BERGMAN COMMUTATORS AND NORM IDEALS

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Abstract. Let P be the orthogonal projection from $L^2(\mathbf{B}, dv)$ onto the Bergman space $L_a^2(\mathbf{B}, dv)$ of the unit ball in $\mathbf{C}^n$. In this paper we characterize the membership of commutators of the form $[M_f, P]$ in the norm ideal $\mathcal{C}_\Phi$, where the symmetric gauge function Φ is allowed to be arbitrary.

1. Introduction

Let $\mathbf{B}$ be the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$ and let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. Recall that the Bergman space $L_a^2(\mathbf{B}, dv)$ is the subspace

$$\{h \in L^2(\mathbf{B}, dv) : h \text{ is analytic on } \mathbf{B}\}$$

of $L^2(\mathbf{B}, dv)$. Note that the symbol $A^2(\mathbf{B})$ is also used for the Bergman space by some authors, but we prefer the notation $L_a^2(\mathbf{B}, dv)$. Let

$$P : L^2(\mathbf{B}, dv) \to L_a^2(\mathbf{B}, dv)$$

be the orthogonal projection. Often, this P is called the Bergman projection. The main result of this paper is a characterization of the membership of $[M_f, P]$ in norm ideals, where M_f denotes the operator of multiplication by the function f. This is a problem with a long and well-known history. Before stating our result, it is necessary to recall the relevant definitions and other background information.

Closely related to the commutator $[M_f, P]$ is the Hankel operator $H_f = (1-P)M_fP$. There is a vast literature on Hankel operators of various kinds, of which we cite [1-4,6,7,9-12,15,17] as a small sample. Because of the well-known relation

$$[M_f, P] = H_f - H_{\bar{f}}^*,$$

the study of the commutator $[M_f, P]$ is equivalent to the so-called "two-sided" theory of Hankel operators, i.e., the simultaneous study of the pair H_f and $H_{\bar{f}}$.

Of particular relevance to this paper are references [15,17]. In [17], Zhu determined the membership of $[M_f, P]$ in the Schatten class $\mathcal{C}_p$ for $2 \le p < \infty$. Later, the author extended the result to the case $2n/(n+1) < p < 2$ [15]. But as it was explained in [15], the *same kind* of characterization cannot be extended to the case $p \le 2n/(n+1)$. The reason

2010 *Mathematics Subject Classification.* Primary 47B10, 47B32, 47B35.
Keywords. Bergman projection, commutator, norm ideal.

for the lower limit $p > 2n/(n+1)$ is that in [15,17], the condition for the membership $[M_f, P] \in \mathcal{C}_p$ involved mean oscillation with respect to the normalized reproducing kernel

$$k_z(w) = \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle w, z\rangle)^{n+1}}, \quad z, w \in \mathbf{B}.$$

More precisely, the condition in [15,17] was stated in terms of the numerical quantity

$$\mathrm{MO}(f)(z) = \|(f - \langle f k_z, k_z\rangle)k_z\|.$$

As it turns out, what is responsible for the lower limit $p > 2n/(n+1)$ in [15] is the fact that the kernel k_z simply "does not decay sufficiently fast". The meaning of the words in the quotation marks will become clear later when we present modified kernel which does "decay sufficiently fast" for our purpose.

On the other hand, obviously there are nonzero commutators in $\mathcal{C}_{2n/(n+1)}$ and in smaller ideals when $n \geq 2$. For example, if f is a bounded measurable function on $\mathbf{B}$ which vanishes on $\mathbf{B}\backslash\{z : |z| \leq r\}$ for some $0 < r < 1$, then the commutator $[M_f, P]$ obviously belongs to the trace class $\mathcal{C}_1$. Thus it is not vacuous to ask, how does one characterize the membership $[M_f, P] \in \mathcal{C}_p$ for $1 \leq p \leq 2n/(n+1)$?

Moreover, Schatten classes are just some of the most familiar examples of a much larger class of operator ideals called norm ideals. Given the discussion above, it is also a legitimate question to ask, how does one characterize the membership $[M_f, P]$ in an arbitrary norm ideal? What makes this question look promising is the fact that the analogous problem has been solved in the setting of the Fock space of the Gaussian measure on $\mathbf{C}^n$ [6]. Of course, each reproducing-kernel Hilbert space has its own peculiarities. The fact that a result can be established on one reproducing-kernel Hilbert space is no guarantee that its analogue can be proved on another.

Before we discuss the membership of $[M_f, P]$ in general norm ideals, let us recall the relevant definitions. We will use [8] as our standard reference for norm ideals, although the term *norm ideal* itself is due to Schatten [14]. Let $\hat{c}$ be the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for each sequence $a_j \neq 0$ only for a finite number of j's. A *symmetric gauge function* (also called *symmetric norming function*) [8,page 71] is a map

$$\Phi : \hat{c} \to [0, \infty)$$

which has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \ldots, 0, \ldots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

Given a bounded operator A, we write $s_1(A), \ldots, s_k(A), \ldots$ for its s-numbers [8,Section II.7] as usual. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{k\geq 1} \Phi(\{s_1(A), \ldots, s_k(A), 0, \ldots, 0, \ldots\})$$

for operators. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\}$$

is a norm ideal [8,page 68]. This term refers to the following properties of $\mathcal{C}_\Phi$:

- For any B, $C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \le \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \le \|A\|_\Phi$, and the equality holds when $\mathrm{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|.\|_\Phi$.

It will be convenient to adopt the convention that for each unbounded operator X, we simply set $\|X\|_\Phi = \infty$.

For each $1 \le p < \infty$, if we define the symmetric gauge function

$$\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{1/p},$$

$\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, then the norm ideal $\mathcal{C}_{\Phi_p}$ is just the Schatten class $\mathcal{C}_p$.

For each $1 \le p < \infty$, we can define the symmetric gauge function $\Phi_p^+ : \hat{c} \to [0,\infty)$ as follows. For each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, let

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{k\ge 1} \frac{|a_{\pi(1)}| + \cdots + |a_{\pi(k)}|}{1^{-1/p} + \cdots + k^{-1/p}},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \ge |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$, which exists because $a_j = 0$ for all but a finite number of j's. Then usually the norm ideal $\mathcal{C}_{\Phi_p^+}$ is simply denoted by the symbol $\mathcal{C}_p^+$. For each $1 < p < \infty$, the ideal $\mathcal{C}_p^+$ is the dual of a certain Lorentz ideal [8,Section III.15]. It is well known that $\mathcal{C}_p \subset \mathcal{C}_p^+$ and that $\mathcal{C}_p \ne \mathcal{C}_p^+$. Another interesting fact is that, as a Banach space, $\mathcal{C}_p^+$ is not separable. It was shown in [4] that $\mathcal{C}_{2n}^+$ plays the role of "critical ideal" in the theory of Hankel operators on the Hardy space of the unit sphere.

The symmetric gauge functions Φ_p and Φ_p^+ are just some of the most familiar examples. For more examples of symmetric gauge functions, see [8]. But these less or more exotic examples aside, this paper only requires a rudimentary knowledge of symmetric gauge functions. In fact, the whole point of the paper is about obtaining estimates from bare definitions.

Given a symmetric gauge Φ, it is a common practice to extend its domain of definition beyond the space $\hat{c}$. Suppose that $\{b_j\}_{j\in\mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., the set $\{j \in \mathbf{N} : b_j \ne 0\}$ is not required to be finite. Then we define

$$\Phi(\{b_j\}_{j\in\mathbf{N}}) = \sup_{k\ge 1} \Phi(\{b_1, \ldots, b_k, 0, \ldots, 0, \ldots\}).$$

For our purpose we also need to deal with sequences indexed by sets other than $\mathbf{N}$. If W is a countable, infinite set, then we define

$$\Phi(\{b_\alpha\}_{\alpha\in W}) = \Phi(\{b_{\pi(j)}\}_{j\in\mathbf{N}}),$$

where $\pi : \mathbf{N} \to W$ is any bijection. The definition of symmetric gauge functions guarantees that the value of $\Phi(\{b_\alpha\}_{\alpha\in W})$ is independent of the choice of the bijection π. To be thorough, let us also mention the case of finite sequences. For a finite index set $F = \{x_1, \dots, x_\ell\}$, we define

$$\Phi(\{b_x\}_{x\in F}) = \Phi(\{b_{x_1}, \dots, b_{x_\ell}, 0, \dots, 0, \dots\}).$$

The main interest of the paper is the following. Suppose that a symmetric gauge function Φ is given. Then for which $f \in L^2(\mathbf{B}, dv)$ does one have the membership $[M_f, P] \in \mathcal{C}_\Phi$? As we explained above, for a general Φ, one cannot hope to characterize the membership $[M_f, P] \in \mathcal{C}_\Phi$ in terms of the normalized reproducing kernel k_z. What we need is a modified version of k_z, a version that "decays faster". This is based an idea that was first used in the study of Hankel operators on the Hardy space [4,Section 3]. Later, this idea again found success in a commutator problem on the Drury-Arveson space [5]. Therefore it is natural to try it here in the setting of Bergman space.

Let $z \in \mathbf{B}$. Following [4], we define the Schur multiplier function

$$m_z(w) = \frac{1-|z|}{1-\langle w, z\rangle}, \quad w \in \mathbf{B}. \tag{1.1}$$

For each integer $i \geq 0$, we define the function

$$\psi_{z,i}(w) = \frac{(1-|z|^2)^{\{(n+1)/2\}+i}}{(1-\langle w, z\rangle)^{n+1+i}}, \quad w \in \mathbf{B}. \tag{1.2}$$

Then we have the relation

$$\psi_{z,i} = (1+|z|)^i m_z^i k_z.$$

The factor $(1+|z|)^i$ is unimportant but it arises from Proposition 4.1 below in a natural way. What is important above is the factor m_z^i, which gives $\psi_{z,i}$ a faster decaying rate than k_z. But if $i \geq 1$, then $\psi_{z,i}$ is not a unit vector. Thus we need to normalized it. Define

$$\tilde{\psi}_{z,i} = \frac{\psi_{z,i}}{\|\psi_{z,i}\|}.$$

The reason for normalizing $\psi_{z,i}$ is that for each $f \in L^2(\mathbf{B}, dv)$, we have

$$\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\| = \inf_{\alpha\in\mathbf{C}} \|(f-\alpha)\psi_{z,i}\|.$$

That is, $\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|$ is the mean oscillation of f with respect to the modified kernel function $\psi_{z,i}$. This is the quantity that will replace $\mathrm{MO}(f)(z) = \|(f - \langle fk_z, k_z\rangle)k_z\|$.

To state our result, we also need to recall the notion of lattice in $\mathbf{B}$, which is defined in terms of the Bergman metric. For each $z \in \mathbf{B}\backslash\{0\}$, we have the Möbius transform

$$\varphi_z(w) = \frac{1}{1-\langle w,z\rangle}\left\{z - \frac{\langle w,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(w - \frac{\langle w,z\rangle}{|z|^2}z\right)\right\}$$

of the unit ball $\mathbf{B}$. Recall that each φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \text{id}$ [13,Theorem 2.2.2]. Also, we define $\varphi_0(w) = -w$. It is well known that the formula

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$$

defines a metric on $\mathbf{B}$. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball

$$D(z,a) = \{w \in \mathbf{B} : \beta(z,w) < a\}.$$

Definition 1.1. (i) Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) Let $0 < a < b < \infty$. A subset Γ of $\mathbf{B}$ is said to be an a,b-lattice if it is a-separated *and* has the property $\cup_{z\in\Gamma}D(z,b) = \mathbf{B}$.

The simplest example of such a lattice is the following. Take any positive number $0 < a < \infty$, and then take any subset M of $\mathbf{B}$ which is *maximal* with respect to the property of being a-separated. Then obviously M is an $a,2a$-lattice in $\mathbf{B}$.

We can now state the main result of the paper.

Theorem 1.2. *Let $0 < a < b < \infty$ be positive numbers such that $b \geq 2a$. Let integer $i \geq 6n+1$ also be given. Then there exist constants $0 < c \leq C < \infty$ which depend only on a, b, i and the complex dimension n such that the inequality*

$$c\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}) \leq \|[M_f,P]\|_\Phi \leq C\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})$$

holds for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every a,b-lattice Γ in $\mathbf{B}$.

The rest of the paper is taken up by the very long proof of this theorem. We would characterize the proof of Theorem 1.2 as being extremely laborious, but not difficult. In other words, most of the work is a matter of verifying details, while the idea behind the proof is actually very simple. Let us give a brief outline of our approach here.

First of all, the reader will find that our approach is fundamentally different from past works on $[M_f,P]$. Our actual estimates of $\|\cdot\|_\Phi$ at the most basic level involve nothing but the definition of Φ-norm. As it turns out, the estimate of $\|[M_f,P]\|_\Phi$ can be ultimately reduced to the estimates of the Φ-norms of *families* of operators of the simple form

$$A = \sum_{j\in\mathbf{N}} a_j x_j \otimes y_j, \tag{1.3}$$

where $\{x_j : j \in \mathbf{N}\}$ and $\{y_j : j \in \mathbf{N}\}$ are orthonormal sets. For such an A, if Φ is a symmetric gauge function and if $0 < s \le 1$, then we have

$$\||A|^s\|_\Phi = \Phi(\{|a_j|^s\}_{j\in\mathbf{N}}), \tag{1.4}$$

where $|A|$ denotes $(A^*A)^{1/2}$, the absolute value of A. But there are numerous steps involved in the reduction process from $[M_f, P]$ to (1.3), which is why the paper is so long.

The first step of reduction involves what we call a quasi-resolution of the Bergman projection P. This is an idea that was first introduced and used successfully in the settings of the Hardy space [4] and the Drury-Arveson space [5]. In the Bergman space setting, it works in the following way. For an appropriate natural number i' we consider the operator

$$R_{i'} = \int \psi_{z,i'} \otimes \psi_{z,i'} d\lambda(z),$$

where $d\lambda$ is the standard Möbius-invariant measure on $\mathbf{B}$. Then there are scalars $0 < c \le C < \infty$ such that the operator inequality $cP \le R_{i'} \le CP$ holds on the Hilbert space $L^2(\mathbf{B}, dv)$. By the Riesz functional calculus, this reduces the estimate of $\|[M_f, P]\|_\Phi$ to that of $\|[M_f, R_{i'}]\|_\Phi$. Using Möbius transform, we can rewrite the operator $R_{i'}$ as

$$R_{i'} = \int_{D(0,2)} T_\zeta d\lambda(\zeta),$$

reducing the estimate of $\|[M_f, R_{i'}]\|_\Phi$ to that of $\|[M_f, T_\zeta]\|_\Phi$, where each T_ζ is a "discrete sum":

$$T_\zeta = \sum_{z\in G} \chi_{E_z}(\zeta)\psi_{\varphi_z(\zeta),i'} \otimes \psi_{\varphi_z(\zeta),i'}.$$

Thus the modified kernel $\psi_{z,i'}$ enters the estimate of $\|[M_f, P]\|_\Phi$. Using the obvious cancellation property of the commutator $[M_f, T_\zeta]$, the estimate of $\|[M_f, T_\zeta]\|_\Phi$ is further reduced to that of the Φ-norm of operators of the form

$$X = \sum_{z\in F} c_z((f - f_{z,i})\psi_{z,i'}) \otimes \psi_{z,i'},$$

where c_z and $f_{z,i}$ are scalars, and F has certain well-defined properties. This, of course, is still quite far from (1.3), for the vectors on the right-hand side lack orthogonality. The next step is to pick any orthonormal set $\{e_z : z \in F\}$ and factor X as $X = X_1X_2^*$, where

$$X_1 = \sum_{z\in F} c_z((f - f_{z,i})\psi_{z,i'}) \otimes e_z \quad \text{and} \quad X_2 = \sum_{z\in F} \psi_{z,i'} \otimes e_z.$$

Then $\|X\|_\Phi \le \|X_1\|_\Phi \|X_2^*\|$. As it turns out, the operator norm $\|X_2\|$ is quite easy to handle. Our main problem is $\|X_1\|_\Phi$. To estimate $\|X_1\|_\Phi$, we start with the identity

$$X_1^*X_1 = \sum_{z,w\in F} c_z\bar{c}_w \langle (f - f_{z,i})\psi_{z,i'}, (f - f_{w,i})\psi_{w,i'}\rangle e_w \otimes e_z.$$

The terms on the right-hand side must then be grouped according to the separation between z and w. In other words, $X_1^*X_1$ needs to be further decomposed. But if one goes through the necessary steps, formula (1.4) can be brought to bear to give us the desired result.

What makes all of this work is the following simple inequality: For z and w satisfying the conditions specified in Lemma 3.5, we have

$$|\langle g\psi_{z,i'}, g\psi_{w,i'}\rangle| \le C_{3.5}2^{-(n+1)\ell}2^{-2im}\|g\psi_{w,i}\|^2,$$

$g \in L^2(\mathbf{B}, dv)$. Here ℓ and m measure the separation of z and w in the radial direction and the spherical direction respectively. This inequality should be read as follows. In the spherical direction, we can achieve arbitrarily fast decaying rate simply by penciling in a large i. In other words, the spherical direction is the good direction. But the decay in the radial direction, i.e., the factor $2^{-(n+1)\ell}$, is unaffected by the value of i. One can think of the factor $2^{-(n+1)\ell}$ as an intrinsic property of the Bergman space that cannot be artificially improved. So the radial direction is the bad direction. Fortunately, the inherent decay $2^{-(n+1)\ell}$ in the radial direction together with fast decay in the spherical direction are sufficient for Theorem 1.2, and our proof grew out of this simple observation.

Although Theorem 1.2 is stated in terms of commutators and norm ideals, it is really a result about the structure of the Bergman space $L_a^2(\mathbf{B}, dv)$. What this paper really shows is that there is enough "almost orthogonality" in the Bergman space to permit a proof of Theorem 1.2.

To conclude the Introduction, let us describe how the paper is organized, which is different from the order of reduction steps described above. Section 2 deals with the radial-spherical decomposition of $\mathbf{B}$ and the equivalence of various mean oscillations. The purpose of Section 3 is to establish the Φ-norm estimate given in Lemma 3.9. The proof of Lemma 3.9 is long because there are several decompositions involved. Unfortunately, there does not appear to be any sensible way to divide Lemma 3.9 into shorter pieces. In Section 4 we introduce the above-mentioned quasi-resolution for the Bergman projection. In Section 5 we combine the results from the preceding sections to establish the upper bound in Theorem 1.2. In Section 6 we prove the lower bound in Theorem 1.2, which is much easier than the upper bound. Finally, in Section 7 we establish an alternate version of Theorem 1.2, a version where the membership $[M_f, P] \in \mathcal{C}_\Phi$ is characterized by the mean oscillations of f over subsets of $\mathbf{B}$.

2. Various Mean Oscillations

As it turns out, the metric β is convenient for the purpose of stating our result, but it is not very useful in many of our proofs. This is mainly due to the disparity between the radial direction and the spherical direction mentioned in the Introduction. Instead of β, our proofs rely much more on the familiar radial-spherical decomposition of $\mathbf{B}$, which we now recall.

Let S denote the unit sphere $\{\xi \in \mathbf{C}^n : |\xi| = 1\}$. Recall that the formula

$$d(u, \xi) = |1 - \langle u, \xi\rangle|^{1/2}, \quad u, \xi \in S, \tag{2.1}$$

defines a metric on S [13,page 66]. Throughout the paper, we denote

$$B(u,r) = \{\xi \in S : |1 - \langle u, \xi \rangle|^{1/2} < r\}$$

for $u \in S$ and $r > 0$. Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. We take the usual normalization $\sigma(S) = 1$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \le \sigma(B(u,r)) \le A_0 r^{2n} \tag{2.2}$$

for all $u \in S$ and $0 < r \le \sqrt{2}$ [13,Proposition 5.1.4]. Note that the upper bound actually holds when $r > \sqrt{2}$.

For each integer $k \ge 0$, let $\{u_{k,1}, \dots, u_{k,m(k)}\}$ be a subset of S which is *maximal* with respect to the property

$$B(u_{k,j}, 2^{-k-1}) \cap B(u_{k,j'}, 2^{-k-1}) = \emptyset \quad \text{for all} \quad 1 \le j < j' \le m(k). \tag{2.3}$$

The maximality of $\{u_{k,1}, \dots, u_{k,m(k)}\}$ implies that

$$\cup_{j=1}^{m(k)} B(u_{k,j}, 2^{-k}) = S. \tag{2.4}$$

For each pair of $k \ge 0$ and $1 \le j \le m(k)$, define the subsets

$$T_{k,j} = \{ru : 1 - 2^{-2k} \le r < 1 - 2^{-2(k+1)}, u \in B(u_{k,j}, 2^{-k})\} \quad \text{and} \tag{2.5}$$

$$Q_{k,j} = \{ru : 1 - 2^{-2k} \le r < 1 - 2^{-2(k+2)}, u \in B(u_{k,j}, 9 \cdot 2^{-k})\} \tag{2.6}$$

of $\mathbf{B}$. Let us also introduce the index set

$$I = \{(k,j) : k \ge 0, 1 \le j \le m(k)\}.$$

For each $(k,j) \in I$, we define the subset

$$F_{k,j} = \{(\ell, i) : \ell > k, 1 \le i \le m(\ell), B(u_{\ell,i}, 2^{-\ell}) \cap B(u_{k,j}, 3 \cdot 2^{-k}) \ne \emptyset\}$$

of I. We then define

$$W_{k,j} = Q_{k,j} \cup \{\cup_{(\ell,i) \in F_{k,j}} Q_{\ell,i}\}. \tag{2.7}$$

Obviously, $W_{k,j} \supset \{ru : 1 - 2^{-2k} \le r < 1, u \in B(u_{k,j}, 3 \cdot 2^{-k})\}$.

As usual, for a Borel subset E of $\mathbf{B}$ with $v(E) > 0$ and for each $f \in L^2(\mathbf{B}, dv)$, we write f_E for the mean value of f on E, i.e., $f_E = (1/v(E)) \int_E f dv$. Furthermore, denote

$$V(f; E) = \frac{1}{v(E)} \int_E |f - f_E|^2 dv,$$

which is the variance of f over the set E. Accordingly, $V^{1/2}(f;E)$ is the standard deviation of f over E. Suppose that E and F are Borel sets in $\mathbf{B}$ such that $v(E\cap F)>0$. Then

$$|f_E - f_F| \le \frac{v(E)}{v(E\cap F)}V^{1/2}(f;E) + \frac{v(F)}{v(E\cap F)}V^{1/2}(f;F). \tag{2.8}$$

See inequality (3.3) in [15].

It is elementary that if c is a complex number with $|c|\le 1$ and if $0\le\rho\le 1$, then

$$2|1-\rho c| \ge |1-c|.$$

This inequality will be used frequently in the paper.

Lemma 2.1. *Given any integer $i\ge 0$, there exists a constant $0<C_{2.1}<\infty$ which depends only on i and n such that for each $(k,j)\in I$ and each $f\in L^2(\mathbf{B},dv)$, the inequality*

$$\|(f-\langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\| \le C_{2.1}\sum_{(t,h)\in H_{k,j}} V^{1/2}(f;W_{t,h})2^{-(n+2i)(k-t)}$$

holds whenever $z\in T_{k,j}$, where

$$H_{k,j} = \{(t,h) : 0\le t\le k, 1\le h\le m(t), B(u_{t,h},2^{-t})\cap B(u_{k,j},2^{-k})\ne\emptyset\}.$$

Proof. Let $(k,j)\in I$ and $z\in T_{k,j}$ be given. Then $z=|z|\xi$ for some $\xi\in S$. By (2.4), for each $0\le\ell\le k$, there is a $\nu(\ell)\in\{1,\dots,m(\ell)\}$ such that $\xi\in B(u_{\ell,\nu(\ell)},2^{-\ell})$. We stipulate that $\nu(k)=j$, which is allowed because $z\in T_{k,j}$. We claim that the inequality

$$|\psi_{z,i}|^2 \le C_1\sum_{\ell=0}^{k} 2^{-2(n+1+2i)(k-\ell)}\frac{1}{v(W_{\ell,\nu(\ell)})}\chi_{W_{\ell,\nu(\ell)}} \tag{2.9}$$

holds on $\mathbf{B}$, where C_1 depends only on n and i.

First of all, by (2.7) we have $W_{0,\nu(0)}=\mathbf{B}$. Suppose that $w\in W_{\ell-1,\nu(\ell-1)}\backslash W_{\ell,\nu(\ell)}$ and let us estimate the value of $|1-\langle w,z\rangle|$. Since $w\notin W_{\ell,\nu(\ell)}$, there are two possibilities. Either $|w|\le 1-2^{-2\ell}$, in which case we have $|1-\langle w,z\rangle|\ge 1-|w|\ge 2^{-2\ell}$. Or $w/|w|\notin B(u_{\ell,\nu(\ell)},3\cdot 2^{-\ell})$, in which case we have $d(w/|w|,\xi)>2\cdot 2^{-\ell}$ since $\xi\in B(u_{\ell,\nu(\ell)},2^{-\ell})$ by the choice of $\nu(\ell)$. In the latter case, $|1-\langle w,z\rangle|\ge(1/2)|1-\langle w/|w|,\xi\rangle|\ge 2\cdot 2^{-2\ell}$. Thus we have shown that if $w\in W_{\ell-1,\nu(\ell-1)}\backslash W_{\ell,\nu(\ell)}$, then $|1-\langle w,z\rangle|^{-1}\le 4\cdot 2^{2(\ell-1)}$. On the other hand, the definition of $T_{k,j}$ gives us $1-|z|\le 2^{-2k}$. By (2.7), (2.2) and the formula $dv=2nr^{2n-1}drd\sigma$, we have $v(W_{\ell-1,\nu(\ell-1)})\le C2^{-2(n+1)(\ell-1)}$. Combining these three inequalities, we see that (2.9) holds on $\mathbf{B}\backslash W_{k,j}$. But on the set $W_{k,j}$, (2.9) follows from the simple fact that $|1-\langle w,z\rangle|\ge 1-|z|\ge 2^{-2(k+1)}=(1/4)2^{-2k}$ since $z\in T_{k,j}$. Thus (2.9) holds on $\mathbf{B}$.

Let $f \in L^2(\mathbf{B}, dv)$ also be given. Then it follows from (2.9) that

$$\begin{aligned}
&\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|^2 \le \|(f - f_{W_{k,j}})\psi_{z,i}\|^2 \\
(2.10) \qquad &\le C_1 \sum_{\ell=0}^{k} 2^{-2(n+1+2i)(k-\ell)} \frac{1}{v(W_{\ell,\nu(\ell)})} \int_{W_{\ell,\nu(\ell)}} |f - f_{W_{k,j}}|^2 dv.
\end{aligned}$$

For $\ell < k$, since $\xi \in B(u_{\ell,\nu(\ell)}, 2^{-\ell}) \cap B(u_{\ell+1,\nu(\ell+1)}, 2^{-\ell-1})$, we have $B(u_{\ell,\nu(\ell)}, 3 \cdot 2^{-\ell}) \supset B(u_{\ell+1,\nu(\ell+1)}, 2^{-\ell-1})$. Therefore $W_{\ell,\nu(\ell)} \cap W_{\ell+1,\nu(\ell+1)} \supset T_{\ell+1,\nu(\ell+1)}$. Hence

$$\frac{v(W_{\ell,\nu(\ell)})}{v(W_{\ell,\nu(\ell)} \cap W_{\ell+1,\nu(\ell+1)})} \le C_2 \quad \text{and} \quad \frac{v(W_{\ell+1,\nu(\ell+1)})}{v(W_{\ell,\nu(\ell)} \cap W_{\ell+1,\nu(\ell+1)})} \le C_2.$$

Applying (2.8), we have

$$|f_{W_{\ell,\nu(\ell)}} - f_{W_{\ell+1,\nu(\ell+1)}}| \le C_2(V^{1/2}(f; W_{\ell,\nu(\ell)}) + V^{1/2}(f; W_{\ell+1,\nu(\ell+1)})).$$

Thus for every $\ell < k$ we have

$$\begin{aligned}
|f - f_{W_{k,j}}| &\le |f - f_{W_{\ell,\nu(\ell)}}| + \sum_{t=\ell}^{k-1} |f_{W_{t,\nu(t)}} - f_{W_{t+1,\nu(t+1)}}| \\
&\le |f - f_{W_{\ell,\nu(\ell)}}| + 2C_2 \sum_{t=\ell}^{k} V^{1/2}(f; W_{t,\nu(t)}).
\end{aligned}$$

Squaring both sides and then applying the Cauchy-Schwarz inequality, we find that

$$|f - f_{W_{k,j}}|^2 \le (2 + k - \ell) \left(|f - f_{W_{\ell,\nu(\ell)}}|^2 + 4C_2^2 \sum_{t=\ell}^{k} V(f; W_{t,\nu(t)}) \right).$$

Substituting the above in (2.10), we see that

$$\begin{aligned}
\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|^2 &\le C_1(1 + 4C_2^2) \sum_{\ell=0}^{k} 2^{-2(n+1+2i)(k-\ell)} (2 + k - \ell) \sum_{t=\ell}^{k} V(f; W_{t,\nu(t)}) \\
&= C_1(1 + 4C_2^2) \sum_{t=0}^{k} V(f; W_{t,\nu(t)}) \sum_{\ell=0}^{t} 2^{-2(n+1+2i)(k-\ell)} (2 + k - \ell) \\
&= C_1(1 + 4C_2^2) \sum_{t=0}^{k} V(f; W_{t,\nu(t)}) \sum_{m=k-t}^{k} 2^{-2(n+1+2i)m} (2 + m) \\
&\le C_1(1 + 4C_2^2) \sum_{t=0}^{k} V(f; W_{t,\nu(t)}) 2^{-2(n+2i)(k-t)} \sum_{m=k-t}^{\infty} 2^{-2m} (2 + m).
\end{aligned}$$

That is,

$$\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|^2 \le C_3 \sum_{t=0}^{k} V(f; W_{t,\nu(t)})2^{-2(n+2i)(k-t)}.$$

By choice, $\xi \in B(u_{t,\nu(t)}, 2^{-t})$ for each $0 \le t \le k$, and $\nu(k) = j$. Thus $(t, \nu(t)) \in H_{k,j}$ for every $0 \le t \le k$. Hence the above implies

$$\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|^2 \le C_3 \sum_{(t,h)\in H_{k,j}} V(f; W_{t,h})2^{-2(n+2i)(k-t)}.$$

Taking the square-root on both sides, the lemma now follows from the elementary fact that if $b_1 \ge 0, \dots, b_m \ge 0$ and $0 < s \le 1$, then $(b_1 + \cdots + b_m)^s \le b_1^s + \cdots + b_m^s$. $\square$

Lemma 2.2. *Suppose that X and Y are countable sets and that N is a natural number. Suppose that $T : X \to Y$ is a map that is at most N-to-1. That is, for every $y \in Y$, $\text{card}\{x \in X : T(x) = y\} \le N$. Then for every set of real numbers $\{b_y\}_{y\in Y}$ and every symmetric gauge function Φ, we have*

$$\Phi(\{b_{T(x)}\}_{x\in X}) \le N\Phi(\{b_y\}_{y\in Y}).$$

Proof. Since T is at most N-to-1, we can decompose X as the union of pairwise disjoint subsets $X_1, \dots, X_N$ such that for each $1 \le j \le N$, the restricted map $T : X_j \to Y$ is injective. The injectivity implies $\Phi(\{b_{T(x)}\}_{x\in X_j}) \le \Phi(\{b_y\}_{y\in Y})$ [8,page 71] for each j. For each j, define $a_x^{(j)} = b_{T(x)}$ for $x \in X_j$ and $a_x^{(j)} = 0$ for $x \in X\backslash X_j$. Then it is obvious that $\Phi(\{a_x^{(j)}\}_{x\in X}) = \Phi(\{b_{T(x)}\}_{x\in X_j})$. Hence

$$\begin{aligned}\Phi(\{b_{T(x)}\}_{x\in X}) &\le \Phi(\{a_x^{(1)}\}_{x\in X}) + \cdots + \Phi(\{a_x^{(N)}\}_{x\in X}) \\ &= \Phi(\{b_{T(x)}\}_{x\in X_1}) + \cdots + \Phi(\{b_{T(x)}\}_{x\in X_N}) \le N\Phi(\{b_y\}_{y\in Y})\end{aligned}$$

as promised. $\square$

Lemma 2.3. *Given any integer $i > n/2$, there exists a constant $0 < C_{2.3} < \infty$ which depends only on i and n such that the following estimate holds: Let $z(k,j) \in T_{k,j}$ for each $(k,j) \in I$. Then for each $f \in L^2(\mathbf{B}, dv)$ and each symmetric gauge function Φ, we have*

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{z(k,j),i}, \tilde{\psi}_{z(k,j),i}\rangle)\psi_{z(k,j),i}\|\}_{(k,j)\in I}) \le C_{2.3}\Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}).$$

Proof. Let $H_{k,j}$ be the set given in Lemma 2.1. For each non-negative integer $\ell \le k$, we further defined the set

$$H_{k,j}^{(\ell)} = \{(\ell, h) : (\ell, h) \in H_{k,j}\}.$$

Let us first show that there is a natural number M such that the inequality

$$\text{card}(H_{k,j}^{(\ell)}) \le M \tag{2.11}$$

for all integers $0 \le \ell \le k$ and $1 \le j \le m(k)$. Indeed if $(\ell, h), (\ell, h') \in H^{(\ell)}_{k,j}$, then

$$B(u_{\ell,h}, 2^{-\ell}) \cap B(u_{k,j}, 2^{-k}) \neq \emptyset \quad \text{and} \quad B(u_{\ell,h'}, 2^{-\ell}) \cap B(u_{k,j}, 2^{-k}) \neq \emptyset$$

by definition. Since $k \ge \ell$, we conclude that $d(u_{\ell,h}, u_{\ell,h'}) \le 4 \cdot 2^{-\ell}$. By (2.3) and (2.2), this clearly implies (2.11).

Let $f \in L^2(\mathbf{B}, dv)$ be given. For each triple of integers $0 \le \ell \le k$ and $1 \le j \le m(k)$, there is an element $(\ell, h(k,j;\ell)) \in H^{(\ell)}_{k,j}$ such that

$$V(f; W_{\ell,h(k,j;\ell)}) \ge V(f; W_{\ell,h}) \quad \text{for every} \quad (\ell, h) \in H^{(\ell)}_{k,j}.$$

Let $z(k,j) \in T_{k,j}$, $(k,j) \in I$, also be given. By Lemma 2.1 and (2.11), we have

$$\begin{aligned}\|(f - \langle f\tilde{\psi}_{z(k,j),i}, \tilde{\psi}_{z(k,j),i}\rangle)\psi_{z(k,j),i}\| &\le C_{2.1} M \sum_{\ell=0}^{k} V^{1/2}(f; W_{\ell,h(k,j;\ell)}) 2^{-(n+2i)(k-\ell)} \\ &= C_{2.1} M \sum_{\nu=0}^{k} V^{1/2}(f; W_{k-\nu,h(k,j;k-\nu)}) 2^{-(n+2i)\nu}\end{aligned}$$

for each $(k,j) \in I$. Thus if we define

$$\eta_{k,j;\nu} = \begin{cases} V^{1/2}(f; W_{k-\nu,h(k,j;k-\nu)}) & \text{if} \quad \nu \le k \\ 0 & \text{if} \quad \nu > k \end{cases}$$

for all $(k,j) \in I$ and all $\nu \ge 0$, then

$$\|(f - \langle f\tilde{\psi}_{z(k,j),i}, \tilde{\psi}_{z(k,j),i}\rangle)\psi_{z(k,j),i}\| \le C_{2.1} M \sum_{\nu=0}^{\infty} \eta_{k,j;\nu} 2^{-(n+2i)\nu}.$$

Consequently, writing $C_1 = C_{2.1}M$, for each symmetric gauge function Φ we have
(2.12)

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{z(k,j),i}, \tilde{\psi}_{z(k,j),i}\rangle)\psi_{z(k,j),i}\|\}_{(k,j)\in I}) \le C_1 \sum_{\nu=0}^{\infty} 2^{-(n+2i)\nu} \Phi(\{\eta_{k,j;\nu}\}_{(k,j)\in I}).$$

Since $\eta_{k,j;\nu} = 0$ whenever $k < \nu$, for each $\nu \ge 0$ we have

$$\Phi(\{\eta_{k,j;\nu}\}_{(k,j)\in I}) = \Phi(\{V^{1/2}(f; W_{k-\nu,h(k,j;k-\nu)})\}_{(k,j)\in I^{(\nu)}}),$$

where $I^{(\nu)} = \{(k,j) : k \ge \nu, 1 \le j \le m(k)\}$.

For each $\nu \ge 0$, consider the map $G_\nu : I^{(\nu)} \to I$ defined by the formula

$$G_\nu(k,j) = (k - \nu, h(k,j;k-\nu)), \quad (k,j) \in I^{(\nu)}.$$

If $k \neq k'$, then, of course, $G_\nu(k,j) \neq G_\nu(k',j')$ for all possible j and j'. Now suppose that integers j and j' are in the set $\{1,\dots,m(k)\}$ such that $G_\nu(k,j) = G_\nu(k,j')$. Then $h(k,j;k-\nu) = h(k,j';k-\nu)$. A chase of definitions gives us

$$\begin{aligned}&B(u_{k-\nu,h(k,j;k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j}, 2^{-k}) \neq \emptyset \quad \text{and}\\ &B(u_{k-\nu,h(k,j';k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j'}, 2^{-k}) \neq \emptyset.\end{aligned}$$

Since $h(k,j;k-\nu) = h(k,j';k-\nu)$, we have $d(u_{k,j}, u_{k,j'}) \leq 4 \cdot 2^{-(k-\nu)}$. Thus we conclude from (2.3) and (2.2) that there is a $C_2 \in \mathbf{N}$ which depends only on n such that for all $\nu \leq k$ and all $1 \leq j \leq m(k)$,

$$\operatorname{card}\{j' \in \{1,\dots,m(k)\} : G_\nu(k,j') = G_\nu(k,j)\} \leq C_2 2^{2n\nu}.$$

That is, the map $G_\nu : I^{(\nu)} \to I$ is at most $C_2 2^{2n\nu}$-to-1. Applying Lemma 2.2, we have

$$\begin{aligned}\Phi(\{\eta_{k,j;\nu}\}_{(k,j)\in I}) &= \Phi(\{V^{1/2}(f; W_{G_\nu(k,j)})\}_{(k,j)\in I^{(\nu)}})\\ &\leq C_2 2^{2n\nu} \Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

Substituting this in (2.12), we find that

$$\begin{aligned}&\Phi(\{\|(f - \langle f\tilde{\psi}_{z(k,j),i}, \tilde{\psi}_{z(k,j),i}\rangle)\psi_{z(k,j),i}\|\}_{(k,j)\in I})\\ &\qquad \leq C_1 C_2 \sum_{\nu=0}^{\infty} 2^{-(2i-n)\nu} \Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

The lemma now follows from the assumption $i > n/2$. □

For each $(k,j) \in I$, we define

$$w(k,j) = (1 - 2^{-2k})u_{k,j}, \tag{2.13}$$

which is an element in $T_{k,j}$. This notation will be fixed for the rest of the paper. As usual, let $d\lambda$ denote the standard Möbius-invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}. \tag{2.14}$$

Lemma 2.4. *Given any $0 < a < \infty$, there exists a natural number K which depends only on a and the complex dimension n such that the following holds true: Suppose that Γ is an a-separated subset of $\mathbf{B}$. Then there exist pairwise disjoint subsets $\Gamma_1,\dots,\Gamma_K$ of Γ such that $\cup_{\mu=1}^K \Gamma_\mu = \Gamma$ and such that $\operatorname{card}(\Gamma_\mu \cap T_{k,j}) \leq 1$ for all $\mu \in \{1,\dots,K\}$ and $(k,j) \in I$.*

Proof. By Theorem 2.2.2 in [13], we have

$$1 - |\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2}. \tag{2.15}$$

Using this formula, it is a routine exercise to show that there is a $0 < C < \infty$ such that $T_{k,j} \subset D(w(k,j), C)$ for each $(k,j) \in I$. Thus $\beta(w, w') < 2C$ for each pair $w, w' \in T_{k,j}$, $(k,j) \in I$. Given $0 < a < \infty$, let K be the smallest integer that is greater than $\lambda(D(0, 2C + a))/\lambda(D(0, a))$.

Suppose that Γ is an a-separated set in $\mathbf{B}$. Then the selection of subsets $\Gamma_1, \ldots, \Gamma_K$ is just a matter of applying the axiom of choice. Indeed one starts with any subset Γ_1 of Γ which is maximal with respect to the property that $\mathrm{card}(\Gamma_1 \cap T_{k,j}) \leq 1$ for each $(k,j) \in I$. Suppose that $1 \leq \mu < K$ and that we have defined the subsets $\Gamma_1, \ldots, \Gamma_\mu$. Then we pick a subset $\Gamma_{\mu+1}$ of $\Gamma \backslash \{\Gamma_1 \cup \cdots \cup \Gamma_\mu\}$ which is maximal with respect to the property that $\mathrm{card}(\Gamma_{\mu+1} \cap T_{k,j}) \leq 1$ for each $(k,j) \in I$. Thus we have inductively defined $\Gamma_1, \ldots, \Gamma_K$.

To complete the proof, we need to show that $\Gamma \backslash \{\Gamma_1 \cup \cdots \cup \Gamma_K\} = \emptyset$. Suppose that there were a $\hat{w} \in \Gamma \backslash \{\Gamma_1 \cup \cdots \cup \Gamma_K\}$. Then for each $1 \leq \mu \leq K$, the maximality of Γ_μ implies that there would be a $(k_\mu, j_\mu) \in I$ such that $\mathrm{card}((\{\hat{w}\} \cup \Gamma_\mu) \cap T_{k_\mu, j_\mu}) = 2$. This happens only if T_{k_μ, j_μ} contains both $\hat{w}$ and some $w_\mu \in \Gamma_\mu$. Since $\hat{w}, w_\mu \in T_{k_\mu, j_\mu}$, we have $\beta(\hat{w}, w_\mu) < 2C$. Thus $D(\hat{w}, 2C)$ contains $\hat{w}, w_1, \ldots, w_K$, $K+1$ distinct elements in Γ. On the other hand, since both β and λ are Möbius invariant, we have

$$\begin{aligned} \lambda(D(0, 2C + a)) &= \lambda(D(\hat{w}, 2C + a)) \\ &\geq \sum_{w \in \Gamma \cap D(\hat{w}, 2C)} \lambda(D(w, a)) = \mathrm{card}(\Gamma \cap D(\hat{w}, 2C))\lambda(D(0, a)). \end{aligned}$$

Hence $\mathrm{card}(\Gamma \cap D(\hat{w}, 2C)) \leq \lambda(D(0, 2C + a))/\lambda(D(0, a)) < K$. This contradicts the statement that $D(\hat{w}, 2C)$ contains at least $K + 1$ distinct elements in Γ. $\square$

Proposition 2.5. *Given $0 < a < \infty$ and integer $i > n/2$, there exists a constant $0 < C_{2.5} < \infty$ which depends only on a, i and n such that the inequality*

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}) \leq C_{2.5}\Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}) \tag{2.16}$$

holds for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every a-separated subset Γ of $\mathbf{B}$.

Proof. Given $0 < a < \infty$, let K be the natural number provided by Lemma 2.4. According to Lemma 2.4, each a-separated set Γ is the union of pairwise disjoint subsets $\Gamma_1, \ldots, \Gamma_K$ such that $\mathrm{card}(\Gamma_\mu \cap T_{k,j}) \leq 1$ for all $\mu \in \{1, \ldots, K\}$ and $(k,j) \in I$. Thus for each $f \in L^2(\mathbf{B}, dv)$, each symmetric gauge function Φ and each $\mu \in \{1, \ldots, K\}$, it follows from Lemma 2.3 that

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma_\mu}) \leq C_{2.3}\Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}).$$

Since $\Gamma_1 \cup \cdots \cup \Gamma_K = \Gamma$, for any set of real numbers $\{a_z\}_{z\in\Gamma}$ we have

$$\Phi(\{a_z\}_{z\in\Gamma}) \leq \Phi(\{a_z\}_{z\in\Gamma_1}) + \cdots + \Phi(\{a_z\}_{z\in\Gamma_K}).$$

Hence (2.16) holds for $C_{2.5} = KC_{2.3}$. $\square$

Lemma 2.6. *Given any positive number* $0 < b < \infty$ *and any integer* $i \geq 0$*, there is a constant* $C_{2.6}$ *which depends only on* b*,* i *and* n *such that if* $z \in \mathbf{B}$ *and* $(k,j) \in I$ *satisfy the condition* $w(k,j) \in D(z,b)$*, then*

$$V^{1/2}(f; W_{k,j}) \leq C_{2.6} \|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|$$

for every $f \in L^2(\mathbf{B}, dv)$*.*

Proof. Obviously, it suffices to show that there is a C such that the inequality

$$\frac{1}{v(W_{k,j})}\chi_{W_{k,j}} \leq C|\psi_{z,i}|^2 \tag{2.17}$$

holds on $\mathbf{B}$ whenever $(k,j) \in I$ and $z \in \mathbf{B}$ satisfy the condition $w(k,j) \in D(z,b)$. Since we know that $v(W_{k,j}) \geq v(T_{k,j}) \geq c2^{-2(n+1)k}$, (2.17) will follow if we can show that there are $0 < c_1 < \infty$ and $0 < C_2 < \infty$ such that for $(k,j) \in I$ and $z \in \mathbf{B}$ satisfying the condition $w(k,j) \in D(z,b)$, we have

$$1 - |z|^2 \geq c_1 2^{-2k} \quad \text{and} \quad |1 - \langle w, z\rangle| \leq C_2 2^{-2k} \quad \text{for each} \quad w \in W_{k,j}. \tag{2.18}$$

To prove this, suppose that $D(z,b)$ contains some $w(k,j)$. Suppose that $1 - |z|^2 < \epsilon 2^{-2k}$ for some $\epsilon > 0$. Then by (2.5) we have

$$\begin{aligned} 1 - |\varphi_{w(k,j)}(z)| \leq 1 - |\varphi_{w(k,j)}(z)|^2 &\leq \frac{(1 - |w(k,j)|^2) \cdot \epsilon 2^{-2k}}{|1 - \langle z, w(k,j)\rangle|^2} \\ &\leq \frac{2(1 - |w(k,j)|) \cdot \epsilon 2^{-2k}}{(1 - |w(k,j)|)^2} = 2\epsilon. \end{aligned}$$

Hence $b \geq \beta(w(k,j), z) \geq (1/2)\log\{(2\epsilon)^{-1}\}$. Solving this inequality, we find that $\epsilon \geq (1/2)e^{-2b}$. Therefore if we set $c_1 = (1/4)e^{-2b}$, then $1 - |z|^2 \geq c_1 2^{-2k}$.

To prove the other half of (2.18), we need an upper bound for $1 - |z|$. Note that $|1 - \langle z, w(k,j)\rangle| \geq 1 - |z|$. Using (2.15) again, we have

$$1 - |\varphi_{w(k,j)}(z)| \leq 1 - |\varphi_{w(k,j)}(z)|^2 \leq \frac{4(1 - |w(k,j)|)(1 - |z|)}{|1 - \langle z, w(k,j)\rangle|^2} \leq \frac{4 \cdot 2^{-2k}}{1 - |z|}.$$

Thus $b \geq (1/2)\log\{(1 - |z|)/(4 \cdot 2^{-2k})\}$, which implies $1 - |z| \leq 4e^{2b}2^{-2k} = C_3 2^{-2k}$.

Let us write $z = |z|\xi$, where $\xi \in S$. We need an upper bound for $d(u_{k,j}, \xi)$. Suppose that $|1 - \langle u_{k,j}, \xi\rangle| > A2^{-2k}$ for some $A > 0$. Then $2|1 - \langle w(k,j), z\rangle| \geq |1 - \langle u_{k,j}, \xi\rangle| \geq A2^{-2k}$. Another application of (2.15) now gives us

$$1 - |\varphi_{w(k,j)}(z)| \leq \frac{4(1 - |w(k,j)|)(1 - |z|)}{|1 - \langle z, w(k,j)\rangle|^2} \leq \frac{4 \cdot 2^{-2k} \cdot C_3 2^{-2k}}{((1/2)A2^{-2k})^2} = \frac{16C_3}{A^2}.$$

Hence $b \geq (1/2)\log\{A^2/(16C_3)\}$. That is, $A \leq 4C_3^{1/2}e^b$. Thus if we set $C_4 = 8C_3^{1/2}e^b$, then $|1 - \langle u_{k,j}, \xi\rangle| \leq C_4 2^{-2k}$. That is, $d(u_{k,j}, \xi) \leq C_4^{1/2}2^{-k}$.

Let $w \in W_{k,j}$ be given. Then by (2.7) we can write $w = |w|u$, where $u \in S$ satisfies the inequality $d(u, u_{k,j}) \leq 13 \cdot 2^{-k}$. Hence $d(u, \xi) \leq (13 + C_4^{1/2})2^{-k}$. Thus if we set $C_5 = (13 + C_4^{1/2})^2$, then $|1 - \langle u, \xi\rangle| \leq C_5 2^{-2k}$. With these estimates in hand, we now have

$$|1 - \langle w, z\rangle| \leq (1 - |w|) + (1 - |z|) + |1 - \langle u, \xi\rangle| \leq 2^{-2k} + C_3 2^{-2k} + C_5 2^{-2k}.$$

This proves the second half of (2.18) and completes the proof of the lemma. □

Lemma 2.7. *Given any $0 < b < \infty$, there is a natural number N such that for every $z \in$ $\mathbf{B}$, we have* $\operatorname{card}\{(k,j) \in I : w(k,j) \in D(z,b)\} \leq N$.

Proof. In the proof of Lemma 2.6 we showed that if $w(k,j) \in D(z,b)$, then $c_1 2^{-2k} \leq 1 - |z| \leq C_3 2^{-2k}$, where c_1 and C_3 depend only on b. In other words, there is an $m \in \mathbf{N}$ which depends only on b such that

$$2^{-2(k+m)} \leq 1 - |z| \leq 2^{-2(k-m)}$$

if $w(k,j) \in D(z,b)$. If $w(k',j')$ also belongs to $D(z,b)$, then $2^{-2(k+m)} \leq 1 - |z| \leq 2^{-2(k'-m)}$. Solving the inequality, we find that $k' \leq k + 2m$ if $w(k,j), w(k',j') \in D(z,a)$.

As in the previous proof, write $z = |z|\xi$, where $\xi \in S$. The previous proof tells us that $d(u_{k,j}, \xi) \leq C_4^{1/2}2^{-k}$ if $w(k,j) \in D(z,b)$. Hence if both $w(k,j)$ and $w(k,\nu)$ belong to $D(z,b)$, then $d(u_{k,j}, u_{k,\nu}) \leq 2C_4^{1/2}2^{-k}$. By (2.3) and (2.2), there is a N_1 which is determined by n and C_4 such that

$$\operatorname{card}\{j \in \{1, \ldots, m(k)\} : w(k,j) \in D(z,b)\} \leq N_1$$

for all $k \geq 0$ and $z \in \mathbf{B}$. Combining this with the conclusion of the preceding paragraph, we see that $\operatorname{card}\{(k,j) \in I : w(k,j) \in D(z,b)\} \leq (2m+1) \cdot N_1$. □

Proposition 2.8. *Given any positive number $0 < b < \infty$ and any integer $i \geq 0$, there is a constant $C_{2.8}$ which depends only on b, i and n such that if Γ is a countable subset of $\mathbf{B}$ with the property that $\cup_{z\in\Gamma} D(z,b) = \mathbf{B}$, then*

$$\Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}) \leq C_{2.8}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})$$

for every $f \in L^2(\mathbf{B}, dv)$ and every symmetric gauge function Φ.

Proof. Given b, let N be the natural number provided by Lemma 2.7. Suppose that Γ has the property that $\cup_{z\in\Gamma} D(z,b) = \mathbf{B}$. Then for each $(k,j) \in I$, pick a $\zeta(k,j) \in \Gamma$ such that $w(k,j) \in D(\zeta(k,j), b)$. By Lemma 2.6, for each $f \in L^2(\mathbf{B}, dv)$ and each symmetric gauge function Φ we have

$$\Phi(\{V^{1/2}(f; W_{k,j})\}_{(k,j)\in I}) \leq C_{2.6}\Phi(\{\|(f - \langle f\tilde{\psi}_{\zeta(k,j),i}, \tilde{\psi}_{\zeta(k,j),i}\rangle)\psi_{\zeta(k,j),i}\|\}_{(k,j)\in I}).$$

Lemma 2.7 tells us that the map $(k,j) \mapsto \zeta(k,j)$ is at most N-to-1. Thus, by Lemma 2.2,

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{\zeta(k,j),i}, \tilde{\psi}_{\zeta(k,j),i}\rangle)\psi_{\zeta(k,j),i}\|\}_{(k,j)\in I}) \le N\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

Hence the constant $C_{2.8} = NC_{2.6}$ suffices for the proposition. □

3. Estimates of Certain Φ-Norms

This section contains the key estimates in the proof of the upper bound in Theorem 1.2. We begin with a general fact about the norm $\|\cdot\|_\Phi$. Recall that for a bounded operator A, $|A|$ denotes $(A^*A)^{1/2}$, the absolute value of A.

Lemma 3.1. *Suppose that $A_1, \dots, A_m$ are finite-rank operators on a Hilbert space $\mathcal{H}$ and let $A = A_1 + \cdots + A_m$. Then for each symmetric gauge function Φ and each $0 < s \le 1$,*

$$\||A|^s\|_\Phi \le 2^{1-s}(\||A_1|^s\|_\Phi + \cdots + \||A_m|^s\|_\Phi).$$

Proof. First we consider the special case where we have $A_j \ge 0$ for each $1 \le j \le m$. Then $A \ge 0$. Assuming that $\dim(\mathcal{H}) = \infty$, we can express A in the form

$$A = \sum_{i=1}^{\infty} s_i(A) e_i \otimes e_i,$$

where $\{e_i : i \in \mathbf{N}\}$ is an orthonormal set in $\mathcal{H}$. Let $0 < s \le 1$ be given. Define

$$X = \sum_{s_i(A)\neq 0} \{s_i(A)\}^{(s-1)/2} e_i \otimes e_i.$$

Since $s_i(A) = 0$ for all but a finite number of i's, X is a bounded operator. We have

$$A^s = XAX = \sum_{j=1}^{m} XA_jX = \sum_{j=1}^{m} B_j^* A_j^s B_j,$$

where $B_j = A_j^{(1-s)/2}X$. Since $0 < s \le 1$, we have $0 \le 1 - s < 1$. Thus for each j, the operator inequality $A_j \le A$ implies $A_j^{1-s} \le A^{1-s}$. Hence for each $h \in \mathcal{H}$,

$$\|B_j h\|^2 = \|A_j^{(1-s)/2}Xh\|^2 = \langle XA_j^{1-s}Xh, h\rangle \le \langle XA^{1-s}Xh, h\rangle = \sum_{s_i(A)\neq 0} |\langle h, e_i\rangle|^2 \le \|h\|^2.$$

That is, $\|B_j\| \le 1$. Therefore for each symmetric gauge function Φ we have

$$\|A^s\|_\Phi \le \sum_{j=1}^{m} \|B_j^* A_j^s B_j\|_\Phi \le \sum_{j=1}^{m} \|B_j^*\|\|A_j^s\|_\Phi\|B_j\| \le \sum_{j=1}^{m} \|A_j^s\|_\Phi.$$

Next we consider the general case. Let $A_j = U_j|A_j|$, $1 \le j \le m$, and $A = U|A|$ be the respective polar decompositions. Then $U_1, \dots, U_m$ and U are partial isometries, and we have $U^*A = |A|$. Thus

$$|A| = U^*U_1|A_1| + \cdots + U^*U_m|A_m| = T_1^*|A_1|^{1/2} + \cdots + T_m^*|A_m|^{1/2},$$

where $T_j = |A_j|^{1/2}U_j^*U$, $1 \le j \le m$. For each $h \in \mathcal{H}$, we have

$$\begin{aligned}|\langle T_j^*|A_j|^{1/2}h, h\rangle| &= |\langle |A_j|^{1/2}h, T_jh\rangle| \le \||A_j|^{1/2}h\|\|T_jh\| \\ &\le \frac{1}{2}(\||A_j|^{1/2}h\|^2 + \|T_jh\|^2) = \frac{1}{2}\langle(|A_j| + T_j^*T_j)h, h\rangle.\end{aligned}$$

Thus if we set

$$\tilde{A} = \frac{1}{2}\sum_{j=1}^{m}(|A_j| + T_j^*T_j),$$

then the operator inequality $|A| \le \tilde{A}$ holds on $\mathcal{H}$. Hence $s_i(|A|) \le s_i(\tilde{A})$ for each $i \in \mathbf{N}$ [8,page 26], and consequently $\||A|^s\|_\Phi \le \|\tilde{A}^s\|_\Phi$. Applying the special case that we already proved to $\tilde{A}$, we have

$$\||A|^s\|_\Phi \le \|\tilde{A}^s\|_\Phi \le \frac{1}{2^s}\sum_{j=1}^{m}(\||A_j|^s\|_\Phi + \|(T_j^*T_j)^s\|_\Phi).$$

On the other hand, for each $1 \le j \le m$ we have $T_j^*T_j = U^*U_j|A_j|U_j^*U$. Therefore for each $i \in \mathbf{N}$ we have $s_i(T_j^*T_j) \le \|U^*U_j\|s_i(|A_j|)\|U_jU^*\| \le s_i(|A_j|)$ [8,page 27]. This implies the inequality $\|(T_j^*T_j)^s\|_\Phi \le \||A_j|^s\|_\Phi$. Substituting this in the above, the lemma is proved. $\square$

Having established the above general lemma, which will not be needed until Lemma 3.9, the rest of the section deals with estimates which are very specific to our setting.

Lemma 3.2. *For each integer $i \ge 0$, there exists a constant $C_{3.2}$ which depends only on i and n such that for each $(k, j) \in I$, if $z, w \in Q_{k,j}$, then the inequality*

$$|\psi_{z,i}| \le C_{3.2}|\psi_{w,i}|$$

holds on $\mathbf{B}$.

Proof. If $z, w \in Q_{k,j}$, $(k, j) \in I$, then $2^{-2(k+2)} < 1 - |z| \le 2^{-2k}$ and $2^{-2(k+2)} < 1 - |w| \le 2^{-2k}$. Thus It suffices to find an absolute constant C such that $|1 - \langle\eta, w\rangle| \le C|1 - \langle\eta, z\rangle|$ for every $\eta \in \mathbf{B}$. By the definition of $Q_{k,j}$ we have $z = |z|\xi$ and $w = |w|u$, where $\xi, u \in B(u_{k,j}, 9 \cdot 2^{-k})$. Given an $\eta \in \mathbf{B}$, let us also write $\eta = |\eta|y$, where $y \in S$. Then

$$|1 - \langle\eta, w\rangle| \le (1 - |\eta|) + (1 - |w|) + |1 - \langle y, u\rangle|.$$

We have $1-|\eta| \le |1-\langle\eta, z\rangle|$ and $1-|w| \le 16(1-|z|) \le 16|1-\langle\eta, z\rangle|$. For $|1-\langle y, u\rangle|$, consider the following two cases. (1) Suppose that $d(y, u_{k,j}) \ge 18 \cdot 2^{-k}$. Then $(1/2)d(y, u_{k,j}) \ge d(u_{k,j}, \xi)$. Applying the triangle inequality, we have

$$d(y, \xi) \ge (1/2)d(y, u_{k,j}) + \{(1/2)d(y, u_{k,j}) - d(u_{k,j}, \xi)\} \ge (1/2)d(y, u_{k,j}).$$

On the other hand, $d(y,u) \le d(y,u_{k,j}) + d(u_{k,j},u) \le 2d(y,u_{k,j})$. Hence $d(y,u) \le 4d(y,\xi)$. Squaring both sides, we find that

$$|1 - \langle y,u\rangle| \le 16|1 - \langle y,\xi\rangle| \le 32|1 - \langle \eta, z\rangle|$$

in this case. (2) Suppose that $d(y,u_{k,j}) < 18 \cdot 2^{-k}$. Then $d(y,u) \le d(y,u_{k,j}) + d(u_{k,j},u) \le 27 \cdot 2^{-k}$. Squaring both sides, we find that

$$|1 - \langle y,u\rangle| \le (27)^2 \cdot 2^{-2k} \le (27)^2 \cdot 16 \cdot (1 - |z|) \le (27)^2 \cdot 16 \cdot |1 - \langle \eta, z\rangle|$$

in this case. This completes the proof. □

For the complicated estimates that are to come, let us introduce the following simplifying notation. For any $f \in L^2(\mathbf{B}, dv)$, $z \in \mathbf{B}$ and integer $i \ge 0$, denote

$$f_{z,i} = \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle.$$

Lemma 3.3. *Given an integer $i \ge 0$, let $C_{3.2}$ be the corresponding constant in Lemma* 3.2. *Then for each pair of $z, w \in Q_{k,j}$, $(k,j) \in I$, and each $f \in L^2(\mathbf{B}, dv)$, we have*

$$|f_{z,i} - f_{w,i}| \le C_{3.2}\|(f - f_{w,i})\psi_{w,i}\|.$$

Proof. First of all, for $i \ge 0$ and $z \in \mathbf{B}$ we have $\langle \psi_{z,i}, k_z\rangle = (1 - |z|^2)^{(n+1)/2}\psi_{z,i}(z) = 1$. Since $\|k_z\| = 1$, this means that $\|\psi_{z,i}\| \ge 1$. Let $z, w \in Q_{k,j}$, $(k,j) \in I$, and $f \in L^2(\mathbf{B}, dv)$ be given. Then

$$\begin{aligned} |f_{z,i} - f_{w,i}| &= \left|\int f|\tilde{\psi}_{z,i}|^2 dv - f_{w,i}\right| \le \int |f - f_{w,i}||\tilde{\psi}_{z,i}|^2 dv \\ &\le \left(\int |f - f_{w,i}|^2|\tilde{\psi}_{z,i}|^2 dv\right)^{1/2} \le \left(\int |f - f_{w,i}|^2|\psi_{z,i}|^2 dv\right)^{1/2}. \end{aligned}$$

Applying Lemma 3.2, we can replace the function $|\psi_{z,i}|^2$ in the last integral by $C_{3.2}^2|\psi_{w,i}|^2$, which gives us the desired conclusion. □

Recall that for each $(k,j) \in I$, $w(k,j)$ was defined by (2.13). We need to further simplify our notation. For any integer $i \ge 0$ and any $f \in L^2(\mathbf{B}, dv)$, denote

$$M_i(f;k,j) = \|(f - f_{w(k,j),i})\psi_{w(k,j),i}\|, \tag{3.1}$$

$(k,j) \in I$. Given integers $\ell \ge 0$ and $m \ge 0$, we define

$$M_i(f;k,j;\ell,m) = \max\{M_i(f;k+\ell,\nu) : d(u_{k+\ell,\nu}, u_{k,j}) \le 2^{-k+m+6}\} \tag{3.2}$$

for $i \ge 0$ and $(k,j) \in I$. Here, ℓ and m indicate how "far" away $u_{k+\ell,\nu}$ is from $u_{k,j}$ in two different ways. The number m, of course, represents an actual distance measurement. But the number ℓ indicates the "generation gap" between $u_{k+\ell,\nu}$ and $u_{k,j}$.

Although the estimate in our next lemma is extremely crude, it suffices for our purpose.

Lemma 3.4. *Given any integer $i \geq 0$, there is a $0 < C_{3.4} < \infty$ which depends only on i and n such that the following estimate holds: Let $\ell \geq 0$ and $m \geq 0$ be integers, and let $(k, j) \in I$. If $w \in T_{k+\ell,\nu}$ where ν satisfies the condition $d(u_{k+\ell,\nu}, u_{k,j}) \leq 2^{-k+m+3}$, then for each $z \in T_{k,j}$ and each $f \in L^2(\mathbf{B}, dv)$ we have*

$$|f_{z,i} - f_{w,i}| \leq C_{3.4} 2^{2nm} \sum_{t=0}^{\ell} M_i(f; k, j; t, m).$$

Proof. Let $\ell \geq 0$, $m \geq 0$ and $(k, j), (k + \ell, \nu) \in I$ be such that $d(u_{k+\ell,\nu}, u_{k,j}) \leq 2^{-k+m+3}$ as in the statement of the lemma. By (2.4), there is a j' such that $u_{k+\ell,\nu} \in B(u_{k,j'}, 2^{-k})$. Note that $d(u_{k,j'}, u_{k,j}) \leq d(u_{k,j'}, u_{k+\ell,\nu},) + d(u_{k+\ell,\nu}, u_{k,j}) \leq 2^{-k+m+4}$. We first show that there are elements $j_1, \ldots, j_r \in \{1, \ldots, m(k)\}$ such that

(i) $u_{k,j'} \in B(u_{k,j_1}, 2^{-k})$ and $u_{k,j} \in B(u_{k,j_r}, 2^{-k})$;

(ii) $B(u_{k,j_s}, 2^{-k}) \cap \{\xi \in S : d(\xi, u_{k,j}) \leq d(u_{k,j'}, u_{k,j})\} \neq \emptyset$ for each $1 \leq s \leq r$;

(iii) $B(u_{k,j_s}, 2^{-k}) \cap B(u_{k,j_{s+1}}, 2^{-k}) \neq \emptyset$ whenever $1 \leq s < r$;

(iv) $j_{s_1} \neq j_{s_2}$ whenever $1 \leq s_1 < s_2 \leq r$.

To prove this, we need to construct a continuous path

$$\eta : [0, 1] \to S$$

such that $\eta(0) = u_{k,j'}$, $\eta(1) = u_{k,j}$, and

$$d(\eta(x), u_{k,j}) \leq d(u_{k,j'}, u_{k,j}) \quad \text{for every} \quad x \in [0, 1]. \tag{3.3}$$

Such construction is trivial if $u_{k,j'}$ and $u_{k,j}$ are linearly dependent as vectors in $\mathbf{C}^n$. Suppose that $u_{k,j'}$ and $u_{k,j}$ are linearly independent. Then we have

$$u_{k,j'} = cu_{k,j} + (1 - |c|^2)^{1/2} u^{\perp},$$

where c is a complex number with $|c| < 1$ and $u^{\perp}$ is a unit vector in $\mathbf{C}^n$ such that $\langle u_{k,j}, u^{\perp} \rangle = 0$. Define

$$\eta(x) = (x + (1 - x)c)u_{k,j} + (1 - |x + (1 - x)c|^2)^{1/2} u^{\perp}, \quad 0 \leq x \leq 1.$$

Then obviously η is a continuous path in S with $\eta(0) = u_{k,j'}$ and $\eta(1) = u_{k,j}$. Moreover, for each $x \in [0, 1]$ we have

$$|1 - \langle \eta(x), u_{k,j} \rangle| = |1 - (x + (1 - x)c)| = (1 - x)|1 - c| \leq |1 - c| = |1 - \langle u_{k,j'}, u_{k,j} \rangle|.$$

Hence (3.3) holds. Once we have such an η, for each $\mu \in \{1, \ldots, m(k)\}$ define the set $U_\mu = \{x \in [0, 1] : \eta(x) \in B(u_{k,\mu}, 2^{-k})\}$, which is open in $[0, 1]$. Then, of course, $\cup_\mu U_\mu = [0, 1]$. We claim that there are $j_1, \ldots, j_r \in \{1, \ldots, m(k)\}$ such that

(1) $0 \in U_{j_1}$ and $1 \in U_{j_r}$;

(2) $U_{j_s} \neq \emptyset$ for each $1 \le s \le r$;
(3) $U_{j_s} \cap U_{j_{s+1}} \neq \emptyset$ whenever $1 \le s < r$;
(4) $j_{s_1} \neq j_{s_2}$ whenever $1 \le s_1 < s_2 \le r$.

The choice of these $j_1, \ldots, j_r$ is easy. We start with any j_1 such that $0 \in U_{j_1}$. Then consider $x_1 = \sup\{x : x \in U_{j_1}\}$. If $x_1 \notin U_{j_1}$, we pick a j_2 such that $x_1 \in U_{j_2}$. Then consider $x_2 = \sup\{x : x \in U_{j_2}\}$, and so on. Obviously, this process must stop after some r steps. Once we have $j_1, \ldots, j_r$ chosen this way, (i), (iii) and (iv) follow from (1), (3) and (4) respectively, while (ii) follows from (2) and (3.3).

Let $z \in T_{k,j}$ and $z' \in T_{k,j'}$. Then

$$|f_{z',i} - f_{z,i}| \le |f_{z',i} - f_{w(k,j_1),i}| + |f_{w(k,j_r),i} - f_{z,i}| + \sum_{1 \le s \le r-1} |f_{w(k,j_s),i} - f_{w(k,j_{s+1}),i}|.$$

By (i), we have $z' \in Q_{k,j_1}$ and $z \in Q_{k,j_r}$. Moreover, (iii) implies $w(k, j_{s+1}) \in Q_{k,j_s}$. Applying Lemma 3.3 to the above, we obtain

$$|f_{z',i} - f_{z,i}| \le 2C_{3.2} \sum_{s=1}^{r} M_i(f; k, j_s).$$

Since $d(u_{k,j'}, u_{k,j}) \le 2^{-k+m+4}$, it follows from (ii) that $M_i(f; k, j_s) \le M_i(f; k, j; 0, m)$ for each $1 \le s \le r$. Therefore

$$|f_{z',i} - f_{z,i}| \le 2C_{3.2} r M_i(f; k, j; 0, m).$$

On the other hand, (ii) and (iv) together imply

$$r \le \operatorname{card}\{\mu \in \{1, \ldots, m(k)\} : d(u_{k,\mu}, u_{k,j}) \le 2^{-k+m+5}\}.$$

By (2.3) and (2.2), this means that $r \le C_1 2^{2nm}$. Thus we have shown that

$$|f_{z',i} - f_{z,i}| \le C_2 2^{2nm} M_i(f; k, j; 0, m) \tag{3.4}$$

for all $z \in T_{k,j}$ and $z' \in T_{k,j'}$. This takes care of any two points in the same "generation" of the decomposition of the ball. Next we consider the "vertical descent" in generations.

For each $0 \le t \le \ell$, there is a $\nu(t) \in \{1, \ldots, m(k+t)\}$ such that

$$u_{k+\ell,\nu} \in B(u_{k+t,\nu(t)}, 2^{-k-t}).$$

In particular, we take $\nu(\ell) = \nu$. Since $u_{k+\ell,\nu} \in B(u_{k,j'}, 2^{-k})$, we can, and do, take $\nu(0) = j'$. Since $B(u_{k+t,\nu(t)}, 2^{-k-t}) \cap B(u_{k+t+1,\nu(t+1)}, 2^{-k-t-1}) \neq \emptyset$ in the case $0 \le t < \ell$, we have $w(k+t+1, \nu(t+1)) \in Q_{k+t,\nu(t)}$. Thus it follows from Lemma 3.3 that

$$|f_{w(k+t+1,\nu(t+1)),i} - f_{w(k+t,\nu(t)),i}| \le C_{3.2} M_i(f; k+t, \nu(t)).$$

Let $w \in T_{k+\ell,\nu}$. Then Lemma 3.3 also gives us $|f_{w,i} - f_{w(k+\ell,\nu),i}| \le C_{3.2} M_i(f; k+\ell, \nu)$. Hence

$$\begin{aligned}|f_{w,i} - f_{w(k,j'),i}| &\le |f_{w,i} - f_{w(k+\ell,\nu),i}| + \sum_{0\le t<\ell} |f_{w(k+t+1,\nu(t+1)),i} - f_{w(k+t,\nu(t)),i}| \\ &\le C_{3.2} \sum_{t=0}^{\ell} M_i(f; k+t, \nu(t)).\end{aligned}$$

For each $0 \le t \le \ell$, we have

$$d(u_{k+t,\nu(t)}, u_{k,j}) \le d(u_{k+t,\nu(t)}, u_{k+\ell,\nu}) + d(u_{k+\ell,\nu}, u_{k,j}) \le 2^{-k+m+4}.$$

Hence $M_i(f; k+t, \nu(t)) \le M_i(f; k, j; t, m)$ for each $0 \le t \le \ell$. Therefore

$$|f_{w,i} - f_{w(k,j'),i}| \le C_{3.2} \sum_{t=0}^{\ell} M_i(f; k, j; t, m).$$

Combining this with the special case of (3.4) where $z' = w(k,j')$, the lemma follows. □

Lemma 3.5. *Given any integer $i \ge 0$, there is a $0 < C_{3.5} < \infty$ which depends only on i and n such that the following estimate holds: Let $\ell \ge 0$ and $m \ge 0$ be integers, and let $(k,j) \in I$. If $w \in T_{k+\ell,\nu}$ and if ν satisfies the condition $d(u_{k+\ell,\nu}, u_{k,j}) \ge 2^{-k+m}$, then for each $z \in T_{k,j}$ and each $g \in L^2(\mathbf{B}, dv)$ we have*

$$|\langle g\psi_{z,3i+n+1}, g\psi_{w,3i+n+1}\rangle| \le C_{3.5} 2^{-(n+1)\ell} 2^{-2im} \|g\psi_{w,i}\|^2 \tag{3.5}$$

and

$$|\langle \psi_{z,3i+n+1}, g\psi_{w,3i+n+1}\rangle| \le C_{3.5} 2^{-(n+1)\ell} 2^{-2im} \|g\psi_{w,i}\|. \tag{3.6}$$

Proof. By (1.2), for each $\eta \in \mathbf{B}$ we have

$$\begin{aligned}&|\psi_{z,3i+n+1}(\eta)\psi_{w,3i+n+1}(\eta)| \\ &\qquad = |\psi_{w,i}(\eta)|^2 \cdot \left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1)/2} \cdot \left|\frac{1-|w|^2}{1-\langle \eta, w\rangle}\right|^i \cdot \left|\frac{1-|z|^2}{1-\langle \eta, z\rangle}\right|^{2n+2+3i} \\ &\qquad \le 2^{2n+2+4i} \left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1)/2} \cdot |m_w(\eta)m_z(\eta)|^i \cdot |\psi_{w,i}(\eta)|^2, \end{aligned} \tag{3.7}$$

where m_w and m_z were defined by (1.1).

Suppose that w and z satisfy the conditions given in the lemma. We claim that $\|m_w m_z\|_\infty \le 72 \cdot 2^{-2m}$. To justify this claim, we only need to consider $m \ge 2$. Write $z = |z|\xi$ and $w = |w|u$, where $\xi \in B(u_{k,j}, 2^{-k})$ and $u \in B(u_{k+\ell,\nu}, 2^{-k-\ell})$. Since $d(u_{k+\ell,\nu}, u_{k,j}) \ge 2^{-k+m}$, it follows from the triangle inequality that $d(\xi, u) \ge (1/3)2^{-k+m}$.

Thus for each $\zeta \in S$, we have either $d(\zeta,\xi) \ge (1/6)2^{-k+m}$ or $d(\zeta,u) \ge (1/6)2^{-k+m}$. In the former case, we have

$$|m_z(\zeta)| = \left|\frac{1-|z|}{1-\langle\zeta,z\rangle}\right| \le 2\frac{1-|z|}{|1-\langle\zeta,\xi\rangle|} \le 2\frac{2^{-2k}}{\{(1/6)2^{-k+m}\}^2} = 72\cdot 2^{-2m}.$$

In the latter case, we similarly have $|m_w(\zeta)| \le 72\cdot 2^{-2m-2\ell} \le 72\cdot 2^{-2m}$. Thus we have shown that $\|m_w m_z\|_\infty \le 72\cdot 2^{-2m}$. For such z and w, we also have $(1-|w|^2)/(1-|z|^2) \le 2(1-|w|)/(1-|z|) \le 8\cdot 2^{-2\ell}$. Combining these facts with (3.7), we see that the inequality

$$|\psi_{z,3i+n+1}\psi_{w,3i+n+1}| \le C2^{-(n+1)\ell}2^{-2im}|\psi_{w,i}|^2$$

holds on $\mathbf{B}$. Obviously, (3.5) is an immediate consequence of this, while (3.6) follows from this inequality and the fact that $\|\psi_{w,i}\| \le 2^i$. □

Lemma 3.6. *Given any integer $i \ge 0$, there is a $0 < C_{3.6} < \infty$ which depends only on i and n such that the following estimate holds: Let $(k,j) \in I$ and $m \in \mathbf{N}$. If $w \in T_{k,\nu}$ and if ν satisfies the condition $d(u_{k,\nu},u_{k,j}) \ge 2^{-k+m}$, then for each $z \in T_{k,j}$ and each pair of $g_1, g_2 \in L^2(\mathbf{B},dv)$ we have*

$$|\langle g_1\psi_{z,3i+n+1}, g_2\psi_{w,3i+n+1}\rangle| \le C_{3.6}2^{-2im}\|g_1\psi_{z,i}\|\|g_2\psi_{w,i}\|.$$

Proof. In the previous proof we showed that $\|m_w m_z\|_\infty \le 72\cdot 2^{-2m}$ for such z and w. This clearly implies the present lemma. □

Lemma 3.7. *Given any integer $i \ge 0$, there is a $0 < C_{3.7} < \infty$ which depends only on i and n such that the following estimate holds: Let $k \ge 0$ and $\ell \ge 0$. If $w \in T_{k+\ell,\nu}$, $1 \le \nu \le m(k+\ell)$, and $z \in T_{k,j}$, $1 \le j \le m(k)$, then for each $g \in L^2(\mathbf{B},dv)$ we have*

$$|\langle g\psi_{z,3i+n+1}, g\psi_{w,3i+n+1}\rangle| \le C_{3.7}2^{-(n+1)\ell}\|g\psi_{w,i}\|^2 \quad \textit{and}$$
$$|\langle \psi_{z,3i+n+1}, g\psi_{w,3i+n+1}\rangle| \le C_{3.7}2^{-(n+1)\ell}\|g\psi_{w,i}\|.$$

Proof. This is an immediate consequence of (3.7). □

It is obviously too long to write $3i+n+1$ for a part of a subscript. To simplify, let us adopt the following convention. For each integer $i \ge 0$, we denote

$$i' = 3i+n+1.$$

We need one more lemma before we get to our main estimate.

Lemma 3.8. [16,Lemma 4.1] *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\mathrm{card}\{y \in X : (x,y) \in E\} \le m \quad \textit{and} \quad \mathrm{card}\{y \in X : (y,x) \in E\} \le m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup ... \cup E_{2m}$$

and such that for each $1 \le j \le 2m$, the conditions $(x, y), (x', y') \in E_j$ and $(x, y) \neq (x', y')$ imply both $x \neq x'$ and $y \neq y'$.

Lemma 3.9. *Let $0 < b < \infty$ and integer $i \ge 6n + 1$ be given. Then there is a constant $C_{3.9}$ which depends only on b, i and n such that the following holds: Let $z(k, j) \in T_{k,j}$ for every $(k, j) \in I$. Let $\{c_{k,j} : (k, j) \in I\}$ be a collection complex numbers such that $|c_{k,j}| \le 1$ for each $(k, j) \in I$, and such that $c_{k,j} = 0$ for all but a finite number of (k, j)'s. Finally, let $\{e_{k,j} : (k, j) \in I\}$ be an orthonormal set. Then for each $f \in L^2(\mathbf{B}, dv)$ and each symmetric function Φ, the operator*

$$A = \sum_{(k,j)\in I} c_{k,j}\{(f - f_{z(k,j),i})\psi_{z(k,j),i'}\} \otimes e_{k,j}$$

satisfies the estimate

$$\|A\|_\Phi \le C_{3.9}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}),$$

where Γ is any countable subset of $\mathbf{B}$ with the property $\cup_{z\in\Gamma} D(z, b) = \mathbf{B}$.

Proof. For the A defined above we have

$$A^*A = \sum_{(k,j),(k',j')\in I} \bar{c}_{k',j'}c_{k,j}\langle(f - f_{z(k,j),i})\psi_{z(k,j),i'}, (f - f_{z(k',j'),i})\psi_{z(k',j'),i'}\rangle e_{k',j'} \otimes e_{k,j}.$$

To simplify our notation, let us denote

$$\begin{aligned}
p(k, j; k', j') &= \bar{c}_{k',j'}c_{k,j}\langle(f - f_{z(k,j),i})\psi_{z(k,j),i'}, (f - f_{z(k',j'),i})\psi_{z(k',j'),i'}\rangle,\\
q(k, j; k', j') &= \bar{c}_{k',j'}c_{k,j}\langle(f - f_{z(k',j'),i})\psi_{z(k,j),i'}, (f - f_{z(k',j'),i})\psi_{z(k',j'),i'}\rangle \quad \text{and}\\
r(k, j; k', j') &= \bar{c}_{k',j'}c_{k,j}(f_{z(k',j'),i} - f_{z(k,j),i})\langle\psi_{z(k,j),i'}, (f - f_{z(k',j'),i})\psi_{z(k',j'),i'}\rangle
\end{aligned}$$

for $(k, j), (k', j') \in I$. Then

$$A^*A = B + \sum_{\ell=1}^{\infty}(B_\ell + B_\ell^*),$$

where

$$B = \sum_{k=0}^{\infty}\sum_{j,j'} p(k, j; k, j')e_{k,j'} \otimes e_{k,j}$$

and

$$B_\ell = \sum_{k=0}^{\infty}\sum_{j,j'} p(k, j; k + \ell, j')e_{k+\ell,j'} \otimes e_{k,j}$$

for each $\ell \geq 1$. Applying Lemma 3.1, for each symmetric gauge function Φ we have

$$\|A\|_\Phi = \|(A^*A)^{1/2}\|_\Phi \leq 2\||B|^{1/2}\|_\Phi + 2\sum_{\ell=1}^{\infty}(\||B_\ell|^{1/2}\|_\Phi + \||B_\ell^*|^{1/2}\|_\Phi). \tag{3.8}$$

Note that $\||B_\ell|^{1/2}\|_\Phi = \||B_\ell^*|^{1/2}\|_\Phi$. Thus our task is to estimate $\||B|^{1/2}\|_\Phi$ and $\||B_\ell|^{1/2}\|_\Phi$. But to carry out these estimates, we need to further decompose B and B_ℓ.

To decompose B, consider the index sets

$$\begin{aligned} E^{(0)} &= \{((k,j),(k,j')) : d(u_{k,j}, u_{k,j'}) < 2^{-k+2}\} \quad \text{and} \\ E^{(m)} &= \{((k,j),(k,j')) : 2^{-k+m+1} \leq d(u_{k,j}, u_{k,j'}) < 2^{-k+m+2}\}, \quad m \geq 1. \end{aligned}$$

Then for each $m \geq 0$ define the operator

$$B^{(m)} = \sum_{((k,j),(k,j')) \in E^{(m)}} p(k,j;k,j') e_{k,j'} \otimes e_{k,j} \tag{3.9}$$

Obviously, we have the decomposition

$$B = \sum_{m=0}^{\infty} B^{(m)}.$$

But even $B^{(m)}$ needs to be further decomposed. By (2.3) and (2.2), there is a natural number C_1 such that for each $(k,j) \in I$ and each $m \geq 0$, we have

$$\text{card}\{j' \in \{1,\dots,m(k)\} : d(u_{k,j}, u_{k,j'}) < 2^{-k+m+2}\} \leq C_1 2^{2nm}. \tag{3.10}$$

By (3.10) and Lemma 3.8, for each $m \geq 0$ we have the partition

$$E^{(m)} = E_1^{(m)} \cup \cdots \cup E_{2C_1 2^{2nm}}^{(m)} \tag{3.11}$$

such that for each $1 \leq \nu \leq 2C_1 2^{2nm}$, if $((k_1,j_1),(k_1,j_1'))$ and $((k_2,j_2),(k_2,j_2'))$ are two distinct elements in $E_\nu^{(m)}$, then we have both $(k_1,j_1) \neq (k_2,j_2)$ and $(k_1,j_1') \neq (k_2,j_2')$. Define

$$B_\nu^{(m)} = \sum_{((k,j),(k,j')) \in E_\nu^{(m)}} p(k,j;k,j') e_{k,j'} \otimes e_{k,j}$$

for $m \geq 0$ and $1 \leq \nu \leq 2C_1 2^{2nm}$. The above-mentioned property of $E_\nu^{(m)}$ implies that the projections $((k,j),(k,j')) \mapsto (k,j)$ and $((k,j),(k,j')) \mapsto (k,j')$ are injective on $E_\nu^{(m)}$. Since $\{e_{k,j} : (k,j) \in I\}$ is an orthonormal set, it follows that

$$\||B_\nu^{(m)}|^{1/2}\|_\Phi = \Phi(\{|p(k,j;k,j')|^{1/2}\}_{((k,j),(k,j')) \in E_\nu^{(m)}}). \tag{3.12}$$

By Lemma 3.6, if $m \geq 1$, then for each $((k,j),(k,j')) \in E^{(m)}$ we have

$$|p(k,j;k,j')| \leq C_{3.6} 2^{-2im} \|(f - f_{z(k,j),i})\psi_{z(k,j),i}\| \|(f - f_{z(k,j'),i})\psi_{z(k,j'),i}\|.$$

On the other hand, by the definition of $p(k,j;k',j')$ and (1.2) we have

$$\begin{aligned} |p(k,j;k',j')| &\leq \|(f - f_{z(k,j),i})\psi_{z(k,j),i'}\| \|(f - f_{z(k',j'),i})\psi_{z(k',j'),i'}\| \\ &\leq C_2 \|(f - f_{z(k,j),i})\psi_{z(k,j),i}\| \|(f - f_{z(k',j'),i})\psi_{z(k',j'),i}\|. \end{aligned}$$

If $x \geq 0$ and $y \geq 0$, then $\sqrt{xy} \leq (1/2)(x+y)$. Hence for each $m \geq 0$, we have

$$(3.13) \quad |p(k,j;k,j')|^{1/2} \leq C_5 2^{-im} (\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\| + \|(f - f_{z(k,j'),i})\psi_{z(k,j'),i}\|)$$

if $((k,j),(k,j')) \in E^{(m)}$. Since the projections $((k,j),(k,j')) \mapsto (k,j)$ and $((k,j),(k,j')) \mapsto (k,j')$ are injective on $E_\nu^{(m)}$, we have

$$\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{((k,j),(k,j')) \in E_\nu^{(m)}}) \leq \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}) \quad \text{and}$$
$$\Phi(\{\|(f - f_{z(k,j'),i})\psi_{z(k,j'),i}\|\}_{((k,j),(k,j')) \in E_\nu^{(m)}}) \leq \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}).$$

Combining this with (3.12) and (3.13), we obtain

$$\||B_\nu^{(m)}|^{1/2}\|_\Phi \leq 2C_5 2^{-im} \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}).$$

By (3.9) and (3.11), $B^{(m)} = B_1^{(m)} + \cdots + B_{2C_1 2^{2nm}}^{(m)}$. Thus it follows from Lemma 3.1 that

$$\begin{aligned} \||B^{(m)}|^{1/2}\|_\Phi &\leq 2(\||B_1^{(m)}|^{1/2}\|_\Phi + \cdots + \||B_{2C_1 2^{2nm}}^{(m)}|^{1/2}\|_\Phi) \\ &\leq 4C_1 2^{2nm} \cdot 2C_5 2^{-im} \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}) \\ &= C_6 2^{-(i-2n)m} \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}). \end{aligned}$$

Since $i \geq 6n+1$, $i - 2n > 0$. Applying Lemma 3.1 again, we have

$$\begin{aligned} \||B|^{1/2}\|_\Phi &\leq 2 \sum_{m=0}^{\infty} \||B^{(m)}|^{1/2}\|_\Phi \leq 2C_6 \sum_{m=0}^{\infty} 2^{-(i-2n)m} \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}) \\ &= C_7 \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j) \in I}). \end{aligned} \tag{3.14}$$

Next we consider the operators B_ℓ, $\ell \geq 1$, which must be handled more carefully.

First of all, by design we have the relation

$$p(k,j;k',j') = q(k,j;k',j') + r(k,j;k',j').$$

Accordingly, for each $\ell \geq 1$ we have $B_\ell = X_\ell + Y_\ell$, where

$$X_\ell = \sum_{k=0}^{\infty} \sum_{j,j'} q(k,j;k+\ell,j') e_{k+\ell,j'} \otimes e_{k,j} \quad \text{and} \quad Y_\ell = \sum_{k=0}^{\infty} \sum_{j,j'} r(k,j;k+\ell,j') e_{k+\ell,j'} \otimes e_{k,j}.$$

We deal with X_ℓ and Y_ℓ separately. But before getting to estimates, we need to group the terms in these operators properly.

Note that for any $k \geq 0$, $\ell \geq 1$ and $1 \leq j' \leq m(k+\ell)$, we have $B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset$ for at least one $t \in \{1, \ldots, m(k)\}$. Thus we can write

$$X_\ell = \sum_{k=0}^{\infty} \sum_{j,t} g_{k,t;j}^{(\ell)} \otimes e_{k,j},$$

where

$$g_{k,t;j}^{(\ell)} = \sum_{B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset} \epsilon(k,t;k+\ell,j') q(k,j;k+\ell,j') e_{k+\ell,j'}, \tag{3.15}$$

where the value of $\epsilon(k,t;k+\ell,j')$ is either 1 or 0. Obviously, if $k \neq k_1$, then we have $\langle g_{k,t;j}^{(\ell)}, g_{k_1,t_1;j_1}^{(\ell)} \rangle = 0$ for all possible t, t_1, j, j_1 and ℓ. For a given $k \geq 0$, if t, t_1, j, j_1 and ℓ are such that $\langle g_{k,t;j}^{(\ell)}, g_{k,t_1;j_1}^{(\ell)} \rangle \neq 0$, then we necessarily have a $j' \in \{1, \ldots, m(k+\ell)\}$ such that $B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset$ and $B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t_1}, 2^{-k}) \neq \emptyset$. Since $k+\ell \geq k$, this implies that $d(u_{k,t}, u_{k,t_1}) < 4 \cdot 2^{-k}$ if $\langle g_{k,t;j}^{(\ell)}, g_{k,t_1;j_1}^{(\ell)} \rangle \neq 0$. Combining this fact with (2.3) and (2.2), we can decompose I as the union of pairwise disjoint subsets $I_1, \ldots, I_N$, where N is determined by n, such that for each $\gamma \in \{1, \ldots, N\}$, if $(k,t), (k_1,t_1) \in I_\gamma$ and if $(k,t) \neq (k_1,t_1)$, then $\langle g_{k,t;j}^{(\ell)}, g_{k_1,t_1;j_1}^{(\ell)} \rangle = 0$ for all possible ℓ, j and j_1.

Accordingly, for each $\gamma \in \{1, \ldots, N\}$ we define

$$X_{\ell,\gamma} = \sum_{(k,t) \in I_\gamma} \sum_{1 \leq j \leq m(k)} g_{k,t;j}^{(\ell)} \otimes e_{k,j}.$$

Then, of course, $X_\ell = X_{\ell,1} + \cdots + X_{\ell,N}$. Next we decompose each $X_{\ell,\gamma}$ in a manner similar to the decomposition of B. Define

$$E^{(\gamma,0)} = \{((k,j),(k,t)) : (k,j) \in I, (k,t) \in I_\gamma, d(u_{k,j}, u_{k,t}) < 2^{-k+2}\} \quad \text{and}$$
$$E^{(\gamma,m)} = \{((k,j),(k,t)) : (k,j) \in I, (k,t) \in I_\gamma, 2^{-k+m+1} \leq d(u_{k,j}, u_{k,t}) < 2^{-k+m+2}\}$$

for $m \geq 1$. For each $m \geq 0$, define

$$X_{\ell,\gamma}^{(m)} = \sum_{((k,j),(k,t)) \in E^{(\gamma,m)}} g_{k,t;j}^{(\ell)} \otimes e_{k,j}.$$

Then we have

$$X_{\ell,\gamma} = \sum_{m=0}^{\infty} X_{\ell,\gamma}^{(m)}. \tag{3.16}$$

By (3.10) and Lemma 3.8, for each $m \geq 0$ we have the partition

$$E^{(\gamma,m)} = E_1^{(\gamma,m)} \cup \cdots \cup E_{2C_1 2^{nm}}^{(\gamma,m)}$$

such that for each $1 \leq \nu \leq 2C_1 2^{2nm}$, if $((k_1, j_1), (k_1, t_1))$ and $((k_2, j_2), (k_2, t_2))$ are two distinct elements in $E_\nu^{(\gamma,m)}$, then we have both $(k_1, j_1) \neq (k_2, j_2)$ and $(k_1, t_1) \neq (k_2, t_2)$. Let

$$X_{\ell,\gamma}^{(m,\nu)} = \sum_{((k,j),(k,t)) \in E_\nu^{(\gamma,m)}} g_{k,t;j}^{(\ell)} \otimes e_{k,j},$$

$1 \leq \nu \leq 2C_1 2^{2nm}$. Then $X_{\ell,\gamma}^{(m)} = X_{\ell,\gamma}^{(m,1)} + \cdots + X_{\ell,\gamma}^{(m,2C_1 2^{2nm})}$. For any two distinct elements $((k_1, j_1), (k_1, t_1))$ and $((k_2, j_2), (k_2, t_2))$ in $E_\nu^{(\gamma,m)}$, we have both $\langle e_{k_1,j_1}, e_{k_2,j_2} \rangle = 0$ and $\langle g_{k_1,t_1;j_1}^{(\ell)}, g_{k_2,t_2;j_2}^{(\ell)} \rangle = 0$. Hence

$$\| |X_{\ell,\gamma}^{(m,\nu)}|^{1/2} \|_\Phi = \Phi(\{\|g_{k,t;j}^{(\ell)}\|^{1/2}\}_{((k,j),(k,t)) \in E_\nu^{(\gamma,m)}}).$$

Next we consider $\|g_{k,t;j}^{(\ell)}\|$.

By (3.15), we have

$$\|g_{k,t;j}^{(\ell)}\|^2 \leq \sum_{B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset} |q(k, j; k+\ell, j')|^2.$$

For $((k, j), (k, t)) \in E_\nu^{(\gamma,m)}$ with $m \geq 1$, if ℓ and j' are such that $B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset$, then $d(u_{k,j}, u_{k+\ell,j'}) \geq d(u_{k,j}, u_{k,t}) - d(u_{k,t}, u_{k+\ell,j'}) \geq 2^{-k+m+1} - 2^{-k+1} \geq 2^{-k+m}$. Thus it follows from inequality (3.5) in Lemma 3.5 that

$$|q(k, j; k+\ell, j')| \leq C 2^{-(n+1)\ell} 2^{-2im} \|(f - f_{z(k+\ell,j'),i}) \psi_{z(k+\ell,j'),i}\|^2.$$

If we apply Lemma 3.7 instead of Lemma 3.5, then the above inequality also holds in the case $m = 0$. If $B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset$ and $((k, j), (k, t)) \in E_\nu^{(\gamma,m)}$, then we also have $d(u_{k,j}, u_{k+\ell,j'}) \leq d(u_{k,j}, u_{k,t}) + d(u_{k,t}, u_{k+\ell,j'}) \leq 2^{-k+m+2} + 2^{-k+1} \leq 2^{-k+m+3}$. Now, for $(k, j) \in I$, $\ell \geq 1$ and $m \geq 0$, define

$$a(k, j; m; \ell) = \max\{\|(f - f_{z(k+\ell,j'),i}) \psi_{z(k+\ell,j'),i}\| : d(u_{k+\ell,j'}, u_{k,j}) \leq 2^{-k+m+3}\}.$$

Combining the above, if $((k, j), (k, t)) \in E_\nu^{(\gamma,m)}$, then,

$$\begin{aligned} \|g_{k,t;j}^{(\ell)}\|^2 \leq (C 2^{-(n+1)\ell} 2^{-2im} a^2(k, j; m; \ell))^2 \times \\ \operatorname{card}\{j' : B(u_{k+\ell,j'}, 2^{-k-\ell}) \cap B(u_{k,t}, 2^{-k}) \neq \emptyset\}. \end{aligned}$$

Applying (2.3) and (2.2) again, for such $((k, j), (k, t)) \in E_\nu^{(\gamma,m)}$ we have

$$\|g_{k,t;j}^{(\ell)}\|^2 \leq (C 2^{-(n+1)\ell} 2^{-2im} a^2(k, j; m; \ell))^2 \cdot C_8 2^{2n\ell} = C_9 2^{-2\ell} 2^{-4im} a^4(k, j; m; \ell).$$

Since the projection $((k,j),(k,t)) \mapsto (k,j)$ is injective on $E_\nu^{(\gamma,m)}$, we now have

$$\||X_{\ell,\gamma}^{(m,\nu)}|^{1/2}\|_\Phi = \Phi(\{\|g_{k,t;j}^{(\ell)}\|^{1/2}\}_{((k,j),(k,t))\in E_\nu^{(\gamma,m)}}) \\ \leq C_9^{1/4} 2^{-\ell/2} 2^{-im} \Phi(\{a(k,j;m;\ell)\}_{(k,j)\in I}).$$

To estimate $\Phi(\{a(k,j;m;\ell)\}_{(k,j)\in I})$, note that given any k,j,m,ℓ, there is a $\tau(k,j;m;\ell) \in \{1,\ldots,m(k+\ell)\}$ such that $d(u_{k+\ell,\tau(k,j;m;\ell)}, u_{k,j}) \leq 2^{-k+m+3}$ and such that

$$a(k,j;m;\ell) = \|(f - f_{z(k+\ell,\tau(k,j;m;\ell)),i})\psi_{z(k+\ell,\tau(k,j;m;\ell)),i}\|.$$

If $j_1, j_2 \in \{1,\ldots,m(k)\}$ are such that $\tau(k,j_1;m;\ell) = \tau(k,j_2;m;\ell)$, then it follows that $d(u_{k,j_1}, u_{k,j_2}) \leq 2^{-k+m+4}$. Combining this fact with (2.3) and (2.2), we see that the map

$$(k,j) \mapsto (k+\ell, \tau(k,j;m;\ell))$$

is at most $C_{10}2^{2nm}$-to-1, where C_{10} depends only on n. Applying Lemma 2.2, we have

$$\Phi(\{a(k,j;m;\ell)\}_{(k,j)\in I}) = \Phi(\{\|(f - f_{z(k+\ell,\tau(k,j;m;\ell)),i})\psi_{z(k+\ell,\tau(k,j;m;\ell)),i}\|\}_{(k,j)\in I}) \\ \leq C_{10}2^{2nm}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}). \tag{3.17}$$

Therefore

$$\||X_{\ell,\gamma}^{(m,\nu)}|^{1/2}\|_\Phi \leq C_{11}2^{-\ell/2}2^{-(i-2n)m}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}).$$

Since $X_{\ell,\gamma}^{(m)} = X_{\ell,\gamma}^{(m,1)} + \cdots + X_{\ell,\gamma}^{(m,2C_1 2^{2nm})}$, by Lemma 3.1 we have

$$\||X_{\ell,\gamma}^{(m)}|^{1/2}\|_\Phi \leq 2(\||X_{\ell,\gamma}^{(m,1)}|^{1/2}\|_\Phi + \cdots + \||X_{\ell,\gamma}^{(m,2C_1 2^{2nm})}|^{1/2}\|_\Phi) \\ \leq 4C_1 2^{2nm} \cdot C_{11}2^{-\ell/2}2^{-(i-2n)m}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}) \\ = C_{12}2^{-\ell/2}2^{-(i-4n)m}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}).$$

Applying Lemma 3.1 again, since $i \geq 6n+1$, we have

$$\||X_{\ell,\gamma}|^{1/2}\|_\Phi \leq 2\sum_{m=0}^{\infty} \||X_{\ell,\gamma}^{(m)}|^{1/2}\|_\Phi \leq C_{14}2^{-\ell/2}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}).$$

Since $X_\ell = X_{\ell,1} + \cdots + X_{\ell,N}$ and since N depends only on n, another application of Lemma 3.1 gives us

$$\||X_\ell|^{1/2}\|_\Phi \leq C_{15}2^{-\ell/2}\Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}). \tag{3.18}$$

Let us now consider Y_ℓ. As the reader can imagine, it must undergo decompositions parallel to those for X_ℓ. First of all, we rewrite Y_ℓ as

$$Y_\ell = \sum_{k=0}^{\infty}\sum_{j,t} h_{k,t;j}^{(\ell)} \otimes e_{k,j},$$

where

$$h_{k,t;j}^{(\ell)} = \sum_{B(u_{k+\ell,j'},2^{-k-\ell})\cap B(u_{k,t},2^{-k})\neq\emptyset} \epsilon(k,t;k+\ell,j')r(k,j;k+\ell,j')e_{k+\ell,j'},$$

where the value of $\epsilon(k,t;k+\ell,j')$ is either 1 or 0. As above, I decomposes as the union of pairwise disjoint subsets $I_1,\dots,I_N$, where N depends only on n, such that for each $\gamma\in\{1,\dots,N\}$, if $(k,t),(k_1,t_1)\in I_\gamma$ and if $(k,t)\neq(k_1,t_1)$, then $\langle h_{k,t;j}^{(\ell)}, h_{k_1,t_1;j_1}^{(\ell)}\rangle = 0$ for all possible ℓ, j and j_1. Accordingly, we have

$$Y_\ell = Y_{\ell,1}+\cdots+Y_{\ell,N}, \quad \text{where } Y_{\ell,\gamma} = \sum_{(k,t)\in I_\gamma}\sum_{1\le j\le m(k)} h_{k,t;j}^{(\ell)}\otimes e_{k,j}$$

for each $\gamma\in\{1,\dots,N\}$. Taking the $E^{(\gamma,m)}$ and $E_\nu^{(\gamma,m)}$ given above, we have

$$Y_{\ell,\gamma} = \sum_{m=0}^{\infty} Y_{\ell,\gamma}^{(m)} \quad\text{and}\quad Y_{\ell,\gamma}^{(m)} = Y_{\ell,\gamma}^{(m,1)}+\cdots+Y_{\ell,\gamma}^{(m,2C_12^{2nm})},$$

where

$$Y_{\ell,\gamma}^{(m,\nu)} = \sum_{((k,j),(k,t))\in E_\nu^{(\gamma,m)}} h_{k,t;j}^{(\ell)}\otimes e_{k,j},$$

$1\le\nu\le 2C_12^{2nm}$. Again, the property of $E_\nu^{(\gamma,m)}$ ensures that

$$\||Y_{\ell,\gamma}^{(m,\nu)}|^{1/2}\|_\Phi = \Phi(\{\|h_{k,t;j}^{(\ell)}\|^{1/2}\}_{((k,j),(k,t))\in E_\nu^{(\gamma,m)}}).$$

Obviously, this is just a repeat of what happened with X_ℓ. The main difference between the case for X_ℓ and the case for Y_ℓ lies in the estimate for $\|h_{k,t;j}^{(\ell)}\|$.

Of course, we still have

$$\|h_{k,t;j}^{(\ell)}\|^2 \le \sum_{B(u_{k+\ell,j'},2^{-k-\ell})\cap B(u_{k,t},2^{-k})\neq\emptyset} |r(k,j;k+\ell,j')|^2.$$

For $((k,j),(k,t))\in E_\nu^{(\gamma,m)}$ with $m\ge 1$, if ℓ and j' are such that $B(u_{k+\ell,j'},2^{-k-\ell})\cap B(u_{k,t},2^{-k})\neq\emptyset$, then $2^{-k+m}\le d(u_{k,j},u_{k+\ell,j'})\le 2^{-k+m+3}$ as before. Thus it follows from inequality (3.6) in Lemma 3.5 that

$$\begin{aligned}|r(k,j;k+\ell,j')| &\le C2^{-(n+1)\ell}2^{-2im}|f_{z(k+\ell,j'),i}-f_{z(k,j),i}|\|(f-f_{z(k+\ell,j'),i})\psi_{z(k+\ell,j'),i}\| \\ &\le C2^{-(n+1)\ell}2^{-2im}(|f_{z(k+\ell,j'),i}-f_{z(k,j),i}|^2+\|(f-f_{z(k+\ell,j'),i})\psi_{z(k+\ell,j'),i}\|^2).\end{aligned}$$

By Lemma 3.7, this inequality also holds in the case $m=0$. Now if we define

$$b(k,j;m;\ell) = \max\{|f_{z(k+\ell,j'),i}-f_{z(k,j),i}| : d(u_{k+\ell,j'},u_{k,j})\le 2^{-k+m+3}\},$$

then, repeating the argument we used in the estimate of $\|g^{(\ell)}_{k,t;j}\|^2$, we have

$$\|h^{(\ell)}_{k,t;j}\|^2 \le C_9 2^{-2\ell} 2^{-4im}(b^2(k,j;m;\ell) + a^2(k,j;m;\ell))^2.$$

Therefore

$$\|h^{(\ell)}_{k,t;j}\|^{1/2} \le C_9^{1/4} 2^{-\ell/2} 2^{-im}(b(k,j;m;\ell) + a(k,j;m;\ell)).$$

Since the projection $((k,j),(k,t)) \mapsto (k,j)$ is injective on $E^{(\gamma,m)}_\nu$, it follows that

$$\begin{aligned} \| |Y^{(m,\nu)}_{\ell,\gamma}|^{1/2}\|_\Phi &= \Phi(\{\|h^{(\ell)}_{k,t;j}\|^{1/2}\}_{((k,j),(k,t))\in E^{(\gamma,m)}_\nu}) \\ &\le C_9^{1/4} 2^{-\ell/2} 2^{-im}(\Phi(\{b(k,j;m;\ell)\}_{(k,j)\in I}) + \Phi(\{a(k,j;m;\ell)\}_{(k,j)\in I})). \end{aligned} \tag{3.19}$$

Since we already have (3.17), we only need to estimate $\Phi(\{b(k,j;m;\ell)\}_{(k,j)\in I})$.

By Lemma 3.4, we have $|f_{z(k+\ell,j'),i} - f_{z(k,j),i}| \le C_{3.4} 2^{2nm} \sum_{t=0}^{\ell} M_i(f;k,j;t,m)$ if ℓ and j' satisfy the condition $d(u_{k+\ell,j'}, u_{k,j}) \le 2^{-k+m+3}$. Hence

$$b(k,j;m;\ell) \le C_{3.4} 2^{2nm} \sum_{t=0}^{\ell} M_i(f;k,j;t,m).$$

Applying Φ, we obtain

$$\Phi(\{b(k,j;m;\ell)\}_{(k,j)\in I}) \le C_{3.4} 2^{2nm} \sum_{t=0}^{\ell} \Phi(\{M_i(f;k,j;t,m)\}_{(k,j)\in I}). \tag{3.20}$$

By (3.2) and (3.1), $\Phi(\{M_i(f;k,j;t,m)\}_{(k,j)\in I})$ can be estimated using the same argument that was used in the estimate of $\Phi(\{a(k,j;m;\ell)\}_{(k,j)\in I})$. Thus, similar to (3.17), we have

$$\Phi(\{M_i(f;k,j;t,m)\}_{(k,j)\in I}) \le C_{16} 2^{2nm} \Phi(\{M_i(f;k,j)\}_{(k,j)\in I}),$$

where $M_i(f;k,j)$ was given by (3.1). Substituting this in (3.20), we find that

$$\Phi(\{b(k,j;m;\ell)\}_{(k,j)\in I}) \le C_{17}(1+\ell) 2^{4nm} \Phi(\{M_i(f;k,j)\}_{(k,j)\in I}). \tag{3.21}$$

Let us write

$$\begin{aligned} \mathcal{V} &= \Phi(\{M_i(f;k,j)\}_{(k,j)\in I}) \quad \text{and} \\ \mathcal{U} &= \Phi(\{\|(f - f_{z(k,j),i})\psi_{z(k,j),i}\|\}_{(k,j)\in I}). \end{aligned}$$

Combining (3.19), (3.21) and (3.17), we see that

$$\| |Y^{(m,\nu)}_{\ell,\gamma}|^{1/2}\|_\Phi \le C_{18}(1+\ell) 2^{-\ell/2} 2^{-(i-4n)m}(\mathcal{V} + \mathcal{U}).$$

Therefore, applying Lemma 3.1,

$$\begin{aligned}\||Y_{\ell,\gamma}^{(m)}|^{1/2}\|_\Phi &\le 2(\||Y_{\ell,\gamma}^{(m,1)}|^{1/2}\|_\Phi + \cdots + \||Y_{\ell,\gamma}^{(m,2C_1 2^{2nm})}|^{1/2}\|_\Phi) \\ &\le 4C_1C_{18}(1+\ell)2^{-\ell/2}2^{-(i-6n)m}(\mathcal{V}+\mathcal{U}).\end{aligned}$$

Since we assume $i \ge 6n+1$, we have $\sum_{m=0}^\infty 2^{-(i-6n)m} < \infty$. Consequently

$$\||Y_{\ell,\gamma}|^{1/2}\|_\Phi \le 2\sum_{m=0}^\infty \||Y_{\ell,\gamma}^{(m)}|^{1/2}\|_\Phi \le C_{19}(1+\ell)2^{-\ell/2}(\mathcal{V}+\mathcal{U}).$$

Since $Y_\ell = Y_{\ell,1} + \cdots + Y_{\ell,N}$, one more application of Lemma 3.1 leads to

$$\||Y_\ell|^{1/2}\|_\Phi \le 2\sum_{\gamma=1}^N \||Y_{\ell,\gamma}|^{1/2}\|_\Phi \le C_{20}(1+\ell)2^{-\ell/2}(\mathcal{V}+\mathcal{U}).$$

Recall from (3.18) that $\||X_\ell|^{1/2}\|_\Phi \le C_{15}2^{-\ell/2}\mathcal{U}$. Since $B_\ell = X_\ell + Y_\ell$, it now follows that

$$\||B_\ell|^{1/2}\|_\Phi \le 2\||X_\ell|^{1/2}\|_\Phi + 2\||Y_\ell|^{1/2}\|_\Phi \le C_{21}(1+\ell)2^{-\ell/2}(\mathcal{V}+\mathcal{U}).$$

Substituting this and (3.14) in (3.8), we have

$$\|A\|_\Phi \le 2C_7\mathcal{U} + 4C_{21}\sum_{\ell=1}^\infty (1+\ell)2^{-\ell/2}(\mathcal{V}+\mathcal{U}) \le C_{22}(\mathcal{V}+\mathcal{U}).$$

On the other hand, by Lemma 2.3 and Proposition 2.8, if Γ is a countable subset of $\mathbf{B}$ which has the property $\cup_{z\in\Gamma}D(z,b) = \mathbf{B}$, then

$$\mathcal{V}+\mathcal{U} \le 2C_{2.3}C_{2.8}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

Thus if we set $C_{3.9} = 2C_{22}C_{2.3}C_{2.8}$, then the lemma holds. $\square$

4. A Quasi-resolution of the Bergman Projection

Recall that $d\lambda$ denotes the Möbius-invariant measure on $\mathbf{B}$ given by (2.14). It is well known that the orthogonal projection $P : L^2(\mathbf{B}, dv) \to L_a^2(\mathbf{B}, dv)$ can be expressed as

$$P = \int k_z \otimes k_z d\lambda(z).$$

One can think of this formula as a "resolution" of the Bergman projection. But as we have seen in the previous sections, the kernel k_z is not good enough for our purposes. What we need is a formula in terms of the modified kernel $\psi_{z,i}$. Such a formula gives us a "quasi-resolution", as we will see. The idea of "resolving" the identity operator in terms of modified kernel first appeared in the study of Hankel operators on the Hardy space [4,Proposition 3.1]. Later, the use of "quasi-resolution" played a crucial role in establishing the Schatten-class membership for certain commutators on the Drury-Arveson space [5]. The same idea will again be crucial for this paper.

The reader will notice that the proof of our next proposition is very similar to the proof of Theorem 3.1 in [5]. Unfortunately, the minor difference in details makes it necessary for us to go through the exercise here again.

Proposition 4.1. *For each integer $i \geq 0$, there exist scalars $0 < c \leq C < \infty$ which are determined by i and n such that the self-adjoint operator*

$$R_i = \int \psi_{z,i} \otimes \psi_{z,i} d\lambda(z)$$

satisfies the operator inequality $cP \leq R_i \leq CP$ on the Hilbert space $L^2(\mathbf{B}, dv)$.

Proof. As in [5], for each $z \in \mathbf{B}$, introduce the function

$$g_z(\zeta) = \langle \zeta, z \rangle.$$

Write C_k^m for the binomial coefficient $m!/(k!(m-k)!)$ as usual. Then

$$\psi_{z,i} = (1 - |z|^2)^{((n+1)/2)+i} \sum_{k=0}^{\infty} C_k^{k+n+i} g_z^k,$$

and consequently

$$\psi_{z,i} \otimes \psi_{z,i} = (1 - |z|^2)^{n+1+2i} \sum_{j,k=0}^{\infty} C_k^{k+n+i} C_j^{j+n+i} g_z^k \otimes g_z^j.$$

For each $0 < \rho < 1$, define $\mathbf{B}_\rho = \{z : |z| < \rho\}$. Since both $d\lambda$ and $\mathbf{B}_\rho$ are invariant under the substitution $z \to e^{\theta\sqrt{-1}} z$, $\theta \in \mathbf{R}$, we have

$$\begin{aligned}\int_{\mathbf{B}_\rho} (1 - |z|^2)^{n+1+2i} g_z^k \otimes g_z^j d\lambda(z) &= \int_{\mathbf{B}_\rho} (1 - |e^{\theta\sqrt{-1}} z|^2)^{n+1+2i} g^k_{e^{\theta\sqrt{-1}} z} \otimes g^j_{e^{\theta\sqrt{-1}} z} d\lambda(z) \\ &= e^{(j-k)\theta\sqrt{-1}} \int_{\mathbf{B}_\rho} (1 - |z|^2)^{n+1+2i} g_z^k \otimes g_z^j d\lambda(z).\end{aligned}$$

This implies that

$$\int_{\mathbf{B}_\rho} (1 - |z|^2)^{n+1+2i} g_z^k \otimes g_z^j d\lambda(z) = 0 \quad \text{if} \quad k \neq j.$$

Therefore

$$\int_{\mathbf{B}_\rho} \psi_{z,i} \otimes \psi_{z,i} d\lambda(z) = \sum_{k=0}^{\infty} (C_k^{k+n+i})^2 \int_{\mathbf{B}_\rho} (1 - |z|^2)^{n+1+2i} g_z^k \otimes g_z^k d\lambda(z).$$

We follow the usual multi-index notation as, for example, given on page 3 in [13]. Then

$$g_z^k(\zeta) = \langle \zeta, z\rangle^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\bar{z}^\alpha \zeta^\alpha.$$

Consequently

$$g_z^k \otimes g_z^k = \sum_{|\alpha|=|\delta|=k} \frac{(k!)^2}{\alpha!\delta!}\bar{z}^\alpha z^\delta \zeta^\alpha \otimes \zeta^\delta.$$

Obviously, we have

$$\int_{\mathbf{B}_\rho} (1-|z|^2)^{n+1+2i}\bar{z}^\alpha z^\delta d\lambda(z) = 0 \quad \text{whenever} \quad \alpha \neq \delta.$$

Therefore

$$\int_{\mathbf{B}_\rho} (1-|z|^2)^{n+1+2i} g_z^k \otimes g_z^k d\lambda(z) = \sum_{|\alpha|=k} \frac{(k!)^2}{(\alpha!)^2} \int_{\mathbf{B}_\rho} (1-|z|^2)^{n+1+2i} |z^\alpha|^2 d\lambda(z) \zeta^\alpha \otimes \zeta^\alpha$$

and, consequently,

$$\begin{aligned}\int_{\mathbf{B}_\rho} \psi_{z,i} \otimes \psi_{z,i} d\lambda(z) &= \sum_{k=0}^\infty (C_k^{k+n+i})^2 \sum_{|\alpha|=k} \frac{(k!)^2}{(\alpha!)^2} \int_{\mathbf{B}_\rho} (1-|z|^2)^{n+1+2i} |z^\alpha|^2 d\lambda(z) \zeta^\alpha \otimes \zeta^\alpha \\ &= \sum_{k=0}^\infty (C_k^{k+n+i})^2 \sum_{|\alpha|=k} \frac{(k!)^2}{(\alpha!)^2} \int_{\mathbf{B}_\rho} (1-|z|^2)^{2i} |z^\alpha|^2 dv(z) \zeta^\alpha \otimes \zeta^\alpha. \qquad (4.1)\end{aligned}$$

By the formula $dv = 2nr^{2n-1}drd\sigma$, if $|\alpha| = k$, then

$$\begin{aligned}\int_{\mathbf{B}_\rho} (1-|z|^2)^{2i} |z^\alpha|^2 dv(z) &= \int_0^\rho (1-r^2)^{2i} 2nr^{2n+2k-1} dr \int_S |\xi^\alpha|^2 d\sigma(\xi) \\ &= \int_0^\rho (1-r^2)^{2i} 2nr^{2n+2k-1} dr \frac{(n-1)!\alpha!}{(n-1+k)!}, \qquad (4.2)\end{aligned}$$

where the second step follows from Proposition 1.4.9 in [13]. On the other hand,

$$2\int_0^1 (1-r^2)^{2i} r^{2n+2k-1} dr = \int_0^1 (1-x)^{2i} x^{n+k-1} dx = \frac{(2i)!(n+k-1)!}{(2i+n+k)!}.$$

Letting $\rho \uparrow 1$ in (4.1) and (4.2), easy algebra yields

$$\int \psi_{z,i} \otimes \psi_{z,i} d\lambda(z) = \sum_{k=0}^\infty b_{k,i} \sum_{|\alpha|=k} \frac{(n+k)!}{n!\alpha!} \zeta^\alpha \otimes \zeta^\alpha,$$

where

$$b_{k,i} = \frac{(n!)^2(2i)!}{((n+i)!)^2} \cdot \frac{((k+n+i)!)^2}{(2i+n+k)!(n+k)!}.$$

By Stirling's formula, there are $0 < c \le C < \infty$ determined by i and n such that

$$c \le b_{k,i} \le C$$

for every $k \ge 0$. Comparing this with the formula for the Bergman projection,

$$P = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{(n+k)!}{n!\alpha!} \zeta^\alpha \otimes \zeta^\alpha,$$

the proposition follows. □

Lemma 4.2. *Given any integer* $i \ge 1$, *there is a constant* $C_{4.2}$ *such that*

$$|\langle \psi_{z,i}, \psi_{w,i} \rangle| \le C_{4.2} e^{-i\beta(z,w)}$$

for all $z, w \in \mathbf{B}$.

Proof. Since z and w are interchangeable, it suffices to consider the case where we have $|w| \ge |z| > 0$. Recall that the formula

$$(U_z h)(\zeta) = h(\varphi_z(\zeta)) k_z(\zeta)$$

defines a unitary operator on $L_a^2(\mathbf{B}, dv)$ [13,Theorem 2.2.6]. Therefore

$$\langle \psi_{z,i}, \psi_{w,i} \rangle = \langle U_z \psi_{z,i}, U_z \psi_{w,i} \rangle.$$

It follows from Theorem 2.2.2 in [13] that

$$(U_z \psi_{z,i})(\zeta) = (1 - \langle \zeta, z \rangle)^i.$$

On the other hand,

$$(U_z \psi_{w,i})(\zeta) = H_{z,w}^i(\zeta)(U_z k_w)(\zeta),$$

where $H_{z,w}(\zeta) = (1 - |w|^2)/(1 - \langle \varphi_z(\zeta), w \rangle)$. Set $\mu = \varphi_z(w)$. Then $w = \varphi_z(\mu)$. Applying [13,Theorem 2.2.2] again, we have

$$H_{z,w}(\zeta) = \frac{(1-|w|^2)(1-\langle z, \mu \rangle)}{1-|z|^2} \cdot \frac{1-\langle \zeta, z \rangle}{1-\langle \zeta, \mu \rangle} = \frac{1-|w|^2}{1-\langle z, w \rangle} \cdot \frac{1-\langle \zeta, z \rangle}{1-\langle \zeta, \mu \rangle}.$$

Hence, if we define $h_z(\zeta) = 1 - \langle \zeta, z \rangle$ and $h_\mu(\zeta) = 1 - \langle \zeta, \mu \rangle$, then

$$\langle \psi_{z,i}, \psi_{w,i} \rangle = \left(\frac{1-|w|^2}{1-\langle w, z \rangle} \right)^i \cdot \langle h_z^i, M_{h_\mu^{-i} h_z^i} U_z k_w \rangle.$$

Note that if η is any monomial in $\zeta_1, \dots, \zeta_n$ of degree $i+1$ or greater, then $\langle h_z^i, \eta \rangle = 0$. Therefore

$$\langle \psi_{z,i}, \psi_{w,i} \rangle = \left(\frac{1-|w|^2}{1-\langle w, z \rangle} \right)^i \cdot \langle h_z^i, M_{q_{\mu,i} h_z^i} U_z k_w \rangle,$$

where

$$q_{\mu,i}(\zeta) = \sum_{j=0}^{i} \frac{(j+i-1)!}{j!(i-1)!} \langle \zeta, \mu \rangle^j.$$

Since $\|h_z\|_\infty \leq 2$, we have $|\langle h_z^i, M_{q_{\mu,i} h_z^i} U_z k_w \rangle| \leq 4^i \sum_{j=0}^{i} \frac{(j+i-1)!}{j!(i-1)!} = C_1$. Thus

$$|\langle \psi_{z,i}, \psi_{w,i} \rangle| \leq C_1 \left| \frac{1-|w|^2}{1-\langle w, z \rangle} \right|^i. \tag{4.3}$$

Using the assumption $|w| \geq |z|$ and (2.15), we have

$$\left| \frac{1-|w|^2}{1-\langle w, z \rangle} \right| \leq \frac{(1-|w|^2)^{1/2}(1-|z|^2)^{1/2}}{|1-\langle w, z \rangle|} = \sqrt{1-|\varphi_z(w)|^2}. \tag{4.4}$$

It is elementary that if $0 \leq x < 1$, then $\sqrt{1-x^2} \leq 2\exp(-(1/2)\log\{(1+x)/(1-x)\})$. Combining this with (4.3) and (4.4), we obtain

$$|\langle \psi_{z,i}, \psi_{w,i} \rangle| \leq 2^i C_1 e^{-i\beta(z,w)}$$

as promised. $\square$

Having gone through the proof of Lemma 3.9, our next proposition is almost trivial.

Proposition 4.3. *Let $0 < a < \infty$ and integer $i \geq 2n+1$ be given. Then there exists a constant $C_{4.3}$ which depends only on a, i and n such that for each a-separated subset Γ of* $\mathbf{B}$, *the operator*

$$A_{\Gamma,i} = \sum_{z \in \Gamma} \psi_{z,i} \otimes \psi_{z,i}$$

satisfies the norm estimate $\|A_{\Gamma,i}\| \leq C_{4.3}$.

Proof. Using the Möbius invariance of both the measure $d\lambda$ and the metric β, it is easy to verify that there is a constant C which such that $\lambda(D(\zeta, r)) \leq Ce^{2nr}$ for all $\zeta \in \mathbf{B}$ and $r > 0$. Thus given any $a > 0$, there is a C_1 which is determined by n and a such that for each a-separated subset Γ of $\mathbf{B}$, the inequality

$$\operatorname{card}(\Gamma \cap D(\zeta, r)) \leq C_1 e^{2nr} \tag{4.5}$$

holds for all $\zeta \in \mathbf{B}$ and $r > 0$.

Given an a-separated subset Γ of $\mathbf{B}$, let $\{e_z : z \in \Gamma\}$ be an orthonormal set indexed by Γ. Then we define

$$B_{\Gamma,i} = \sum_{z \in \Gamma} \psi_{z,i} \otimes e_z.$$

Since $\|A_{\Gamma,i}\| = \|B_{\Gamma,i}B^*_{\Gamma,i}\| = \|B^*_{\Gamma,i}B_{\Gamma,i}\|$, it suffices to estimate the latter. We have

$$B^*_{\Gamma,i}B_{\Gamma,i} = \sum_{z,w\in\Gamma} \langle \psi_{z,i}, \psi_{w,i}\rangle e_w \otimes e_z = Y + \sum_{m=1}^{\infty} Y^{(m)}, \tag{4.6}$$

where

$$Y = \sum_{z\in\Gamma} \|\psi_{z,i}\|^2 e_z \otimes e_z \qquad \text{and} \qquad Y^{(m)} = \sum_{\substack{ma\le\beta(z,w)<(m+1)a \\ z,w\in\Gamma}} \langle \psi_{z,i}, \psi_{w,i}\rangle e_w \otimes e_z.$$

Obviously, we have $\|Y\| \le \sup\{\|\psi_{z,i}\|^2 : z \in \Gamma\} \le 2^{2i}$. To estimate $\|Y^{(m)}\|$, let us define

$$E^{(m)} = \{(z,w) : z,w \in \Gamma, ma \le \beta(z,w) < (m+1)a\}.$$

By (4.5), for each $z \in \Gamma$ we have

$$\text{card}\{w \in \Gamma : (z,w) \in E^{(m)}\} \le C_1 e^{2n(m+1)a}.$$

Let $\nu(m) = 1+[C_1e^{2n(m+1)a}]$, where $[C_1e^{2n(m+1)a}]$ denotes the integer part of $C_1e^{2n(m+1)a}$. By Lemma 3.8, we have the partition $E^{(m)} = E^{(m)}_1 \cup \cdots \cup E^{(m)}_{2\nu(m)}$ such that for each $1 \le j \le 2\nu(m)$, if $(z,w),(z',w') \in E^{(m)}_j$ and if $(z,w) \ne (z',w')$, then we have both $z \ne z'$ and $w \ne w'$. Accordingly, $Y^{(m)} = Y^{(m)}_1 + \cdots + Y^{(m)}_{2\nu(m)}$, where

$$Y^{(m)}_j = \sum_{(z,w)\in E^{(m)}_j} \langle \psi_{z,i}, \psi_{w,i}\rangle e_w \otimes e_z$$

for each $1 \le j \le 2\nu(m)$. It follows from the property of $E^{(m)}_j$ that

$$\|Y^{(m)}_j\| = \sup\{|\langle \psi_{z,i}, \psi_{w,i}\rangle| : (z,w) \in E^{(m)}_j\}.$$

But for each $(z,w) \in E^{(m)}$, Lemma 4.2 gives us $|\langle \psi_{z,i}, \psi_{w,i}\rangle| \le C_{4.2}e^{-i\beta(z,w)} \le C_{4.2}e^{-ima}$. Hence $\|Y^{(m)}_j\| \le C_{4.2}e^{-ima}$ for each $1 \le j \le 2\nu(m)$. Consequently,

$$\begin{aligned}\|Y^{(m)}\| &\le \|Y^{(m)}_1\| + \cdots + \|Y^{(m)}_{2\nu(m)}\| \le 2\nu(m)C_{4.2}e^{-ima} \\ &\le 2(1 + C_1e^{2n(m+1)a})C_{4.2}e^{-ima} \le C_2e^{-(i-2n)ma}.\end{aligned}$$

Combining this with (4.6) and with the fact that $\|Y\| \le 2^{2i}$, we see that the constant $C_{4.3} = 2^{2i} + C_2\sum_{m=1}^{\infty} e^{-(i-2n)ma}$ will do for our purpose. $\square$

5. Upper Bound

The purpose of this section is to establish the upper bound for $\|[M_f, P]\|_\Phi$ given in Theorem 1.2. This requires all the preparations up to this point. In addition, we also need to recall a few elementary facts about symmetric gauge functions.

Given a symmetric gauge function Φ, the formula

$$\Phi^*(\{b_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^\infty a_j b_j\right| : \{a_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi(\{a_j\}_{j\in\mathbf{N}}) \le 1\right\}, \quad \{b_j\}_{j\in\mathbf{N}} \in \hat{c},$$

defines the symmetric gauge function that is dual to Φ [8,page 125]. For any $A \in \mathcal{C}_\Phi$ and $B \in \mathcal{C}_{\Phi^*}$, we have

$$|\mathrm{tr}(AB)| \le \|A\|_\Phi \|B\|_{\Phi^*}. \tag{5.1}$$

This follows from inequality (7.9) on page 63 of [8]. Moreover, we have the relation $\Phi^{**} = \Phi$ [8,page 125]. This relation implies that

$$\Phi(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^\infty a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi^*(\{b_j\}_{j\in\mathbf{N}}) \le 1\right\} \tag{5.2}$$

for each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$. Thus for each operator A, we have

$$\|A\|_\Phi = \sup\{|\mathrm{tr}(AB)| : \mathrm{rank}(B) < \infty, \|B\|_{\Phi^*} \le 1\}. \tag{5.3}$$

From (5.3) and (5.1) we immediately obtain

Lemma 5.1. *Let $\{A_k\}$ be a sequence of bounded operators on a separable Hilbert space $\mathcal{H}$. If $\{A_k\}$ weakly converges to an operator A, then the inequality*

$$\|A\|_\Phi \le \sup_k \|A_k\|_\Phi$$

holds for each symmetric gauge function Φ.

To prove the upper bound in Theorem 1.2, we begin with a variant of it involving the quasi-resolution introduced in Section 4.

Proposition 5.2. *Let integer $i \ge 6n+1$ be given and denote $i' = 3i+n+1$ as before. Let $0 < b < \infty$ also be given. Then there exists a constant $0 < C_{5.2} < \infty$ which depends only on b, i and n such that the inequality*

$$\|[M_f, R_{i'}]\|_\Phi \le C_{5.2}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})$$

holds for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every countable subset Γ of $\mathbf{B}$ which has the property $\cup_{z\in\Gamma} D(z,b) = \mathbf{B}$.

Proof. Set $\omega = (e^4 - 1)/(e^4 + 1)$. Then $D(0,2) = \mathbf{B}_\omega = \{\zeta \in \mathbf{C}^n : |\zeta| < \omega\}$. Let G be a subset of $\mathbf{B}$ which is maximal with respect to the property of being 1-separated. The maximality implies that $\cup_{z\in G} D(z,2) = \mathbf{B}$. Hence there are pairwise disjoint Borel sets $\{\Delta_z : z \in G\}$ such that $\Delta_z \subset D(z,2)$ for each $z \in G$ and $\cup_{z\in G}\Delta_z = \mathbf{B}$. We have $D(z,2) = \varphi_z(D(0,2))$ by the Möbius invariance of β. Thus for each $z \in G$, there is a Borel subset E_z of $D(0,2) = \mathbf{B}_\omega$ such that $\Delta_z = \varphi_z(E_z)$. By the Möbius invariance of $d\lambda$, we have

$$(5.4)\quad R_{i'} = \sum_{z\in G}\int_{\Delta_z} \psi_{\zeta,i'} \otimes \psi_{\zeta,i'} d\lambda(\zeta) = \sum_{z\in G}\int_{E_z} \psi_{\varphi_z(\zeta),i'} \otimes \psi_{\varphi_z(\zeta),i'} d\lambda(\zeta) = \int_{\mathbf{B}_\omega} T_\zeta d\lambda(\zeta),$$

where

$$T_\zeta = \sum_{z\in G} \chi_{E_z}(\zeta)\psi_{\varphi_z(\zeta),i'} \otimes \psi_{\varphi_z(\zeta),i'}.$$

This needs to be further decomposed.

Since G is 1-separated, there is a natural number N_0 such that for each $z \in G$,

$$(5.5)\qquad \operatorname{card}\{w \in G : \beta(w,z) < 6\} \le N_0 - 1.$$

We claim that for each $\zeta \in \mathbf{B}_\omega$ and each $z \in G$,

$$(5.6)\qquad \operatorname{card}\{w \in G : \beta(\varphi_z(\zeta), \varphi_w(\zeta)) < 2\} \le N_0 - 1.$$

Indeed if $\zeta \in \mathbf{B}_\omega$ and $z, w \in G$ are such that $\beta(\varphi_z(\zeta), \varphi_w(\zeta)) < 2$, then

$$\begin{aligned}\beta(z,w) = \beta(\varphi_z(0), \varphi_w(0)) &\le \beta(\varphi_z(0), \varphi_z(\zeta)) + \beta(\varphi_z(\zeta), \varphi_w(\zeta)) + \beta(\varphi_w(\zeta), \varphi_w(0)) \\ &= \beta(0,\zeta) + \beta(\varphi_z(\zeta), \varphi_w(\zeta)) + \beta(\zeta, 0) < 2 + 2 + 2 = 6.\end{aligned}$$

Hence (5.6) follows from (5.5). As a consequence of (5.6), for each $\zeta \in \mathbf{B}_\omega$ there is a partition

$$(5.7)\qquad G = G_\zeta^{(1)} \cup \cdots \cup G_\zeta^{(N_0)}$$

such that for each $1 \le \nu \le N_0$, if $z, w \in G_\zeta^{(\nu)}$ and if $z \ne w$, then $\beta(\varphi_z(\zeta), \varphi_w(\zeta)) \ge 2$. Applying Lemma 2.4 to the case $a = 1$, we obtain a $K \in \mathbf{N}$ for which the following holds: For each pair of $\zeta \in \mathbf{B}_\omega$ and $1 \le \nu \le N_0$, the set $G_\zeta^{(\nu)}$ admits a partition

$$(5.8)\qquad G_\zeta^{(\nu)} = G_\zeta^{(\nu,1)} \cup \cdots \cup G_\zeta^{(\nu,K)}$$

such that for each $1 \le \ell \le K$, the subset $G_\zeta^{(\nu,\ell)}$ has the property that

$$(5.9)\qquad \operatorname{card}\{z \in G_\zeta^{(\nu,\ell)} : \varphi_z(\zeta) \in T_{k,j}\} \le 1$$

for every $(k,j) \in I$.

Now consider any finite subset F of G. Accordingly, we define

$$T_{\zeta,F} = \sum_{z\in F} \chi_{E_z}(\zeta)\psi_{\varphi_z(\zeta),i'} \otimes \psi_{\varphi_z(\zeta),i'} \tag{5.10}$$

for each $\zeta \in \mathbf{B}_\omega$, and then define

$$R_{i',F} = \int_{\mathbf{B}_\omega} T_{\zeta,F} d\lambda(\zeta). \tag{5.11}$$

By (5.7) and (5.8) we have

$$T_{\zeta,F} = \sum_{\nu=1}^{N_0}\sum_{\ell=1}^{K} T_{\zeta,F}^{(\nu,\ell)}, \tag{5.12}$$

where

$$T_{\zeta,F}^{(\nu,\ell)} = \sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \chi_{E_z}(\zeta)\psi_{\varphi_z(\zeta),i'} \otimes \psi_{\varphi_z(\zeta),i'}$$

for each pair of $1 \le \nu \le N_0$ and $1 \le \ell \le K$.

Let $\{e_z : z \in G\}$ be an orthonormal set. Given any $f \in L^2(\mathbf{B}, dv)$, we have

$$[M_f, T_{\zeta,F}^{(\nu,\ell)}] =$$
$$\sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \chi_{E_z}(\zeta)\{((f - f_{\varphi_z(\zeta),i})\psi_{\varphi_z(\zeta),i'}) \otimes \psi_{\varphi_z(\zeta),i'} - \psi_{\varphi_z(\zeta),i'} \otimes ((\bar f - \bar f_{\varphi_z(\zeta),i})\psi_{\varphi_z(\zeta),i'})\}$$
$$= X_{\zeta,F}^{(\nu,\ell)} - (Y_{\zeta,F}^{(\nu,\ell)})^*,$$

where

$$X_{\zeta,F}^{(\nu,\ell)} = \sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \chi_{E_z}(\zeta)((f - f_{\varphi_z(\zeta),i})\psi_{\varphi_z(\zeta),i'}) \otimes \psi_{\varphi_z(\zeta),i'},$$
$$Y_{\zeta,F}^{(\nu,\ell)} = \sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \chi_{E_z}(\zeta)((\bar f - \bar f_{\varphi_z(\zeta),i})\psi_{\varphi_z(\zeta),i'}) \otimes \psi_{\varphi_z(\zeta),i'}.$$

Using the orthonormal set $\{e_z : z \in G\}$, we can factor $X_{\zeta,F}^{(\nu,\ell)}$ as

$$X_{\zeta,F}^{(\nu,\ell)} = A_{\zeta,F}^{(\nu,\ell)}(B_{\zeta,F}^{(\nu,\ell)})^*,$$

where

$$A_{\zeta,F}^{(\nu,\ell)} = \sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \chi_{E_z}(\zeta)((f - f_{\varphi_z(\zeta),i})\psi_{\varphi_z(\zeta),i'}) \otimes e_z \quad \text{and}$$
$$B_{\zeta,F}^{(\nu,\ell)} = \sum_{z\in F\cap G_\zeta^{(\nu,\ell)}} \psi_{\varphi_z(\zeta),i'} \otimes e_z.$$

Let Φ be a symmetric gauge function, and let Γ be a countable subset of $\mathbf{B}$ which has the property $\cup_{z\in\Gamma}D(z,b) = \mathbf{B}$. Because of (5.9), we can apply Lemma 3.9 to $A^{(\nu,\ell)}_{\zeta,F}$ to obtain

$$\|A^{(\nu,\ell)}_{\zeta,F}\|_\Phi \le C_{3.9}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

Since the set $\{\varphi_z(\zeta) : z \in F \cap G^{(\nu,\ell)}_\zeta\}$ is 1-separated and since the map $z \mapsto \varphi_z(\zeta)$ is injective on $F \cap G^{(\nu,\ell)}_\zeta$, it follows from Proposition 4.3 that $\|(B^{(\nu,\ell)}_{\zeta,F})^*\|^2 = \|B^{(\nu,\ell)}_{\zeta,F}(B^{(\nu,\ell)}_{\zeta,F})^*\| \le C_{4.3}$. Thus

$$\|X^{(\nu,\ell)}_{\zeta,F}\|_\Phi \le \|A^{(\nu,\ell)}_{\zeta,F}\|_\Phi\|(B^{(\nu,\ell)}_{\zeta,F})^*\| \le C_{3.9}C^{1/2}_{4.3}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

Obviously, the same argument is applicable to $\|Y^{(\nu,\ell)}_{\zeta,F}\|_\Phi = \|(Y^{(\nu,\ell)}_{\zeta,F})^*\|_\Phi$. Therefore

$$\|[M_f, T^{(\nu,\ell)}_{\zeta,F}]\|_\Phi \le 2C_{3.9}C^{1/2}_{4.3}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

Combining this with (5.12), we find that

$$\begin{aligned}\|[M_f, T_{\zeta,F}]\|_\Phi &\le 2N_0KC_{3.9}C^{1/2}_{4.3}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})\\ &= C_1\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).\end{aligned}\tag{5.13}$$

Let B be a finite-rank operator. Then it follows from (5.11) that

$$|\mathrm{tr}([M_f, R_{i',F}]B)| = \left|\int_{\mathbf{B}_\omega}\mathrm{tr}([M_f, T_{\zeta,F}]B)d\lambda(\zeta)\right| \le \lambda(\mathbf{B}_\omega)\sup_{\zeta\in\mathbf{B}_\omega}|\mathrm{tr}([M_f, T_{\zeta,F}]B)|.$$

Applying (5.1) and (5.13), we obtain

$$\begin{aligned}|\mathrm{tr}([M_f, R_{i',F}]B)| &\le \lambda(\mathbf{B}_\omega)\sup_{\zeta\in\mathbf{B}_\omega}\|[M_f, T_{\zeta,F}]\|_\Phi\|B\|_{\Phi^*}\\ &\le \lambda(\mathbf{B}_\omega)C_1\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})\|B\|_{\Phi^*}.\end{aligned}$$

Since this holds for every finite-rank operator B, by (5.3) this implies

$$\|[M_f, R_{i',F}]\|_\Phi \le \lambda(\mathbf{B}_\omega)C_1\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).\tag{5.14}$$

To complete the proof of the proposition, let us pick a sequence of finite subsets $\{F_\mu\}$ of G such that $F_\mu \subset F_{\mu+1}$ for every μ and such that $\cup_{\mu=1}^\infty F_\mu = G$. Then we have

$$\lim_{\mu\to\infty} R_{i',F_\mu} = R_{i'}$$

in the strong operator topology. Thus it follows from Lemma 5.1 and (5.14) that

$$\|[M_f, R_{i'}]\|_\Phi \le \sup_{\mu\ge1}\|[M_f, R_{i',F_\mu}]\|_\Phi \le \lambda(\mathbf{B}_\omega)C_1\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}).$$

This completes the proof of the proposition. □

Proposition 5.3. *Given any integer $i \geq 0$, there is a constant $C_{5.3}(i)$ which depends only on i and n such that the inequality*

$$\|[M_f, P]\|_\Phi \leq C_{5.3}(i)\|[M_f, R_i]\|_\Phi$$

holds for every $f \in L^2(\mathbf{B}, dv)$ and every symmetric gauge function Φ.

Proof. By Proposition 4.1, there are $0 < c \leq C < \infty$ such that $cP \leq R_i \leq CP$. This means that the spectrum of the self-adjoint operator R_i is contained in $\{0\} \cup [c, C]$. Let $\mathcal{T}$ be the circle in $\mathbf{C}$ with center located at the point $(c + C)/2$ and with radius equal to $C/2 = \{(C - c)/2\} + (c/2)$. Furthermore, let $\mathcal{T}$ be oriented in the counter-clockwise direction. From the spectral decomposition of R_i we obtain

$$P = \frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{T}} (\tau - R_i)^{-1} d\tau.$$

Thus for each $f \in L^2(\mathbf{B}, dv)$ we have

$$[M_f, P] = \frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{T}} (\tau - R_i)^{-1}[M_f, R_i](\tau - R_i)^{-1} d\tau. \tag{5.15}$$

Since the spectrum of R_i is contained in $\{0\} \cup [c, C]$, the function

$$\tau \mapsto (\tau - R_i)^{-1}$$

is continuous with respect to the operator norm on the contour $\mathcal{T}$. Approximating the right-hand side of (5.15) by, for example, Riemann sums and then applying Lemma 5.1, for each symmetric gauge function Φ we have

$$\|[M_f, P]\|_\Phi \leq \frac{|\mathcal{T}|}{2\pi} \sup_{\tau \in \mathcal{T}} \|(\tau - R_i)^{-1}\|\|[M_f, R_i]\|_\Phi\|(\tau - R_i)^{-1}\|,$$

where $|\mathcal{T}|$ denotes the length of $\mathcal{T}$. This gives us the desired conclusion. □

Proof of the upper bound in Theorem 1.2. Given integer $i \geq 6n+1$, write $i' = 3i+n+1$ as before. Then it follows from Propositions 5.3 and 5.2 that for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every a, b-lattice Γ in $\mathbf{B}$, we have

$$\|[M_f, P]\|_\Phi \leq C_{5.3}(i')\|[M_f, R_{i'}]\|_\Phi \leq C_{5.3}(i')C_{5.2}\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})$$

This establishes the desired upper bound for $\|[M_f, P]\|_\Phi$. □

Remark 5.4. A minor issue in Theorem 1.2 is the stated lower limit $6n+1$ for the integer i. The number $6n + 1$ came up naturally in the proof of Lemma 3.9, as we saw. But Theorem 1.2 actually holds for smaller i for the following reason. Suppose that i_1 and i_2

are integers greater than $n/2$. Let a, b be positive numbers with $b \geq 2a$. Then it follows from Propositions 2.5 and 2.8 that there is a constant $C(i_1, i_2; a, b)$ such that

$$\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i_1}, \tilde{\psi}_{z,i_1}\rangle)\psi_{z,i_1}\|\}_{z\in\Gamma}) \leq C(i_1, i_2; a, b)\Phi(\{\|(f - \langle f\tilde{\psi}_{z,i_2}, \tilde{\psi}_{z,i_2}\rangle)\psi_{z,i_2}\|\}_{z\in\Gamma})$$

for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every a, b-lattice Γ in $\mathbf{B}$.

6. Lower Bound

The proof of the lower bound in Theorem 1.2 involves estimates in the trace class. Following the usual practice, we will write $\|\cdot\|_1$ for the norm of the trace class, while the norm of the Hilbert-Schmidt class will be denoted by $\|\cdot\|_2$.

Lemma 6.1. *There is a constant $C_{6.1}$ such that for each $f \in L^2(\mathbf{B}, dv)$ and each $(k, j) \in I$, we have*

$$\|M_{\chi_{Q_{k,j}}}[M_f, P]M_{\chi_{Q_{k,j}}}\|_1 \leq C_{6.1}V^{1/2}(f; Q_{k,j}).$$

Proof. Given any $f \in L^2(\mathbf{B}, dv)$ and $(k, j) \in I$, we have

$$M_{\chi_{Q_{k,j}}}[M_f, P]M_{\chi_{Q_{k,j}}} = M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}PM_{\chi_{Q_{k,j}}} - M_{\chi_{Q_{k,j}}}PM_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}.$$

By the relation $\|AB\|_1 \leq \|A\|_2\|B\|_2$, we have

$$\|M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}PM_{\chi_{Q_{k,j}}}\|_1 \leq \|M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}P\|_2\|PM_{\chi_{Q_{k,j}}}\|_2. \tag{6.1}$$

But

$$\begin{aligned}\|M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}P\|_2^2 &= \int_{Q_{k,j}} |f(w) - f_{Q_{k,j}}|^2 \left(\int \frac{dv(z)}{|1 - \langle w, z\rangle|^{2(n+1)}}\right) dv(w)\\ &= \int_{Q_{k,j}} \frac{|f(w) - f_{Q_{k,j}}|^2}{(1 - |w|^2)^{n+1}} dv(w) \leq \int_{Q_{k,j}} \frac{|f(w) - f_{Q_{k,j}}|^2}{2^{-2(n+1)(k+2)}} dv(w).\end{aligned}$$

There is a C such that $v(Q_{k,j}) \leq C2^{-2(n+1)k}$ for every $(k, j) \in I$. Therefore

$$\|M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}P\|_2^2 \leq C_1V(f; Q_{k,j}). \tag{6.2}$$

For the same reason, we have

$$\|PM_{\chi_{Q_{k,j}}}\|_2^2 = \int_{Q_{k,j}} \left(\int \frac{dv(w)}{|1 - \langle w, z\rangle|^{2(n+1)}}\right) dv(z) = \int_{Q_{k,j}} \frac{dv(z)}{(1 - |z|^2)^{n+1}} \leq C_1. \tag{6.3}$$

Combining (6.1), (6.2) and (6.3), we find that

$$\|M_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}PM_{\chi_{Q_{k,j}}}\|_1 \leq C_1V^{1/2}(f; Q_{k,j}).$$

Obviously, the same estimate also holds for $\|M_{\chi_{Q_{k,j}}}PM_{(f-f_{Q_{kj}})\chi_{Q_{k,j}}}\|_1$. Therefore the lemma follows. □

We need the following "condensation inequality" for symmetric gauge functions:

Lemma 6.2. *If $A_1, \dots, A_m, \dots$ are trace-class operators, then the inequality*

$$\|A_1 \oplus \cdots \oplus A_m \oplus \cdots\|_\Phi \le \Phi(\{\|A_1\|_1, \dots, \|A_m\|_1, \dots\})$$

holds for every symmetric gauge function Φ.

This lemma was first established in [6] (see Lemma 4.2 in that paper). But since its proof is really simple, let us produce it here anyway.

Consider a sequence of the form $\{a, b, c_3, \dots, c_k, \dots\}$, where $a > 0$ and $b > 0$. Since

$$\{a, b, c_3, \dots, c_k, \dots\} = \frac{a}{a+b}\{a+b, 0, c_3, \dots, c_k, \dots\} + \frac{b}{a+b}\{0, a+b, c_3, \dots, c_k, \dots\},$$

for each symmetric gauge function Φ we have

$$\begin{aligned}\Phi(\{a, b, c_3, \dots, c_k, \dots\}) \le &\frac{a}{a+b}\Phi(\{a+b, 0, c_3, \dots, c_k, \dots\}) + \\ &\frac{b}{a+b}\Phi(\{0, a+b, c_3, \dots, c_k, \dots\}).\end{aligned}$$

Since $\Phi(\{a+b, 0, c_3, \dots, c_k, \dots\}) = \Phi(\{0, a+b, c_3, \dots, c_k, \dots\})$, it follows that

$$\Phi(\{a, b, c_3, \dots, c_k, \dots\}) \le \Phi(\{a+b, 0, c_3, \dots, c_k, \dots\}).$$

Applying this inequality repeatedly, we see that

$$\|A_1 \oplus \cdots \oplus A_m \oplus 0 \oplus \cdots \oplus 0 \oplus \dots\|_\Phi \le \Phi(\{\|A_1\|_1, \dots, \|A_m\|_1, 0, \dots, 0, \dots\})$$

for any number of finite-rank operators $A_1, \dots, A_m$. Once this is established, applying Lemma 5.1, the general case of Lemma 6.2 follows.

The proof of our next lemma will again use the duality between symmetric gauge functions discussed in Section 5.

Lemma 6.3. *There is a constant $C_{6.3}$ such that the inequality*

$$\Phi(\{V^{1/2}(f; Q_{k,j})\}_{(k,j)\in I}) \le C_{6.3}\|[M_f, P]\|_\Phi$$

holds for every $f \in L^2(\mathbf{B}, dv)$ and every symmetric gauge function Φ.

Proof. By design, we have $Q_{k,j} \cap Q_{k',j'} = \emptyset$ whenever $k' \ge k+2$. Hence, by (2.6), (2.3) and (2.2), there is a partition

$$I = I_1 \cup \cdots \cup I_N$$

of the index set I such that for each $1 \le \nu \le N$, if $(k,j),(k',j') \in I_\nu$ and if $(k,j) \ne (k',j')$, then $Q_{k,j} \cap Q_{k',j'} = \emptyset$. It suffices to show that there is a C such that the inequality

$$\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I_\nu}) \le C\|[M_f,P]\|_\Phi \tag{6.4}$$

holds for all $1 \le \nu \le N$, $f \in L^2(\mathbf{B},dv)$, and symmetric gauge functions Φ.

Let $\{b_{k,j}\}_{(k,j)\in I_\nu}$ be a set of non-negative numbers such that $b_{k,j} = 0$ for all but a finite number of (k,j)'s. Let $f \in L^2(\mathbf{B},dv)$ be given. For each $(k,j) \in I_\nu$, define

$$c_{k,j} = \begin{cases} V^{-1/2}(f;Q_{k,j}) & \text{if} \quad V(f;Q_{k,j}) \ne 0 \\ \\ \quad 0 & \text{if} \quad V(f;Q_{k,j}) = 0 \end{cases}.$$

By Lemma 6.1, we have $c_{k,j}\|M_{\chi_{Q_{k,j}}}[M_f,P]M_{\chi_{Q_{k,j}}}\|_1 \le C_{6.1}$. Define the operator

$$B = \sum_{(k,j)\in I_\nu} b_{k,j}c_{k,j}M_{\chi_{Q_{k,j}}}[M_f,P]M_{\chi_{Q_{k,j}}}.$$

The property of I_ν ensures that $M_{\chi_{Q_{k,j}}}M_{\chi_{Q_{k',j'}}} = 0$ if $(k,j),(k',j') \in I_\nu$ and $(k,j) \ne (k',j')$. This allows us to apply Lemma 6.2 to obtain

$$\|B\|_{\Phi^*} \le \Phi^*(\{b_{k,j}c_{k,j}\|M_{\chi_{Q_{k,j}}}[M_f,P]M_{\chi_{Q_{k,j}}}\|_1\}_{(k,j)\in I_\nu}) \le C_{6.1}\Phi^*(\{b_{k,j}\}_{(k,j)\in I_\nu}). \tag{6.5}$$

Since we assume that $b_{k,j} = 0$ for all but a finite number of (k,j)'s, there is a $0 < \rho < 1$ which depends on the choice of $\{b_{k,j}\}_{(k,j)\in I_\nu}$ such that $Q_{k,j} \subset \mathbf{B}_\rho = \{\zeta : |\zeta| < \rho\}$ whenever $b_{k,j} \ne 0$. Now we have

$$\operatorname{tr}(M_{\chi_{\mathbf{B}_\rho}}[M_f,P]^*M_{\chi_{\mathbf{B}_\rho}}B) = \sum_{(k,j)\in I_\nu} b_{k,j}c_{k,j}\iint_{Q_{k,j}\times Q_{k,j}} \frac{|f(w)-f(z)|^2}{|1-\langle w,z\rangle|^{2(n+1)}}dv(w)dv(z).$$

For $w,z \in Q_{k,j}$, we have $z = |z|\xi$ and $w = |w|u$ with $\xi,u \in B(u_{k,j}, 9\cdot 2^{-k})$. Therefore

$$|1-\langle w,z\rangle| \le 1-|w|+1-|z|+|1-\langle u,\xi\rangle| \le 2^{-2k}+2^{-2k}+(18\cdot 2^{-k})^2 = 326\cdot 2^{-2k}.$$

Hence there is a $\delta > 0$ such that for each $(k,j) \in I$, if $w,z \in Q_{k,j}$, then $|1-\langle w,z\rangle|^{-2(n+1)} \ge \delta/v^2(Q_{k,j})$. Therefore

$$\begin{aligned}\operatorname{tr}(M_{\chi_{\mathbf{B}_\rho}}[M_f,P]^*M_{\chi_{\mathbf{B}_\rho}}B) &\ge \sum_{(k,j)\in I_\nu} b_{k,j}c_{k,j}\iint_{Q_{k,j}\times Q_{k,j}} \frac{\delta|f(w)-f(z)|^2}{v^2(Q_{k,j})}dv(w)dv(z) \\ &= 2\delta\sum_{(k,j)\in I_\nu} b_{k,j}c_{k,j}V(f;Q_{k,j}) = 2\delta\sum_{(k,j)\in I_\nu} b_{k,j}V^{1/2}(f;Q_{k,j}).\end{aligned}$$

On the other hand, by (5.1) and (6.5), we have

$$\begin{aligned}\operatorname{tr}(M_{\chi_{\mathbf{B}_\rho}}[M_f,P]^* M_{\chi_{\mathbf{B}_\rho}} B) &\le \|M_{\chi_{\mathbf{B}_\rho}}[M_f,P]^* M_{\chi_{\mathbf{B}_\rho}}\|_\Phi \|B\|_{\Phi^*} \\ &\le \|[M_f,P]^*\|_\Phi C_{6.1}\Phi^*(\{b_{k,j}\}_{(k,j)\in I_\nu}) = C_{6.1}\|[M_f,P]\|_\Phi \Phi^*(\{b_{k,j}\}_{(k,j)\in I_\nu}).\end{aligned}$$

Combining the above two inequalities, we obtain

$$2\delta \sum_{(k,j)\in I_\nu} b_{k,j} V^{1/2}(f;Q_{k,j}) \le C_{6.1}\|[M_f,P]\|_\Phi \Phi^*(\{b_{k,j}\}_{(k,j)\in I_\nu})$$

for every set of non-negative numbers $\{b_{k,j}\}_{(k,j)\in I_\nu}$ which has the property that $b_{k,j} = 0$ for all but a finite number of (k,j)'s. By (5.2), this means $2\delta\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I_\nu}) \le C_{6.1}\|[M_f,P]\|_\Phi$, i.e., (6.4). This completes the proof. □

Our next lemma is an improvement of Lemma 6 in [15]. But this improvement actually involves something subtle.

Lemma 6.4. *There is a constant $C_{6.4}$ such that the inequality*

$$\Phi(\{V^s(f;W_{k,j})\}_{(k,j)\in I}) \le \frac{C_{6.4}}{1-2^{-s}}\Phi(\{V^s(f;Q_{k,j})\}_{(k,j)\in I})$$

holds for every $f \in L^2(\mathbf{B}, dv)$, every symmetric gauge function Φ, and every $0 < s \le 1$.

Proof. By (2.7), for each $(k,j) \in I$, we have

$$\begin{aligned}V(f;W_{k,j}) &\le \frac{1}{v(W_{k,j})}\int_{W_{k,j}} |f - f_{Q_{k,j}}|^2 dv \\ &\le \frac{v(Q_{k,j})}{v(W_{k,j})} V(f;Q_{k,j}) + \sum_{(\ell,i)\in F_{k,j}} \frac{v(Q_{\ell,i})}{v(W_{k,j})}\cdot\frac{1}{v(Q_{\ell,i})}\int_{Q_{\ell,i}} |f - f_{Q_{k,j}}|^2 dv.\end{aligned}$$

Since $v(Q_{\ell,i}) \le C_1 2^{-2(n+1)\ell}$ and $v(W_{k,j}) \ge v(Q_{k,j}) \ge C_2 2^{-2(n+1)k}$, it follows that

$$V(f;W_{k,j}) \le V(f;Q_{k,j}) + C_3 \sum_{(\ell,i)\in F_{k,j}} 2^{-2(n+1)(\ell-k)}\frac{1}{v(Q_{\ell,i})}\int_{Q_{\ell,i}} |f - f_{Q_{k,j}}|^2 dv. \tag{6.6}$$

Consider any $(\ell,i) \in F_{k,j}$. Pick an $x \in B(u_{k,j}, 3\cdot 2^{-k}) \cap B(u_{\ell,i}, 2^{-\ell})$, which is possible since the intersection is non-empty by the definition of $F_{k,j}$. Then by (2.4) there is a chain of elements $\{(t,i(t)) : k \le t \le \ell\}$ in I such that $(\ell, i(\ell)) = (\ell, i)$, $(k, i(k)) = (k,j)$, and $x \in B(u_{t,i(t)}, 2^{-t})$ for each $k < t \le \ell$. This implies that

$$Q_{t,i(t)} \cap Q_{t+1,i(t+1)} \supset T_{t+1,i(t+1)} \qquad \text{for each } k \le t < \ell.$$

Indeed, since $x \in B(u_{t,i(t)}, 3\cdot 2^{-t}) \cap B(u_{t+1,i(t+1)}, 2^{-t-1})$, we have $B(u_{t+1,i(t+1)}, 2^{-t-1}) \subset B(u_{t,i(t)}, 9\cdot 2^{-t})$. The above assertion now follows from (2.5) and (2.6). Since $v(T_{t+1,i(t+1)})$

$\geq C_4 2^{-2(n+1)(t+1)}$, we have $v(Q_{t,i(t)})/v(Q_{t,i(t)} \cap Q_{t+1,i(t+1)}) \leq 2^{2(n+1)} C_1/C_4$. Applying (2.8), we find that

$$|f_{Q_{t,i(t)}} - f_{Q_{t+1,i(t+1)}}| \leq C_5(V^{1/2}(f; Q_{t,i(t)}) + V^{1/2}(f; Q_{t+1,i(t+1)}))$$

if $k \leq t < \ell$. Therefore

$$\begin{aligned} |f_{Q_{k,j}} - f_{Q_{\ell,i}}|^2 &\leq \left(\sum_{t=k}^{\ell-1} |f_{Q_{t,i(t)}} - f_{Q_{t+1,i(t+1)}}| \right)^2 \leq \left(2C_5 \sum_{t=k}^{\ell} V^{1/2}(f; Q_{t,i(t)}) \right)^2 \\ &\leq 4C_5^2(1+\ell-k) \sum_{t=k}^{\ell} V(f; Q_{t,i(t)}). \end{aligned}$$

Let $G_{k,j;\ell,i} = \{(\nu,h) : k \leq \nu \leq \ell,\ 1 \leq h \leq m(\nu),\ B(u_{\nu,h}, 2^{-\nu}) \cap B(u_{\ell,i}, 2^{-\ell}) \neq \emptyset$ and $B(u_{\nu,h}, 2^{-\nu}) \cap B(u_{k,j}, 3 \cdot 2^{-k}) \neq \emptyset\}$. By the above choice of $(t, i(t))$, we have $(t, i(t)) \in G_{k,j;\ell,i}$ for all $k \leq t \leq \ell$. Therefore

$$|f_{Q_{\ell,i}} - f_{Q_{k,j}}|^2 \leq 4C_5^2(1+\ell-k) \sum_{(\nu,h) \in G_{k,j;\ell,i}} V(f; Q_{\nu,h}). \tag{6.7}$$

Substituting $2|f - f_{Q_{\ell,i}}|^2 + 2|f_{Q_{\ell,i}} - f_{Q_{k,j}}|^2$ for $|f - f_{Q_{k,j}}|^2$ in (6.6) and then applying (6.7), we obtain

$$V(f; W_{k,j}) \leq V(f; Q_{k,j}) + C_6(A_{k,j} + B_{k,j}),$$

where

$$\begin{aligned} A_{k,j} &= \sum_{(\ell,i) \in F_{k,j}} 2^{-2(n+1)(\ell-k)} V(f; Q_{\ell,i}), \\ B_{k,j} &= \sum_{(\ell,i) \in F_{k,j}} 2^{-2(n+1)(\ell-k)} (1+\ell-k) \sum_{(\nu,h) \in G_{k,j;\ell,i}} V(f; Q_{\nu,h}). \end{aligned}$$

Since $(\ell, i) \in G_{k,j;\ell,i}$, we have $A_{k,j} \leq B_{k,j}$. Therefore

$$V(f; W_{k,j}) \leq V(f; Q_{k,j}) + 2C_6 B_{k,j}. \tag{6.8}$$

Let us estimate $B_{k,j}$. First of all, if we set $C_7 = \sup_{m \geq 0} 2^{-m/2}(1+m)$, then

$$B_{k,j} \leq C_7 \sum_{(\ell,i) \in F_{k,j}} \sum_{(\nu,h) \in G_{k,j;\ell,i}} 2^{-2(n+(3/4))(\ell-k)} V(f; Q_{\nu,h}). \tag{6.9}$$

For each (ν, h), if $\nu \leq \ell$ and $B(u_{\nu,h}, 2^{-\nu}) \cap B(u_{\ell,i}, 2^{-\ell}) \neq \emptyset$, then $B(u_{\nu,h}, 3 \cdot 2^{-\nu}) \supset B(u_{\ell,i}, 2^{-\ell})$. Thus by (2.3) and (2.2), for each (ν, h) with $\nu \leq \ell$, the cardinality of the set $\{i : 1 \leq i \leq m(\ell),\ B(u_{\ell,i}, 2^{-\ell}) \cap B(u_{\nu,h}, 2^{-\nu}) \neq \emptyset\}$ is at most $C_8 2^{2n(\ell-\nu)}$. Set

$$G_{k,j} = \{(\nu,h) : \nu \geq k, 1 \leq h \leq m(\nu), B(u_{\nu,h}, 2^{-\nu}) \cap B(u_{k,j}, 3 \cdot 2^{-k}) \neq \emptyset\}.$$

Then a change of the order of summation in (6.9) yields

$$
\begin{aligned}
B_{k,j} &\le C_7 \sum_{(\nu,h)\in G_{k,j}} V(f;Q_{\nu,h})\times \\
&\qquad \sum_{\ell=\nu}^{\infty} 2^{-2(n+(3/4))(\ell-k)} \mathrm{card}\{i : B(u_{\ell,i},2^{-\ell}) \cap B(u_{\nu,h},2^{-\nu}) \ne \emptyset\} \\
&\le C_7C_8 \sum_{(\nu,h)\in G_{k,j}} V(f;Q_{\nu,h}) \sum_{\ell=\nu}^{\infty} 2^{-2(n+(3/4))(\ell-k)} \cdot 2^{2n(\ell-\nu)} \\
&\le C_7C_8 \sum_{(\nu,h)\in G_{k,j}} V(f;Q_{\nu,h}) 2^{-2(n+(1/2))(\nu-k)} \sum_{\ell=\nu}^{\infty} 2^{-(1/2)(\ell-k)} \\
&\le C_9 \sum_{(\nu,h)\in G_{k,j}} V(f;Q_{\nu,h}) 2^{-2(n+(1/2))(\nu-k)}. \qquad (6.10)
\end{aligned}
$$

Up to this point, the proof is basically a repeat of a part of the proof of Lemma 6 in [15]. The more subtle part comes next when we bring in general symmetric gauge functions.

The idea is to further analyze $B_{k,j}$. First of all, we claim that there is a natural number N such that for each $(k,j) \in I$, we have

$$
\mathrm{card}\{j' \in \{1,\dots m(k)\} : G_{k,j'} \cap G_{k,j} \ne \emptyset\} \le N. \qquad (6.11)
$$

To prove this, consider any $1 \le j' \le m(k)$ such that $G_{k,j'} \cap G_{k,j} \ne \emptyset$. This means that there exist a $\nu \ge k$ and a $1 \le h \le m(\nu)$ such that

$$
B(u_{\nu,h},2^{-\nu}) \cap B(u_{k,j'},3\cdot 2^{-k}) \ne \emptyset \quad \text{and} \quad B(u_{\nu,h},2^{-\nu}) \cap B(u_{k,j},3\cdot 2^{-k}) \ne \emptyset.
$$

Since $k \le \nu$, this gives us $d(u_{\nu,h},u_{k,j'}) \le 4\cdot 2^{-k}$ and $d(u_{\nu,h},u_{k,j}) \le 4\cdot 2^{-k}$. That is, if $G_{k,j'} \cap G_{k,j} \ne \emptyset$, then $d(u_{k,j'},u_{k,j}) \le 8\cdot 2^{-k}$. Thus (6.11) follows from (2.3) and (2.2).

Next we subdivide each $G_{k,j}$. For each integer $\ell \ge 0$, define the subset

$$
G_{k,j}^{(\ell)} = \{(k+\ell,h) : 1 \le h \le m(k+\ell), B(u_{k+\ell,h},2^{-k-\ell}) \cap B(u_{k,j},3\cdot 2^{-k}) \ne \emptyset\}
$$

of $G_{k,j}$. By (2.3) and (2.2), there is a natural number M such that

$$
\mathrm{card}(G_{k,j}^{(\ell)}) \le M2^{2n\ell} \qquad (6.12)
$$

for all $(k,j) \in I$ and $\ell \ge 0$. It follows from (6.10) that

$$
B_{k,j} \le C_9 \sum_{\ell=0}^{\infty} B_{k,j}^{(\ell)}, \qquad (6.13)
$$

where

$$B^{(\ell)}_{k,j} = 2^{-2(n+(1/2))\ell} \sum_{(k+\ell,h)\in G^{(\ell)}_{k,j}} V(f;Q_{k+\ell,h}). \tag{6.14}$$

Given any $(k,j) \in I$ and $\ell \geq 0$, there is a $(k+\ell, h(k,j;\ell)) \in G^{(\ell)}_{k,j}$ such that

$$V(f;Q_{k+\ell,h(k,j;\ell)}) \geq V(f;Q_{k+\ell,h}) \quad \text{for every} \quad (k+\ell,h) \in G^{(\ell)}_{k,j}.$$

By (6.12) and (6.14), we have

$$B^{(\ell)}_{k,j} \leq M2^{-\ell} V(f;Q_{k+\ell,h(k,j;\ell)}).$$

For each $\ell \geq 0$, define the map $F_\ell : I \to I$ by the formula

$$F_\ell(k,j) = (k+\ell, h(k,j;\ell)).$$

If $k \neq k_1$, then $F_\ell(k,j) \neq F_\ell(k_1,j_1)$ for all possible j and j_1. Since $(k+\ell, h(k,j;\ell)) \in G^{(\ell)}_{k,j} \subset G_{k,j}$, (6.11) tells us that for each ℓ, the map F_ℓ is at most N-to-1. Hence, by Lemma 2.2, for each $0 < s \leq 1$ and each symmetric gauge function Φ, we have

$$\begin{aligned}\Phi(\{(B^{(\ell)}_{k,j})^s\}_{(k,j)\in I}) &\leq M^s 2^{-s\ell} \Phi(\{V^s(f;Q_{k+\ell,h(k,j;\ell)})\}_{(k,j)\in I}) \\ &= M^s 2^{-s\ell} \Phi(\{V^s(f;Q_{F_\ell(k,j)})\}_{(k,j)\in I}) \\ &\leq NM^s 2^{-s\ell} \Phi(\{V^s(f;Q_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

Since $0 < s \leq 1$, (6.13) gives us

$$B^s_{k,j} \leq C_9^s \sum_{\ell=0}^{\infty} (B^{(\ell)}_{k,j})^s \leq (1+C_9) \sum_{\ell=0}^{\infty} (B^{(\ell)}_{k,j})^s,$$

$(k,j) \in I$. Thus if we set $C_{10} = NM(1+C_9)$, then the above leads to

$$\begin{aligned}\Phi(\{B^s_{k,j}\}_{(k,j)\in I}) &\leq (1+C_9) \sum_{\ell=0}^{\infty} \Phi(\{(B^{(\ell)}_{k,j})^s\}_{(k,j)\in I}) \\ &\leq C_{10} \sum_{\ell=0}^{\infty} 2^{-s\ell} \Phi(\{V^s(f;Q_{k,j})\}_{(k,j)\in I}) = \frac{C_{10}}{1-2^{-s}} \Phi(\{V^s(f;Q_{k,j})\}_{(k,j)\in I}). \end{aligned} \tag{6.15}$$

By (6.8), we have

$$V^s(f;W_{k,j}) \leq V^s(f;Q_{k,j}) + (2C_6 B_{k,j})^s \leq V^s(f;Q_{k,j}) + (1+2C_6) B^s_{k,j}. \tag{6.16}$$

The lemma now follows from (6.16) and (6.15). □

Proof of the lower bound in Theorem 1.2. Given $i \geq 6n+1$, a,b-lattice Γ, $f \in L^2(\mathbf{B}, dv)$ and symmetric gauge function Φ, we apply Proposition 2.5, Lemma 6.4 and Lemma 6.3, in that order. This gives us

$$\begin{aligned}\Phi(\{\|(f-\langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}) &\leq C_{2.5}\Phi(\{V^{1/2}(f;W_{k,j})\}_{(k,j)\in I}) \\ &\leq \frac{C_{2.5}C_{6.4}}{1-2^{-1/2}}\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I}) \leq \frac{C_{2.5}C_{6.4}C_{6.3}}{1-2^{-1/2}}\|[M_f, P]\|_\Phi\end{aligned}$$

as desired. □

7. An Alternate Version of Theorem 1.2

Obviously, Lemma 6.3 provides an alternate lower bound for $\|[M_f, P]\|_\Phi$. Combining that with the upper bound in Theorem 1.2 and with the argument at the end of Section 6, we obtain another characterization of the membership $[M_f, P] \in \mathcal{C}_\Phi$.

Theorem 7.1. *There are constants* $0 < c \leq C < \infty$ *such that the inequality*

$$c\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I}) \leq \|[M_f, P]\|_\Phi \leq C\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I})$$

holds for every $f \in L^2(\mathbf{B}, dv)$ *and every symmetric gauge function* Φ.

Proof. Let $C_{6.3}$ be the constant provided by Lemma 6.3. By what we just mentioned, the constant $c = 1/C_{6.3}$ suffices for the lower bound.

To established the upper bound, take $i = 6n+1$, and pick a $1,2$-lattice Γ in $\mathbf{B}$. Then Theorem 1.2 provides a constant C_1 such that the inequality

$$\|[M_f, P]\|_\Phi \leq C_1\Phi(\{\|(f-\langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma})$$

holds for every $f \in L^2(\mathbf{B}, dv)$ and every symmetric gauge function Φ. Applying Proposition 2.5 and Lemma 6.4, we have

$$\begin{aligned}\Phi(\{\|(f-\langle f\tilde{\psi}_{z,i}, \tilde{\psi}_{z,i}\rangle)\psi_{z,i}\|\}_{z\in\Gamma}) &\leq C_{2.5}\Phi(\{V^{1/2}(f;W_{k,j})\}_{(k,j)\in I}) \\ &\leq \frac{C_{2.5}C_{6.4}}{1-2^{-1/2}}\Phi(\{V^{1/2}(f;Q_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

Hence the constant $C = C_1C_{2.5}C_{6.4}(1-2^{-1/2})^{-1}$ suffices for the upper bound. □

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