_y\rangle$$

for all $z \in \mathbf{B}\backslash\{0\}$ and $x, y \in \mathbf{B}$. Therefore each U_z is a unitary operator on H^2_n.

Recall the elementary fact that if c is a complex number with $|c| \le 1$ and if $0 < t < 1$, then

$$2|1-tc| \ge |1-c|. \tag{2.10}$$

This equality will be used frequently in the sequel.

Lemma 2.4. *Let $z, w \in \mathbf{B}$ be such that $|z| = |w|$. Then*

$$\|M_{m_w m_z}\| \le 48\frac{1-|z|^2}{|1-\langle z,w\rangle|}.$$

Proof. If $z = w$, then the conclusion is a trivial consequence of Lemma 2.1. So let us assume $z \neq w$. Using the unitary operator defined by (2.9), we see that

$$\|M_{m_w m_z}\| = \|M_{(m_w m_z)\circ\varphi_z}\|.$$

Thus we only need to estimate the norm of $M_{(m_w m_z)\circ\varphi_z}$. By Theorem 2.2.2 in [14],

$$1 - \langle \varphi_z(\zeta), z\rangle = 1 - \langle \varphi_z(\zeta), \varphi_z(0)\rangle = \frac{1-|z|^2}{1-\langle \zeta, z\rangle},$$

which leads to

$$m_z(\varphi_z(\zeta)) = 1 - \langle \zeta, z\rangle.$$

Write $\lambda = \varphi_z(w)$. Then $w = \varphi_z(\lambda)$. Using the above-cited theorem,

$$1 - \langle \varphi_z(\zeta), w\rangle = 1 - \langle \varphi_z(\zeta), \varphi_z(\lambda)\rangle = \frac{(1-|z|^2)(1-\langle \zeta, \lambda\rangle)}{(1-\langle \zeta, z\rangle)(1-\langle z, \lambda\rangle)}.$$

Since $1 - |w|^2 = 1 - |z|^2$, this gives us

$$m_w(\varphi_z(\zeta))m_z(\varphi_z(\zeta)) = (1-\langle z, \lambda\rangle)\frac{(1-\langle \zeta, z\rangle)^2}{1-\langle \zeta, \lambda\rangle}. \tag{2.11}$$

Since we know that

$$1 - \langle z, \lambda\rangle = 1 - \langle \varphi_z(0), \varphi_z(w)\rangle = \frac{1-|z|^2}{1-\langle z, w\rangle},$$

we only need to consider the operator of multiplication by $F(\zeta) = (1-\langle \zeta, z\rangle)^2/(1-\langle \zeta, \lambda\rangle)$.

Write $s = |\lambda| = |\varphi_z(w)|$. Let $U : \mathbf{C}^n \to \mathbf{C}^n$ be a unitary transformation such that

$$U^*\lambda = (s, 0, 0, \dots, 0) \quad \text{and}$$
$$U^*z = (\overline{a}, \overline{b}, 0, \dots, 0),$$

where $\overline{a} = \langle z, \lambda/s\rangle$ and $|b|^2 = |z|^2 - |\langle z, \lambda/s\rangle|^2$. Since $1 - |w|^2 = 1 - |z|^2$, we have

$$2(1-s) \geq 1 - s^2 = 1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)^2}{|1-\langle w, z\rangle|^2}. \tag{2.12}$$

Since

$$1 - sa = 1 - \langle \lambda, z\rangle = \frac{1-|z|^2}{1-\langle w, z\rangle},$$

(2.10) gives us

$$|1-a| \leq 2|1-sa| = 2\frac{1-|z|^2}{|1-\langle w, z\rangle|}. \tag{2.13}$$

Also,

$$|b|^2 \leq 1 - |\langle z, \lambda\rangle|^2 \leq 2(1-|\langle z, \lambda\rangle|) \leq 2\frac{1-|z|^2}{|1-\langle z, w\rangle|}. \tag{2.14}$$

Since $\|M_F\| = \|M_{F\circ U}\|$, it suffices to estimate the latter. We have

$$\begin{aligned}\frac{(1-\langle U\zeta, z\rangle)^2}{1-\langle U\zeta, \lambda\rangle} &= \frac{(1-a\zeta_1-b\zeta_2)^2}{1-s\zeta_1} = \frac{(1-a\zeta_1)^2}{1-s\zeta_1} - 2b\frac{(1-a\zeta_1)\zeta_2}{1-s\zeta_1} + \frac{b^2\zeta_2^2}{1-s\zeta_1} \\ &= G_1(\zeta) - 2bG_2(\zeta) + G_3(\zeta). \end{aligned}\tag{2.15}$$

Write the first term in (2.15) as

$$\begin{aligned} G_1(\zeta) &= \frac{(1-a)^2}{1-s\zeta_1} + 2a(1-a)\frac{1-\zeta_1}{1-s\zeta_1} + a^2\frac{(1-\zeta_1)^2}{1-s\zeta_1} \\ &= G_{11}(\zeta) + 2a(1-a)G_{12}(\zeta) + a^2 G_{13}(\zeta). \end{aligned}$$

By (2.10) and Lemma 2.1, we have $\|M_{G_{12}}\| \le 2$. Similarly, $\|M_{G_{13}}\| \le 4$. For G_{11}, Lemma 2.1 yields

$$\|M_{G_{11}}\| \le \frac{|1-a|^2}{1-s} \le 8,$$

where the second $\le$ follows from (2.12) and (2.13). Therefore we conclude that

$$\|M_{G_1}\| \le 20. \tag{2.16}$$

For the second term in (2.15), we have

$$G_2(\zeta) = \frac{(1-a\zeta_1)\zeta_2}{1-s\zeta_1} = (1-a)\frac{\zeta_2}{1-s\zeta_1} + a\frac{(1-\zeta_1)}{1-s\zeta_1}\zeta_2 = G_{21}(\zeta) + G_{22}(\zeta).$$

By Lemma 2.3, (2.12) and (2.13),

$$\|M_{G_{21}}\| \le \frac{2|1-a|}{\sqrt{1-s}} \le 4\sqrt{2} < 8.$$

By (2.10) and Lemma 2.1, $\|M_{G_{22}}\| \le 2$. Therefore

$$\|M_{G_2}\| \le 10. \tag{2.17}$$

Since

$$G_3(\zeta) = b^2\frac{\zeta_2}{1-s\zeta_1}\cdot\zeta_2,$$

by Lemma 2.3, (2.12) and (2.14),

$$\|M_{G_3}\| \le \frac{2|b|^2}{\sqrt{1-s}} \le 8. \tag{2.18}$$

Combining (2.16),(2.17),(2.18), and (2.15), we now have $\|M_F\| = \|M_{F\circ U}\| \le 48$. Recalling (2.11), the proof is complete. □

Lemma 2.5. *For every $z \in \mathbf{B}$ and every $i \in \{1,\dots,n\}$, the norm of the operator of multiplication by the function*

$$(\zeta_i - z_i)m_z(\zeta)$$

on H^2_n does not exceed

$$3n\sqrt{1-|z|^2},$$

where z_i is the i-th component of z.

Proof. Let $z \in \mathbf{B}$ and $i \in \{1,\dots,n\}$ be given, and write $G(\zeta) = (\zeta_i - z_i)m_z(\zeta)$. Let $\hat{z} = (|z|, 0, \dots, 0)$. Then there is a unitary operator $U : \mathbf{C}^n \to \mathbf{C}^n$ such that $U^*z = \hat{z}$. Since $\|M_G\| = \|M_{G\circ U}\|$, it suffices to estimate the latter. We have

$$\begin{aligned} G(U\zeta) &= ((U\zeta)_i - z_i)m_z(U\zeta) = ((U\zeta)_i - (U\hat{z})_i)m_{\hat{z}}(\zeta) = (U(\zeta-\hat{z}))_i\frac{1-|z|^2}{1-|z|\zeta_1} \\ &= (u_{i1}(\zeta_1 - |z|) + u_{i2}\zeta_2 + \cdots + u_{in}\zeta_n)\frac{1-|z|^2}{1-|z|\zeta_1}, \end{aligned} \tag{2.19}$$

where $\sum_{k=1}^n |u_{ik}|^2 = 1$. By Lemma 2.1, the norm of the operator of multiplication by $(\zeta_1 - |z|)/(1-|z|\zeta_1)$ does not exceed 1. By Lemma 2.3, for each $2 \le j \le n$, the norm of the operator of multiplication by $\zeta_j/(1-|z|\zeta_1)$ does not exceed $2(1-|z|)^{-1/2}$. Therefore

$$\|M_{G\circ U}\| \le (1-|z|^2) + (n-1)(1-|z|^2)\cdot \frac{2}{\sqrt{1-|z|}} \le 3n\sqrt{1-|z|^2}.$$

This completes the proof. □

The next lemma will be needed in Section 6 when we deal with localization.

Lemma 2.6. *For each $h \in H^2_n$, we have*

$$\lim_{|z|\uparrow 1} \|s_z h\| = 0,$$

where s_z was defined in (1.2).

Proof. Write

$$b_r(\zeta) = \frac{1-r}{1-r\zeta_1}$$

for each $0 \le r < 1$. We first show that for each $h \in H^2_n$,

$$\lim_{r\uparrow 1} \|b_r h\| = 0. \tag{2.20}$$

For this, we use the orthogonal decomposition $H^2_n = \oplus_{\beta\in\mathcal{B}} H_\beta$ introduced at the beginning of the section. First consider any

$$h_0(\zeta) = \sum_{k=0}^\infty c_k \zeta_1^k$$

in H_0. Then

$$\|b_r h_0\|^2 = \int_T \left| \frac{1-r}{1-r\tau} \sum_{k=0}^\infty c_k \tau^k \right|^2 dm(\tau).$$

As $r \uparrow 1$, $(1-r)/(1-r\tau) \to 0$ for every $\tau \in T\backslash\{1\}$. Thus it follows from the dominated convergence theorem that

$$\lim_{r\uparrow 1} \|b_r h_0\| = 0. \tag{2.21}$$

Next we consider an $h_\beta \in H_\beta$, where $\beta \in \mathcal{B}\backslash\{0\}$. Suppose that

$$h_\beta(\zeta) = \sum_{k=0}^\infty a_k \zeta_1^k \zeta^\beta.$$

As we saw in the proof of Lemma 2.3,

$$\|b_r h_\beta\|^2 = \frac{\beta!}{(|\beta|-1)!} \int_D \left| \frac{1-r}{1-rz} \sum_{k=0}^\infty a_k z^k \right|^2 (1-|z|^2)^{|\beta|-1} dA(z).$$

As $r \uparrow 1$, $(1-r)/(1-rz) \to 0$ for every $z \in D$. Thus it follows from the dominated convergence theorem that

$$\lim_{r\uparrow 1} \|b_r h_\beta\| = 0. \tag{2.22}$$

For each $\beta \in \mathcal{B}$, $b_r H_\beta \subset H_\beta$. Therefore (2.20) follows from (2.21) and (2.22).

Recall that we denote the collection of unitary transformations on $\mathbf{C}^n$ by $\mathcal{U}$. For each $h \in H_n^2$, the collection of vectors $\{h \circ U : U \in \mathcal{U}\}$ is a compact subset of H_n^2. Therefore (2.20) implies that

$$\lim_{r \uparrow 1} \sup_{U \in \mathcal{U}} \|b_r \cdot h \circ U\| = 0. \tag{2.23}$$

For each $z \in \mathbf{B}$, there is a $V_z \in \mathcal{U}$ such that $V_z^* z = (|z|, 0, \ldots, 0)$. Hence

$$\|s_z h\| = \|(s_z h) \circ V_z\| = \|s_{V_z^* z} \cdot h \circ V_z\| = \|b_{|z|} \cdot h \circ V_z\|.$$

The lemma obviously follows from this identity and (2.23). □

3. A Quasi-resolution of the Identity Operator

Let N be an integer greater than or equal to $n/2$. For each $z \in \mathbf{B}$, define the function

$$\psi_{z,N}(\zeta) = \frac{(1 - |z|^2)^{(1/2)+N}}{(1 - \langle \zeta, z \rangle)^{1+N}}.$$

Then we have the relation

$$\psi_{z,N} = m_z^N k_z,$$

where m_z and k_z were given by (2.6) and (2.8) respectively. In this sense $\psi_{z,N}$ is a modified version of k_z. The main difference between these two functions is that $\psi_{z,N}$ "decays much faster". The reader will clearly see the meaning of this statement in the subsequent proofs.

Let $d\lambda$ be the Möbius invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}},$$

where dv is the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. Let $d\sigma$ be the positive, regular Borel measure on the unit sphere S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. We normalize σ such that $\sigma(S) = 1$.

Theorem 3.1. *Let N be an integer greater than or equal to $n/2$. Then the self-adjoint operator*

$$R_N = \int \psi_{z,N} \otimes \psi_{z,N} d\lambda(z)$$

is both bounded and invertible on the Drury-Arveson space H_n^2. In other words, there exist constants $0 < a(N) \le b(N) < \infty$ which only depend on N and the complex dimension n such that

$$a(N) \le R_N \le b(N)$$

on H_n^2.

Proof. For each $z \in \mathbf{B}$, define the function $g_z \in H_n^2$ by the formula

$$g_z(\zeta) = \langle \zeta, z \rangle.$$

Write C_k^m for the binomial coefficient $m!/(k!(m-k)!)$ as usual. Then

$$\psi_{z,N} = (1 - |z|^2)^{(1/2)+N} \sum_{k=0}^{\infty} C_k^{k+N} g_z^k,$$

and consequently

$$\psi_{z,N} \otimes \psi_{z,N} = (1 - |z|^2)^{1+2N} \sum_{j,k=0}^{\infty} C_k^{k+N} C_j^{j+N} g_z^k \otimes g_z^j.$$

For each $0 < \rho < 1$, define $\mathbf{B}_\rho = \{z : |z| < \rho\}$. Since both $d\lambda$ and $\mathbf{B}_\rho$ are invariant under the substitution $z \to e^{i\theta}z$, $\theta \in \mathbf{R}$, we have

$$\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N} g_z^k \otimes g_z^j d\lambda(z) = \int_{\mathbf{B}_\rho}(1-|e^{i\theta}z|^2)^{1+2N} g_{e^{i\theta}z}^k \otimes g_{e^{i\theta}z}^j d\lambda(z)$$
$$= e^{i(j-k)\theta}\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N} g_z^k \otimes g_z^j d\lambda(z).$$

This implies that

$$\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N} g_z^k \otimes g_z^j d\lambda(z) = 0 \quad \text{if} \quad k \neq j.$$

Therefore

$$\int_{\mathbf{B}_\rho}\psi_{z,N}\otimes\psi_{z,N} d\lambda(z) = \sum_{k=0}^\infty (C_k^{k+N})^2 \int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N} g_z^k \otimes g_z^k d\lambda(z).$$

Since

$$g_z^k(\zeta) = \langle \zeta, z\rangle^k = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\bar{z}^\alpha\zeta^\alpha,$$

we have

$$g_z^k \otimes g_z^k = \sum_{|\alpha|=|\beta|=k}\frac{(k!)^2}{\alpha!\beta!}\bar{z}^\alpha z^\beta \zeta^\alpha \otimes \zeta^\beta.$$

By the radial-spherical decomposition of $d\lambda$, it is obvious that

$$\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N}\bar{z}^\alpha z^\beta d\lambda(z) = 0 \quad \text{if} \quad \alpha \neq \beta.$$

Therefore

$$\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N} g_z^k \otimes g_z^k d\lambda(z) = \sum_{|\alpha|=k}\frac{(k!)^2}{(\alpha!)^2}\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N}|z^\alpha|^2 d\lambda(z)\zeta^\alpha\otimes\zeta^\alpha.$$

Consequently

$$\int_{\mathbf{B}_\rho}\psi_{z,N}\otimes\psi_{z,N} d\lambda(z) = \sum_{k=0}^\infty (C_k^{k+N})^2 \sum_{|\alpha|=k}\frac{(k!)^2}{(\alpha!)^2}\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N}|z^\alpha|^2 d\lambda(z)\zeta^\alpha\otimes\zeta^\alpha. \tag{3.1}$$

Notice that if $|\alpha| = k$, then

$$\int_{\mathbf{B}_\rho}(1-|z|^2)^{1+2N}|z^\alpha|^2 d\lambda(z) = \int_{\mathbf{B}_\rho}(1-|z|^2)^{2N-n}|z^\alpha|^2 dv(z)$$
$$= \int_0^\rho (1-r^2)^{2N-n} 2nr^{2k+2n-1}dr\int_S |\xi^\alpha|^2 d\sigma(\xi)$$
$$= \int_0^\rho (1-r^2)^{2N-n} 2nr^{2k+2n-1}dr\frac{(n-1)!\alpha!}{(n-1+k)!}, \tag{3.2}$$

where the third = follows from Proposition 1.4.9 in [14]. Since $2N - n \geq 0$, we can integrate by parts to obtain

$$2\int_0^1 (1-r^2)^{2N-n}r^{2k+2n-1}dr = \int_0^1 (1-x)^{2N-n}x^{n-1+k}dx = \frac{(2N-n)!(n-1+k)!}{(2N+k)!}.$$

Letting $\rho \uparrow 1$ in (3.1) and (3.2), we see that

$$\int \psi_{z,N} \otimes \psi_{z,N} d\lambda(z) = \sum_{k=0}^{\infty} b_{k,N} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \zeta^\alpha \otimes \zeta^\alpha, \tag{3.3}$$

where

$$b_{k,N} = (C_k^{k+N})^2 k! \frac{(2N-n)!n!}{(2N+k)!} = \frac{(2N-n)!n!}{(N!)^2} \cdot \frac{\{(k+N)!\}^2}{k!(2N+k)!}.$$

Using Stirling's formula, it is straightforward to verify that there exist $0 < a(N) \leq b(N) < \infty$ which depend only on N and n such that

$$a(N) \leq b_{k,N} \leq b(N) \tag{3.4}$$

for every $k \geq 0$. Since we can write the identity operator on H^2_n as

$$1 = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \zeta^\alpha \otimes \zeta^\alpha,$$

the lemma follows from (3.3) and (3.4). □

4. Three Lemmas

It is well known that the formula

$$d(x,y) = |1 - \langle x, y\rangle|^{1/2}, \qquad x, y \in S,$$

defines a metric on the unit sphere S [14]. Throughout the paper, we write

$$B(x,r) = \{y \in S : |1 - \langle x, y\rangle|^{1/2} < r\}$$

for $x \in S$ and $r > 0$. By Proposition 5.1.4 in [14], there is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \leq \sigma(B(x,r)) \leq A_0 r^{2n} \tag{4.1}$$

for all $x \in S$ and $0 < r \leq \sqrt{2}$. Note that the upper bound actually holds for all $r > 0$.

Before getting to the main estimates of the section, let us recall:

Lemma 4.1 (Lemma 4.1 in [15]). *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\text{card}\{y \in X : (x,y) \in E\} \leq m \quad \text{and} \quad \text{card}\{y \in X : (y,x) \in E\} \leq m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup \ldots \cup E_{2m}$$

and such that for each $1 \leq j \leq 2m$, the conditions $(x,y), (x',y') \in E_j$ and $(x,y) \neq (x',y')$ imply both $x \neq x'$ and $y \neq y'$.

For each $z \in \mathbf{B}$, define the functions

$$u_z(\zeta) = m_z^{n+3}(\zeta) = \left(\frac{1-|z|^2}{1-\langle \zeta, z\rangle}\right)^{n+3} \quad \text{and} \quad v_z(\zeta) = m_z^{n+4}(\zeta) = \left(\frac{1-|z|^2}{1-\langle \zeta, z\rangle}\right)^{n+4}. \tag{4.2}$$

The proofs of our next three lemmas have much in common. More specifically, they all use a counting argument based on Lemma 4.1. However, because the estimates involved vary in details, it is difficult to reduce them to one. Therefore we present all three proofs.

It should be pretty clear from Lemma 2.1 that $\|M_{s_z}\| = 1$ for each $z \in \mathbf{B}$. Therefore $\|M_{m_z}\| = 1 + |z|$. This fact will be used several times in this section.

Lemma 4.2. *Let $2n < p < \infty$. Then there is a $C_{4.2}(p)$ which depends only on p and n such that the following estimate holds: Suppose that $0 < t < 1$ and that $\{\xi_j : j \in J\}$ is a subset of S satisfying the condition*

$$B(\xi_i, t) \cap B(\xi_j, t) = \emptyset \qquad \text{for all } \ i \neq j. \tag{4.3}$$

Define $z_j = (1 - t^2)^{1/2}\xi_j$, $j \in J$. Let $\{f_j : j \in J\}$ be a set of vectors in H^2_n with norm at most 1, and let $\{e_j : j \in J\}$ be an orthonormal set. For each $\nu \in \{1, \dots, n\}$, define the operator

$$E_\nu = \sum_{j\in J} (M^*_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j) \otimes e_j,$$

where $(z_j)_\nu$ denotes the ν-th component of z_j. Then $\|E_\nu\|_p \leq C_{4.2}(p)t^{1-(2n/p)}$.

Proof. Let $\nu \in \{1, \dots, n\}$ be given. By Lemma 2.1, M_{ζ_ν} has a normal extension. More precisely, there is a Hilbert space L_ν containing H^2_n and a normal operator M_ν on L_ν such that

$$M_\nu h = M_{\zeta_\nu} h, \qquad \text{for each } h \in H^2_n. \tag{4.4}$$

Let $P_\nu : L_\nu \to H^2_n$ be the orthogonal projection. Define the operator

$$\tilde{E}_\nu = \sum_{j\in J} \{(M^*_\nu - \overline{(z_j)_\nu}) v_{z_j} f_j\} \otimes e_j.$$

Since $M^*_{\zeta_\nu} = P_\nu M^*_\nu | H^2_n$, we have

$$E_\nu = P_\nu \tilde{E}_\nu.$$

Thus it suffices to estimate $\|\tilde{E}_\nu\|_p$.

For the convenience of the reader, we will denote the inner product and the norm on L_ν by $\langle \cdot, \cdot \rangle_{L_\nu}$ and $\| \cdot \|_{L_\nu}$ respectively, whereas those on the subspace H^2_n will still be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. We have

$$\tilde{E}^*_\nu \tilde{E}_\nu = \sum_{i,j\in J} \langle (M^*_\nu - \overline{(z_j)_\nu}) v_{z_j} f_j, (M^*_\nu - \overline{(z_i)_\nu}) v_{z_i} f_i \rangle_{L_\nu} e_i \otimes e_j = B + \sum_{k=0}^{\infty} Y_k, \tag{4.5}$$

where

$$B = \sum_{j\in J} \|(M^*_\nu - \overline{(z_j)_\nu}) v_{z_j} f_j\|^2_{L_\nu} e_j \otimes e_j$$

and

$$Y_k = \sum_{2^k t \leq d(\xi_i,\xi_j) < 2^{k+1}t} \langle (M^*_\nu - \overline{(z_j)_\nu}) v_{z_j} f_j, (M^*_\nu - \overline{(z_i)_\nu}) v_{z_i} f_i \rangle_{L_\nu} e_i \otimes e_j,$$

$k \in \mathbf{Z}_+$. Next we estimate $\|B\|_{p/2}$ and $\|Y_k\|_{p/2}$.

For $\|B\|_{p/2}$, note that by the normality of M_ν and (4.4), we have

$$\begin{aligned} \|(M^*_\nu - \overline{(z_j)_\nu}) v_{z_j} f_j\|_{L_\nu} &= \|(M_\nu - (z_j)_\nu) v_{z_j} f_j\|_{L_\nu} = \|M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j\| \\ &= \|M_{\zeta_\nu - (z_j)_\nu} M_{m_{z_j}} u_{z_j} f_j\| \leq 2^{n+3} \|M_{(\zeta_\nu - (z_j)_\nu) m_{z_j}}\|. \end{aligned}$$

Applying Lemma 2.5, the above yields

$$\|(M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j\|_{L_\nu} \le 2^{n+3}3n\sqrt{1-|z_j|^2} = 2^{n+3}3nt.$$

By (4.1) and (4.3), $\mathrm{card}(J) \le 2^n t^{-2n}$. Therefore

$$\|B\|_{p/2}^{p/2} = \sum_{j\in J} \|(M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j\|_{L_\nu}^p \le (2^{n+3}3nt)^p \cdot \mathrm{card}(J) \le (2^{n+3}3n)^p 2^n t^{p-2n}.$$

If we set $C = (2^{n+3}3n)^2 2^{2n/p}$, then

$$\|B\|_{p/2} \le ((2^{n+3}3n)^p 2^n t^{p-2n})^{2/p} = Ct^{2(1-(2n/p))}. \tag{4.6}$$

For $\|Y_k\|_{p/2}$, note that by the normality of M_ν and (4.4), we have

$$\begin{aligned}
\langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu} \\
&= \langle (M_\nu - (z_i)_\nu)v_{z_j}f_j, (M_\nu - (z_j)_\nu)v_{z_i}f_i\rangle_{L_\nu} \\
&= \langle M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j, M_{\zeta_\nu-(z_j)_\nu}v_{z_i}f_i\rangle \\
&= \langle M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j, M_{u_{z_i}}M_{\zeta_\nu-(z_j)_\nu}m_{z_i}f_i\rangle \\
&= \langle M_{u_{z_i}}^* M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j, M_{\zeta_\nu-(z_j)_\nu}m_{z_i}f_i\rangle.
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$|\langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu}| \le \|M_{u_{z_i}}^* M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j\|\|M_{\zeta_\nu-(z_j)_\nu}m_{z_i}f_i\|. \tag{4.7}$$

The two norms above need to be estimated separately, which is the most subtle part of the proof.

For the first norm in (4.7), we use Corollary 2.2. Since $M_{u_{z_i}}$ is subnormal, it is hyponormal. Therefore

$$\begin{aligned}
\|M_{u_{z_i}}^* M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j\| &\le \|M_{u_{z_i}} M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j\| \\
&= \|M_{m_{z_i}m_{z_j}}^{n+2} M_{(\zeta_\nu-(z_i)_\nu)m_{z_i}} m_{z_j}^2 f_j\| \\
&\le 4\|M_{m_{z_i}m_{z_j}}\|^{n+2}\|M_{(\zeta_\nu-(z_i)_\nu)m_{z_i}}\|.
\end{aligned}$$

Applying Lemma 2.4 to the first factor and Lemma 2.5 to the second factor, we have

$$\|M_{u_{z_i}}^* M_{\zeta_\nu-(z_i)_\nu}v_{z_j}f_j\| \le 4(48)^{n+2}\left(\frac{t^2}{|1-\langle z_i, z_j\rangle|}\right)^{n+2} \cdot 3nt \le 12n(96)^{n+2}\left(\frac{t^2}{|1-\langle \xi_i, \xi_j\rangle|}\right)^{n+2} t. \tag{4.8}$$

For the second norm in (4.7), we use Lemma 2.5 again:

$$\begin{aligned}
\|M_{\zeta_\nu-(z_j)_\nu}m_{z_i}f_i\| &\le \|M_{(\zeta_\nu-(z_i)_\nu)m_{z_i}}f_i\| + \|M_{((z_i)_\nu-(z_j)_\nu)m_{z_i}}f_i\| \\
&\le 3n\sqrt{1-|z_i|^2} + |(z_i)_\nu - (z_j)_\nu|\|M_{m_{z_i}}\| \\
&\le 3nt + 2|z_i - z_j| \le 3nt + 4|1-\langle \xi_i, \xi_j\rangle|^{1/2} \\
&= 3nt + 4\frac{|1-\langle \xi_i, \xi_j\rangle|^{1/2}}{t} \cdot t.
\end{aligned} \tag{4.9}$$

Bringing (4.8) and (4.9) into (4.7), we obtain

$$|\langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu}|$$
$$\leq C_1\left\{\left(\frac{t^2}{|1-\langle \xi_i,\xi_j\rangle|}\right)^{n+2} + \left(\frac{t^2}{|1-\langle \xi_i,\xi_j\rangle|}\right)^{n+(3/2)}\right\}t^2,$$

where $C_1 = 48n^2(96)^{n+2}$. For any pair of i, j such that $d(\xi_i,\xi_j) \geq 2^k t$, the above gives us

$$|\langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu}| \leq \frac{2C_1}{2^{2k(n+(3/2))}}t^2. \tag{4.10}$$

For each $i \in J$, if $d(\xi_i,\xi_j) < 2^{k+1}t$, then $B(\xi_j,t) \subset B(\xi_i,2^{k+2}t)$. By (4.3) and the fact that $\sigma(B(x,t)) = \sigma(B(y,t))$ for all $x,y \in S$, for each $i \in J$ we have

$$\mathrm{card}\{j \in J : d(\xi_i,\xi_j) < 2^{k+1}t\} \leq \frac{\sigma(B(\xi_i,2^{k+2}t))}{\sigma(B(\xi_i,t))} \leq \frac{A_0(2^{k+2}t)^{2n}}{2^{-n}t^{2n}} = C_2 2^{2nk}, \tag{4.11}$$

where A_0 is the constant that appears in (4.1) and $C_2 = 2^{5n}A_0$. Set

$$\ell(k) = \min\{\ell \in \mathbf{N} : \ell \geq C_2 2^{2nk}\}. \tag{4.12}$$

According to Lemma 4.1, we can decompose

$$\mathcal{E}^{(k)} = \{(i,j) \in J \times J : 2^k t \leq d(\xi_i,\xi_j) < 2^{k+1}t\}$$

as the union of pairwise disjoint subsets

$$\mathcal{E}_1^{(k)}, ..., \mathcal{E}_{2\ell(k)}^{(k)}$$

such that for each $m \in \{1,...,2\ell(k)\}$, if $(i,j),(i',j') \in \mathcal{E}_m^{(k)}$ and if $(i,j) \neq (i',j')$, then we have both $i \neq i'$ and $j \neq j'$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$Y_k = Y_{k,1} + ... + Y_{k,2\ell(k)}, \tag{4.13}$$

where

$$Y_{k,m} = \sum_{(i,j)\in\mathcal{E}_m^{(k)}} \langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu} e_i \otimes e_j,$$

$1 \leq m \leq 2\ell(k)$.

The property of $\mathcal{E}_m^{(k)}$ simply means that the projection onto the first component, $(i,j) \mapsto i$, is injective on $\mathcal{E}_m^{(k)}$. Similarly, the projection onto the second component, $(i,j) \mapsto j$, is also injective on each $\mathcal{E}_m^{(k)}$. Combining these injectivities with the fact that $\{e_j : j \in J\}$ is an orthonormal set and with (4.10), we obtain

$$\|Y_{k,m}\|_{p/2}^{p/2} = \sum_{(i,j)\in\mathcal{E}_m^{(k)}} |\langle (M_\nu^* - \overline{(z_j)_\nu})v_{z_j}f_j, (M_\nu^* - \overline{(z_i)_\nu})v_{z_i}f_i\rangle_{L_\nu}|^{p/2}$$
$$\leq \left(\frac{2C_1}{2^{2k(n+(3/2))}}t^2\right)^{p/2} \cdot \mathrm{card}(J)$$
$$\leq \left(\frac{2C_1}{2^{2k(n+(3/2))}}\right)^{p/2} \cdot t^p \cdot 2^n t^{-2n} = \frac{C_3}{2^{pk(n+(3/2))}}t^{p-2n},$$

where $C_3 = (2C_1)^{p/2}2^n$. Setting $C_4 = C_3^{2/p}$, the above yields

$$\|Y_{k,m}\|_{p/2} \le \frac{C_4}{2^{2k(n+(3/2))}} t^{2(1-(2n/p))}$$

for each $m \in \{1, ..., 2\ell(k)\}$. Recalling (4.12) and (4.13), we now have

$$\|Y_k\|_{p/2} \le \frac{C_4}{2^{2k(n+(3/2))}} t^{2(1-(2n/p))} \cdot 2(1 + C_2 2^{2nk}) \le \frac{2C_4(1+C_2)}{2^{3k}} t^{2(1-(2n/p))}.$$

Combining this with (4.5) and (4.6), we see that

$$\|\tilde{E}_\nu^* \tilde{E}_\nu\|_{p/2} \le \left(C + \sum_{k=0}^{\infty} \frac{2C_4(1+C_2)}{2^{3k}} \right) t^{2(1-(2n/p))}.$$

Since $\|\tilde{E}_\nu\|_p = \|\tilde{E}_\nu^* \tilde{E}_\nu\|_{p/2}^{1/2}$ and $\|E_\nu\|_p \le \|\tilde{E}_\nu\|_p$, this completes the proof. □

If we replace the operator $M^*_{\zeta_\nu - (z_j)_\nu}$ in the above lemma by $M_{\zeta_\nu - (z_j)_\nu}$, with an easier proof, we obtain the same type of estimate:

Lemma 4.3. *Let $2n < p < \infty$. Then there is a $C_{4.3}(p)$ which depends only on p and n such that the following estimate holds: Suppose that $0 < t < 1$ and that $\{\xi_j : j \in J\}$ is a subset of S satisfying the condition*

$$B(\xi_i, t) \cap B(\xi_j, t) = \emptyset \qquad \text{for all } \ i \ne j. \tag{4.14}$$

Define $z_j = (1-t^2)^{1/2}\xi_j$, $j \in J$. Let $\{f_j : j \in J\}$ be a set of vectors in H^2_n with norm at most 1, and let $\{e_j : j \in J\}$ be an orthonormal set. For each $\nu \in \{1, \dots, n\}$, define

$$E_\nu = \sum_{j \in J} (M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j) \otimes e_j,$$

where $(z_j)_\nu$ denotes the ν-th component of z_j. Then $\|E_\nu\|_p \le C_{4.3}(p) t^{1-(2n/p)}$.

Proof. We have

$$E_\nu^* E_\nu = \sum_{i,j \in J} \langle M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu - (z_i)_\nu} v_{z_i} f_i \rangle e_i \otimes e_j = B + \sum_{k=0}^{\infty} Y_k, \tag{4.15}$$

where

$$B = \sum_{j \in J} \|M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j\|^2 e_j \otimes e_j$$

and

$$Y_k = \sum_{2^k t \le d(\xi_i, \xi_j) < 2^{k+1} t} \langle M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu - (z_i)_\nu} v_{z_i} f_i \rangle e_i \otimes e_j,$$

$k \in \mathbf{Z}_+$. As in the previous lemma, we need to estimate $\|B\|_{p/2}$ and $\|Y_k\|_{p/2}$.

For $\|B\|_{p/2}$, by Lemma 2.5 we have

$$\|M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j\| \le 2^{n+3} \|M_{(\zeta_\nu - (z_j)_\nu) m_{z_j}}\| \le 2^{n+3} 3n \sqrt{1 - |z_j|^2} = 2^{n+3} 3nt.$$

By (4.1) and (4.14), $\mathrm{card}(J) \le 2^n t^{-2n}$. Therefore

$$\|B\|_{p/2}^{p/2} = \sum_{j \in J} \|M_{\zeta_\nu - (z_j)_\nu} v_{z_j} f_j\|^p \le (2^{n+3} 3nt)^p \cdot \mathrm{card}(J) \le (2^{n+3} 3n)^p 2^n t^{p-2n}.$$

Consequently,

$$\|B\|_{p/2} \le \left((2^{n+3}3n)^p 2^n t^{p-2n}\right)^{2/p} = Ct^{2(1-(2n/p))}. \tag{4.16}$$

For $\|Y_k\|_{p/2}$, note that

$$\langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle = \langle M^*_{u_{z_i}} M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} m_{z_i} f_i\rangle.$$

By the Cauchy-Schwarz inequality,

$$|\langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle| \le \|M^*_{u_{z_i}} M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j\| \|M_{\zeta_\nu-(z_i)_\nu} m_{z_i} f_i\|. \tag{4.17}$$

As before, we will estimate the two norms above separately.

For the first norm in (4.17), it follows from Corollary 2.2 that

$$\begin{aligned}\|M^*_{u_{z_i}} M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j\| &\le \|M_{u_{z_i}} M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j\| \\ &= \|M^{n+3}_{m_{z_i} m_{z_j}} M_{(\zeta_\nu-(z_j)_\nu) m_{z_j}} f_j\| \\ &\le \|M_{m_{z_i} m_{z_j}}\|^{n+3} \|M_{(\zeta_\nu-(z_j)_\nu) m_{z_j}}\|.\end{aligned}$$

Applying Lemma 2.4 to the first factor and Lemma 2.5 to the second factor, we have

$$\|M^*_{u_{z_i}} M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j\| \le (48)^{n+3} \left(\frac{t^2}{|1-\langle z_i, z_j\rangle|}\right)^{n+3} \cdot 3nt \le 3n(96)^{n+3} \left(\frac{t^2}{|1-\langle \xi_i, \xi_j\rangle|}\right)^{n+3} t. \tag{4.18}$$

For the second norm in (4.17), we use Lemma 2.5 again:

$$\|M_{\zeta_\nu-(z_i)_\nu} m_{z_i} f_i\| \le \|M_{(\zeta_\nu-(z_i)_\nu) m_{z_i}}\| \le 3n\sqrt{1-|z_i|^2} = 3nt. \tag{4.19}$$

Bringing (4.18) and (4.19) into (4.17), we obtain

$$|\langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle| \le C_1 \left(\frac{t^2}{|1-\langle \xi_i, \xi_j\rangle|}\right)^{n+3} t^2,$$

where $C_1 = 9n^2(96)^{n+3}$. For any pair of i, j such that $d(\xi_i, \xi_j) \ge 2^k t$, the above gives us

$$|\langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle| \le \frac{C_1}{2^{2k(n+3)}} t^2. \tag{4.20}$$

Set

$$\ell(k) = \min\{\ell \in \mathbf{N} : \ell \ge C_2 2^{2nk}\},$$

where $C_2 = 2^{5n} A_0$. Then, by (4.11),

$$\mathrm{card}\{j \in J : d(\xi_i, \xi_j) < 2^{k+1} t\} \le \ell(k). \tag{4.21}$$

According to Lemma 4.1, we can decompose

$$\mathcal{E}^{(k)} = \{(i,j) \in J \times J : 2^k t \le d(\xi_i, \xi_j) < 2^{k+1} t\}$$

as the union of pairwise disjoint subsets

$$\mathcal{E}^{(k)}_1, ..., \mathcal{E}^{(k)}_{2\ell(k)}$$

such that for each $m \in \{1, ..., 2\ell(k)\}$, if $(i,j), (i',j') \in \mathcal{E}^{(k)}_m$ and if $(i,j) \ne (i',j')$, then we have both $i \ne i'$ and $j \ne j'$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$Y_k = Y_{k,1} + ... + Y_{k,2\ell(k)}, \tag{4.22}$$

where

$$Y_{k,m} = \sum_{(i,j)\in\mathcal{E}_m^{(k)}} \langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle e_i \otimes e_j,$$

$1 \le m \le 2\ell(k)$.

By the property of $\mathcal{E}_m^{(k)}$ and (4.20), we have

$$\begin{aligned}\|Y_{k,m}\|_{p/2}^{p/2} &= \sum_{(i,j)\in\mathcal{E}_m^{(k)}} |\langle M_{\zeta_\nu-(z_j)_\nu} v_{z_j} f_j, M_{\zeta_\nu-(z_i)_\nu} v_{z_i} f_i\rangle|^{p/2} \\ &\le \left(\frac{C_1}{2^{2k(n+3)}}t^2\right)^{p/2} \cdot \mathrm{card}(J) \le \left(\frac{C_1}{2^{2k(n+3)}}\right)^{p/2} \cdot t^p \cdot 2^n t^{-2n} = \frac{C_3}{2^{pk(n+3)}} t^{p-2n},\end{aligned}$$

where $C_3 = {C_1}^{p/2} 2^n$. Setting $C_4 = C_3^{2/p}$, the above yields

$$\|Y_{k,m}\|_{p/2} \le \frac{C_4}{2^{2k(n+3)}} t^{2(1-(2n/p))}$$

for each $m \in \{1, ..., 2\ell(k)\}$. Recalling (4.21) and (4.22), we now have

$$\|Y_k\|_{p/2} \le \frac{C_4}{2^{2k(n+3)}} t^{2(1-(2n/p))} \cdot 2(1 + C_2 2^{2nk}) \le \frac{2C_4(1+C_2)}{2^{6k}} t^{2(1-(2n/p))}.$$

Combining this with (4.16) and (4.15), we see that

$$\|E_\nu^* E_\nu\|_{p/2} \le \left(C + \sum_{k=0}^{\infty} \frac{2C_4(1+C_2)}{2^{6k}}\right) t^{2(1-(2n/p))}.$$

Since $\|E_\nu\|_p = \|E_\nu^* E_\nu\|_{p/2}^{1/2}$, this completes the proof. □

The last lemma of this section is about operator norm.

Lemma 4.4. *There is a $C_{4.4}$ which depends only on n such that the following estimate holds: Suppose that $0 < t < 1$ and that $\{\xi_j : j \in J\}$ is a subset of S satisfying the condition*

$$B(\xi_i, t) \cap B(\xi_j, t) = \emptyset \qquad \text{for all } \ i \ne j.$$

Define $z_j = (1-t^2)^{1/2}\xi_j$, $j \in J$. Let $\{f_j : j \in J\}$ be a set of vectors in H_n^2 with norm at most 1, and let $\{e_j : j \in J\}$ be an orthonormal set. Then the operator

$$E = \sum_{j\in J} (v_{z_j} f_j) \otimes e_j$$

satisfies the estimate $\|E\| \le C_{4.4}$.

Proof. It suffices to estimate $\|E^*E\|$. We have

$$E^*E = \sum_{i,j\in J} \langle v_{z_j} f_j, v_{z_i} f_i\rangle e_i \otimes e_j = B + \sum_{k=0}^{\infty} Y_k, \tag{4.23}$$

where

$$B = \sum_{j\in J} \|v_{z_j} f_j\|^2 e_j \otimes e_j$$

and

$$Y_k = \sum_{2^k t \le d(\xi_i, \xi_j) < 2^{k+1} t} \langle v_{z_j} f_j, v_{z_i} f_i \rangle e_i \otimes e_j,$$

$k \in \mathbf{Z}_+$. By Lemma 2.1 and (4.2), $\|M_{v_z}\| \le (1 + |z|)^{n+4} \le 2^{n+4}$ for each $z \in \mathbf{B}$. Since $\|f_j\| \le 1$, we have $\|v_{z_j} f_j\| \le 2^{n+4}$, $j \in J$. Since $\{e_j : j \in J\}$ is an orthonormal set, we conclude that

$$\|B\| \le 4^{n+4}. \tag{4.24}$$

Next we estimate $\|Y_k\|$. For each $k \in \mathbf{Z}_+$, define

$$\mathcal{E}^{(k)} = \{(i, j) \in J \times J : 2^k t \le d(\xi_i, \xi_j) < 2^{k+1} t\}.$$

Now, since $\|f_j\| \le 1$ and $\|f_i\| \le 1$, from Corollary 2.2 we obtain

$$|\langle v_{z_j} f_j, v_{z_i} f_i \rangle| = |\langle M^*_{v_{z_i}} M_{v_{z_j}} f_j, f_i \rangle| \le \|M^*_{v_{z_i}} M_{v_{z_j}}\| \le \|M_{v_{z_i}} M_{v_{z_j}}\| = \|M^{n+4}_{m_{z_i} m_{z_j}}\|.$$

For each $(i, j) \in \mathcal{E}^{(k)}$, it follows from Lemma 2.4 and the condition $d(\xi_i, \xi_j) \ge 2^k t$ that

$$\|M^{n+4}_{m_{z_i} m_{z_j}}\| \le \left(48 \frac{1 - |z_i|^2}{|1 - \langle z_i, z_j \rangle|}\right)^{n+4} \le \left(96 \frac{1 - |z_i|^2}{|1 - \langle \xi_i, \xi_j \rangle|}\right)^{n+4} \le \frac{C_1}{2^{2k(n+4)}},$$

where $C_1 = (96)^{n+4}$. Hence

$$|\langle v_{z_j} f_j, v_{z_i} f_i \rangle| \le \frac{C_1}{2^{2k(n+4)}} \quad \text{for each} \quad (i, j) \in \mathcal{E}^{(k)}. \tag{4.25}$$

Set $\ell(k) = \min\{\ell \in \mathbf{N} : \ell \ge C_2 2^{2nk}\}$ as before, where $C_2 = 2^{5n} A_0$. Then, by (4.11),

$$\text{card}\{j \in J : d(\xi_i, \xi_j) < 2^{k+1} t\} \le \ell(k).$$

According to Lemma 4.1, we can decompose $\mathcal{E}^{(k)}$ as the union of pairwise disjoint subsets

$$\mathcal{E}^{(k)}_1, \ldots, \mathcal{E}^{(k)}_{2\ell(k)}$$

such that for each $m \in \{1, \ldots, 2\ell(k)\}$, if $(i, j), (i', j') \in \mathcal{E}^{(k)}_m$ and if $(i, j) \ne (i', j')$, then we have both $i \ne i'$ and $j \ne j'$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$Y_k = Y_{k,1} + \ldots + Y_{k,2\ell(k)}, \tag{4.26}$$

where

$$Y_{k,m} = \sum_{(i,j) \in \mathcal{E}^{(k)}_m} \langle v_{z_j} f_j, v_{z_i} f_i \rangle e_i \otimes e_j,$$

$1 \le m \le 2\ell(k)$. By the property of $\mathcal{E}^{(k)}_m$ and (4.25), we have

$$\|Y_{k,m}\| \le \frac{C_1}{2^{2k(n+4)}}$$

for each $m \in \{1, \ldots, 2\ell(k)\}$. By (4.26) and the definition of $\ell(k)$,

$$\|Y_k\| \le \frac{C_1}{2^{2k(n+4)}} \cdot 2\ell(k) \le \frac{C_1}{2^{2k(n+4)}} \cdot 2(C_2 + 1) 2^{2nk} = \frac{2C_1(C_2 + 1)}{2^{8k}}.$$

Combining this estimate with (4.23) and (4.24), we see that if we set

$$C_{4.4} = \left\{ 4^{n+4} + 2C_1(C_2 + 1) \sum_{k=0}^{\infty} \frac{1}{2^{8k}} \right\}^{1/2},$$

then $\|E\| \le C_{4.4}$.

□

5. Spherical Decomposition

Before we get to the proof of Theorem 1.1, we want to recall an elementary fact:

Lemma 5.1. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose that $\{\mathcal{X}, \mu\}$ is a measure space and that A is a weakly measurable $\mathcal{B}(\mathcal{H})$-valued function on $\mathcal{X}$. If $A(x) \in \mathcal{C}_p$ for every x, $1 < p < \infty$, then*

$$\left\| \int_{\mathcal{X}} A(x) d\mu(x) \right\|_p \le \int_{\mathcal{X}} \|A(x)\|_p d\mu(x).$$

This lemma follows easily from the duality between $\mathcal{C}_p$ and $\mathcal{C}_{p/(p-1)}$. We omit the details.

Proof of Theorem 1.1. Recall that for each integer $N \ge n/2$, Theorem 3.1 provides an operator

$$R_N = \int \psi_{z,N} \otimes \psi_{z,N} d\lambda(z)$$

which is both bounded and invertible on H^2_n. We will only use the case where $N = n + 4$. That is, for the rest of the section, we will denote

$$R = R_{n+4}.$$

Similarly, we write

$$\psi_z = \psi_{z,n+4}.$$

This gives us the relation

$$\psi_z = v_z k_z, \tag{5.1}$$

where v_z was given in (4.2). Next we express R in a slightly different form, a form which is more convenient for subsequent estimates. Since

$$R = \int_0^1 2nr^{2n-1} \int \psi_{r\xi} \otimes \psi_{r\xi} d\sigma(\xi) \frac{dr}{(1-r^2)^{n+1}},$$

making the substitution $t = (1-r^2)^{1/2}$, we have

$$R = \int_0^1 2n(1-t^2)^{n-1} T_t \frac{dt}{t}, \tag{5.2}$$

where

$$T_t = \frac{1}{t^{2n}} \int \psi_{(1-t^2)^{1/2}\xi} \otimes \psi_{(1-t^2)^{1/2}\xi} d\sigma(\xi), \tag{5.3}$$

$0 < t < 1$. We then decompose each T_t, which involves spherical decomposition.

Let a $0 < t < 1$ be given. Then there is a subset $\{x_1, \dots, x_{m(t)}\}$ of S which is *maximal* with respect to the property

$$B(x_i, t/2) \cap B(x_j, t/2) = \emptyset \qquad \text{whenever} \quad i \ne j.$$

The maximality implies that

$$\bigcup_{j=1}^{m(t)} B(x_j, t) = S.$$

There are Borel sets $G_1, \dots, G_{m(t)}$ in S such that

$$\begin{aligned} &G_j \subset B(x_j, t) \quad \text{for each} \quad j \in \{1, \dots, m(t)\}, \\ &G_i \cap G_j = \emptyset \quad \text{whenever} \quad i \neq j, \quad \text{and} \\ &\cup_{j=1}^{m(t)} G_j = S. \end{aligned} \tag{5.4}$$

For any i, j, if $B(x_i, 2t) \cap B(x_j, 2t) \neq \emptyset$, then $d(x_i, x_j) < 4t$, which implies $B(x_j, t/2) \subset B(x_i, 5t)$. It follows that for each $i \in \{1, \dots, m(t)\}$,

$$\begin{aligned} &\operatorname{card}\{j : 1 \leq j \leq m(t),\ B(x_i, 2t) \cap B(x_j, 2t) \neq \emptyset\} \\ &\qquad \leq \frac{\sigma(B(x_i, 5t))}{\sigma(B(x_i, t/2))} \leq \frac{A_0(5t)^{2n}}{2^{-n}(t/2)^{2n}} = 2^{3n}5^{2n}A_0 = C_1. \end{aligned} \tag{5.5}$$

Let L be the smallest integer which is greater than C_1. Then we have the decomposition

$$\{1, \dots, m(t)\} = J_1 \cup \cdots \cup J_L,$$

where $J_1, \dots, J_L$ are pairwise disjoint and, for each $1 \leq \ell \leq L$, J_ℓ has the property that

$$B(x_i, 2t) \cap B(x_j, 2t) = \emptyset \quad \text{if} \quad i, j \in J_\ell \quad \text{and} \quad i \neq j. \tag{5.6}$$

The J_ℓ's are obtained through a well-known method. One starts with a maximal subset J_1 of $\{1, \dots, m(t)\}$ which has property (5.6). If $\{1, \dots, m(t)\}\backslash J_1 \neq \emptyset$, one similarly picks a maximal subset J_2 of $\{1, \dots, m(t)\}\backslash J_1$, and so on. The maximality of each J_ℓ and (5.5) ensure that this process stops after at most L steps.

There exist an $x \in S$ and unitary transformations $U_1, \dots, U_{m(t)}$ on $\mathbf{C}^n$ such that $x_j = U_j x$ for $j = 1, \dots, m(t)$. Then we can write

$$\frac{1}{t^{2n}} = \frac{C(t)}{\sigma(B(x, t))}, \quad \text{where} \quad C(t) \leq A_0$$

by (4.1). Therefore by (5.3) and (5.4),

$$\begin{aligned} T_t &= \frac{C(t)}{\sigma(B(x, t))} \sum_{j=1}^{m(t)} \int_{B(x_j, t)} \chi_{G_j}(\xi) \psi_{(1-t^2)^{1/2}\xi} \otimes \psi_{(1-t^2)^{1/2}\xi} d\sigma(\xi) \\ &= \frac{C(t)}{\sigma(B(x, t))} \int_{B(x, t)} \sum_{j=1}^{m(t)} \chi_{G_j}(U_j\xi) \psi_{(1-t^2)^{1/2}U_j\xi} \otimes \psi_{(1-t^2)^{1/2}U_j\xi} d\sigma(\xi) \\ &= \frac{C(t)}{\sigma(B(x, t))} \int_{B(x, t)} \sum_{\ell=1}^{L} Y_\ell(\xi) d\sigma(\xi), \end{aligned} \tag{5.7}$$

where

$$Y_\ell(\xi) = \sum_{j \in J_\ell} \chi_{G_j}(U_j\xi) \psi_{(1-t^2)^{1/2}U_j\xi} \otimes \psi_{(1-t^2)^{1/2}U_j\xi}.$$

If $\xi \in B(x, t)$, then $U_j\xi \in B(x_j, t)$. Therefore by (5.6), for each $\xi \in B(x, t)$ we have

$$B(U_i\xi, t) \cap B(U_j\xi, t) = \emptyset \quad \text{if} \quad i, j \in J_\ell \quad \text{and} \quad i \neq j. \tag{5.8}$$

To ease the notation, let us denote

$$z_j(\xi) = (1 - t^2)^{1/2} U_j \xi \tag{5.9}$$

for $j = 1, \dots, m(t)$ and $\xi \in B(x,t)$. Thus

$$Y_\ell(\xi) = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)\psi_{z_j(\xi)} \otimes \psi_{z_j(\xi)}. \tag{5.10}$$

Now let a multiplier f of H^2_n be given. Then by (5.1),

$$M_f Y_\ell(\xi) = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(f\psi_{z_j(\xi)}) \otimes \psi_{z_j(\xi)} = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} f_{z_j(\xi)}) \otimes \psi_{z_j(\xi)},$$

where $f_{z_j(\xi)} = f k_{z_j(\xi)}$. We have $\|f_{z_j(\xi)}\| \le \|M_f\|$. Let a $\nu \in \{1, \dots, n\}$ be given. Then

$$\begin{aligned}[M^*_{\zeta_\nu}, M_f Y_\ell(\xi)] &= \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(M^*_{\zeta_\nu - (z_j(\xi))_\nu} v_{z_j(\xi)} f_{z_j(\xi)}) \otimes \psi_{z_j(\xi)} \\ &\quad - \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} f_{z_j(\xi)}) \otimes (M_{\zeta_\nu - (z_j(\xi))_\nu} \psi_{z_j(\xi)}),\end{aligned} \tag{5.11}$$

where $(z_j(\xi))_\nu$ denotes the ν-th component of $z_j(\xi)$. Let $2n < p < \infty$ also be given. We will estimate the Schatten p-norm of the above two terms.

Let $\{e_j : j \in J_\ell\}$ be an orthonormal set. We have

$$\sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(M^*_{\zeta_\nu - (z_j(\xi))_\nu} v_{z_j(\xi)} f_{z_j(\xi)}) \otimes \psi_{z_j(\xi)} = E_\nu A^*,$$

where

$$E_\nu = \sum_{j\in J_\ell} (M^*_{\zeta_\nu - (z_j(\xi))_\nu} v_{z_j(\xi)} f_{z_j(\xi)}) \otimes e_j$$

and

$$A = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)\psi_{z_j(\xi)} \otimes e_j = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} k_{z_j(\xi)}) \otimes e_j.$$

Conditions (5.8) and (5.9) enable us to apply the lemmas in Section 4 here. By Lemma 4.2, we have $\|E_\nu\|_p \le C_{4.2}(p) t^{1-(2n/p)} \|M_f\|$. On the other hand, Lemma 4.4 tells us $\|A\| \le C_{4.4}$. Therefore

$$\left\| \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(M^*_{\zeta_\nu - (z_j(\xi))_\nu} v_{z_j(\xi)} f_{z_j(\xi)}) \otimes \psi_{z_j(\xi)} \right\|_p \le C_{4.4} C_{4.2}(p) \|M_f\| t^{1-(2n/p)}. \tag{5.12}$$

Similarly,

$$\sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} f_{z_j(\xi)}) \otimes (M_{\zeta_\nu - (z_j(\xi))_\nu} \psi_{z_j(\xi)}) = B F^*_\nu,$$

where

$$F_\nu = \sum_{j\in J_\ell} (M_{\zeta_\nu - (z_j(\xi))_\nu} \psi_{z_j(\xi)}) \otimes e_j = \sum_{j\in J_\ell} (M_{\zeta_\nu - (z_j(\xi))_\nu} v_{z_j(\xi)} k_{z_j(\xi)}) \otimes e_j$$

and

$$B = \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} f_{z_j(\xi)}) \otimes e_j.$$

By Lemma 4.3, $\|F_\nu\|_p \le C_{4.3}(p) t^{1-(2n/p)}$. By Lemma 4.4, $\|B\| \le C_{4.4}\|M_f\|$. Therefore

$$\left\| \sum_{j\in J_\ell} \chi_{G_j}(U_j\xi)(v_{z_j(\xi)} f_{z_j(\xi)}) \otimes (M_{\zeta_\nu - (z_j(\xi))_\nu} \psi_{z_j(\xi)}) \right\|_p \le C_{4.4} C_{4.3}(p) \|M_f\| t^{1-(2n/p)}. \tag{5.13}$$

Let $C_2 = C_{4.4}C_{4.2}(p) + C_{4.4}C_{4.3}(p)$. Then, combining (5.11), (5.12) and (5.13), we have

$$\|[M_{\zeta_\nu}^*, M_f Y_\ell(\xi)]\|_p \le C_2\|M_f\|t^{1-(2n/p)}.$$

Thus

$$\left\|\left[M_{\zeta_\nu}^*, M_f \sum_{\ell=1}^{L} Y_\ell(\xi)\right]\right\|_p \le C_2 L\|M_f\|t^{1-(2n/p)}.$$

Recalling (5.7) and the fact that $C(t) \le A_0$, and using Lemma 5.1, we obtain

$$\|[M_{\zeta_\nu}^*, M_f T_t]\|_p = \frac{C(t)}{\sigma(B(x,t))}\left\|\int_{B(x,t)}\left[M_{\zeta_\nu}^*, M_f \sum_{\ell=1}^{L} Y_\ell(\xi)\right]d\sigma(\xi)\right\|_p \le A_0 C_2 L\|M_f\|t^{1-(2n/p)}.$$

Recalling (5.2) and using Lemma 5.1 again, we find that

$$\begin{aligned}\|[M_{\zeta_\nu}^*, M_f R]\|_p &= \left\|\int_0^1 2n(1-t^2)^{n-1}[M_{\zeta_\nu}^*, M_f T_t]\frac{dt}{t}\right\|_p \\ &\le \int_0^1 2n(1-t^2)^{n-1}\|[M_{\zeta_\nu}^*, M_f T_t]\|_p\frac{dt}{t} \\ &\le A_0 C_2 L\|M_f\|\int_0^1 2n(1-t^2)^{n-1}t^{1-(2n/p)}\frac{dt}{t} = C(n,p)\|M_f\|. \end{aligned} \tag{5.14}$$

Note that the condition $p > 2n$ ensures $C(n,p) < \infty$. The above in particular implies

$$\|[M_{\zeta_\nu}^*, R]\|_p \le C(n,p). \tag{5.15}$$

Getting to the commutator that we are interested in, we have

$$[M_{\zeta_\nu}^*, M_f] = [M_{\zeta_\nu}^*, M_f RR^{-1}] = [M_{\zeta_\nu}^*, M_f R]R^{-1} + M_f[R, M_{\zeta_\nu}^*]R^{-1}.$$

Now it follows from (5.14) and (5.15) that

$$\|[M_{\zeta_\nu}^*, M_f]\|_p \le C(n,p)\|M_f\|\|R^{-1}\| + \|M_f\|C(n,p)\|R^{-1}\| = 2C(n,p)\|M_f\|\|R^{-1}\|.$$

Since Theorem 3.1 asserts that $\|R^{-1}\| < \infty$, this completes the proof of Theorem 1.1. □

6. Localization

Let us recall Arveson's exact sequence (1.1), in particular the homomorphism τ. According to Theorem 5.7 in [3],

$$\tau(M_{\zeta_j}) = \zeta_j \tag{6.1}$$

for each $j \in \{1, \ldots, n\}$. Define the quotient C^*-algebras

$$\hat{\mathcal{T}}_n = \mathcal{T}_n/\mathcal{K} \quad \text{and} \quad \widehat{\mathcal{TM}_n} = \mathcal{TM}_n/\mathcal{K}.$$

Let $\hat{\tau} : \hat{\mathcal{T}}_n \to C(S)$ be the isomorphism induced by τ. Thus S is the maximal ideal space of $\hat{\mathcal{T}}_n$. Theorem 1.1 asserts that $\hat{\mathcal{T}}_n$ is contained in the center of $\widehat{\mathcal{TM}_n}$.

For each $\xi \in S$, let $\hat{\mathcal{I}}_\xi$ be the ideal in $\widehat{\mathcal{TM}_n}$ generated by

$$\{b \in \hat{\mathcal{T}}_n : \hat{\tau}(b)(\xi) = 0\}. \tag{6.2}$$

By Douglas' localization theorem (see Theorem 7.47 in [8]), we have

$$\bigcap_{\xi\in S}\hat{\mathcal{I}}_\xi = \{0\}.$$

An elementary C^*-algebraic argument then yields

$$\|a\| = \sup_{\xi\in S}\|a+\hat{\mathcal{I}}_\xi\| \tag{6.3}$$

for every $a \in \widehat{\mathcal{TM}_n}$. Let $\pi : \mathcal{TM}_n \to \widehat{\mathcal{TM}_n}$ be the quotient map. For each $\xi \in S$, let $\mathcal{I}_\xi$ be the inverse image of $\hat{\mathcal{I}}_\xi$ under π. Since $\mathcal{TM}_n \supset \mathcal{K}$ and since $\mathcal{I}_\xi \neq \{0\}$, we have $\mathcal{I}_\xi \supset \mathcal{K}$. By (6.2), $\mathcal{I}_\xi$ is the ideal in $\mathcal{TM}_n$ generated by

$$\{B \in \mathcal{T}_n : \tau(B)(\xi) = 0\}. \tag{6.4}$$

It follows from (6.3) that

$$\|A\|_{\mathcal{Q}} = \sup_{\xi\in S}\|A+\mathcal{I}_\xi\| \tag{6.5}$$

for every $A \in \mathcal{TM}_n$.

Lemma 6.1. *Let $\xi \in S$. Then the linear span of operators of the form*

$$TM_{\zeta_j-\xi_j} + K,$$

where $j \in \{1,\dots,n\}$, ξ_j is the j-th component of ξ, $T \in \mathcal{TM}_n$, and $K \in \mathcal{K}$, is dense in $\mathcal{I}_\xi$.

Proof. Let $Z \in \mathcal{I}_\xi$ and $\epsilon > 0$ be given. By (6.4), there are $B_1,\dots,B_m \in \mathcal{T}_n$ with $\tau(B_1)(\xi) = \cdots = \tau(B_m)(\xi) = 0$ and $X_1,\dots,X_m,Y_1,\dots,Y_m \in \mathcal{TM}_n$ such that

$$\|Z - (X_1B_1Y_1 + \cdots + X_mB_mY_m)\| \le \epsilon.$$

But, by Theorem 1.1, there is a $K_1 \in \mathcal{K}$ such that

$$X_1B_1Y_1 + \cdots + X_mB_mY_m = X_1Y_1B_1 + \cdots + X_mY_mB_m + K_1.$$

Therefore

$$\|Z - (X_1Y_1B_1 + \cdots + X_mY_mB_m + K_1)\| \le \epsilon. \tag{6.6}$$

Let $B \in \mathcal{T}_n$ be such that $\tau(B)(\xi) = 0$. Then $\tau(B)$ lies in the ideal in $C(S)$ generated by $\zeta_1-\xi_1,\dots,\zeta_n-\xi_n$. Let $\delta > 0$ be given. By (6.1), there exist $T_1,\dots,T_n \in \mathcal{T}_n$ and a $K \in \mathcal{K}$ such that

$$\|B - (T_1M_{\zeta_1-\xi_1} + \cdots + T_nM_{\zeta_n-\xi_n} + K)\| \le \delta. \tag{6.7}$$

The conclusion of the lemma follows from (6.6) and (6.7). □

Proposition 6.2. *For every $A \in \mathcal{TM}_n$ and every $\xi \in S$, we have*

$$\lim_{r\uparrow 1}\|AM_{s_{r\xi}}\| = \|A+\mathcal{I}_\xi\|.$$

Proof. Let $\xi \in S$ be given. We first show that

$$\lim_{r\uparrow 1} \|WM_{s_{r\xi}}\| = 0 \tag{6.8}$$

for every $W \in \mathcal{I}_\xi$. Applying Lemma 2.5, we have

$$\lim_{r\uparrow 1} \|M_{(\zeta_j-\xi_j)}M_{s_{r\xi}}\| = \lim_{r\uparrow 1} \|M_{(\zeta_j-\xi_j)s_{r\xi}}\| \le \lim_{r\uparrow 1}\{\|M_{(\zeta_j-r\xi_j)s_{r\xi}}\| + |r\xi_j - \xi_j|\|M_{s_{r\xi}}\|\} = 0 \tag{6.9}$$

for each $j \in \{1, \dots, n\}$, where ξ_j is the j-th component of ξ. Let K be a compact operator. Then Corollary 2.2 gives us $\|KM_{s_{r\xi}}\| = \|M^*_{s_{r\xi}}K^*\| \le \|M_{s_{r\xi}}K^*\|$. Since K^* is also compact, it follows from Lemma 2.6 that

$$\lim_{r\uparrow 1} \|KM_{s_{r\xi}}\| \le \lim_{r\uparrow 1} \|M_{s_{r\xi}}K^*\| = 0. \tag{6.10}$$

Combining (6.9), (6.10) and Lemma 6.1, (6.8) is proved.

Let $A \in \mathcal{TM}_n$ be given. Then by (6.8), for every $W \in \mathcal{I}_\xi$ we have

$$\limsup_{r\uparrow 1} \|AM_{s_{r\xi}}\| = \limsup_{r\uparrow 1} \|(A+W)M_{s_{r\xi}}\| \le \|A+W\|.$$

Since this holds for every $W \in \mathcal{I}_\xi$, it follows that

$$\limsup_{r\uparrow 1} \|AM_{s_{r\xi}}\| \le \|A + \mathcal{I}_\xi\|. \tag{6.11}$$

Next we show that

$$\|AM_{s_{r\xi}}\| \ge \|A + \mathcal{I}_\xi\| \tag{6.12}$$

for every $0 < r < 1$. Note that, since $|\xi| = 1$,

$$1 - s_{r\xi}(\zeta) = 1 - \frac{1-r}{1-r\langle\zeta,\xi\rangle} = \frac{r\langle\xi-\zeta,\xi\rangle}{1-r\langle\zeta,\xi\rangle}. \tag{6.13}$$

This and (6.1) together imply $1 - M_{s_{r\xi}} \in \mathcal{I}_\xi$. Thus $A - AM_{s_{r\xi}} \in \mathcal{I}_\xi$, which clearly implies (6.12). The proposition follows from (6.11) and (6.12). □

Proof of Theorem 1.2. It follows immediately from Proposition 6.2 and (6.5). □

Proof of Theorem 1.3. Let $A \in \mathcal{TM}_n$ be given. Then we obviously have

$$\sup_{\xi\in S}\lim_{r\uparrow 1} \|AM_{s_{r\xi}}\| \le \lim_{r\uparrow 1} \sup_{r\le|z|<1} \|AM_{s_z}\|.$$

By Theorem 1.2, the proof of Theorem 1.3 is reduced to the proof of the inequality

$$\lim_{r\uparrow 1} \sup_{r\le|z|<1} \|AM_{s_z}\| \le \|A\|_{\mathcal{Q}}. \tag{6.14}$$

Let K be a compact operator. By the subnormality of M_{s_z} and Lemma 2.6, we have

$$\lim_{|z|\uparrow 1} \|KM_{s_z}\| = \lim_{|z|\uparrow 1} \|M^*_{s_z}K^*\| \le \lim_{|z|\uparrow 1} \|M_{s_z}K^*\| = 0.$$

Therefore for each compact operator K we have

$$\lim_{r\uparrow 1} \sup_{r\le|z|<1} \|AM_{s_z}\| = \lim_{r\uparrow 1} \sup_{r\le|z|<1} \|(A+K)M_{s_z}\| \le \|A+K\|.$$

This clearly implies (6.14). Hence the theorem follows. □

Remark 1. There is a "left version" for Theorems 1.2 and 1.3. That is, if we replace $AM_{s_{r\xi}}$ by $M_{s_{r\xi}}A$ in Theorem 1.2 (resp. AM_{s_z} by $M_{s_z}A$ in Theorem 1.3), then the same statement holds. The point is that by Lemma 6.1 and Theorem 1.1, the linear span of operators of the form $M_{\zeta_j-\xi_j}T+K$ is also dense in $\mathcal{I}_\xi$. Using this "left version" of Lemma 6.1, the proof of the left version of Theorems 1.2 and 1.3 is the same as the right version.

Remark 2. Theorems 1.2 and 1.3 are better suited for application to the problem of determining compactness than their left version.

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