A CLOSER LOOK AT A POISSON-LIKE CONDITION ON THE DRURY-ARVESON SPACE

Quanlei Fang and Jingbo Xia

Abstract. Let $\mathcal{M}$ be the collection of the multipliers of the Drury-Arveson space H^2_n, $n \geq 2$. In a recent paper [1], Aleman et al showed that for $f \in H^2_n$, the condition $\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty$ is sufficient for the membership $f \in \mathcal{M}$. We show that this condition is not necessary for $f \in \mathcal{M}$. Moreover, we show that the condition $\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty$ only captures a nowhere dense subset of $\mathcal{M}$.

1. Introduction

Denote $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$, the unit ball in $\mathbf{C}^n$. In this paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H^2_n is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle}, \tag{1.1}$$

$z, \zeta \in \mathbf{B}$, as its reproducing kernel [2,3,7]. Equivalently, H^2_n can be described as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle h, g\rangle = \sum_{\alpha \in \mathbf{Z}^n_+} \frac{\alpha!}{|\alpha|!} a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha \in \mathbf{Z}^n_+} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}^n_+} b_\alpha \zeta^\alpha.$$

Here and throughout, we follow the standard multi-index notation [9,page 3].

Perhaps, the most fascinating aspect of the Drury-Arveson space is its collection of *multipliers*, which were introduced by Arveson. A function $f \in H^2_n$ is said to be a multiplier of the Drury-Arveson space if $fh \in H^2_n$ for every $h \in H^2_n$ [2]. We will write $\mathcal{M}$ for the collection of the multipliers of H^2_n. If $f \in \mathcal{M}$, then the multiplication operator M_f is bounded on H^2_n [2], and the multiplier norm $\|f\|_{\mathcal{M}}$ is defined to be the operator norm $\|M_f\|$ on H^2_n.

2010 *Mathematics Subject Classification.* Primary 46E25, 47B32.
Key words and phrases. Drury-Arveson space, Multiplier.

An enduring challenge in the theory of the Drury-Arveson space, since its very inception, has been the quest for a *good* characterization of the membership in $\mathcal{M}$. In other words, we are asking a very instinctive question, what does a general $f \in \mathcal{M}$ look like?

One's first instinct is to turn to the normalized reproducing kernel for possible answers. By (1.1), the normalized reproducing kernel for H^2_n is given by the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{1-\langle \zeta, z\rangle},$$

$z, \zeta \in \mathbf{B}$. For example, anyone who gives any thought about multipliers is likely to examine the condition

$$\sup_{|z|<1} \|fk_z\| < \infty \tag{1.2}$$

for $f \in H^2_n$. In other words, one might ask, does (1.2) imply the membership $f \in \mathcal{M}$? Conditions of this type are now called "reproducing-kernel thesis" [8] and are among the first things that one would check when it comes to boundedness. But as it turns out, (1.2) is not sufficient for the membership $f \in \mathcal{M}$ [5].

Recently in [1], Aleman et al examined a different condition, one that is in terms of the *unnormalized* reproducing kernel K_z. They showed that for $f \in H^2_n$, the condition

$$\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty \tag{1.3}$$

is sufficient to imply the membership $f \in \mathcal{M}$ [1,Corollary 4.6]. This naturally leads to the question, is (1.3) necessary for the membership $f \in \mathcal{M}$?

In the same paper, Aleman et al showed that on the Dirichlet space D_α, $0 < \alpha < 1$, on the unit disc in $\mathbf{C}$, the analogue of condition (1.3) is not necessary for the multipliers of D_α [1,Proposition 4.8]. But that does not settle the question for the Drury-Arveson space H^2_n, particularly in view of the fact that [1,Proposition 4.8] deals with a one-variable situation. We will settle this question for the Drury-Arveson space.

Theorem 1.1. *The function*

$$\varphi(\zeta) = \frac{\zeta_2}{\sqrt{1-\zeta_1}}, \tag{1.4}$$

$\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{B}$, *is a multiplier of the Drury-Arveson space* H^2_n. *Moreover, there is a constant* $c_{1.1} > 0$ *such that*

$$\sup_{|z|=r} \mathrm{Re}\langle \varphi, K_z\varphi\rangle \ge c_{1.1}\left(1 + \log\frac{1}{1-r}\right) \tag{1.5}$$

for every $0 \le r < 1$. *In particular,*

$$\sup_{|z|<1} \mathrm{Re}\langle \varphi, K_z\varphi\rangle = \infty.$$

We will see that the function φ given by (1.4), simple as it is, is already "extremal" among the multipliers of H^2_n in that lower bound (1.5) is actually sharp. The fact that there are such extremal functions in $\mathcal{M}$ has consequences.

Definition 1.2. Let $\mathcal{F}$ denote the collection of $f \in \mathcal{M}$ satisfying the condition

$$\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty.$$

Theorem 1.3. *With respect to the multiplier norm $\|\cdot\|_{\mathcal{M}}$, $\mathcal{F}$ is nowhere dense in $\mathcal{M}$.*

The actual situation is even more shocking. We will see that for each $f \in \mathcal{M}$, there are just two possibilities: either f belongs to the interior of $\mathcal{M}\backslash\mathcal{F}$ outright, or $f + \xi\varphi$ belongs to the interior of $\mathcal{M}\backslash\mathcal{F}$ for every $\xi \in \mathbf{C}\backslash\{0\}$.

The rest of the paper is organized as follows. We will prove Theorem 1.1 in Section 2. Then in Section 3, we prove an upper bound for the growth of $\mathrm{Re}\langle f, K_z f\rangle$, $|z| \uparrow 1$, for $f \in \mathcal{M}$. Using this upper bound and (1.5), we prove Theorem 1.3 in Section 4.

2. Estimates on the unit disc

Let D denote the unit disc $\{w \in \mathbf{C} : |w| < 1\}$ in the complex plane $\mathbf{C}$. We write dA for the area measure on $\mathbf{C}$ with the normalization $A(D) = 1$.

Proposition 2.1. *The measure*

$$d\mu(w) = \frac{1}{|1-w|} dA(w) \tag{2.1}$$

defined on the unit disc D is a Carleson measure for the one-variable Hardy space $H^2 = H^2_1$.

Proof. For each pair of $\theta \in \mathbf{R}$ and $0 < \rho \le 1$, define the sector

$$S(\theta, \rho) = \{re^{it} : 1 - \rho \le r < 1 \ \text{ and } \ |t - \theta| < \rho\}$$

in D. It is well known that, to show that $d\mu$ is a Carleson measure for the one-variable Hardy space $H^2 = H^2_1$, it suffices to find a constant $0 < C < \infty$ such that

$$\mu(S(\theta, \rho)) \le C\rho \tag{2.2}$$

for all $\theta \in \mathbf{R}$ and $0 < \rho \le 1$. See, e.g., [6,pages 238, 239].

To prove (2.2), for $\xi \in \mathbf{C}$ and $a > 0$ we define

$$\Delta(\xi, a) = \{w \in \mathbf{C} : |w - \xi| < a\}.$$

Consider any $w \in S(\theta, \rho)$. That is, $w = re^{it}$ with $1 - \rho \le r < 1$ and $|t - \theta| < \rho$. Then

$$|w - e^{i\theta}| \le |w - e^{it}| + |e^{it} - e^{i\theta}| = 1 - r + |1 - e^{i(\theta - t)}| \le \rho + |\theta - t| < 2\rho.$$

That is, we have

$$S(\theta,\rho) \subset \Delta(e^{i\theta}, 2\rho) \cap D \tag{2.3}$$

for all $\theta \in \mathbf{R}$ and $0 < \rho \leq 1$. For any $\theta \in \mathbf{R}$ and $a > 0$, we have $\Delta(e^{i\theta}, a) \cap D = e^{i\theta}\{\Delta(1,a) \cap D\}$, consequently

$$A(\Delta(e^{i\theta}, a) \cap D) = A(\Delta(1,a) \cap D).$$

This implies that for any $\theta \in \mathbf{R}$ and $a > 0$, we have

$$A(\{\Delta(e^{i\theta}, a)\backslash\Delta(1,a)\} \cap D) = A(\{\Delta(1,a)\backslash\Delta(e^{i\theta}, a)\} \cap D). \tag{2.4}$$

Obviously, if $w \in \Delta(e^{i\theta}, a)\backslash\Delta(1,a)$ and $w' \in \Delta(1,a)\backslash\Delta(e^{i\theta}, a)$, then $|1-w| \geq a > |1-w'|$. Combining this fact with (2.3), (2.1) and (2.4), for $\theta \in \mathbf{R}$ and $0 < \rho \leq 1$ we have

$$\begin{aligned}
\mu(S(\theta,\rho)) &\leq \mu(\Delta(e^{i\theta}, 2\rho) \cap D) \\
&= \mu(\Delta(e^{i\theta}, 2\rho) \cap \Delta(1, 2\rho) \cap D) + \mu(\{\Delta(e^{i\theta}, 2\rho)\backslash\Delta(1,2\rho)\} \cap D) \\
&\leq \mu(\Delta(e^{i\theta}, 2\rho) \cap \Delta(1, 2\rho) \cap D) + \mu(\{\Delta(1,2\rho)\backslash\Delta(e^{i\theta}, 2\rho)\} \cap D) \\
&= \mu(\Delta(1,2\rho) \cap D). \qquad (2.5)
\end{aligned}$$

On the other hand, by (2.1) and the translation invariance of dA, we have

$$\mu(\Delta(1,2\rho) \cap D) \leq \int_{\Delta(1,2\rho)} \frac{dA(w)}{|1-w|} = \int_{\Delta(0,2\rho)} \frac{1}{|w|} dA(w) = 2\int_0^{2\rho} \frac{1}{r} r dr = 4\rho.$$

Combining this with (2.5), we obtain $\mu(S(\theta,\rho)) \leq 4\rho$ for all $\theta \in \mathbf{R}$ and $0 < \rho \leq 1$. This proves (2.2) and completes the proof of the proposition. $\square$

As it turns out, the key to the proof of Theorem 1.1 is an orthogonal decomposition for the Drury-Arveson space H_n^2 that we introduced in [4], which we now recall.

Define the subset $\mathcal{B} = \{(0,\beta_2,\dots,\beta_n) : \beta_2,\dots,\beta_n \in \mathbf{Z}_+\}$ of $\mathbf{Z}_+^n$. As before, write $\zeta = (\zeta_1,\dots,\zeta_n)$. The definition of $\mathcal{B}$ ensures that for $\beta, \beta' \in \mathcal{B}$ and $k, k' \in \mathbf{Z}_+$, we have

$$\langle \zeta_1^k \zeta^\beta, \zeta_1^{k'} \zeta^{\beta'} \rangle = 0 \quad \text{whenever} \quad (k,\beta) \neq (k',\beta'). \tag{2.6}$$

For each $\beta \in \mathcal{B}$, define the closed linear subspace

$$H_\beta = \overline{\text{span}\{\zeta_1^k \zeta^\beta : k \geq 0\}}$$

of H_n^2. Then we have the orthogonal decomposition

$$H_n^2 = \bigoplus_{\beta \in \mathcal{B}} H_\beta. \tag{2.7}$$

Obviously, H_0 is the one-variable Hardy space $H^2 = H_1^2$, which is where Proposition 2.1 will be applied.

If $\beta \in \mathcal{B}\backslash\{0\}$, H_β can be naturally identified with a weighted Bergman space on D. Indeed it is elementary to verify that if $\beta \in \mathcal{B}\backslash\{0\}$, then

$$\|\zeta_1^k \zeta^\beta\|^2 = \frac{\beta!}{(|\beta|-1)!}\int_D |w^k|^2(1-|w|^2)^{|\beta|-1}dA(w) \tag{2.8}$$

for every $k \in \mathbf{Z}_+$.

Let $\mathbf{T}$ denote the unit circle $\{\tau \in \mathbf{C} : |\tau| = 1\}$. Write dm for the Lebesgue measure on $\mathbf{T}$ with the normalization $m(\mathbf{T}) = 1$. Given $h, g \in H_n^2$, (2.7) gives us the representation

$$h(\zeta) = \sum_{\beta\in\mathcal{B}} h_\beta(\zeta_1)\zeta^\beta \quad \text{and} \quad g(\zeta) = \sum_{\beta\in\mathcal{B}} g_\beta(\zeta_1)\zeta^\beta,$$

where h_β and g_β are one-variable analytic functions, $\beta \in \mathcal{B}$. By (2.6) and (2.8), we have

$$\langle h, g\rangle = \int_{\mathbf{T}} h_0\overline{g_0}dm + \sum_{\beta\in\mathcal{B}\backslash\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_D h_\beta(w)\overline{g_\beta(w)}(1-|w|^2)^{|\beta|-1}dA(w). \tag{2.9}$$

Arveson taught us that $\langle h, g\rangle$ cannot be expressed as a *single* integral [2,Corollary 2]. That notwithstanding, (2.9) expresses $\langle h, g\rangle$ as a *sum* of convenient integrals, which was one of the crucial observations in [4].

Another ingredient in the proof of Theorem 1.1 is a particular Forelli-Rudin estimate. Recall that we have

$$\int_D \frac{1-|w|^2}{|1-rw|^3}dA(w) \approx 1 + \log\frac{1}{1-r} \tag{2.10}$$

for $0 \le r < 1$. See, e.g., [9,Proposition 1.4.10].

In our analysis of $\text{Re}\langle f, K_z f\rangle$, the identity

$$\text{Re}\frac{1}{1-w} = \frac{1}{2}\cdot\frac{1-|w|^2}{|1-w|^2} + \frac{1}{2}, \quad w \in D, \tag{2.11}$$

plays a special role. In fact, (2.11) plays a role that is very much like, but not exactly the same as, the Poisson kernel. This explains the phrase "Poisson-like condition" in the title of the paper.

Proof of Theorem 1.1. Let us first show that $\varphi \in \mathcal{M}$. Denote $e_2 = (0, 1, 0, \ldots, 0)$. Given any $h \in H_n^2$, (2.7) provides the representation

$$h(\zeta) = \sum_{\beta\in\mathcal{B}} h_\beta(\zeta_1)\zeta^\beta. \tag{2.12}$$

Then

$$(\varphi h)(\zeta) = \sum_{\beta \in \mathcal{B}} \frac{h_\beta(\zeta_1)}{\sqrt{1-\zeta_1}} \zeta^{\beta+e_2},$$

and (2.9) gives us

$$\|\varphi h\|^2 = \int_D \frac{|h_0(w)|^2}{|1-w|} dA(w) + \sum_{\beta \in \mathcal{B}\backslash\{0\}} \frac{(\beta+e_2)!}{|\beta|!} \int_D \frac{|h_\beta(w)|^2}{|1-w|}(1-|w|^2)^{|\beta|} dA(w). \tag{2.13}$$

By Proposition 2.1, there is a constant $0 < C < \infty$ such that

$$\int_D \frac{|h_0(w)|^2}{|1-w|} dA(w) \le C \int_{\mathbf{T}} |h_0|^2 dm.$$

Obviously, $1 - |w| \le |1-w|$ for every $w \in D$. Hence if $\beta \in \mathcal{B}\backslash\{0\}$, then

$$\int_D \frac{|h_\beta(w)|^2}{|1-w|}(1-|w|^2)^{|\beta|} dA(w) \le 2 \int_D |h_\beta(w)|^2 (1-|w|^2)^{|\beta|-1} dA(w).$$

For $\beta \in \mathcal{B}\backslash\{0\}$, if we write $\beta = (0, \beta_2, \ldots, \beta_n)$, then

$$\frac{(\beta+e_2)!}{|\beta|!} = \frac{\beta_2+1}{|\beta|} \cdot \frac{\beta!}{(|\beta|-1)!} \le 2 \frac{\beta!}{(|\beta|-1)!}.$$

Substituting these three inequalities in (2.13), we find that

$$\|\varphi h\|^2 \le C \int_{\mathbf{T}} |h_0|^2 dm + 4 \sum_{\beta \in \mathcal{B}\backslash\{0\}} \frac{\beta!}{(|\beta|-1)!} \int_D |h_\beta(w)|^2 (1-|w|^2)^{|\beta|-1} dA(w).$$

By (2.9), this means $\|\varphi h\|^2 \le C_1 \|h\|^2$, where $C_1 = \max\{C, 4\}$. Hence φ is a multiplier of the Drury-Arveson space H_n^2.

To prove (1.5), let us denote $e_1 = (1, 0, \ldots, 0)$. For $0 \le r < 1$, we have

$$K_{re_1}(\zeta) = \frac{1}{1-r\zeta_1} \quad \text{and} \quad (K_{re_1}\varphi)(\zeta) = \frac{\zeta_2}{(1-r\zeta_1)\sqrt{1-\zeta_1}}.$$

Thus (2.9) gives us

$$\langle \varphi, K_{re_1}\varphi \rangle = \int_D \frac{dA(w)}{(1-r\overline{w})|1-w|}.$$

Applying (2.11), we obtain

$$\mathrm{Re}\langle \varphi, K_{re_1}\varphi \rangle = \int_D \mathrm{Re}\left(\frac{1}{1-r\overline{w}}\right) \frac{dA(w)}{|1-w|} = \frac{1}{2} \int_D \left(\frac{1-|rw|^2}{|1-rw|^2} + 1\right) \frac{dA(w)}{|1-w|}.$$

For $0 \le r < 1$ and $w \in D$, it is elementary that $|1-w| \le 2|1-rw|$. Hence

$$\mathrm{Re}\langle \varphi, K_{re_1}\varphi\rangle \ge \frac{1}{4}\int_D \frac{1-|rw|^2}{|1-rw|^3}dA(w) + \frac{1}{4} \ge \frac{1}{4}\int_D \frac{1-|w|^2}{|1-rw|^3}dA(w) + \frac{1}{4}.$$

Combining this with (2.10), (1.5) follows. This completes the proof of Theorem 1.1. $\square$

3. A necessary condition for the membership $f \in \mathcal{M}$

Having proved Theorem 1.1, our next task is to derive an upper bound for the growth of $\mathrm{Re}\langle f, K_z f\rangle$ as $|z| \uparrow 1$. For each $j \in \mathbf{Z}_+$, define

$$\rho_j = 1 - 2^{-j}.$$

Proposition 3.1. *Let $h \in H^2_n$. If $z \in \mathbf{B}$ satisfies the condition $1 - 2^{-k} \le |z| < 1 - 2^{-k-1}$ for some $k \in \mathbf{Z}_+$, then*

$$\mathrm{Re}\langle h, K_z h\rangle \le 10\left(\|hk_z\|^2 + \sum_{j=0}^{k}\|hk_{\rho_j z}\|^2\right).$$

Proof. First, consider $z = re_1$, where $e_1 = (1,0,\dots,0)$ and $1 - 2^{-k} \le r < 1 - 2^{-k-1}$ for some $k \in \mathbf{Z}_+$. Given an $h \in H^2_n$, we again represent it in the form (2.12). Then by (2.9),

$$\langle h, K_{re_1}h\rangle = \int_{\mathbf{T}} \frac{|h_0(\tau)|^2}{1-r\overline{\tau}}dm(\tau) + \sum_{\beta\in\mathcal{B}\backslash\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_D \frac{|h_\beta(w)|^2}{1-r\overline{w}}(1-|w|^2)^{|\beta|-1}dA(w).$$

Combining this with (2.11), we find that

$$\begin{aligned} \mathrm{Re}\langle h, K_{re_1}h\rangle &= \frac{1}{2}\|h\|^2 + \frac{1}{2}\|h_0 k_{re_1}\|^2 \\ &\quad + \frac{1}{2}\sum_{\beta\in\mathcal{B}\backslash\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_D |h_\beta(w)|^2\frac{1-|rw|^2}{|1-rw|^2}(1-|w|^2)^{|\beta|-1}dA(w). \end{aligned} \tag{3.1}$$

Since $1 - |rw|^2 = 1 - r^2 + r^2(1-|w|^2)$, the above gives us

$$\begin{aligned} \mathrm{Re}\langle h, K_{re_1}h\rangle &= \frac{1}{2}\|h\|^2 + \frac{1}{2}\|hk_{re_1}\|^2 \\ &\quad + \frac{r^2}{2}\sum_{\beta\in\mathcal{B}\backslash\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_D |h_\beta(w)|^2\frac{1-|w|^2}{|1-rw|^2}(1-|w|^2)^{|\beta|-1}dA(w). \end{aligned} \tag{3.2}$$

To proceed further, we decompose the unit disc D. For each $j \in \mathbf{Z}_+$, we define

$$R_j = \{w \in \mathbf{C} : 1 - 2^{-j-1} \le |w| < 1\} \quad \text{and} \quad D_j = \{w \in \mathbf{C} : 1 - 2^{-j} \le |w| < 1 - 2^{-j-1}\}.$$

Since $D = D_0 \cup \cdots \cup D_k \cup R_k$, from (3.2) we obtain

$$\mathrm{Re}\langle h, K_{re_1} h\rangle = \frac{1}{2}\|h\|^2 + \frac{1}{2}\|hk_{re_1}\|^2 + \frac{r^2}{2}A_k + \frac{r^2}{2}\sum_{j=0}^{k} B_j, \tag{3.3}$$

where

$$A_k = \sum_{\beta \in \mathcal{B}\setminus\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_{R_k} |h_\beta(w)|^2 \frac{1-|w|^2}{|1-rw|^2}(1-|w|^2)^{|\beta|-1} dA(w) \quad \text{and}$$
$$B_j = \sum_{\beta \in \mathcal{B}\setminus\{0\}} \frac{\beta!}{(|\beta|-1)!}\int_{D_j} |h_\beta(w)|^2 \frac{1-|w|^2}{|1-rw|^2}(1-|w|^2)^{|\beta|-1} dA(w)$$

for $j = 0, \ldots, k$. Let us first consider A_k. For $w \in R_k$, we have $1 - |w|^2 \leq 2 \cdot 2^{-k-1} < 2(1-r) \leq 2(1-r^2)$. Combining this with (2.9), we see that

$$A_k \leq 2\|hk_{re_1}\|^2. \tag{3.4}$$

On the other hand, if $w \in D_j$, then $1 - |w|^2 \leq 2 \cdot 2^{-j} = 2(1-\rho_j) \leq 2(1-(\rho_j r)^2)$. Also, for $w \in D_j$, we have $|1 - rw| \geq 1 - |w| > 2^{-j-1} = (1/2)(1-\rho_j)$. Thus if $w \in D_j$, then $|1 - \rho_j rw| \leq 1 - \rho_j + |1 - rw| \leq 3|1 - rw|$. Hence the inequality

$$\frac{1-|w|^2}{|1-rw|^2} \leq 18 \frac{1-(\rho_j r)^2}{|1-\rho_j rw|^2}$$

holds when $w \in D_j$. Applying (2.9) once more, we have

$$B_j \leq 18\|hk_{\rho_j re_1}\|^2 \tag{3.5}$$

for $j = 0, \ldots, k$. Combining (3.3), (3.4) and (3.5), we obtain

$$\mathrm{Re}\langle h, K_{re_1} h\rangle \leq \|h\|^2 + 2\|hk_{re_1}\|^2 + 9\sum_{j=0}^{k} \|hk_{\rho_j re_1}\|^2.$$

Since $\rho_0 = 0$ and k_0 is the constant function 1, this proves the proposition in the case where $z = re_1$ and $1 - 2^{-k} \leq r < 1 - 2^{-k-1}$ for some $k \in \mathbf{Z}_+$.

Now consider the general case. That is $z = ru$, where $1 - 2^{-k} \leq r < 1 - 2^{-k-1}$ for some $k \in \mathbf{Z}_+$ and u is a unit vector in $\mathbf{C}^n$. Let $U : \mathbf{C}^n \to \mathbf{C}^n$ be a unitary transformation such that $Ue_1 = u$. Then it gives rise to the unitary operator W on H^2_n by the formula

$$(Wg)(\zeta) = g(U\zeta), \tag{3.6}$$

$g \in H^2_n$. We have $WK_z = K_{re_1}$, $Wk_z = k_{re_1}$ and $Wk_{\rho_j z} = k_{\rho_j re_1}$ for $j = 0, \ldots, k$. Given an $h \in H^2_n$, we write $\eta = Wh$. Applying the special case proved above, we have

$$\begin{aligned}\mathrm{Re}\langle h, K_z h\rangle &= \mathrm{Re}\langle Wh, WK_z h\rangle = \mathrm{Re}\langle \eta, K_{re_1}\eta\rangle \\ &\le 10\left(\|\eta k_{re_1}\|^2 + \sum_{j=0}^{k} \|\eta k_{\rho_j re_j}\|^2\right) \\ &= 10\left(\|W(hk_z)\|^2 + \sum_{j=0}^{k} \|W(hk_{\rho_j z})\|^2\right) = 10\left(\|hk_z\|^2 + \sum_{j=0}^{k} \|hk_{\rho_j z}\|^2\right).\end{aligned}$$

This completes the proof of the proposition. □

Note that if $1 - 2^{-k} \le |z| < 1 - 2^{-k-1}$, $k \in \mathbf{Z}_+$, then $k + 2 \approx 1 - \log(1 - |z|)$. Thus from Proposition 3.1 we immediately obtain

Corollary 3.2. *There is a constant* $0 < C_{3.2} < \infty$ *such that*

$$\mathrm{Re}\langle f, K_z f\rangle \le C_{3.2}\|f\|^2_{\mathcal{M}}\left(1 + \log\frac{1}{1 - |z|}\right)$$

for all $f \in \mathcal{M}$ *and* $z \in \mathbf{B}$.

Thus lower bound (1.5) is sharp. The above upper bound motivates us to introduce

Definition 3.3. An element $h \in H^2_n$ is said to be in the class $(H^2_n)_{\log}$ if there is a constant $C = C(h) \in (0, \infty)$ such that

$$\mathrm{Re}\langle h, K_z h\rangle \le C\left(1 + \log\frac{1}{1 - |z|}\right)$$

for every $z \in \mathbf{B}$.

Proposition 3.4. *The condition* $f \in (H^2_n)_{\log}$ *is necessary, but not sufficient, for the membership* $f \in \mathcal{M}$. *That is,* $(H^2_n)_{\log} \supset \mathcal{M}$ *and* $(H^2_n)_{\log} \neq \mathcal{M}$.

Proof. Obviously, the inclusion $(H^2_n)_{\log} \supset \mathcal{M}$ follows from Corollary 3.2. To prove that $(H^2_n)_{\log} \neq \mathcal{M}$, we apply [5,Theorem 1.2], which provides an $f \in H^2_n$ such that $f \notin \mathcal{M}$ and yet $\|f\|' < \infty$, where

$$\|f\|' = \sup_{|z|<1} \|fk_z\|.$$

By Proposition 3.1, we have

$$\mathrm{Re}\langle f, K_z f\rangle \le 10(\|f\|')^2(k + 2)$$

if $1 - 2^{-k} \le |z| < 1 - 2^{-k-1}$, $k \in \mathbf{Z}_+$. Since $\|f\|' < \infty$, this implies $f \in (H^2_n)_{\log}$. □

4. A non-commutative Poisson kernel

Recall that when $f \in \mathcal{M}$, we write M_f for the operator of multiplication by f on H_n^2. In particular, $(M_{\langle \zeta, z\rangle} h)(\zeta) = \langle \zeta, z\rangle h(\zeta)$, $h \in H_n^2$. For $z \in \mathbf{B}$, we have $\|M_{\langle \zeta, z\rangle}\| = |z| < 1$. Thus for each $z \in \mathbf{B}$, we can define the "defect operator"

$$Q_z = (1 - M^*_{\langle \zeta, z\rangle} M_{\langle \zeta, z\rangle})^{1/2}.$$

Proposition 4.1. *For $h \in H_n^2$ and $z \in \mathbf{B}$, we have*

$$\mathrm{Re}\langle h, K_z h\rangle = \frac{1}{2}(\|h\|^2 + \|Q_z M_{K_z} h\|^2).$$

Proof. Again, we first consider the case $z = re_1$, where $0 \le r < 1$ and $e_1 = (1, 0, \ldots, 0)$. In this case $M_{\langle \zeta, z\rangle} = M_{r\zeta_1}$. Given an $h \in H_n^2$, we write it in the form (2.12). Then the corresponding decompositions for $M_{K_{re_1}} h$ and $M_{r\zeta_1} M_{K_{re_1}} h$ are

$$(M_{K_{re_1}} h)(\zeta) = \sum_{\beta \in \mathcal{B}} \frac{h_\beta(\zeta_1)}{1 - r\zeta_1}\zeta^\beta \quad \text{and} \quad (M_{r\zeta_1} M_{K_{re_1}} h)(\zeta) = \sum_{\beta \in \mathcal{B}} \frac{r\zeta_1 h_\beta(\zeta_1)}{1 - r\zeta_1}\zeta^\beta. \tag{4.1}$$

Since the restriction of M_{ζ_1} to $H_0 = H^2$ is an isometry, we have

$$\|h_0 k_{re_1}\|^2 = \|M_{K_{re_1}} h_0\|^2 - r^2\|M_{K_{re_1}} h_0\|^2 = \|M_{K_{re_1}} h_0\|^2 - \|M_{r\zeta_1} M_{K_{re_1}} h_0\|^2. \tag{4.2}$$

For each $\beta \in \mathcal{B}\backslash\{0\}$, we have

$$\begin{aligned} &\int_D |h_\beta(w)|^2 \frac{1 - |rw|^2}{|1 - rw|^2}(1 - |w|^2)^{|\beta|-1} dA(w) \\ &= \int_D \left|\frac{h_\beta(w)}{1 - rw}\right|^2 (1 - |w|^2)^{|\beta|-1} dA(w) - \int_D \left|\frac{rw h_\beta(w)}{1 - rw}\right|^2 (1 - |w|^2)^{|\beta|-1} dA(w). \end{aligned} \tag{4.3}$$

Substituting (4.2) and (4.3) in (3.1), it now follows from (2.9) and (4.1) that

$$\begin{aligned} \mathrm{Re}\langle h, K_{re_1} h\rangle &= \frac{1}{2}(\|h\|^2 + \|M_{K_{re_1}} h\|^2 - \|M_{r\zeta_1} M_{K_{re_1}} h\|^2) \\ &= \frac{1}{2}(\|h\|^2 + \langle (1 - M^*_{r\zeta_1} M_{r\zeta_1}) M_{K_{re_1}} h, M_{K_{re_1}} h\rangle) \\ &= \frac{1}{2}(\|h\|^2 + \|Q_{re_1} M_{K_{re_1}} h\|^2). \end{aligned} \tag{4.4}$$

This proves the proposition the special case $z = re_1$, where $0 \le r < 1$ and $e_1 = (1, 0, \ldots, 0)$.

Now consider the general case. That is, $z = ru$, where $0 \le r < 1$ and u is a unit vector in $\mathbf{C}^n$. Again, let $U : \mathbf{C}^n \to \mathbf{C}^n$ be a unitary transformation such that $Ue_1 = u$, and let W be the unitary operator on H_n^2 defined by (3.6). We have $WM_{\langle \zeta, z\rangle} = M_{r\zeta_1} W$. Taking adjoints, since W is a unitary operator, we see that $WM^*_{\langle \zeta, z\rangle} = M^*_{r\zeta_1} W$. Thus

$$WQ_z = Q_{re_1} W.$$

Given an $h \in H^2_n$, we write $\eta = Wh$. Repeating the argument in the proof of Proposition 3.1 and applying (4.4), we obtain

$$\begin{aligned}\mathrm{Re}\langle h, K_z h\rangle = \mathrm{Re}\langle Wh, WK_z h\rangle = \mathrm{Re}\langle \eta, K_{re_1}\eta\rangle &= \frac{1}{2}(\|\eta\|^2 + \|Q_{re_1}M_{K_{re_1}}\eta\|^2) \\ = \frac{1}{2}(\|Wh\|^2 + \|WQ_z M_{K_z}h\|^2) &= \frac{1}{2}(\|h\|^2 + \|Q_z M_{K_z}h\|^2).\end{aligned}$$

This completes the proof of the proposition. □

For any normed space $\mathcal{N}$ and any $x, y \in \mathcal{N}$, we always have

$$\|x + y\|^2 \le 2\|x\|^2 + 2\|y\|^2.$$

Thus Proposition 4.1 immediately implies the following "quasi triangle inequality":

Corollary 4.2. *For all $h, g \in H^2_n$ and $z \in \mathbf{B}$, we have*

$$\mathrm{Re}\langle h + g, K_z(h + g)\rangle \le 2\mathrm{Re}\langle h, K_z h\rangle + 2\mathrm{Re}\langle g, K_z g\rangle.$$

Proposition 4.3. *Let $f \in \mathcal{M}$. Suppose that there is a $c > 0$ and a sequence $\{r_k\}$ in $(0, 1)$ such that $\lim_{k\to\infty} r_k = 1$ and*

$$\sup_{|z|=r_k} \mathrm{Re}\langle f, K_z f\rangle \ge c \log \frac{1}{1 - r_k} \tag{4.5}$$

for every $k \ge 1$. Then f belongs to the interior of $\mathcal{M}\backslash\mathcal{F}$.

Proof. We need to find an $\epsilon > 0$ such that if $\gamma \in \mathcal{M}$ and $\|\gamma\|_{\mathcal{M}} < \epsilon$, then $f + \gamma \in \mathcal{M}\backslash\mathcal{F}$. To do this, consider any $\gamma \in \mathcal{M}$. Applying Corollary 4.2 to the case where $h = f + \gamma$ and $g = -\gamma$, we have

$$\mathrm{Re}\langle f, K_z f\rangle \le 2\mathrm{Re}\langle f + \gamma, K_z(f + \gamma)\rangle + 2\mathrm{Re}\langle \gamma, K_z\gamma\rangle$$

for every $z \in \mathbf{B}$. Applying Corollary 3.2 to $\mathrm{Re}\langle \gamma, K_z\gamma\rangle$, we obtain

$$\mathrm{Re}\langle f, K_z f\rangle \le 2\mathrm{Re}\langle f + \gamma, K_z(f + \gamma)\rangle + 2C_{3.2}\|\gamma\|^2_{\mathcal{M}}\left(1 + \log \frac{1}{1 - |z|}\right),$$

$z \in \mathbf{B}$. Combining this with (4.5), we find that

$$2 \sup_{|z|=r_k} \mathrm{Re}\langle f + \gamma, K_z(f + \gamma)\rangle \ge (c - 2C_{3.2}\|\gamma\|^2_{\mathcal{M}}) \log \frac{1}{1 - r_k} - 2C_{3.2}\|\gamma\|^2_{\mathcal{M}}$$

for every $k \ge 1$. Now pick an $\epsilon > 0$ such that $2C_{3.2}\epsilon^2 < c/2$. For $\gamma \in \mathcal{M}$ satisfying the condition $\|\gamma\|_{\mathcal{M}} < \epsilon$, the above gives us

$$2 \sup_{|z|=r_k} \mathrm{Re}\langle f + \gamma, K_z(f + \gamma)\rangle \ge \frac{c}{2} \log \frac{1}{1 - r_k} - 2C_{3.2}\epsilon^2.$$

Since this holds for every $k \geq 1$ and since $\lim_{k\to\infty} r_k = 1$, we conclude that $f + \gamma \in \mathcal{M}\backslash\mathcal{F}$. This proves the proposition. $\square$

Proposition 4.4. *Let $f \in \mathcal{M}$. Suppose that f has the property that for every $\epsilon > 0$, there is an $r(\epsilon) \in (0, 1)$ such that*

$$\mathrm{Re}\langle f, K_z f\rangle \leq \epsilon \log \frac{1}{1 - |z|} \quad \textit{whenever} \;\; r(\epsilon) \leq |z| < 1. \tag{4.6}$$

Then for every $\xi \in \mathbf{C}\backslash\{0\}$, $f + \xi\varphi$ belongs to the interior of $\mathcal{M}\backslash\mathcal{F}$.

Proof. Given any $\xi \in \mathbf{C}\backslash\{0\}$, we pick an $\epsilon = \epsilon(\xi) > 0$ such that $2\epsilon < |\xi|^2 c_{1.1}/2$, where $c_{1.1}$ is the constant provided by Theorem 1.1. Applying Corollary 4.2 to the case where $h = f + \xi\varphi$ and $g = -f$, we obtain

$$|\xi|^2 \mathrm{Re}\langle \varphi, K_z\varphi\rangle \leq 2\mathrm{Re}\langle f + \xi\varphi, K_z(f + \xi\varphi)\rangle + 2\mathrm{Re}\langle f, K_z f\rangle,$$

$z \in \mathbf{B}$. Applying Theorem 1.1 on the left and (4.6) on the right, if $r(\epsilon) \leq r < 1$, then

$$\begin{aligned} |\xi|^2 c_{1.1} \log \frac{1}{1 - r} &\leq |\xi|^2 \sup_{|z|=r} \mathrm{Re}\langle \varphi, K_z\varphi\rangle \\ &\leq 2 \sup_{|z|=r} \mathrm{Re}\langle f + \xi\varphi, K_z(f + \xi\varphi)\rangle + 2\epsilon \log \frac{1}{1 - r}. \end{aligned}$$

Since $2\epsilon < |\xi|^2 c_{1.1}/2$, the obvious cancellation leads to

$$\frac{|\xi|^2 c_{1.1}}{2} \log \frac{1}{1 - r} \leq 2 \sup_{|z|=r} \mathrm{Re}\langle f + \xi\varphi, K_z(f + \xi\varphi)\rangle$$

for every $r(\epsilon) \leq r < 1$. This shows that the function $f + \xi\varphi$ satisfies condition (4.5). By Proposition 4.3, $f + \xi\varphi$ is in the interior of $\mathcal{M}\backslash\mathcal{F}$ as promised. $\square$

Proof of Theorem 1.3. Let $\mathcal{U}$ denote the interior of $\mathcal{M}\backslash\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{M}\backslash\mathcal{U}$, it suffices to show that $\mathcal{M}\backslash\mathcal{U}$ is nowhere dense in $\mathcal{M}$. Since $\mathcal{M}\backslash\mathcal{U}$ is closed, the desired conclusion will follow if we can show that $\mathcal{U}$ is dense in $\mathcal{M}$.

Consider any $f \in \mathcal{M}$. If f satisfies condition (4.5), then Proposition 4.3 tells us that $f \in \mathcal{U}$. If f fails condition (4.5), then f has no choice but to satisfy condition (4.6), in which case Proposition 4.4 provides the inclusion $\{f + \xi\varphi : \xi \in \mathbf{C}\backslash\{0\}\} \subset \mathcal{U}$. Thus we see that in either case, f is in the closure of $\mathcal{U}$. This completes the proof. $\square$

References

1. A. Aleman, M. Hartz, J. McCarthy and S. Richter, Factorizations induced by complete Nevanlinna-Pick factors, Adv. Math. **335** (2018), 372-404.
2. W. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory, Acta Math. **181** (1998), 159-228.

3. S. Drury, A generalization of von Neumann's inequality to the complex ball, Proc. Amer. Math. Soc. **68** (1978), 300-304.
4. Q. Fang and J. Xia, Commutators and localization on the Drury-Arveson space, J. Funct. Anal. **260** (2011), 639-673.
5. Q. Fang and J. Xia, On the problem of characterizing multipliers for the Drury-Arveson space, Indiana Univ. Math. J. **64** (2015), 663-696.
6. J. Garnett, Bounded analytic functions, Academic Press, New York, 1981.
7. A. Lubin, Weighted shifts and products of subnormal operators, Indiana Univ. Math. J. **26** (1977), 839-845.
8. M. Mitkovski and B. Wick, A reproducing kernel thesis for operators on Bergman-type function spaces, J. Funct. Anal. **267** (2014), 2028-2055.
9. W. Rudin, Function theory in the unit ball of $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

Quanlei Fang
Department of Mathematics and Computer Science, Bronx Community College, CUNY, Bronx, NY 10453
E-mail: quanlei.fang@bcc.cuny.edu

Jingbo Xia
Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260
E-mail: jxia@acsu.buffalo.edu