DIXMIER TRACES, CESARO MEANS AND LOGARITHMS

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Abstract. This paper addresses a subtle issue arising from the measurability of operators with respect to the Dixmier trace.

1. Measurability and convergence

In the last few decades, Dixmier trace [3] has played an increasingly important role in non-commutative geometry, operator theory, and the study of operator algebras. We cite [1,4-7,9,11] as a sample of recent developments. This paper stems from the concept of *measurability* of an operator with respect to the Dixmier trace, and the reason why this is even an issue can be traced to a subtlety in how Dixmier trace is introduced in various books and papers. To explain this subtlety, let us start from scratch.

We first recall that the domain of every Dixmier trace is the Lorentz ideal $\mathcal{C}_1^+$, which consists of operators A satisfying the condition

$$\|A\|_1^+ = \sup_{k\geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1} + 2^{-1} + \cdots + k^{-1}} < \infty,$$

where $s_1(A), s_2(A), \ldots, s_k(A), \ldots$ are the s-numbers of A. See, e.g., [8]. Alternatively, the symbol $\mathcal{L}^{(1,\infty)}$ is often used to denote this ideal.

To define the Dixmier trace, one starts with a linear form (also called an *extended limit*) $\omega : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ that has the following three properties:

(α) $\omega(\{a_k\}_{k\in\mathbf{N}}) \geq 0$ if $a_k \geq 0$ for every $k \in \mathbf{N}$.

(β) $\omega(\{a_k\}_{k\in\mathbf{N}}) = \lim_{k\to\infty} a_k$ whenever the sequence $\{a_k\}$ converges.

(γ) For each $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, $\omega(\{a_k\}_{k\in\mathbf{N}}) = \omega(\{a_1, a_1, a_2, a_2, \ldots, a_k, a_k, \ldots\})$.

Given such an ω, for any positive operator $A \in \mathcal{C}_1^+$, its Dixmier trace is defined to be

$$\mathrm{Tr}_\omega(A) = \omega\left(\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} s_j(A)\right\}_{k\in\mathbf{N}}\right).$$

The doubling property (γ) ensures the additivity $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for positive operators $A, B \in \mathcal{C}_1^+$. Thus Tr_ω naturally extends to a linear functional on $\mathcal{C}_1^+$. This ensures the unitary invariance of Tr_ω, and consequently $\mathrm{Tr}_\omega(XT) = \mathrm{Tr}_\omega(TX)$ for every $T \in \mathcal{C}_1^+$ and every bounded operator X, which is what one expects of a trace.

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At this point, it will be convenient to introduce

Definition 1.1. Let Ω denote the collection of linear forms $\omega : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ that have properties (α), (β) and (γ).

Obviously, there are plenty of ω with properties (α) and (β). The problem lies with how to obtain property (γ), which is the main issue in defining the Dixmier trace. In much of the literature, most noticeably in Alain Connes' seminal book [2], property (γ) is obtained through the interposition of Cesàro mean. Recall from [2, Section 4.2β] that for a sequence $a = \{a_k\}$, its *Cesàro mean* is the sequence of numbers

$$M_k(a) = \frac{1}{\log(k+1)} \sum_{j=1}^{k} a_j \log\left(\frac{j+1}{j}\right), \quad k \in \mathbf{N}.$$

Let $L : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ be any linear form that has properties (α) and (β). Then the formula

$$\omega(a) = L\left(\{M_k(a)\}_{k\in\mathbf{N}}\right), \quad a \in \ell^\infty(\mathbf{N}), \tag{1.1}$$

defines a linear form on $\ell^\infty(\mathbf{N})$ that has all three properties (α), (β) and (γ).

Definition 1.2. Let $\mathcal{M}$ denote the collection of linear forms $\omega : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ given by (1.1), where $L : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ is any linear form that has properties (α) and (β).

Thus we have two collections of Dixmier traces, $\{\mathrm{Tr}_\omega : \omega \in \Omega\}$ and $\{\mathrm{Tr}_\omega : \omega \in \mathcal{M}\}$, and the latter is a subset of the former. This leads to the issue of measurability of operators. Recall that an operator $A \in \mathcal{C}_1^+$ is said to be measurable if its Dixmier trace $\mathrm{Tr}_\omega(A)$ is independent of ω. Here, one might want to be a little more careful by asking, independent of *which set of* ω? Prima facie, the condition that $\mathrm{Tr}_\omega(A)$ is independent of $\omega \in \mathcal{M}$ appears to be weaker than the condition that $\mathrm{Tr}_\omega(A)$ is independent of $\omega \in \Omega$.

A careful reading of Section 4.2β in [2] tells us that the Dixmier traces considered in [2] are all in the collection $\{\mathrm{Tr}_\omega : \omega \in \mathcal{M}\}$. Therefore the measurability of an operator $A \in \mathcal{C}_1^+$ in [2] means that its Dixmier trace $\mathrm{Tr}_\omega(A)$ is independent of $\omega \in \mathcal{M}$. Accordingly, we have

Theorem 1.3. (Proposition 6(a) in [2, Section 4.2β]) *Let A be a positive operator in $\mathcal{C}_1^+$. Then $\mathrm{Tr}_\omega(A)$ is independent of $\omega \in \mathcal{M}$ if and only if the Cesàro mean of the sequence $\{\sum_{j=1}^k s_j(A)/\log(k+1)\}_{k\in\mathbf{N}}$ converges, i.e., the limit*

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} \left(\frac{1}{\log(j+1)} \sum_{i=1}^{j} s_i(A)\right) \log\left(\frac{j+1}{j}\right)$$

exists.

In contrast, the entire collection of Dixmier traces $\{\mathrm{Tr}_\omega : \omega \in \Omega\}$ is considered in [10], and more. Accordingly, measurability in [10] yields an apparently stronger result:

Theorem 1.4. [10, Theorem 9.2.1] *Let A be a positive operator in $\mathcal{C}_1^+$. Then* $\mathrm{Tr}_\omega(A)$ *is independent of $\omega \in \Omega$ if and only if the limit*

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A)$$

exists.

As the first result of this paper, we report that for a positive operator $A \in \mathcal{C}_1^+$, its measurability with respect the more restricted set of Dixmier traces $\{\mathrm{Tr}_\omega : \omega \in \mathcal{M}\}$ actually implies its measurability with respect to $\{\mathrm{Tr}_\omega : \omega \in \Omega\}$, the whole set of Dixmier traces:

Theorem 1.5. *Let A be a positive operator in $\mathcal{C}_1^+$. If the limit*

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} \left(\frac{1}{\log(j+1)} \sum_{i=1}^{j} s_i(A) \right) \log\left(\frac{j+1}{j} \right)$$

exists, then the limit

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A)$$

exists.

Given Theorem 1.5, one has to wonder, is the subset $\{\mathrm{Tr}_\omega : \omega \in \mathcal{M}\}$ of Dixmier traces sufficient for all practical purposes? The answer is negative. To see this, we first observe, based on properties (α) and (β), that for every positive operator $A \in \mathcal{C}_1^+$ and every $\omega \in \Omega$,

$$\mathrm{Tr}_\omega(A) \leq \limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A).$$

But the above is actually an equality for at least one $\omega \in \Omega$:

Theorem 1.6. [10, page 275] *Let A be any positive operator in $\mathcal{C}_1^+$. Then there is an $\omega \in \Omega$ such that*

$$\mathrm{Tr}_\omega(A) = \limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A).$$

As the second result of the paper, we show that there is a substantive difference between the subset $\{\mathrm{Tr}_\omega : \omega \in \mathcal{M}\}$ and the full set $\{\mathrm{Tr}_\omega : \omega \in \Omega\}$ of Dixmier traces:

Theorem 1.7. *There exists a positive operator $A \in \mathcal{C}_1^+$ such that*

$$\sup_{\omega\in\mathcal{M}} \mathrm{Tr}_\omega(A) < \limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A).$$

The rest of the paper consists of the proofs of Theorems 1.5 and 1.7. Specifically, we prove Theorem 1.5 in Section 2 and Theorem 1.7 in Section 3. From the proofs the reader will see that both results are due to the involvement of logarithm in Dixmier trace.

2. Limit of Cesàro mean

To prove Theorem 1.5, it will be convenient to introduce a particular collection of sequences. Let d_1^+ denote the collection of sequences $\{x_k\}$ of non-negative terms such that

$$\sup_{k\geq 1} \frac{x_1 + x_2 + \cdots + x_k}{1^{-1} + 2^{-1} + \cdots + k^{-1}} < \infty$$

and such that $x_k \geq x_{k+1}$ for every $k \in \mathbf{N}$. Thus a sequence $\{x_k\}$ is in d_1^+ if and only if there is an operator $A \in \mathcal{C}_1^+$ such that $s_k(A) = x_k$ for every $k \in \mathbf{N}$.

Lemma 2.1. *Let $\{x_k\} \in d_1^+$. If the limit*

$$\mathcal{L} = \lim_{N\to\infty} \frac{1}{\log(N+1)} \sum_{k=1}^{N} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right)$$

exists, then

$$\mathcal{L} \geq \limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j.$$

Proof. If it were true that

$$\limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j > \mathcal{L},$$

then there would be a $c > 0$ and a sequence

$$k_1 < k_2 < \cdots < k_i < \cdots$$

in $\mathbf{N}$ such that

$$\frac{1}{\log(k_i+1)} \sum_{j=1}^{k_i} x_j \geq \mathcal{L} + c \tag{2.1}$$

for every $i \in \mathbf{N}$. We will show that this leads to a contradiction.

Let $\epsilon > 0$ be such that

$$\frac{\mathcal{L} + c}{1+\epsilon} \geq \mathcal{L} + (c/2). \tag{2.2}$$

For each i, let m_i be the largest integer that is less than or equal to $(k_i+1)^{1+\epsilon}-1$. If $k \in \mathbf{N}$ satisfies the condition $k_i < k \le m_i$, then by (2.1) and (2.2) we have

$$\begin{aligned}\frac{1}{\log(k+1)}\sum_{j=1}^{k} x_j &\ge \frac{1}{\log(m_i+1)}\sum_{j=1}^{k_i} x_j \\ &= \frac{\log(k_i+1)}{\log(m_i+1)}\cdot\frac{1}{\log(k_i+1)}\sum_{j=1}^{k_i} x_j \ge \frac{\mathcal{L}+c}{1+\epsilon} \ge \mathcal{L}+(c/2). \end{aligned} \tag{2.3}$$

For each $i \in \mathbf{N}$, we also have

$$\frac{1}{\log(m_i+1)}\sum_{k=1}^{m_i}\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} x_j\right\}\log\left(\frac{k+1}{k}\right) = \frac{\log(k_i+1)}{\log(m_i+1)}a_i + \frac{\log\left(\frac{m_i+1}{k_i+1}\right)}{\log(m_i+1)}b_i, \tag{2.4}$$

where

$$a_i = \frac{1}{\log(k_i+1)}\sum_{k=1}^{k_i}\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} x_j\right\}\log\left(\frac{k+1}{k}\right) \quad \text{and}$$

$$b_i = \frac{1}{\log\left(\frac{m_i+1}{k_i+1}\right)}\sum_{k=k_i+1}^{m_i}\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} x_j\right\}\log\left(\frac{k+1}{k}\right).$$

By assumption, we have

$$\lim_{i\to\infty} a_i = \mathcal{L}. \tag{2.5}$$

It follows from (2.3) that

$$b_i \ge \mathcal{L}+(c/2) \tag{2.6}$$

for large i. It is obvious that

$$\lim_{i\to\infty}\frac{\log(k_i+1)}{\log(m_i+1)} = \frac{1}{1+\epsilon} \quad \text{and} \quad \lim_{i\to\infty}\frac{\log\left(\frac{m_i+1}{k_i+1}\right)}{\log(m_i+1)} = \frac{\epsilon}{1+\epsilon}. \tag{2.7}$$

Thus it follows from (2.4-7) that

$$\lim_{i\to\infty}\frac{1}{\log(m_i+1)}\sum_{k=1}^{m_i}\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} x_j\right\}\log\left(\frac{k+1}{k}\right) \ge \frac{\mathcal{L}}{1+\epsilon} + \frac{\epsilon\{\mathcal{L}+(c/2)\}}{1+\epsilon} > \mathcal{L},$$

which is the contradiction promised earlier. □

The proof of our next lemma uses an argument similar to the one in the proof of Lemma 2.1. Nonetheless, the proof is included here for completeness.

Lemma 2.2. *Let* $\{x_k\} \in d_1^+$. *If the limit*

$$\mathcal{L} = \lim_{N\to\infty} \frac{1}{\log(N+1)} \sum_{k=1}^{N} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right)$$

exists, then

$$\mathcal{L} \le \liminf_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j.$$

Proof. If it were true that

$$\liminf_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j < \mathcal{L},$$

then there would be a $c > 0$ and a sequence

$$k_1 < k_2 < \cdots < k_i < \cdots$$

in $\mathbf{N}$ such that

$$\frac{1}{\log(k_i+1)} \sum_{j=1}^{k_i} x_j \le \mathcal{L} - c \tag{2.8}$$

for every $i \in \mathbf{N}$. We will show that this leads to a contradiction.

Let $0 < \epsilon < 1$ be such that

$$\frac{\mathcal{L} - c}{1 - \epsilon} \le \mathcal{L} - (c/2). \tag{2.9}$$

For each i, let ℓ_i be the smallest integer that is greater than or equal to $(k_i + 1)^{1-\epsilon} - 1$. If $k \in \mathbf{N}$ satisfies the condition $\ell_i < k \le k_i$, then by (2.8) and (2.9) we have

$$\begin{aligned} \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j &\le \frac{1}{\log(\ell_i+1)} \sum_{j=1}^{k_i} x_j \\ &= \frac{\log(k_i+1)}{\log(\ell_i+1)} \cdot \frac{1}{\log(k_i+1)} \sum_{j=1}^{k_i} x_j \le \frac{\mathcal{L} - c}{1 - \epsilon} \le \mathcal{L} - (c/2). \end{aligned} \tag{2.10}$$

For sufficiently large i, we also have

$$\frac{1}{\log(k_i+1)} \sum_{k=1}^{k_i} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right) = \frac{\log(\ell_i+1)}{\log(k_i+1)} a_i + \frac{\log\left(\frac{k_i+1}{\ell_i+1}\right)}{\log(k_i+1)} b_i, \tag{2.11}$$

where

$$a_i = \frac{1}{\log(\ell_i + 1)} \sum_{k=1}^{\ell_i} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right) \quad \text{and}$$

$$b_i = \frac{1}{\log\left(\frac{k_i+1}{\ell_i+1}\right)} \sum_{k=\ell_i+1}^{k_i} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right).$$

By assumption, we have

$$\lim_{i\to\infty} a_i = \mathcal{L}. \tag{2.12}$$

It follows from (2.10) that

$$b_i \le \mathcal{L} - (c/2) \tag{2.13}$$

for sufficiently large i. It is obvious that

$$\lim_{i\to\infty} \frac{\log(\ell_i + 1)}{\log(k_i + 1)} = 1 - \epsilon \quad \text{and} \quad \lim_{i\to\infty} \frac{\log\left(\frac{k_i+1}{\ell_i+1}\right)}{\log(k_i + 1)} = \epsilon. \tag{2.14}$$

Thus it follows from (2.11-14) that

$$\lim_{i\to\infty} \frac{1}{\log(k_i + 1)} \sum_{k=1}^{k_i} \left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} x_j \right\} \log\left(\frac{k+1}{k}\right) \le (1-\epsilon)\mathcal{L} + \epsilon(\mathcal{L} - (c/2)) < \mathcal{L},$$

which is the contradiction promised earlier. □

Proof of Theorem 1.5. Let $A \in \mathcal{C}_1^+$ be a positive operator such that the limit

$$\mathcal{L} = \lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} \left(\frac{1}{\log(j+1)} \sum_{i=1}^{j} s_i(A) \right) \log\left(\frac{j+1}{j}\right)$$

exists. Applying Lemmas 2.1 and 2.2 to the sequence $\{x_k\} = \{s_k(A)\}$, we obtain

$$\limsup_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A) \le \mathcal{L} \le \liminf_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A).$$

This implies, of course, that the limit

$$\lim_{k\to\infty} \frac{1}{\log(k+1)} \sum_{j=1}^{k} s_j(A)$$

exists and equals $\mathcal{L}$. $\square$

3. Supremum over $\mathcal{M}$

From properties (α) and (β) of the linear form ω we see that for every positive operator $A \in \mathcal{C}_1^+$ and every $\omega \in \mathcal{M}$, we have

$$\mathrm{Tr}_\omega(A) \leq \limsup_{n\to\infty} \frac{1}{\log(n+1)} \sum_{\nu=1}^{n} \left\{ \frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} s_j(A) \right\} \log\left(\frac{\nu+1}{\nu}\right).$$

For each sequence $\xi = \{\xi_j\}$ and each $n \in \mathbf{N}$, we define

$$(T\xi)(n) = \frac{1}{\log(n+1)} \sum_{\nu=1}^{n} \left\{ \frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} \xi_j \right\} \log\left(\frac{\nu+1}{\nu}\right).$$

Thus to prove Theorem 1.7, it suffices to produce a sequence $y = \{y_j\}$ in d_1^+ that satisfies the conditions

$$(Ty)(n) \leq \frac{257}{300} \quad \text{for every} \quad n \in \mathbf{N} \tag{3.1}$$

and

$$\limsup_{\nu\to\infty} \frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} y_j \geq 1. \tag{3.2}$$

To construct such a sequence, we consider any $k \geq 100$ such that

$$\frac{\log(k+1)}{\log k} \leq \frac{101}{100}.$$

For such a k, define the sequence $x^{(k)} = \{x_j^{(k)}\}$ by the rules that $x_j^{(k)} = 1$ if $j \leq k^2$ and $x_j^{(k)} = 0$ if $j > k^2$. Write

$$a_\nu^{(k)} = \frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} x_j^{(k)}, \quad \nu \in \mathbf{N}.$$

By differentiation we see that the function $x/\log(x+1)$ is increasing on $[2,\infty)$. Thus

$$a_1^{(k)} < a_2^{(k)} < \cdots < a_{k^2}^{(k)} = \frac{k^2}{\log(k^2+1)}.$$

Write

$$b_n^{(k)} = \frac{1}{\log(n+1)} \sum_{\nu=1}^{n} a_\nu^{(k)} \log\left(\frac{\nu+1}{\nu}\right),$$

$n \in \mathbf{N}$. If $1 \le n \le k$, then

$$b_n^{(k)} \le a_k^{(k)} = \frac{a_k^{(k)}}{a_{k^2}^{(k)}} a_{k^2}^{(k)} = \frac{1}{k} \cdot \frac{\log(k^2+1)}{\log(k+1)} a_{k^2}^{(k)} \le \frac{2}{k} a_{k^2}^{(k)} \le \frac{1}{50} a_{k^2}^{(k)}.$$

If $k < n \le k^2$, then

$$\begin{aligned} b_n^{(k)} &= \frac{\log(k+1)}{\log(n+1)} b_k^{(k)} + \frac{\log\left(\frac{n+1}{k+1}\right)}{\log(n+1)} \cdot \frac{1}{\log\left(\frac{n+1}{k+1}\right)} \sum_{\nu=k+1}^{n} a_\nu^{(k)} \log\left(\frac{\nu+1}{\nu}\right) \\ &\le \frac{\log(k+1)}{\log(n+1)} \cdot \frac{1}{50} a_{k^2}^{(k)} + \frac{\log\left(\frac{n+1}{k+1}\right)}{\log(n+1)} \cdot a_{k^2}^{(k)} \\ &\le \left(\frac{1}{50} + \frac{\log\left(\frac{k^2+1}{k+1}\right)}{\log(k^2+1)}\right) a_{k^2}^{(k)} \le \left(\frac{1}{50} + \frac{1}{2}\right) a_{k^2}^{(k)} = \frac{26}{50} a_{k^2}^{(k)}, \end{aligned}$$

where the last $\le$ follows from the fact that $k+1 \ge \sqrt{k^2+1}$. If $k^2 < n \le k^3$, then

$$\begin{aligned} b_n^{(k)} &= \frac{\log(k^2+1)}{\log(n+1)} b_{k^2}^{(k)} + \frac{\log\left(\frac{n+1}{k^2+1}\right)}{\log(n+1)} \cdot \frac{1}{\log\left(\frac{n+1}{k^2+1}\right)} \sum_{\nu=k^2+1}^{n} a_\nu^{(k)} \log\left(\frac{\nu+1}{\nu}\right) \\ &\le \left(\frac{26}{50} + \frac{\log\left(\frac{n+1}{k^2+1}\right)}{\log(n+1)}\right) a_{k^2}^{(k)} \le \left(\frac{26}{50} + \frac{\log\left(\frac{k^3+1}{k^2+1}\right)}{\log(k^3+1)}\right) a_{k^2}^{(k)}. \end{aligned}$$

Since $(k^3+1)^{2/3} \le (k^3)^{2/3} + 1^{2/3} = k^2+1$, we have $\log(k^2+1)/\log(k^3+1) \ge 2/3$. Thus for $k^2 < n \le k^3$,

$$b_n^{(k)} \le \left(\frac{26}{50} + \frac{1}{3}\right) a_{k^2}^{(k)} = \frac{128}{150} a_{k^2}^{(k)}.$$

If $n > k^3$, then

$$\begin{aligned} b_n^{(k)} &= \frac{\log(k^3+1)}{\log(n+1)} b_{k^3}^{(k)} + \frac{\log\left(\frac{n+1}{k^3+1}\right)}{\log(n+1)} \cdot \frac{1}{\log\left(\frac{n+1}{k^3+1}\right)} \sum_{\nu=k^3+1}^{n} a_\nu^{(k)} \log\left(\frac{\nu+1}{\nu}\right) \\ &\le \frac{\log(k^3+1)}{\log(n+1)} \cdot \frac{128}{150} a_{k^2}^{(k)} + \frac{\log\left(\frac{n+1}{k^3+1}\right)}{\log(n+1)} a_{k^3}^{(k)} \\ &= \frac{\log(k^3+1)}{\log(n+1)} \cdot \frac{128}{150} a_{k^2}^{(k)} + \frac{\log\left(\frac{n+1}{k^3+1}\right)}{\log(n+1)} \cdot \frac{\log(k^2+1)}{\log(k^3+1)} a_{k^2}^{(k)} \\ &\le \frac{\log(k^3+1)}{\log(n+1)} \cdot \frac{128}{150} a_{k^2}^{(k)} + \frac{\log\left(\frac{n+1}{k^3+1}\right)}{\log(n+1)} \cdot \frac{2}{3} \cdot \frac{\log(k+1)}{\log k} a_{k^2}^{(k)} \le \frac{128}{150} a_{k^2}^{(k)}. \end{aligned}$$

Summarizing the above, for every $n \geq 1$ we have

$$b_n^{(k)} \leq \frac{128}{150} a_{k^2}^{(k)} = \frac{128}{150} \cdot \frac{k^2}{\log(k^2+1)}. \tag{3.3}$$

Now, for each $k \geq 100$ such that $\log(k+1)/\log k \leq 101/100$, we define the sequence

$$y^{(k)} = \{y_j^{(k)}\} = \frac{\log(k^2+1)}{k^2} x^{(k)}.$$

That is, $y_j^{(k)} = x_j^{(k)} k^{-2} \log(k^2+1)$ for every $j \geq 1$. Obviously,

$$\frac{1}{\log(k^2+1)} \sum_{j=1}^{k^2} y_j^{(k)} = 1, \tag{3.4}$$

and (3.3) translates to

$$(Ty^{(k)})(n) \leq \frac{128}{150} \tag{3.5}$$

for every $n \geq 1$. For every $i \in \mathbf{N}$, there is a $k_i \in \mathbf{N}$ satisfying the conditions $k_i \geq 100$, $\log(k_i+1)/\log k_i \leq 101/100$, and

$$\frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} y_j^{(k_i)} \leq \frac{2^{-i}}{300} \quad \text{for every} \quad 1 \leq \nu \leq i. \tag{3.6}$$

Obviously, this implies that

$$(Ty^{(k_i)})(n) \leq \frac{2^{-i}}{300} \quad \text{for every} \quad 1 \leq n \leq i. \tag{3.7}$$

Since $y_j^{(k_i)} = 0$ for $j > k_i^2$, for each i, there is an $n_i > i$ such that

$$\frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} y_j^{(k_i)} \leq \frac{2^{-i}}{300} \quad \text{for every} \quad \nu \geq n_i \tag{3.8}$$

and

$$(Ty^{(k_i)})(n) \leq \frac{2^{-i}}{300} \quad \text{for every} \quad n \geq n_i. \tag{3.9}$$

Inductively, we now select a sequence

$$i(1) < i(2) < \cdots < i(s) < \cdots$$

such that

$$n_{i(s)} < i(s+1)$$

for every $s \in \mathbf{N}$. Thus if we define $I_s = [i(s), n_{i(s)}]$ for $s \in \mathbf{N}$, then $I_s \cap I_{s'} = \emptyset$ for all $s \neq s'$ in $\mathbf{N}$. Now define

$$y = \{y_j\} = \sum_{s=1}^{\infty} y^{(k_{i(s)})}.$$

Consider any $n \in \mathbf{N}$. If $n \notin \cup_{s=1}^{\infty} I_s$, then it follows from (3.7) and (3.9) that

$$(Ty)(n) = \sum_{s=1}^{\infty} (Ty^{(k_{i(s)})})(n) \leq \frac{1}{300} \sum_{s=1}^{\infty} 2^{-i(s)} \leq \frac{1}{300}.$$

If $n \in \cup_{s=1}^{\infty} I_s$, then there is an $r \in \mathbf{N}$ such that $n \in I_r$, and consequently $n \notin \cup_{s \neq r} I_s$. In this case, it follows from (3.5), (3.7) and (3.9) that

$$(Ty)(n) = (Ty^{(k_{i(r)})})(n) + \sum_{s \neq r} (Ty^{(k_{i(s)})})(n) \leq \frac{128}{150} + \frac{1}{300} \sum_{s \neq r} 2^{-i(s)} \leq \frac{257}{300}.$$

Combining these two inequalities, we see that (3.1) holds. From (3.6), (3.8) and the fact that $I_s \cap I_{s'} = \emptyset$ for all $s \neq s'$ in $\mathbf{N}$ we similarly deduce that

$$\frac{1}{\log(\nu+1)} \sum_{j=1}^{\nu} y_j \leq 1 + \frac{1}{300} \quad \text{for every} \quad \nu \in \mathbf{N}.$$

That is, $y \in d_1^+$. On the other hand, it follows from (3.4) that

$$\frac{1}{\log(k_{i(s)}^2 + 1)} \sum_{j=1}^{k_{i(s)}^2} y_j \geq 1 \quad \text{for every} \quad s \in \mathbf{N}.$$

Hence (3.2) also holds. This completes the construction of the sequence $y = \{y_j\}$ and proves Theorem 1.7. □

References

1. A. Connes, The action functional in noncommutative geometry, Comm. Math. Phys. **117** (1988), 673-683.

2. A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.

3. J. Dixmier, Existence de traces non normales, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1107-A1108.

4. M. Engliš, K. Guo and G. Zhang, Toeplitz and Hankel operators and Dixmier traces on the unit ball of $\mathbf{C}^n$, Proc. Amer. Math. Soc. **137** (2009), 3669-3678.

5. M. Engliš and R. Rochberg, The Dixmier trace of Hankel operators on the Bergman space, J. Funct. Anal. **257** (2009), 1445-1479.

6. M. Engliš and G. Zhang, Hankel operators and the Dixmier trace on the Hardy space, J. Lond. Math. Soc. (2) **94** (2016), 337-356.
7. M. Goffeng and A. Usachev, Estimating Dixmier traces of Hankel operators in Lorentz ideals, J. Funct. Anal. **279** (2020), no. 7, 108688.
8. I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
9. L. Jiang, Y. Wang and J. Xia, Toeplitz operators associated with measures and the Dixmier trace on the Hardy space, Complex Anal. Oper. Theory **14** (2020), no. 2, Paper No. 30.
10. L. Steven, F. Sukochev and D. Zanin, Singular traces: theory and applications, De Gruyter Studies in Mathematics **46**, De Gruyter, Berlin, 2013.
11. H. Upmeier and K. Wang, Dixmier trace for Toeplitz operators on symmetric domains, J. Funct. Anal. **271** (2016), 532-565.

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