A CLOSER LOOK AT A POISSON-LIKE CONDITION
ON THE DRURY-ARVESON SPACE

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Abstract. Let $\mathcal{M}$ be the collection of the multipliers of the Drury-Arveson space $H^2_n$, $n \geq 2$. In a recent paper [1], Aleman et al showed that for $f \in H^2_n$, the condition $\sup_{|z|<1} \text{Re}(f, K_z f) < \infty$ is sufficient for the membership $f \in \mathcal{M}$. We show that this condition is not necessary for $f \in \mathcal{M}$. Moreover, we show that the condition $\sup_{|z|<1} \text{Re}(f, K_z f) < \infty$ only captures a nowhere dense subset of $\mathcal{M}$.

1. Introduction

Denote $B = \{z \in \mathbb{C}^n : |z| < 1\}$, the unit ball in $\mathbb{C}^n$. In this paper, the complex dimension $n$ is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space $H^2_n$ is the Hilbert space of analytic functions on $B$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z \rangle},$$

$z, \zeta \in B$, as its reproducing kernel [2,3,7]. Equivalently, $H^2_n$ can be described as the Hilbert space of analytic functions on $B$ where the inner product is given by

$$\langle h, g \rangle = \sum_{\alpha \in \mathbb{Z}_+^n} \alpha! a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^n} b_\alpha \zeta^\alpha.$$

Here and throughout, we follow the standard multi-index notation [9,page 3].

Perhaps, the most fascinating aspect of the Drury-Arveson space is its collection of multipliers, which were introduced by Arveson. A function $f \in H^2_n$ is said to be a multiplier of the Drury-Arveson space if $fh \in H^2_n$ for every $h \in H^2_n$ [2]. We will write $\mathcal{M}$ for the collection of the multipliers of $H^2_n$. If $f \in \mathcal{M}$, then the multiplication operator $M_f$ is bounded on $H^2_n$ [2], and the multiplier norm $\|f\|_{\mathcal{M}}$ is defined to be the operator norm $\|M_f\|$ on $H^2_n$.
An enduring challenge in the theory of the Drury-Arveson space, since its very inception, has been the quest for a good characterization of the membership in $\mathcal{M}$. In other words, we are asking a very instinctive question, what does a general $f \in \mathcal{M}$ look like?

One’s first instinct is to turn to the normalized reproducing kernel for possible answers. By (1.1), the normalized reproducing kernel for $H^2_\mathcal{N}$ is given by the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{1/2}}{1 - \langle \zeta, z \rangle},$$

$z, \zeta \in \mathcal{B}$. For example, anyone who gives any thought about multipliers is likely to examine the condition

$$\sup_{|z|<1} \|fk_z\| < \infty$$

for $f \in H^2_\mathcal{N}$. In other words, one might ask, does (1.2) imply the membership $f \in \mathcal{M}$? Conditions of this type are now called “reproducing-kernel thesis” [8] and are among the first things that one would check when it comes to boundedness. But as it turns out, (1.2) is not sufficient for the membership $f \in \mathcal{M}$ [5].

Recently in [1], Aleman et al examined a different condition, one that is in terms of the unnormalized reproducing kernel $K_z$. They showed that for $f \in H^2_\mathcal{N}$, the condition

$$\sup_{|z|<1} \text{Re}\langle f, K_z f \rangle < \infty$$

is sufficient to imply the membership $f \in \mathcal{M}$ [1, Corollary 4.6]. This naturally leads to the question, is (1.3) necessary for the membership $f \in \mathcal{M}$?

In the same paper, Aleman et al showed that on the Dirichlet space $D_\alpha$, $0 < \alpha < 1$, on the unit disc in $\mathbb{C}$, the analogue of condition (1.3) is not necessary for the multipliers of $D_\alpha$ [1, Proposition 4.8]. But that does not settle the question for the Drury-Arveson space $H^2_\mathcal{N}$, particularly in view of the fact that [1, Proposition 4.8] deals with a one-variable situation. We will settle this question for the Drury-Arveson space.

**Theorem 1.1.** The function

$$\varphi(\zeta) = \frac{\zeta_2}{\sqrt{1 - \zeta_1}},$$

$\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathcal{B}$, is a multiplier of the Drury-Arveson space $H^2_\mathcal{N}$. Moreover, there is a constant $c_{1,1} > 0$ such that

$$\sup_{|z|=r} \text{Re}\langle \varphi, K_z \varphi \rangle \geq c_{1,1} \left(1 + \log \frac{1}{1 - r}\right)$$

for every $0 \leq r < 1$. In particular,

$$\sup_{|z|<1} \text{Re}\langle \varphi, K_z \varphi \rangle = \infty.$$
We will see that the function \( \varphi \) given by (1.4), simple as it is, is already “extremal” among the multipliers of \( H^2_n \) in that lower bound (1.5) is actually sharp. The fact that there are such extremal functions in \( \mathcal{M} \) has consequences.

**Definition 1.2.** Let \( \mathcal{F} \) denote the collection of \( f \in \mathcal{M} \) satisfying the condition

\[
\sup_{|z|<1} \Re \langle f, K_z f \rangle < \infty.
\]

**Theorem 1.3.** With respect to the multiplier norm \( \| \cdot \|_\mathcal{M} \), \( \mathcal{F} \) is nowhere dense in \( \mathcal{M} \).

The actual situation is even more shocking. We will see that for each \( f \in \mathcal{M} \), there are just two possibilities: either \( f \) belongs to the interior of \( \mathcal{M} \setminus \mathcal{F} \) outright, or \( f + \xi \varphi \) belongs to the interior of \( \mathcal{M} \setminus \mathcal{F} \) for every \( \xi \in \mathbb{C} \setminus \{0\} \).

The rest of the paper is organized as follows. We will prove Theorem 1.1 in Section 2. Then in Section 3, we prove an upper bound for the growth of \( \Re \langle f, K_z f \rangle, |z| \uparrow 1 \), for \( f \in \mathcal{M} \). Using this upper bound and (1.5), we prove Theorem 1.3 in Section 4.

**2. Estimates on the unit disc**

Let \( D \) denote the unit disc \( \{ w \in \mathbb{C} : |w| < 1 \} \) in the complex plane \( \mathbb{C} \). We write \( dA \) for the area measure on \( \mathbb{C} \) with the normalization \( A(D) = 1 \).

**Proposition 2.1.** The measure

\[
(2.1) \quad d\mu(w) = \frac{1}{|1-w|}dA(w)
\]

defined on the unit disc \( D \) is a Carleson measure for the one-variable Hardy space \( H^2 = H^2_1 \).

**Proof.** For each pair of \( \theta \in \mathbb{R} \) and \( 0 < \rho \leq 1 \), define the sector

\[
S(\theta, \rho) = \{ re^{it} : 1 - \rho \leq r < 1 \text{ and } |t - \theta| < \rho \}
\]
in \( D \). It is well known that, to show that \( d\mu \) is a Carleson measure for the one-variable Hardy space \( H^2 = H^2_1 \), it suffices to find a constant \( 0 < C < \infty \) such that

\[
(2.2) \quad \mu(S(\theta, \rho)) \leq C\rho
\]

for all \( \theta \in \mathbb{R} \) and \( 0 < \rho \leq 1 \). See, e.g., [6, pages 238, 239].

To prove (2.2), for \( \xi \in \mathbb{C} \) and \( a > 0 \) we define

\[
\Delta(\xi, a) = \{ w \in \mathbb{C} : |w - \xi| < a \}.
\]

Consider any \( w \in S(\theta, \rho) \). That is, \( w = re^{it} \) with \( 1 - \rho \leq r < 1 \) and \( |t - \theta| < \rho \). Then

\[
|w - e^{i\theta}| \leq |w - e^{it}| + |e^{it} - e^{i\theta}| = 1 - r + |1 - e^{i(\theta-t)}| \leq \rho + |\theta - t| < 2\rho.
\]
That is, we have

\[ S(\theta, \rho) \subset \Delta(e^{i\theta}, 2\rho) \cap D \]

for all \( \theta \in \mathbb{R} \) and \( 0 < \rho \leq 1 \). For any \( \theta \in \mathbb{R} \) and \( a > 0 \), we have \( \Delta(e^{i\theta}, a) \cap D = e^{i\theta} \{ \Delta(1, a) \cap D \} \), consequently

\[ A(\Delta(e^{i\theta}, a) \cap D) = A(\Delta(1, a) \cap D). \]

This implies that for any \( \theta \in \mathbb{R} \) and \( a > 0 \), we have

\[ A(\{ \Delta(e^{i\theta}, a) \} \cap D) = A(\{ \Delta(1, a) \} \cap D). \]

Obviously, if \( w \in \Delta(e^{i\theta}, a) \Delta(1, a) \) and \( w' \in \Delta(1, a) \Delta(e^{i\theta}, a) \), then \(|1 - w| \geq a > |1 - w'|\).

Combining this fact with (2.3), (2.1) and (2.4), for \( \theta \in \mathbb{R} \) and \( 0 < \rho \leq 1 \) we have

\[ \mu(S(\theta, \rho)) \leq \mu(\Delta(e^{i\theta}, 2\rho) \cap D) \]

\[ = \mu(\Delta(e^{i\theta}, 2\rho) \cap \Delta(1, 2\rho) \cap D) + \mu(\{ \Delta(e^{i\theta}, 2\rho) \} \cap D) \]

\[ \leq \mu(\Delta(e^{i\theta}, 2\rho) \cap \Delta(1, 2\rho) \cap D) + \mu(\Delta(1, 2\rho) \cap \Delta(e^{i\theta}, 2\rho) \cap D) \]

\[ = \mu(\Delta(1, 2\rho) \cap D). \]

On the other hand, by (2.1) and the translation invariance of \( dA \), we have

\[ \mu(\Delta(1, 2\rho) \cap D) \leq \int_{\Delta(1, 2\rho)} dA(w) = \int_{\Delta(0, 2\rho)} \frac{1}{|w|} dA(w) = 2 \int_{0}^{2\rho} \frac{1}{r} dr = 4 \rho. \]

Combining this with (2.5), we obtain \( \mu(S(\theta, \rho)) \leq 4 \rho \) for all \( \theta \in \mathbb{R} \) and \( 0 < \rho \leq 1 \). This proves (2.2) and completes the proof of the proposition. \( \square \)

As it turns out, the key to the proof of Theorem 1.1 is an orthogonal decomposition for the Drury-Arveson space \( H_n^2 \) that we introduced in [4], which we now recall.

Define the subset \( \mathcal{B} = \{ (\beta_2, \ldots, \beta_n) : \beta_2, \ldots, \beta_n \in Z_+ \} \) of \( Z_+^n \). As before, write \( \zeta = (\zeta_1, \ldots, \zeta_n) \). The definition of \( \mathcal{B} \) ensures that for \( \beta, \beta' \in \mathcal{B} \) and \( k, k' \in Z_+ \), we have

\[ \langle \zeta_1^k \zeta, \zeta_1^{k'} \zeta^{\beta'} \rangle = 0 \quad \text{whenever} \quad (k, \beta) \neq (k', \beta'). \]

For each \( \beta \in \mathcal{B} \), define the closed linear subspace

\[ H_\beta = \text{span}\{ \zeta_1^k \zeta^{\beta} : k \geq 0 \} \]

of \( H_n^2 \). Then we have the orthogonal decomposition

\[ H_n^2 = \bigoplus_{\beta \in \mathcal{B}} H_\beta. \]
Obviously, $H_0$ is the one-variable Hardy space $H^2 = H^2_1$, which is where Proposition 2.1 will be applied.

If $\beta \in \mathcal{B}\setminus\{0\}$, $H_\beta$ can be naturally identified with a weighted Bergman space on $D$. Indeed it is elementary to verify that if $\beta \in \mathcal{B}\setminus\{0\}$, then

$$\|\zeta_1^k \zeta^\beta\|^2 = \frac{\beta!}{(|\beta| - 1)!} \int_D |w^k|^2 (1 - |w|^2)^{|\beta| - 1} dA(w)$$

for every $k \in \mathbb{Z}_+$.

Let $T$ denote the unit circle $\{\tau \in \mathbb{C} : |\tau| = 1\}$. Write $dm$ for the Lebesgue measure on $T$ with the normalization $m(T) = 1$. Given $h, g \in H^2_n$, (2.7) gives us the representation

$$h(\zeta) = \sum_{\beta \in \mathcal{B}} h_\beta(\zeta_1) \zeta_1^\beta \quad \text{and} \quad g(\zeta) = \sum_{\beta \in \mathcal{B}} g_\beta(\zeta_1) \zeta_1^\beta,$$

where $h_\beta$ and $g_\beta$ are one-variable analytic functions, $\beta \in \mathcal{B}$. By (2.6) and (2.8), we have

$$\langle h, g \rangle = \int_T h_0 \overline{g_0} dm + \sum_{\beta \in \mathcal{B}\setminus\{0\}} \frac{\beta!}{(|\beta| - 1)!} \int_D h_\beta(w) \overline{g_\beta(w)} (1 - |w|^2)^{|\beta| - 1} dA(w).$$

Arveson taught us that $\langle h, g \rangle$ cannot be expressed as a single integral [2, Corollary 2]. That notwithstanding, (2.9) expresses $\langle h, g \rangle$ as a sum of convenient integrals, which was one of the crucial observations in [4].

Another ingredient in the proof of Theorem 1.1 is a particular Forelli-Rudin estimate. Recall that we have

$$\int_D \frac{1 - |w|^2}{|1 - rw|^3} dA(w) \approx 1 + \log \frac{1}{1 - r}$$

for $0 \leq r < 1$. See, e.g., [9, Proposition 1.4.10].

In our analysis of $\text{Re} \langle f, K_z f \rangle$, the identity

$$\text{Re} \frac{1}{1 - w} = \frac{1}{2} \cdot \frac{1 - |w|^2}{|1 - w|^2} + \frac{1}{2}, \quad w \in D,$$

plays a special role. In fact, (2.11) plays a role that is very much like, but not exactly the same as, the Poisson kernel. This explains the phrase “Poisson-like condition” in the title of the paper.

**Proof of Theorem 1.1.** Let us first show that $\varphi \in \mathcal{M}$. Denote $e_2 = (0, 1, 0, \ldots, 0)$. Given any $h \in H^2_n$, (2.7) provides the representation

$$h(\zeta) = \sum_{\beta \in \mathcal{B}} h_\beta(\zeta_1) \zeta_1^\beta.$$
Then
\[(\varphi h)(\zeta) = \sum_{\beta \in B} \frac{h_\beta(\zeta_1)}{\sqrt{1 - \zeta_1}} \zeta^{\beta + e_2},\]
and (2.9) gives us
\[
\|\varphi h\|^2 = \int_D \frac{|h_0(w)|^2}{|1 - w|} dA(w) + \sum_{\beta \in \mathcal{B} \setminus \{0\}} \frac{(\beta + e_2)!}{|\beta|!} \int_D \frac{|h_\beta(w)|^2}{|1 - w|} (1 - |w|^2)^{|\beta|} dA(w).
\]
By Proposition 2.1, there is a constant $0 < C < \infty$ such that
\[
\int_D \frac{|h_0(w)|^2}{|1 - w|} dA(w) \leq C \int_T |h_0|^2 dm.
\]
Obviously, $1 - |w| \leq |1 - w|$ for every $w \in D$. Hence if $\beta \in \mathcal{B} \setminus \{0\}$, then
\[
\int_D \frac{|h_\beta(w)|^2}{|1 - w|} (1 - |w|^2)^{|\beta|} dA(w) \leq 2 \int_D |h_\beta(w)|^2 (1 - |w|^2)^{|\beta| - 1} dA(w).
\]
For $\beta \in \mathcal{B} \setminus \{0\}$, if we write $\beta = (0, \beta_2, \ldots, \beta_n)$, then
\[
\frac{(\beta + e_2)!}{|\beta|!} = \frac{\beta_2 + 1}{|\beta|} \cdot \frac{\beta!}{(|\beta| - 1)!} \leq 2 \frac{\beta!}{(|\beta| - 1)!}.
\]
Substituting these three inequalities in (2.13), we find that
\[
\|\varphi h\|^2 \leq C \int_T |h_0|^2 dm + 4 \sum_{\beta \in \mathcal{B} \setminus \{0\}} \frac{\beta!}{(|\beta| - 1)!} \int_D |h_\beta(w)|^2 (1 - |w|^2)^{|\beta| - 1} dA(w).
\]
By (2.9), this means $\|\varphi h\|^2 \leq C_1 \|h\|^2$, where $C_1 = \max\{C, 4\}$. Hence $\varphi$ is a multiplier of the Drury-Arveson space $H^2_n$.

To prove (1.5), let us denote $e_1 = (1, 0, \ldots, 0)$. For $0 \leq r < 1$, we have
\[
K_{re_1}(\zeta) = \frac{1}{1 - r\zeta_1} \quad \text{and} \quad (K_{re_1} \varphi)(\zeta) = \frac{\zeta_2}{(1 - r\zeta_1)^{\sqrt{1 - \zeta_1}}}.\]

Thus (2.9) gives us
\[
\langle \varphi, K_{re_1} \varphi \rangle = \int_D \frac{dA(w)}{(1 - rw)|1 - w|}.
\]
Applying (2.11), we obtain
\[
\text{Re}\langle \varphi, K_{re_1} \varphi \rangle = \int_D \text{Re}\left(\frac{1}{|1 - rw|} \right) \frac{dA(w)}{|1 - w|} = \frac{1}{2} \int_D \left(\frac{1 - |rw|^2}{1 - |w|^2} + 1\right) \frac{dA(w)}{|1 - w|}.
\]
For $0 \leq r < 1$ and $w \in D$, it is elementary that $|1 - w| \leq 2|1 - rw|$. Hence
\[
\text{Re}(\varphi, K_{re_1}\varphi) \geq \frac{1}{4} \int_D \frac{1 - |rw|^2}{|1 - rw|^3} dA(w) + \frac{1}{4} \geq \frac{1}{4} \int_D \frac{1 - |w|^2}{|1 - rw|^3} dA(w) + \frac{1}{4}.
\]
Combining this with (2.10), (1.5) follows. This completes the proof of Theorem 1.1. □

3. A necessary condition for the membership $f \in M$

Having proved Theorem 1.1, our next task is to derive an upper bound for the growth of $\text{Re}(f, K_z f)$ as $|z| \uparrow 1$. For each $j \in \mathbb{Z}_+$, define
\[
\rho_j = 1 - 2^{-j}.
\]

**Proposition 3.1.** Let $h \in H^2_n$. If $z \in B$ satisfies the condition $1 - 2^{-k} \leq |z| < 1 - 2^{-k-1}$ for some $k \in \mathbb{Z}_+$, then
\[
\text{Re}(h, K_z h) \leq 10 \left( \|hk_z\|^2 + \sum_{j=0}^k \|hk_{\rho_j}z\|^2 \right).
\]

**Proof.** First, consider $z = re_1$, where $e_1 = (1, 0, \ldots, 0)$ and $1 - 2^{-k} \leq r < 1 - 2^{-k-1}$ for some $k \in \mathbb{Z}_+$. Given an $h \in H^2_n$, we again represent it in the form (2.12). Then by (2.9),
\[
\langle h, K_{re_1} h \rangle = \int_T \frac{|h_0(\tau)|^2}{1 - r^2} dm(\tau) + \sum_{\beta \in B \setminus \{0\}} \frac{\beta!}{(|\beta|-1)!} \int_D \frac{|h_\beta(w)|^2}{1 - |rw|^2} (1 - |w|^2)^{|\beta|-1} dA(w).
\]
Combining this with (2.11), we find that
\[
\text{Re}\langle h, K_{re_1} h \rangle = \frac{1}{2} \|h\|^2 + \frac{1}{2} \|h_0k_{re_1}\|^2
\]
\[
+ \frac{1}{2} \sum_{\beta \in B \setminus \{0\}} \frac{\beta!}{(|\beta|-1)!} \int_D |h_\beta(w)|^2 \frac{1 - |rw|^2}{|1 - rw|^2} (1 - |w|^2)^{|\beta|-1} dA(w).
\]
Since $1 - |rw|^2 = 1 - r^2 + r^2(1 - |w|^2)$, the above gives us
\[
\text{Re}\langle h, K_{re_1} h \rangle = \frac{1}{2} \|h\|^2 + \frac{1}{2} \|hk_{re_1}\|^2
\]
\[
+ \frac{r^2}{2} \sum_{\beta \in B \setminus \{0\}} \frac{\beta!}{(|\beta|-1)!} \int_D |h_\beta(w)|^2 \frac{1 - |w|^2}{|1 - rw|^2} (1 - |w|^2)^{|\beta|-1} dA(w).
\]
To proceed further, we decompose the unit disc $D$. For each $j \in \mathbb{Z}_+$, we define
\[
R_j = \{ w \in \mathbb{C} : 1 - 2^{-j-1} \leq |w| < 1 \} \quad \text{and} \quad D_j = \{ w \in \mathbb{C} : 1 - 2^{-j} \leq |w| < 1 - 2^{-j-1} \}.
\]
Since \( D = D_0 \cup \cdots \cup D_k \cup R_k \), from (3.2) we obtain

\[
(3.3) \quad \text{Re} \langle h, K_{re_1} h \rangle = \frac{1}{2} \| h \|^2 + \frac{1}{2} \| hk_{re_1} \|^2 + \frac{r^2}{2} A_k + \frac{r^2}{2} \sum_{j=0}^{k} B_j,
\]

where

\[
A_k = \sum_{\beta \in B \setminus \{0\}} \frac{\beta!}{(|\beta| - 1)!} \int_{R_k} |h_\beta(w)|^2 \frac{1-|w|^2}{|1-rw|^2} (1-|w|^2)^{|\beta|-1} dA(w) \quad \text{and}
\]

\[
B_j = \sum_{\beta \in B \setminus \{0\}} \frac{\beta!}{(|\beta| - 1)!} \int_{D_j} |h_\beta(w)|^2 \frac{1-|w|^2}{|1-rw|^2} (1-|w|^2)^{|\beta|-1} dA(w)
\]

for \( j = 0, \ldots, k \). Let us first consider \( A_k \). For \( w \in R_k \), we have \( 1-|w|^2 \leq 2 \cdot 2^{-k-1} < 2(1-r) \leq 2(1-r^2) \). Combining this with (2.9), we see that

\[
(3.4) \quad A_k \leq 2 \| hk_{re_1} \|^2.
\]

On the other hand, if \( w \in D_j \), then \( 1-|w|^2 \leq 2 \cdot 2^{-j} = 2(1-\rho_j) \leq 2(1-(\rho_j r)^2) \). Also, for \( w \in D_j \), we have \( |1-rw| \geq 1-|w| > 2^{-j-1} = (1/2)(1-\rho_j) \). Thus if \( w \in D_j \), then \( |1-\rho_j rw| \leq 1-\rho_j + |1-rw| \leq 3|1-rw| \). Hence the inequality

\[
\frac{1-|w|^2}{|1-rw|^2} \leq 18 \frac{1-(\rho_j r)^2}{|1-\rho_j rw|^2}
\]

holds when \( w \in D_j \). Applying (2.9) once more, we have

\[
(3.5) \quad B_j \leq 18 \| hk_{\rho_j re_1} \|^2
\]

for \( j = 0, \ldots, k \). Combining (3.3), (3.4) and (3.5), we obtain

\[
\text{Re} \langle h, K_{re_1} h \rangle \leq \| h \|^2 + 2 \| hk_{re_1} \|^2 + 9 \sum_{j=0}^{k} \| hk_{\rho_j re_1} \|^2.
\]

Since \( \rho_0 = 0 \) and \( k_0 \) is the constant function 1, this proves the proposition in the case where \( z = re_1 \) and \( 1-2^{-k} \leq r < 1-2^{-k-1} \) for some \( k \in \mathbb{Z}_+ \).

Now consider the general case. That is \( z = ru \), where \( 1-2^{-k} \leq r < 1-2^{-k-1} \) for some \( k \in \mathbb{Z}_+ \) and \( u \) is a unit vector in \( \mathbb{C}^n \). Let \( U : \mathbb{C}^n \to \mathbb{C}^n \) be a unitary transformation such that \( Ue_1 = u \). Then it gives rise to the unitary operator \( W \) on \( H_n^2 \) by the formula

\[
(3.6) \quad (Wg)(\zeta) = g(U\zeta),
\]
Given $h \in H^2_n$, we have $WK_z = K_{re_1}$, $Wk_z = k_{re_1}$ and $Wk_{\rho_j} = k_{\rho_j re_1}$ for $j = 0, \ldots, k$. Given an $h \in H^2_n$, we write $\eta = Wh$. Applying the special case proved above, we have

$$\text{Re}\langle h, K_z h \rangle = \text{Re}\langle Wh, WK_z h \rangle = \text{Re}\langle \eta, K_{re_1} \rangle \leq 10 \left( \left\| \eta k_{re_1} \right\|^2 + \sum_{j=0}^{k} \left\| \eta k_{\rho_j re_1} \right\|^2 \right).$$

This completes the proof of the proposition. \( \square \)

Note that if $1 - 2^{-k} \leq |z| < 1 - 2^{-k-1}$, $k \in \mathbb{Z}_+$, then $k + 2 \approx 1 - \log(1 - |z|)$. Thus from Proposition 3.1 we immediately obtain

**Corollary 3.2.** There is a constant $0 < C_{3.2} < \infty$ such that

$$\text{Re}\langle f, K_z f \rangle \leq C_{3.2} \| f \|_M^2 \left( 1 + \log \frac{1}{1 - |z|} \right)$$

for all $f \in M$ and $z \in B$.

Thus lower bound (1.5) is sharp. The above upper bound motivates us to introduce

**Definition 3.3.** An element $h \in H^2_n$ is said to be in the class $(H^2_n)_\log$ if there is a constant $C = C(h) \in (0, \infty)$ such that

$$\text{Re}\langle h, K_z h \rangle \leq C \left( 1 + \log \frac{1}{1 - |z|} \right)$$

for every $z \in B$.

**Proposition 3.4.** The condition $f \in (H^2_n)_\log$ is necessary, but not sufficient, for the membership $f \in M$. That is, $(H^2_n)_\log \supset M$ and $(H^2_n)_\log \neq M$.

**Proof.** Obviously, the inclusion $(H^2_n)_\log \supset M$ follows from Corollary 3.2. To prove that $(H^2_n)_\log \neq M$, we apply [5,Theorem 1.2], which provides an $f \in H^2_n$ such that $f \notin M$ and yet $\| f \|' < \infty$, where

$$\| f \|' = \sup_{|z| < 1} \| f k_z \|.$$

By Proposition 3.1, we have

$$\text{Re}\langle f, K_z f \rangle \leq 10(\| f \|')^2(k + 2)$$

if $1 - 2^{-k} \leq |z| < 1 - 2^{-k-1}$, $k \in \mathbb{Z}_+$. Since $\| f \|' < \infty$, this implies $f \in (H^2_n)_\log$. \( \square \)

4. A non-commutative Poisson kernel
Recall that when $f \in \mathcal{M}$, we write $M_f$ for the operator of multiplication by $f$ on $H^2_n$. In particular, $(M_{\langle \zeta, z \rangle}h)(\zeta) = \langle \zeta, z \rangle h(\zeta)$, $h \in H^2_n$. For $z \in \mathbf{B}$, we have $\|M_{\langle \zeta, z \rangle}\| = |z| < 1$. Thus for each $z \in \mathbf{B}$, we can define the “defect operator”

$$Q_z = (1 - M^*_{\langle \zeta, z \rangle} M_{\langle \zeta, z \rangle})^{1/2}.$$ 

**Proposition 4.1.** For $h \in H^2_n$ and $z \in \mathbf{B}$, we have

$$\text{Re}\langle h, K_z h \rangle = \frac{1}{2}(\|h\|^2 + \|Q_z M_{K_z} h\|^2).$$

**Proof.** Again, we first consider the case $z = re_1$, where $0 \leq r < 1$ and $e_1 = (1, 0, \ldots, 0)$. In this case $M_{\langle \zeta, z \rangle} = M_{r\zeta}$. Given an $h \in H^2_n$, we write it in the form (2.12). Then the corresponding decompositions for $M_{K_{re_1}} h$ and $M_{r\zeta} M_{K_{re_1}} h$ are

$$(4.1) \quad (M_{K_{re_1}} h)(\zeta) = \sum_{\beta \in \mathcal{B}} \frac{h_\beta(\zeta_1)}{1 - r\zeta_1^2} \zeta^\beta$$

and

$$(4.2) \quad (M_{r\zeta} M_{K_{re_1}} h)(\zeta) = \sum_{\beta \in \mathcal{B}} \frac{r\zeta_1 h_\beta(\zeta_1)}{1 - r\zeta_1^2} \zeta^\beta.$$

Since the restriction of $M_{\zeta}$ to $H_0 = H^2$ is an isometry, we have

$$(4.3) \quad \|h_{0K_{re_1}}\|^2 = \|M_{K_{re_1}} h_{0}\|^2 - r^2 \|M_{K_{re_1}} h_{0}\|^2 = \|M_{K_{re_1}} h_{0}\|^2 - \|M_{r\zeta} M_{K_{re_1}} h_{0}\|^2.$$

For each $\beta \in \mathcal{B}\setminus\{0\}$, we have

$$\int_D |h_\beta(w)|^2 \frac{1 - |rw|^2}{|1 - rw|^2} (1 - |w|^2)^{|eta|-1} dA(w)$$

$$= \int_D \left| \frac{h_\beta(w)}{1 - rw} \right|^2 (1 - |w|^2)^{|eta|-1} dA(w) - \int_D \left| \frac{rwh_\beta(w)}{1 - rw} \right|^2 (1 - |w|^2)^{|eta|-1} dA(w).$$

Substituting (4.2) and (4.3) in (3.1), it now follows from (2.9) and (4.1) that

$$\text{Re}\langle h, K_{re_1} h \rangle = \frac{1}{2}(\|h\|^2 + \|M_{K_{re_1}} h\|^2 - \|M_{r\zeta} M_{K_{re_1}} h\|^2)$$

$$= \frac{1}{2}(\|h\|^2 + \|(1 - M^*_{r\zeta} M_{r\zeta}) M_{K_{re_1}} h, M_{K_{re_1}} h\|)$$

$$= \frac{1}{2}(\|h\|^2 + \|Q_{re_1} M_{K_{re_1}} h\|^2).$$

This proves the proposition the special case $z = re_1$, where $0 \leq r < 1$ and $e_1 = (1, 0, \ldots, 0)$.

Now consider the general case. That is, $z = ru$, where $0 \leq r < 1$ and $u$ is a unit vector in $\mathbf{C}^n$. Again, let $U : \mathbf{C}^n \to \mathbf{C}^n$ be a unitary transformation such that $U e_1 = u$, and let $W$ be the unitary operator on $H^2_n$ defined by (3.6). We have $WM_{\langle \zeta, z \rangle} = M_{r\zeta} W$. Taking adjoints, since $W$ is a unitary operator, we see that $WM^*_{\langle \zeta, z \rangle} = M^*_{r\zeta} W$. Thus

$$WQ_z = Q_{re_1} W.$$
Given an $h \in H^2_n$, we write $\eta = Wh$. Repeating the argument in the proof of Proposition 3.1 and applying (4.4), we obtain

$$\text{Re}\langle h, Kzh \rangle = \text{Re}\langle \eta, K\text{re}_1 \eta \rangle = \frac{1}{2} (\|\eta\|^2 + \|Q\text{re}_1 M K\text{re}_1 \eta\|^2)$$

$$= \frac{1}{2} (\|Wh\|^2 + \|WQz MKzh\|^2) = \frac{1}{2} (\|h\|^2 + \|QzMKzh\|^2).$$

This completes the proof of the proposition. □

For any normed space $N$ and any $x, y \in N$, we always have

$$\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2.$$ 

Thus Proposition 4.1 immediately implies the following “quasi triangle inequality”:

**Corollary 4.2.** For all $h, g \in H^2_n$ and $z \in B$, we have

$$\text{Re}\langle h + g, Kzh + g \rangle \leq 2\text{Re}\langle h, Kzh \rangle + 2\text{Re}\langle g, Kzh \rangle.$$ 

**Proposition 4.3.** Let $f \in M$. Suppose that there is a $c > 0$ and a sequence $\{r_k\}$ in $(0, 1)$ such that $\lim_{k \to \infty} r_k = 1$ and

$$\sup_{|z| = r_k} \text{Re}\langle f, Kzf \rangle \geq c \log \frac{1}{1 - r_k}$$
for every $k \geq 1$. Then $f$ belongs to the interior of $M \setminus F$.

**Proof.** We need to find an $\epsilon > 0$ such that if $\gamma \in M$ and $\|\gamma\|_M < \epsilon$, then $f + \gamma \in M \setminus F$. To do this, consider any $\gamma \in M$. Applying Corollary 4.2 to the case where $h = f + \gamma$ and $g = -\gamma$, we have

$$\text{Re}\langle f, Kzf \rangle \leq 2\text{Re}\langle f + \gamma, Kzf + \gamma \rangle + 2\text{Re}\langle \gamma, Kzf \rangle$$
for every $z \in B$. Applying Corollary 3.2 to $\text{Re}\langle \gamma, Kzf \rangle$, we obtain

$$\text{Re}\langle f, Kzf \rangle \leq 2\text{Re}\langle f + \gamma, Kzf + \gamma \rangle + 2C_{3,2}\|\gamma\|_M^2 \left(1 + \log \frac{1}{1 - |z|}\right),$$

$z \in B$. Combining this with (4.5), we find that

$$2 \sup_{|z| = r_k} \text{Re}\langle f + \gamma, Kzf + \gamma \rangle \geq (c - 2C_{3,2}\|\gamma\|_M^2) \log \frac{1}{1 - r_k} - 2C_{3,2}\|\gamma\|_M^2$$

for every $k \geq 1$. Now pick an $\epsilon > 0$ such that $2C_{3,2}\epsilon^2 < c/2$. For $\gamma \in M$ satisfying the condition $\|\gamma\|_M < \epsilon$, the above gives us

$$2 \sup_{|z| = r_k} \text{Re}\langle f + \gamma, Kzf + \gamma \rangle \geq \frac{c}{2} \log \frac{1}{1 - r_k} - 2C_{3,2}\epsilon^2.$$
Since this holds for every $k \geq 1$ and since $\lim_{k \to \infty} r_k = 1$, we conclude that $f + \gamma \in \mathcal{M}\setminus\mathcal{F}$. This proves the proposition. □

**Proposition 4.4.** Let $f \in \mathcal{M}$. Suppose that $f$ has the property that for every $\epsilon > 0$, there is an $r(\epsilon) \in (0, 1)$ such that

\[
\text{Re}\langle f, K_z f \rangle \leq \epsilon \log \frac{1}{1 - |z|} \quad \text{whenever } r(\epsilon) \leq |z| < 1.
\]

Then for every $\xi \in \mathbb{C}\setminus\{0\}$, $f + \xi \varphi$ belongs to the interior of $\mathcal{M}\setminus\mathcal{F}$.

**Proof.** Given any $\xi \in \mathbb{C}\setminus\{0\}$, we pick an $\epsilon = \epsilon(\xi) > 0$ such that $2\epsilon < |\xi|^2 c_{1,1}/2$, where $c_{1,1}$ is the constant provided by Theorem 1.1. Applying Corollary 4.2 to the case where $h = f + \xi \varphi$ and $g = -f$, we obtain

\[
|\xi|^2 \text{Re}\langle \varphi, K_z \varphi \rangle \leq 2\text{Re}\langle f + \xi \varphi, K_z(f + \xi \varphi) \rangle + 2\text{Re}\langle f, K_z f \rangle,
\]

$z \in \mathcal{B}$. Applying Theorem 1.1 on the left and (4.6) on the right, if $r(\epsilon) \leq r < 1$, then

\[
|\xi|^2 c_{1,1} \log \frac{1}{1 - r} \leq |\xi|^2 \sup_{|z|=r} \text{Re}\langle \varphi, K_z \varphi \rangle
\]

\[
\leq 2 \sup_{|z|=r} \text{Re}\langle f + \xi \varphi, K_z(f + \xi \varphi) \rangle + 2\epsilon \log \frac{1}{1 - r}.
\]

Since $2\epsilon < |\xi|^2 c_{1,1}/2$, the obvious cancellation leads to

\[
\frac{|\xi|^2 c_{1,1}}{2} \log \frac{1}{1 - r} \leq 2 \sup_{|z|=r} \text{Re}\langle f + \xi \varphi, K_z(f + \xi \varphi) \rangle
\]

for every $r(\epsilon) \leq r < 1$. This shows that the function $f + \xi \varphi$ satisfies condition (4.5). By Proposition 4.3, $f + \xi \varphi$ is in the interior of $\mathcal{M}\setminus\mathcal{F}$ as promised. □

**Proof of Theorem 1.3.** Let $\mathcal{U}$ denote the interior of $\mathcal{M}\setminus\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{M}\setminus\mathcal{U}$, it suffices to show that $\mathcal{M}\setminus\mathcal{U}$ is nowhere dense in $\mathcal{M}$. Since $\mathcal{M}\setminus\mathcal{U}$ is closed, the desired conclusion will follow if we can show that $\mathcal{U}$ is dense in $\mathcal{M}$.

Consider any $f \in \mathcal{M}$. If $f$ satisfies condition (4.5), then Proposition 4.3 tells us that $f \in \mathcal{U}$. If $f$ fails condition (4.5), then $f$ has no choice but to satisfy condition (4.6), in which case Proposition 4.4 provides the inclusion $\{f + \xi \varphi : \xi \in \mathbb{C}\setminus\{0\} \} \subset \mathcal{U}$. Thus we see that in either case, $f$ is in the closure of $\mathcal{U}$. This completes the proof. □

**References**


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