# BOYD INDICES AND THE BERGER-COBURN PHENOMENON

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**Abstract.** We settle the issue of Berger-Coburn phenomenon on the Fock space completely for general symmetrically normed ideals $\mathcal{C}_\Phi$, where $\Phi$ is not equivalent to $\Phi_\infty$. We show that if the Boyd indices of $\mathcal{C}_\Phi$ satisfy the condition $1 < p_\Phi \le q_\Phi < \infty$, then for $f \in L^\infty(\mathbf{C}^n)$, we have $H_f \in \mathcal{C}_\Phi$ if and only if $H_{\bar{f}} \in \mathcal{C}_\Phi$. We further show that if either $p_\Phi = 1$ or $q_\Phi = \infty$, then there is an $f \in L^\infty(\mathbf{C}^n)$ such that $H_f \in \mathcal{C}_\Phi$ while $H_{\bar{f}} \notin \mathcal{C}_\Phi$.

## 1. Introduction

Let $d\mu$ denote the Gaussian measure on $\mathbf{C}^n$. More precisely, we write

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z),$$

where $dV$ is the standard volume measure on $\mathbf{C}^n$. Recall that the Fock space $H^2(\mathbf{C}^n, d\mu)$ is the norm closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(\mathbf{C}^n, d\mu)$. Let $P : L^2(\mathbf{C}^n, d\mu) \to H^2(\mathbf{C}^n, d\mu)$ be the orthogonal projection. Given an appropriate symbol function $f$, the Hankel operator $H_f : H^2(\mathbf{C}^n, d\mu) \to L^2(\mathbf{C}^n, d\mu) \ominus H^2(\mathbf{C}^n, d\mu)$ is defined by the formula

$$H_f h = (1 - P)(fh),$$

$h \in H^2(\mathbf{C}^n, d\mu)$.

For a general $f$, very little about $H_{\bar{f}}$ can be inferred from the properties of $H_f$. Consequently, the so-called one-sided theory of Hankel operators, namely the study of $H_f$ alone, is generally more difficult than the so-called two-sided theory, the study of the pair $H_f$ and $H_{\bar{f}}$, which is equivalent to the study of the commutator $[M_f, P]$. And this is true not only on the Fock space, but also on the Bergman space and the Hardy space.

Therefore it was all the more remarkable that Berger and Coburn proved the following result in [2]: for $f \in L^\infty(\mathbf{C}^n)$, $H_f$ is compact if and only if $H_{\bar{f}}$ is compact. From the author's conversations with Lew Coburn about this result in the late 1990s and early 2000s arose a natural question:

**Question 1.1.** [15, page 1384] For $f \in L^\infty(\mathbf{C}^n)$ and $1 \le p < \infty$, does the membership $H_f \in \mathcal{C}_p$ imply $H_{\bar{f}} \in \mathcal{C}_p$?

Here, $\mathcal{C}_p$ denotes the Schatten $p$-class. That is, $\mathcal{C}_p$ is the collection of operators $A$ satisfying the condition $\|A\|_p < \infty$, where $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$.

In [1], Bauer answered Question 1.1 for the Hilbert-Schmidt class $\mathcal{C}_2$: for $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_2$ if and only if $H_{\bar{f}} \in \mathcal{C}_2$. Then, after a long period that saw no progress on this question, Hu and Virtanen answered it for $1 < p < \infty$:

*Keywords*: Fock space, Hankel operator, Boyd indices.

**Theorem 1.2.** [9,10] *Let $1 < p < \infty$. Then there is a $0 < C < \infty$ such that*

$$\|H_{\bar{f}}\|_p \le C\|H_f\|_p$$

*for every $f \in L^\infty(\mathbf{C}^n)$. In particular, for $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p$ if and only if $H_{\bar{f}} \in \mathcal{C}_p$.*

Hu and Virtanen referred to Theorem 1.2 as the Berger-Coburn phenomenon for the Schatten classes $\mathcal{C}_p$, $1 < p < \infty$. We will adopt their terminology. In [16,8], the root cause of the Berger-Coburn phenomenon was ascribed to the absence of bounded, non-constant analytic functions on $\mathbf{C}^n$.

Then in [14], the Berger-Coburn phenomenon was proved for Lorentz ideals. These ideals are defined in the following way.

Let $\mathcal{H}$ be a Hilbert space. For any given $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{j\ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

defines a norm for bounded operators on $\mathcal{H}$. Here and in what follows, we write $s_1(A)$, $s_2(A)$, $\dots$, $s_j(A)$, $\dots$ for the $s$-numbers [7] of the operator $A$. It is well known that the collection of operators

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

form a norm ideal [7].

For each $1 \le p < \infty$, the formula

$$\|A\|_p^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j^{(p-1)/p}}$$

also defines a norm for bounded operators on $\mathcal{H}$. Denote

$$\mathcal{C}_p^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^- < \infty\},$$

which is also a norm ideal of operators on $\mathcal{H}$ [7].

For these ideals, we recall the following:

**Theorem 1.3.** [14, Theorem 1.3] *Let $1 < p < \infty$. For $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p^+$, if and only if $H_{\bar{f}} \in \mathcal{C}_p^+$.*

**Theorem 1.4.** [14, Theorem 1.4] *Let $1 < p < \infty$. For $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p^-$, if and only if $H_{\bar{f}} \in \mathcal{C}_p^-$.*

For the case of the complex plane $\mathbf{C}$, in [14] we showed that the function

$$g(z) = \begin{cases} z^{-1} & \text{if} \quad |z| \ge 1 \\ 0 & \text{if} \quad |z| < 1 \end{cases}$$

has the property that $H_g \in \mathcal{C}_1$ while $H_{\bar{g}} \notin \mathcal{C}_1$. Thus there is no Berger-Coburn phenomenon for the trace class $\mathcal{C}_1$. Furthermore, we showed in [14] that there is no Berger-Coburn phenomenon for the famous Macaev ideal $\mathcal{C}_\infty^-$, for its dual $\mathcal{C}_1^+$, and for a class of ideals $\mathcal{C}_\alpha$.

But the ideals mentioned above are only some of examples of a much broader class called *symmetrically normed ideals*, or norm ideals for short. The purpose of this paper is to settle the issue of Berger-Coburn phenomenon completely for this entire class of ideals. Let us recall the definition of these ideals.

Following [7], let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ for every $j \in \mathbf{N}$, and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A symmetric gauge function (also called *symmetric norming function*) is a map

$$\Phi : \hat{c} \to [0, \infty)$$

that has the following properties:

(a) $\Phi$ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \dots, 0, \dots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [7, page 71]. Each symmetric gauge function $\Phi$ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \dots, s_j(A), 0, \dots, 0, \dots\})$$

for operators. On any Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{1.1}$$

is a symmetrically normed ideal [7, page 68].

Let us recall some familiar examples. First of all, the formula

$$\Phi_\infty(a) = \sup_{j\in\mathbf{N}} |a_j|, \quad a = \{a_1, \dots, a_j, \dots\} \in \hat{c},$$

gives us what may be the most familiar symmetric gauge function. It is obvious that $\|\cdot\|_{\Phi_\infty} = \|\cdot\|$, the operator norm. It is also obvious that a symmetric gauge function $\Phi$ is equivalent to $\Phi_\infty$ if and only if there is a $0 < C < \infty$ such that

$$\Phi(\{\overbrace{1, \dots, 1}^{k}, 0, \dots, 0 \dots\}) \leq C \quad \text{for every} \quad k \in \mathbf{N}.$$

For each $1 \leq p < \infty$, the formula

$$\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = (|a_1|^p + |a_2|^p + \cdots + |a_j|^p + \cdots)^{1/p}$$

defines a symmetric gauge function on $\hat{c}$, and the ideal $\mathcal{C}_{\Phi_p}$ defined by (1.1) is just the Schatten class $\mathcal{C}_p$.

For each $1 \le p < \infty$, define the symmetric gauge functions $\Phi_p^+$ and $\Phi_p^-$ by the formulas

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\ge 1} \frac{|a_{\pi(1)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + \cdots + j^{-1/p}} \quad \text{and} \quad \Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^{\infty} \frac{|a_{\pi(j)}|}{j^{(p-1)/p}},$$

$\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \ge |a_{\pi(2)}| \ge \cdots \ge |a_{\pi(j)}| \ge \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. Then the ideals $\mathcal{C}_{\Phi_p^+}$ and $\mathcal{C}_{\Phi_p^-}$ defined by (1.1) using $\Phi_p^+$ and $\Phi_p^-$ are none other than the Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$ introduced earlier.

For a general symmetric gauge function $\Phi$, let $\mathcal{C}_\Phi^{(0)}$ denote the $\|\cdot\|_\Phi$-closure of the collection of finite-rank operators in $\mathcal{C}_\Phi$. We always have $\mathcal{C}_\Phi^{(0)} \subset \mathcal{C}_\Phi$ of course, but we can have either $\mathcal{C}_\Phi^{(0)} = \mathcal{C}_\Phi$ or $\mathcal{C}_\Phi^{(0)} \ne \mathcal{C}_\Phi$. For example, on any Hilbert space $\mathcal{H}$ we have $\mathcal{C}_{\Phi_\infty} = \mathcal{B}(\mathcal{H})$ and $\mathcal{C}_{\Phi_\infty}^{(0)} = \mathcal{K}(\mathcal{H})$. For every $1 \le p < \infty$, it is well known that $\mathcal{C}_{\Phi_p}^{(0)} = \mathcal{C}_{\Phi_p}$ and $\mathcal{C}_{\Phi_p^-}^{(0)} = \mathcal{C}_{\Phi_p^-}$ while $\mathcal{C}_{\Phi_p^+}^{(0)} \ne \mathcal{C}_{\Phi_p^+}$ [7].

Next we recall the *Boyd indices* [3,11] for $\Phi$. For any $a = \{a_j\}_{j\in\mathbf{N}}$ and $m \in \mathbf{N}$, define the sequence $a^{[m]} = \{a_j^m\}_{j\in\mathbf{N}}$ by the formula

$$a_j^m = a_i \quad \text{if} \;\; (i-1)m + 1 \le j \le im, \; i \in \mathbf{N}.$$

In other words, $a^{[m]}$ is obtained from $a$ by repeating each term $m$ times. Alternately, we can think of $a^{[m]}$ as $a \oplus \cdots \oplus a$, the "direct sum" of $m$ copies of $a$.

For each $m \in \mathbf{N}$, the formula $D_m a = a^{[m]}$ defines a linear operator on $\hat{c}$. Related to $D_m$ is the operator $D_{1/m}$ defined by the formula

$$D_{1/m}a = \frac{1}{m}\left\{ \sum_{j=1}^{m} a_j, \sum_{j=1}^{m} a_{m+j}, \ldots, \sum_{j=1}^{m} a_{(k-1)m+j}, \ldots \right\}, \quad a = \{a_1, \ldots, a_k, \ldots\} \in \hat{c}.$$

We obviously have $D_{1/m}D_m b = b$ for all $m \in \mathbf{N}$ and $b \in \hat{c}$. Moreover, if $a, b \in \hat{c}$ are such that $D_{1/m}a = b$, and if $\Phi$ is any symmetric gauge function, then it follows from the properties of $\Phi$ that $\Phi(b^{[m]}) \le \Phi(a)$. Consequently,

$$\sup\left\{ \frac{\Phi(b)}{\Phi(b^{[m]})} : b \in \hat{c}\backslash\{0\} \right\} = \sup\left\{ \frac{\Phi(D_{1/m}a)}{\Phi(a)} : a \in \hat{c}\backslash\{0\} \right\}.$$

It is well known that for any symmetric gauge function $\Phi$ on $\hat{c}$, the limits

$$p_\Phi = \lim_{m\to\infty} \frac{\log m}{\log\left(\sup\left\{\frac{\Phi(a^{[m]})}{\Phi(a)} : a \in \hat{c}\backslash\{0\}\right\}\right)}, \quad q_\Phi = \lim_{m\to\infty} \frac{\log m}{\log\left(\inf\left\{\frac{\Phi(a^{[m]})}{\Phi(a)} : a \in \hat{c}\backslash\{0\}\right\}\right)}$$

exist and satisfy the condition $1 \le p_\Phi \le q_\Phi \le \infty$. The quantities $p_\Phi$ and $q_\Phi$ are called the Boyd indices of $\Phi$. See [3,11].

The results of this paper can be simply summarized thus: the Berger-Coburn phenomenon is completely determined by the Boyd indices of the ideal in question.

**Theorem 1.5.** *Let $\Phi$ be a symmetric gauge function such that $1 < p_\Phi \le q_\Phi < \infty$. Then there is a constant $0 < C < \infty$ such that*

$$\|H_{\bar f}\|_\Phi \le C\|H_f\|_\Phi \tag{1.2}$$

*for every $f \in L^\infty(\mathbf{C}^n)$. In particular, for $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_\Phi$ if and only if $H_{\bar f} \in \mathcal{C}_\Phi$.*

The symmetric gauge function $\Phi_\infty$ is worth special attention. Since $\mathcal{C}^{(0)}_{\Phi_\infty}$ is the collection of all compact operators, the original theorem of Berger and Coburn tells us that for $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}^{(0)}_{\Phi_\infty}$ if and only if $H_{\bar f} \in \mathcal{C}^{(0)}_{\Phi_\infty}$. On the other hand, since $\|\cdot\|_{\Phi_\infty} = \|\cdot\|$, the operator norm, we now know that there is no constant $0 < C < \infty$ such that

$$\|H_{\bar f}\|_{\Phi_\infty} \le C\|H_f\|_{\Phi_\infty}$$

for every $f \in L^\infty(\mathbf{C})$ [14, Proposition 13.2]. This makes $\Phi_\infty$ unique in the context of Berger-Coburn phenomenon. For any $\Phi$ not equivalent to $\Phi_\infty$, if the condition $1 < p_\Phi \le q_\Phi < \infty$ is not satisfied, then there is no Berger-Coburn phenomenon for $\mathcal{C}_\Phi$:

**Theorem 1.6.** *Let $\Phi$ be a symmetric gauge function not equivalent to $\Phi_\infty$. If either $p_\Phi = 1$ or $q_\Phi = \infty$, then there is an $f \in L^\infty(\mathbf{C}^n)$ such that $H_f \in \mathcal{C}^{(0)}_\Phi$ while $H_{\bar f} \notin \mathcal{C}_\Phi$.*

The rest of the paper is organized as follows. Sections 2-5 are devoted to the proof of Theorem 1.5. Specifically, in Section 2 we deal with Boyd interpolation for general symmetric gauge functions, which is the most crucial step in the proof of Theorem 1.5. Using the result in Section 2, we show in Section 3 that if the condition $1 < p_\Phi \le q_\Phi < \infty$ holds, then the integral operators $T_1, \dots, T_n$ defined by (3.5) below are bounded on the space $\mathcal{L}^{2,\Phi}_n$. Section 4 deals with commutators $[M_f, P]$. Then, after the preparations in Sections 2, 3 and 4, we prove Theorem 1.5 in Section 5.

After that, we turn to the proof of Theorem 1.6. In Section 6 we produce symbol functions $\varphi$ which are both bounded and boundedly supported, and for which $H_\varphi$ and $H_{\bar\varphi}$ exhibit quantitatively different behaviors. Then, using such $\varphi$ as building blocks, in Section 7 we construct the $f \in L^\infty(\mathbf{C}^n)$ promised in Theorem 1.6.

To conclude the paper, we take a closer look at the condition $1 < p_\Phi \le q_\Phi < \infty$ itself. It is easy to show that if a symmetric gauge function $\Phi$ satisfies the condition $1 < p_\Phi \le q_\Phi < \infty$, then for any $q_\Phi < s < \infty$ and $1 < r < p_\Phi$ we have

$$\mathcal{C}_r \subset \mathcal{C}_\Phi \subset \mathcal{C}_s. \tag{1.3}$$

That is, for such a $\Phi$ we can bound the size of $\mathcal{C}_\Phi$ by Schatten classes. This immediately raises the question, is the converse also true? In other words, if (1.3) holds for some $1 < r < s < \infty$, does it follow that $1 < p_\Phi \le q_\Phi < \infty$? This question is important because an affirmative answer would give us a very convenient characterization of the condition $1 < p_\Phi \le q_\Phi < \infty$. But the actual answer is decidedly negative. In Section 8 we will show that given any $1 < r < s < \infty$, there is a symmetric gauge function $\Phi$ that simultaneously satisfies (1.3) and the conditions that $q_\Phi = \infty$ and that $p_\Phi = 1$. Consequently, there exists a symmetric gauge function $\Phi$ such that (1.3) holds, and yet there is no Berger-Coburn phenomenon for the ideal $\mathcal{C}_\Phi$.

## 2. Boyd interpolation

Recall from [7, page 125] that given a symmetric gauge function $\Phi$, the formula

$$\Phi^*(\{b_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{a_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi(\{a_j\}_{j\in\mathbf{N}}) \le 1\right\}, \quad \{b_j\}_{j\in\mathbf{N}} \in \hat{c},$$

defines the symmetric gauge function that is dual to $\Phi$. Moreover, we have the relation $\Phi^{**} = \Phi$ [7, page 125]. This relation implies that for every $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, we have

$$\Phi(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi^*(\{b_j\}_{j\in\mathbf{N}}) \le 1\right\}.$$

In terms of operators, this duality is manifested in the form of the following trace inequality. If $F$ is a finite-rank operator and $A$ is an arbitrary operator, then for every symmetric gauge function $\Phi$ we have

$$|\mathrm{tr}(AF)| \le \|A\|_\Phi \|F\|_{\Phi^*}. \tag{2.1}$$

See inequality (II.7.9) in [7].

We need to extend the domain of definition of a symmetric gauge $\Phi$ beyond the space $\hat{c}$. Suppose that $\{b_j\}_{j\in\mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., the set $\{j \in \mathbf{N} : b_j \neq 0\}$ is not necessarily finite. Then we define

$$\Phi(\{b_j\}_{j\in\mathbf{N}}) = \sup_{k\ge 1} \Phi(\{b_1, \dots, b_k, 0, \dots, 0, \dots\}). \tag{2.2}$$

More generally, for any countable, infinite index set $A$, we define

$$\Phi(\{b_\alpha\}_{\alpha\in A}) = \Phi(\{b_{h(j)}\}_{j\in\mathbf{N}}), \tag{2.3}$$

where $h : \mathbf{N} \to A$ is a bijection. By the properties of symmetric gauge functions, the value of $\Phi(\{b_\alpha\}_{\alpha\in A})$ is independent of the choice of the bijection $h : \mathbf{N} \to A$.

Let $X$ be a Banach space. We will now define spaces of $X$-valued sequences.

As in [14], we define $\ell_{00}(\mathbf{N}, X)$ to be the collection of $a = \{a_j\}$ satisfying the conditions that $a_j \in X$ for every $j \in \mathbf{N}$ and that

$$\mathrm{card}\{j \in \mathbf{N} : a_j \neq 0\} < \infty.$$

That is, if $a = \{a_j\} \in \ell_{00}(\mathbf{N}, X)$, then the sequence $\{a_j\}$ has at most a finite number of nonzero terms. In other words, $\ell_{00}(\mathbf{N}, X)$ is the $X$-valued version of $\hat{c}$.

**Definition 2.1.** Let $\Phi$ be a symmetric gauge function. Then $\ell_\Phi(\mathbf{N}, X)$ denotes the collection of sequences $a = \{a_j\}_{j\in\mathbf{N}}$, where $a_j \in X$ for every $j \in \mathbf{N}$, such that

$$\|a\|_\Phi = \Phi(\{\|a_j\|\}_{j\in\mathbf{N}}) < \infty.$$

Let us conisder some examples. Recall the symmetric gauge function $\Phi_p(\{b_j\}_{j\in\mathbf{N}}) = (\sum_{j=1}^\infty |b_j|^p)^{1/p}$, $1 \le p < \infty$. Then obviously $\ell_{\Phi_p}(\mathbf{N}, X) = \ell^p(\mathbf{N}, X)$, the collection of $X$-valued $\ell^p$-sequences. Particularly important to this paper is the symmetric gauge function

$$\Phi_p^-(\{b_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|b_{\pi(j)}|}{j^{(p-1)/p}}, \quad \{b_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|b_{\pi(1)}| \ge |b_{\pi(2)}| \ge \cdots \ge |b_{\pi(j)}| \ge \cdots$, $1 < p < \infty$. As we have mentioned, we write $\mathcal{C}_p^- = \mathcal{C}_{\Phi_p^-}$. In this spirit, we will write

$$\ell_-^p(\mathbf{N}, X) = \ell_{\Phi_p^-}(\mathbf{N}, X) \quad \text{and} \quad \|a\|_p^- = \|a\|_{\Phi_p^-} \ \text{ for } \ a \in \ell_{\Phi_p^-}(\mathbf{N}, X).$$

It is well known that if $1 \le r < p < s < \infty$, then

$$\ell^r(\mathbf{N}, X) \subset \ell_-^p(\mathbf{N}, X) \subset \ell^s(\mathbf{N}, X). \tag{2.4}$$

**Lemma 2.2.** *Let $\Phi$ be a symmetric gauge function such that $q_\Phi < q < \infty$. Then there is a $c > 0$ such that $\Phi(\xi^{[m]}) \ge cm^{1/q}\Phi(\xi)$ for all $m \in \mathbf{N}$ and $\xi \in \hat{c}$.*

*Proof.* By the definition of $q_\Phi$, the condition $q_\Phi < q$ implies that there is an $N \in \mathbf{N}$ such that if $m \ge N$, then

$$\frac{\log m}{\log\left(\frac{\Phi(\xi^{[m]})}{\Phi(\xi)}\right)} < q \quad \text{for every } \ \xi \in \hat{c}\backslash\{0\}.$$

Elementary manipulation leads to the inequality

$$m^{1/q}\Phi(\xi) \le \Phi(\xi^{[m]})$$

for all $m \ge N$ and $\xi \in \hat{c}$. Thus the constant $c = N^{-1/q}$ will do for the lemma. □

**Lemma 2.3.** *Let $\Phi$ be a symmetric gauge function such that $q_\Phi < q < \infty$. Then $\ell_\Phi(\mathbf{N}, X) \subset \ell^q(\mathbf{N}, X)$.*

*Proof.* Pick a $q_0 \in (q_\Phi, q)$. By Lemma 2.2 there is a $c > 0$ such that $\Phi(\xi^{[m]}) \ge cm^{1/q_0}\Phi(\xi)$ for all $m \in \mathbf{N}$ and $\xi \in \hat{c}$. Since $q > q_0$, by [13, Lemma 3.1], there is a $0 < B < \infty$ such that

$$\left(\sum_{j=1}^\infty |\alpha_j|^q\right)^{1/q} \le B\Phi(\alpha) \quad \text{for every } \alpha = \{\alpha_1, \dots, \alpha_j, \dots\} \in \hat{c}.$$

Recalling Definition 2.1, from the above we deduce

$$\left(\sum_{j=1}^{\infty}\|a_j\|^q\right)^{1/q} \le B\|a\|_\Phi$$

for every $a = \{a_1, \dots, a_j, \dots\} \in \ell_\Phi(\mathbf{N}, X)$. This completes the proof. □

**Lemma 2.4.** *Let $\Phi$ be a symmetric gauge function such that $p_\Phi > p > 1$. Then $q_{\Phi^*} < p/(p-1)$.*

*Proof.* Let $s$ be such that $p_\Phi > s > p$. Then there is an $N \in \mathbf{N}$ such that

$$\frac{\log m}{\log\left(\frac{\Phi(\xi^{[m]})}{\Phi(\xi)}\right)} > s$$

for all $m \ge N$ and $\xi \ne 0$ in $\hat{c}$. This obviously implies that

$$\Phi(\xi^{[m]}) \le m^{1/s}\Phi(\xi) \quad \text{for all} \quad m \ge N \quad \text{and} \quad \xi \in \hat{c}.$$

Let $b = \{b_j\}_{j\in\mathbf{N}} \in \hat{c}$. Since $\Phi^{**} = \Phi$, there is an $a = \{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ with $\Phi(a) = 1$ such that $\Phi^*(b) = \sum_{j=1}^{\infty} b_j a_j$. Therefore for each $m \ge N$,

$$m\Phi^*(b) = m\sum_{j=1}^{\infty} b_j a_j \le \Phi^*(b^{[m]})\Phi(a^{[m]}) \le m^{1/s}\Phi^*(b^{[m]})\Phi(a) = m^{1/s}\Phi^*(b^{[m]}).$$

Thus we conclude that if $m \ge N$, then $m^{(s-1)/s}\Phi^*(b) \le \Phi^*(b^{[m]})$ for every $b \in \hat{c}$. In other words, for every $m \ge N$ we have

$$m^{(s-1)/s} \le \inf\left\{\frac{\Phi^*(b^{[m]})}{\Phi^*(b)} : b \in \hat{c} \text{ and } b \ne 0\right\}.$$

From this we deduce that $q_{\Phi^*} \le s/(s-1)$. Since $s > p > 1$, we have $q_{\Phi^*} < p/(p-1)$. □

Next we perform Boyd interpolation [3,4] for these spaces.

**Proposition 2.5.** *Let $1 < r' < r < \infty$. Suppose that $A : \ell^r_-(\mathbf{N}, X) \to \ell^r_-(\mathbf{N}, X)$ is a bounded operator. Furthermore, suppose that there is a $0 < B_{r'} < \infty$ such that*

$$\|Ax\|^-_{r'} \le B_{r'}\|x\|^-_{r'} \tag{2.5}$$

*for every $x \in \ell_{00}(\mathbf{N}, X)$. Let $\Phi$ be a symmetric gauge function such that $r' < p_\Phi \le q_\Phi < r$. Then $A$ maps $\ell_\Phi(\mathbf{N}, X)$ into itself, and there is a $0 < C < \infty$ such that*

$$\|Aa\|_\Phi \le C\|a\|_\Phi \tag{2.6}$$

*for every $a \in \ell_\Phi(\mathbf{N}, X)$.*

*Proof.* By the condition $q_\Phi < r$, Lemma 2.3 and inclusion (2.4), we have $\ell_\Phi(\mathbf{N}, X) \subset \ell^r_-(\mathbf{N}, X)$. Thus our task is to find a constant $0 < C < \infty$ such that (2.6) holds.

To find such a $0 < C < \infty$, we pick $p$ and $q$ such that

$$r' < p < p_\Phi \le q_\Phi < q < r. \tag{2.7}$$

Let an $a = \{a_j\}_{j\in\mathbf{N}} \in \ell_\Phi(\mathbf{N}, X)$ be given. Then

$$Aa = \{(Aa)_1, (Aa)_2, \dots, (Aa)_k, \dots\}.$$

If there are infinitely many nonzero terms among $(Aa)_1, (Aa)_2, \dots, (Aa)_k, \dots,$ we let $z_1, z_2, \dots, z_j, \dots$ be an enumeration of all the nonzero terms such that

$$\|z_1\| \ge \|z_2\| \ge \cdots \ge \|z_k\| \ge \cdots . \tag{2.8}$$

If there are only finitely many nonzero terms among $(Aa)_1, (Aa)_2, \dots, (Aa)_k, \dots,$ we let $z_1, z_2, \dots, z_k, \dots$ be an enumeration of $(Aa)_1, (Aa)_2, \dots, (Aa)_k, \dots$ such that (2.8) holds. This defines the sequence $\{z_k\}_{k\in\mathbf{N}}$. We call $z = \{z_k\}_{k\in\mathbf{N}}$ a *descending rearrangement* of $Aa$, and we will use this terminology below. Our goal is to show that $\|z\|_\Phi \le C\|a\|_\Phi$.

Define $z^{(1)} = \{z_{1+2(k-1)}\}_{k\in\mathbf{N}}$ and $z^{(2)} = \{z_{2k}\}_{k\in\mathbf{N}}$. Then $\|z\|_\Phi \le \|z^{(1)}\|_\Phi + \|z^{(2)}\|_\Phi$ and $\|z^{(2)}\|_\Phi \le \|z^{(1)}\|_\Phi$. Thus it suffices to show that $\|z^{(1)}\|_\Phi \le C\|a\|_\Phi$.

There is an injective map $\pi : \mathbf{N} \to \mathbf{N}$ such that $\|a_{\pi(i)}\| \ge \|a_{\pi(i+1)}\|$ for every $i \in \mathbf{N}$ and such that $a_k = 0$ if $k \in \mathbf{N}\backslash\pi(\mathbf{N})$. For each $j \in \mathbf{N}$, we define $v_{j,k} = a_k$ if $k \in \{\pi(i) : 1 \le i \le j\}$ and $v_{j,k} = 0$ if $k \notin \{\pi(i) : 1 \le i \le j\}$. We then define the sequences

$$v(j) = \{v_{j,1}, v_{j,2}, \dots, v_{j,k}, \dots\} \quad \text{and} \quad u(j) = a - v(j),$$

$j \in \mathbf{N}$. For each $j \in \mathbf{N}$, let $\xi_{j,1}, \dots, \xi_{j,k}, \dots$ be a descending rearrangement of $Au(j)$. Similarly, let $\eta_{j,1}, \dots, \eta_{j,k}, \dots$ be a descending rearrangement of $Av(j)$, $j \in \mathbf{N}$. From the relation $Aa = Au(j) + Av(j)$ it is easy to see that

$$\|z_{1+2(k-1)}\| \le \|\xi_{j,k}\| + \|\eta_{j,k}\| \tag{2.9}$$

for all $j, k \in \mathbf{N}$.

Because $\sum_{i=1}^{j} 1/i^{(r-1)/r} \ge c_1 j^{1/r}$ and because $\xi_{j,1}, \dots, \xi_{j,k}, \dots$ are a descending rearrangement of $Au(j)$, we have $c_1 j^{1/r}\|\xi_{j,j}\| \le \|Au(j)\|^-_r$, $j \in \mathbf{N}$. Applying the boundedness of $A$ on $\ell^r_-(\mathbf{N}, X)$, for every $j \in \mathbf{N}$ we have

$$\begin{aligned}
\|\xi_{j,j}\| &\le \frac{1}{c_1 j^{1/r}}\|Au(j)\|^-_r \le \frac{C_1}{j^{1/r}}\|u(j)\|^-_r = \frac{C_1}{j^{1/r}}\sum_{k=1}^{\infty}\frac{\|a_{\pi(j+k)}\|}{k^{(r-1)/r}} \\
&= \frac{C_1}{j^{1/r}}\sum_{\nu=1}^{\infty}\sum_{k=1}^{j}\frac{\|a_{\pi(j\nu+k)}\|}{(j(\nu-1)+k)^{(r-1)/r}} \\
&\le \frac{C_1}{j^{1/r}}\sum_{\nu=1}^{\infty}\|a_{\pi(j\nu)}\|\sum_{k=1}^{j}\frac{1}{(j(\nu-1)+k)^{(r-1)/r}} \\
&\le C_2\sum_{\nu=1}^{\infty}\frac{\|a_{\pi(j\nu)}\|}{\nu^{(r-1)/r}} = C_2\sum_{\nu=1}^{\infty}\frac{\|(x(\nu))_j\|}{\nu^{(r-1)/r}},
\end{aligned}$$

where, for each $\nu \in \mathbf{N}$, we define the element

$$x(\nu) = \{a_{\pi(\nu)}, a_{\pi(2\nu)}, \ldots, a_{\pi(i\nu)}, \ldots\} \in \ell_\Phi(\mathbf{N}, X).$$

Let $b = \{b_j\}_{j\in\mathbf{N}}$ be an element in $\hat{c}$ satisfying the conditions $b_j \geq 0$ for every $j$ and

$$b_1 \geq b_2 \geq \cdots \geq b_j \geq \cdots . \tag{2.10}$$

We have

$$\sum_{j=1}^{\infty} b_j \|\xi_{j,j}\| \leq C_2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^{(r-1)/r}} \sum_{j=1}^{\infty} b_j \|(x(\nu))_j\| \leq C_2 \Phi^*(b) \sum_{\nu=1}^{\infty} \frac{\|x(\nu)\|_\Phi}{\nu^{(r-1)/r}}. \tag{2.11}$$

We have

$$\|x(\nu)\|_\Phi \leq (C_3/\nu^{1/q})\|x(\nu)^{[\nu]}\|_\Phi \leq (C_3/\nu^{1/q})\|a\|_\Phi \quad \text{for} \quad \nu \in \mathbf{N},$$

where the first $\leq$ follows from Lemma 2.2 and the second $\leq$ follows from the condition $\|a_{\pi(i)}\| \geq \|a_{\pi(i+1)}\|$, $i \in \mathbf{N}$. Substituting this in (2.11) and applying the condition $q < r$, we find that

$$\sum_{j=1}^{\infty} b_j \|\xi_{j,j}\| \leq C_4 \Phi^*(b) \sum_{\nu=1}^{\infty} \frac{\|a\|_\Phi}{\nu^{1/q}\nu^{(r-1)/r}} = C_5 \Phi^*(b) \|a\|_\Phi. \tag{2.12}$$

For each $j \in \mathbf{N}$, we also have

$$\|\eta_{j,j}\| \leq \frac{1}{c_2 j^{1/r'}} \|Av(j)\|_{r'}^{-} \leq \frac{B_{r'}}{c_2 j^{1/r'}} \|v(j)\|_{r'}^{-} = \frac{C_6}{j^{1/r'}} \sum_{i=1}^{j} \frac{\|a_{\pi(i)}\|}{i^{(r'-1)/r'}},$$

where for the second $\leq$ we apply (2.5). Thus

$$\sum_{j=1}^{\infty} b_j \|\eta_{j,j}\| \leq \sum_{j=1}^{\infty} b_j \frac{C_6}{j^{1/r'}} \sum_{i=1}^{j} \frac{\|a_{\pi(i)}\|}{i^{(r'-1)/r'}} = C_6 \sum_{i=1}^{\infty} \frac{\|a_{\pi(i)}\|}{i^{(r'-1)/r'}} \sum_{j=i}^{\infty} \frac{b_j}{j^{1/r'}}.$$

We now define the element $\beta = \{\beta_1, \ldots, \beta_i, \ldots\} \in \hat{c}$, where

$$\beta_i = \frac{1}{i^{(r'-1)/r'}} \sum_{j=i}^{\infty} \frac{b_j}{j^{1/r'}}$$

for each $i \geq 1$. Then

$$\sum_{j=1}^{\infty} b_j \|\eta_{j,j}\| \leq C_6 \|a\|_\Phi \Phi^*(\beta).$$

Combining this with (2.9) and (2.12), we obtain the inequality

$$\sum_{j=1}^{\infty} b_j \|z_{1+2(j-1)}\| \le (C_5 \Phi^*(b) + C_6 \Phi^*(\beta))\|a\|_\Phi.$$

Recall the descending conditions (2.8) and (2.10). Thus we will have $\|z^{(1)}\|_\Phi \le C\|a\|_\Phi$ if we can find a constant $\tilde{C}$ such that $\Phi^*(\beta) \le \tilde{C}\Phi^*(b)$.

We have

$$\begin{aligned}
\beta_i &\le \frac{b_i}{i} + \frac{1}{i^{(r'-1)/r'}} \sum_{j=1}^{\infty} \frac{b_{j+i}}{j^{1/r'}} = \frac{b_i}{i} + \frac{1}{i^{(r'-1)/r'}} \sum_{\nu=1}^{\infty}\sum_{k=1}^{i} \frac{b_{i\nu+k}}{((\nu-1)i+k)^{1/r'}} \\
&\le \frac{b_i}{i} + \frac{1}{i^{(r'-1)/r'}} \sum_{\nu=1}^{\infty} b_{i\nu} \sum_{k=1}^{i} \frac{1}{((\nu-1)i+k)^{1/r'}} \\
&\le \frac{b_i}{i} + C_7 \sum_{\nu=1}^{\infty} \frac{b_{i\nu}}{\nu^{1/r'}}, \qquad (2.13)
\end{aligned}$$

where the second $\le$ uses (2.10). Since $p_\Phi > p > r' > 1$, Lemma 2.4 tells us that $q_{\Phi^*} < p/(p-1) < r'/(r'-1)$. Thus for any $y = \{y_i\}_{i\in\mathbf{N}} \in \hat{c}$ such that $y_i \ge 0$ for every $i$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} y_i \sum_{\nu=1}^{\infty} \frac{b_{i\nu}}{\nu^{1/r'}} &= \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1/r'}} \sum_{i=1}^{\infty} y_i b_{i\nu} \le \Phi(y) \sum_{\nu=1}^{\infty} \frac{\Phi^*(\{b_{i\nu}\}_{i\in\mathbf{N}})}{\nu^{1/r'}} \\
&\le C_8 \Phi(y) \sum_{\nu=1}^{\infty} \frac{\Phi^*((\{b_{i\nu}\}_{i\in\mathbf{N}})^{[\nu]})}{\nu^{(p-1)/p}\nu^{1/r'}} \\
&\le C_8 \Phi(y) \sum_{\nu=1}^{\infty} \frac{\Phi^*(b)}{\nu^{(p-1)/p}\nu^{1/r'}} = C_9 \Phi(y)\Phi^*(b),
\end{aligned}$$

where the second $\le$ is obtained from Lemma 2.2 and the third $\le$ uses (2.10). Combining this with (2.13), we see that $\Phi^*(\beta) \le (1 + C_9 C_7)\Phi^*(b)$. This completes the proof. □

## 3. The integral operators $T_1, \ldots, T_n$

For each $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$, we define the square

$$I_{(\alpha_1,\alpha_2)} = \{\alpha_1 + x + i(\alpha_2 + y) : x, y \in [0, 1)\} \qquad (3.1)$$

in $\mathbf{C}$. We now consider the standard partition of $\mathbf{C}^n$ by cubes of the size $1 \times 1 \times \cdots \times 1 \times 1$. That is, for each $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbf{Z}^{2n}$, we introduce the cube

$$Q_\alpha = I_{(\alpha_1,\alpha_2)} \times \cdots \times I_{(\alpha_{2n-1},\alpha_{2n})}. \qquad (3.2)$$

Let $\Phi$ be a symmetric gauge function. For a measurable function $\varphi$ on $\mathbf{C}^n$, we define

$$(3.3) \qquad \|\varphi\|_{2,\Phi} = \Phi\left(\left\{\left(\int_{Q_\alpha} |\varphi(\zeta)|^2 dV(\zeta)\right)^{1/2}\right\}_{\alpha\in\mathbf{Z}^{2n}}\right)$$

(see (2.3)). Here, the 2 in the subscript of $\|\cdot\|_{2,\Phi}$ refers to the fact that one first computes the $L^2$-norm of $\varphi$ on each $Q_\alpha$, $\alpha \in \mathbf{Z}^{2n}$. Indeed $\|\cdot\|_{2,\Phi}$ is a kind of "hybrid" norm. But by what we know from previous investigations, this is the right kind of norm when one deals with the symbol of a Hankel operator on the Fock space $H^2(\mathbf{C}^n, d\mu)$.

**Definition 3.1.** Let $\Phi$ be a symmetric gauge function. Then $\mathcal{L}_n^{2,\Phi}$ denotes the collection of measurable functions $\varphi$ on $\mathbf{C}^n$ satisfying the condition $\|\varphi\|_{2,\Phi} < \infty$.

Recall that in [14], we used the notation $\mathcal{L}_n^{2,p}$ for $\mathcal{L}_n^{2,\Phi_p}$, and we used the notation $\mathcal{L}_n^{2,p,-}$ for $\mathcal{L}_n^{2,\Phi_p^-}$, $1 < p < \infty$. We will continue to do so in this paper.

It will be convenient to identify $\mathbf{Z}^{2n}$ with the standard lattice in $\mathbf{C}^n$. That is, for $\alpha_1, \alpha_2, \ldots, \alpha_{2n} \in \mathbf{Z}$, we will

$$(3.4) \qquad \text{identify } (\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \text{ with } (\alpha_1 + i\alpha_2, \ldots, \alpha_{2n-1} + i\alpha_{2n}).$$

Let $dA$ be the area measure on $\mathbf{C}$. For each $1 \leq j \leq n$ we define the operator

$$(3.5) \qquad (T_j\varphi)(\zeta_1, \ldots, \zeta_n) = \text{p.v.}\int_{\mathbf{C}} \frac{\varphi(\zeta_1, \cdots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n)}{(\zeta_j - z)^2} dA(z),$$

$(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$. See Lemma 5.3 below for the purpose of $T_1, \ldots, T_n$. But this section deals with the boundedness of these operators.

**Proposition 3.2.** [14, Proposition 7.4] *On each* $\mathcal{L}_n^{2,p,-} = \mathcal{L}_n^{2,\Phi_p^-}$, $1 < p < \infty$, *the operators* $T_1, \ldots, T_n$ *are bounded.*

Combining the interpolation in Proposition 2.5 with Proposition 3.2, we have

**Proposition 3.3.** *Let* $\Phi$ *be a symmetric gauge function such that* $1 < p_\Phi \leq q_\Phi < \infty$. *Then* $T_1, \ldots, T_n$ *are bounded operators on* $\mathcal{L}_n^{2,\Phi}$.

*Proof.* Let $1 < p < q < \infty$ be such that $p < p_\Phi \leq q_\Phi < q$.

By (3.1), (3.2) and (3.4), for each $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbf{Z}^{2n}$ we have $Q_\alpha = Q_0 + \alpha$. Let $X = L^2(Q_0)$. Let $\pi : \mathbf{N} \to \mathbf{Z}^{2n}$ be a bijection. Then any function $\varphi$ on $\mathbf{C}^n$ is naturally identified with the sequence $\{\varphi_k\}_{k\in\mathbf{N}}$, where

$$\varphi_k(z) = \varphi(z + \pi(k)), \quad z \in Q_0,$$

$k \in \mathbf{N}$. This naturally identifies $\mathcal{L}_n^{2,\Phi}$ with $\ell_\Phi(\mathbf{N}, X)$. This also naturally identifies $\mathcal{L}_n^{2,p,-} = \mathcal{L}_n^{2,\Phi_p^-}$ with $\ell_{\Phi_p^-}(\mathbf{N}, X) = \ell_-^p(\mathbf{N}, X)$, and $\mathcal{L}_n^{2,q,-} = \mathcal{L}_n^{2,\Phi_q^-}$ with $\ell_{\Phi_q^-}(\mathbf{N}, X) = \ell_-^q(\mathbf{N}, X)$. Under this identification, Proposition 3.2 tells us that the maps

$$T_j : \ell_-^q(\mathbf{N}, X) \to \ell_-^q(\mathbf{N}, X) \quad \text{and} \quad T_j : \ell_-^p(\mathbf{N}, X) \to \ell_-^p(\mathbf{N}, X)$$

are bounded, $1 \le j \le n$. Therefore it follows from Proposition 2.5 that each map

$$T_j : \ell_\Phi(\mathbf{N}, X) \to \ell_\Phi(\mathbf{N}, X)$$

is bounded, $1 \le j \le n$. This completes the proof. $\square$

## 4. Commutators and norm ideals

For each $z \in \mathbf{C}^n$, let $\tau_z$ be the translation $\tau_z(\zeta) = \zeta - z$, $\zeta \in \mathbf{C}^n$. Denote

$$\mathcal{T}(\mathbf{C}^n) = \{f \in L^2(\mathbf{C}^n, d\mu) : f \circ \tau_z \in L^2(\mathbf{C}^n, d\mu) \text{ for every } z \in \mathbf{C}^n\}$$

as in [6,14,15]. We define the open cube

$$W = \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, y_1, \dots, x_n, y_n \in (-1, 2)\} \tag{4.1}$$

in $\mathbf{C}^n$. For $f \in \mathcal{T}(\mathbf{C}^n)$ and $u \in \mathbf{Z}^{2n}$, we define the quantity

$$J(f; u) = \left\{ \int_{W+u} \int_{W+u} |f(z) - f(w)|^2 dV(w) dV(z) \right\}^{1/2}.$$

We need the following result:

**Proposition 4.1.** [6, Lemma 5.6] *Let $\Phi$ be an arbitrary symmetric gauge function. Then there is a constant $0 < C < \infty$ such that*

$$\|[M_f, P]\|_\Phi \le C\Phi(\{J(f; u)\}_{u \in \mathbf{Z}^{2n}})$$

*for every $f \in \mathcal{T}(\mathbf{C}^n)$.*

For $f \in \mathcal{T}(\mathbf{C}^n)$ and $\alpha \in \mathbf{Z}^{2n}$, we define the quantities

$$A(f; \alpha) = \left( \int_{Q_\alpha} |f(z)|^2 dV(z) \right)^{1/2} \quad \text{and} \quad B(f; \alpha) = \left( \int_{W+\alpha} |f(z)|^2 dV(z) \right)^{1/2}. \tag{4.3}$$

We further define the set

$$\mathcal{E} = \{(j_1 + ik_1, \dots, j_n + ik_n) : j_1, k_1, \dots j_n, k_n \in \{-1, 0, 1\}\}. \tag{4.4}$$

**Lemma 4.2.** [14, Lemma 8.2] *For any set of non-negative numbers $\{x_\alpha\}_{\alpha \in \mathbf{Z}^{2n}}$ and any symmetric gauge function $\Phi$, we have*

$$\Phi\left( \left\{ \sum_{\epsilon \in \mathcal{E}} x_{\alpha+\epsilon} \right\}_{\alpha \in \mathbf{Z}^{2n}} \right) \le 3^{2n} \Phi(\{x_\alpha\}_{\alpha \in \mathbf{Z}^{2n}}).$$

**Lemma 4.3.** [14, Lemma 8.4] *Let $\varphi$ be any non-negative, measurable function on $W$. Then*

$$\int_W \int_W \int_0^1 \varphi(tz + (1-t)w)dtdV(w)dV(z) \le 6^{2n} \int_W \varphi(x)dV(x).$$

**Proposition 4.4.** *Let $\Phi$ be an arbitrary symmetric gauge function. Then there is a constant $0 < C_{4.4} < \infty$ such that*

$$\|[M_f, P]\|_\Phi \le C_{4.4}\Phi(\{A(|\nabla f|; u)\}_{u\in\mathbf{Z}^{2n}})$$

*for every $f \in \mathcal{T}(\mathbf{C}^n) \cap C^1(\mathbf{C}^n)$.*

*Proof.* For each $\alpha \in \mathbf{Z}^{2n}$, $W + \alpha \subset \cup_{\epsilon\in\mathcal{E}} Q_{\alpha+\epsilon}$. Hence by (4.3) and Lemma 4.2,

$$\Phi(\{B(|\nabla f|; u)\}_{u\in\mathbf{Z}^{2n}}) \le 3^{2n}\Phi(\{A(|\nabla f|; u)\}_{u\in\mathbf{Z}^{2n}}). \tag{4.5}$$

Thus, by Proposition 4.1, it suffices to show that there is a $0 < C < \infty$ such that

$$J(f; u) \le CB(|\nabla f|; u) \tag{4.6}$$

for all $f \in \mathcal{T}(\mathbf{C}^n) \cap C^1(\mathbf{C}^n)$ and $u \in \mathbf{Z}^{2n}$.

To prove (4.6), it will be convenient to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the natural way. Since our $f$ is in $C^1$, for any $u \in \mathbf{Z}^{2n}$ and any $z, w \in W + u$, we have

$$f(z) - f(w) = \int_0^1 \frac{d}{dt} f(tz + (1-t)w)dt = \int_0^1 \langle(\nabla f)(tz + (1-t)w), z - w\rangle dt,$$

where the $\langle\cdot,\cdot\rangle$ is taken in the sense of the inner product on $\mathbf{R}^{2n}$. Since $z, w \in W + u$, we have $|z - w| \le 3\sqrt{2n}$. Hence the above implies

$$|f(z) - f(w)|^2 \le 18n \int_0^1 |(\nabla f)(tz + (1-t)w)|^2 dt.$$

Applying Lemma 4.3, we have

$$\begin{aligned} J^2(f; u) &\le 18n \int_{W+u} \int_{W+u} \int_0^1 |(\nabla f)(tz + (1-t)w)|^2 dtdV(w)dV(z) \\ &\le 18n6^{2n} \int_{W+u} |(\nabla f)(x)|^2 dV(x) = 18n6^{2n} B^2(|\nabla f|; u). \end{aligned}$$

This proves (4.6) and completes the proof of the proposition. □

**Proposition 4.5.** *Let $\Phi$ be an arbitrary symmetric gauge function. Then there is a constant $0 < C_{4.5} < \infty$ such that*

$$\|[M_f, P]\|_\Phi \le C_{4.5}\Phi(\{A(f; u)\}_{u\in\mathbf{Z}^{2n}}) \tag{4.7}$$

*for every* $f \in \mathcal{T}(\mathbf{C}^n)$.

*Proof.* It is obvious that for any $u \in \mathbf{Z}^{2n}$,

$$J(f;u) \le 2\{V(W)\}^{1/2} B(f;u) = 2 \cdot 3^n B(f;u). \tag{4.8}$$

Similar to (4.5), Lemma 4.2 now gives us the inequality

$$\Phi(\{B(f;u)\}_{u\in\mathbf{Z}^{2n}}) \le 3^{2n}\Phi(\{A(f;u)\}_{u\in\mathbf{Z}^{2n}}).$$

Combining this with (4.8) and with Proposition 4.1, we obtain (4.7). □

## 5. Proof of Theorem 1.5

The proof of Theorem 1.5 involves a well-known decomposition (see [12,5,9,14]) of the symbol function of a Hankel operator, which we now review. We begin with the sets

$$Q = \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, \dots, x_n, y_1, \dots, y_n \in [0,1)\} \quad \text{and}$$
$$S = \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, \dots, x_n, y_1, \dots, y_n \in (-1/2, 3/2)\}.$$

Thus $Q = Q_0$ (see (3.2)). Fix an $\eta \in C^\infty(\mathbf{C}^n)$ satisfying the following three conditions:

(1) $0 \le \eta \le 1$ on $\mathbf{C}^n$.

(2) $\eta = 1$ on $Q$.

(3) $\eta = 0$ on $\mathbf{C}^n \backslash S$.

For each $z \in \mathbf{C}^n$, we define the function $\eta_z(\zeta) = \eta(\zeta - z)$ on $\mathbf{C}^n$. By (3), for $\zeta \in \mathbf{C}^n$ and $u \in \mathbf{Z}^{2n}$, if $\eta_u(\zeta) \ne 0$, then $\zeta - u \in S$, i.e., $u \in \zeta - S$. This ensures that the function

$$\varphi = \sum_{u\in\mathbf{Z}^{2n}} \eta_u$$

belongs to $C^\infty(\mathbf{C}^n)$. Also, by (1)-(3), the inequality $1 \le \varphi \le 3^{2n}$ holds on $\mathbf{C}^n$. Note that the identity $\varphi(\zeta) = \varphi(\zeta - u)$ holds for all $u \in \mathbf{Z}^{2n}$ and $\zeta \in \mathbf{C}^n$. Now we define

$$\gamma_z = \varphi^{-1}\eta_z$$

for every $z \in \mathbf{Z}^{2n}$. Then $\{\gamma_z : z \in \mathbf{Z}^{2n}\}$ is a set of $C^\infty$-partition of the unity on $\mathbf{C}^n$. Moreover, for every $z \in \mathbf{Z}^{2n}$, we have $\gamma_z = 0$ on the set $\mathbf{C}^n \backslash \{S + z\}$.

For an open set $U$ in $\mathbf{C}^n$, let $\mathrm{Hol}(U)$ denote the collection of analytic functions on $U$. For $f \in \mathcal{T}(\mathbf{C}^n)$ and $z \in \mathbf{Z}^{2n}$, we define

$$M(f;z) = \inf_{h\in\mathrm{Hol}(W+z)} \left( \int_{W+z} |f(\zeta) - h(\zeta)|^2 dV(\zeta) \right)^{1/2},$$

where $W$ is given by (4.1).

Let $f \in \mathcal{T}(\mathbf{C}^n)$. For each $z \in \mathbf{Z}^{2n}$, there is an $h_{f,z} \in \text{Hol}(W + z)$ such that

$$\int_{W+z} |f(\zeta) - h_{f,z}(\zeta)|^2 dV(\zeta) \le 2M^2(f; z).$$

Note that this is true even if $M(f; z) = 0$. We extend the definition of $h_{f,z}$ to the entire $\mathbf{C}^n$ by setting $h_{f,z} = 0$ on $\mathbf{C}^n \backslash \{W + z\}$. Now define the functions

$$f^{(1)} = \sum_{z \in \mathbf{Z}^{2n}} (f - h_{f,z})\gamma_z \quad \text{and} \quad f^{(2)} = \sum_{z \in \mathbf{Z}^{2n}} h_{f,z}\gamma_z. \tag{5.1}$$

We have $f = f^{(1)} + f^{(2)}$ because $\{\gamma_z : z \in \mathbf{Z}^{2n}\}$ is a partition of the unity on $\mathbf{C}^n$. Also note that $f^{(2)} \in C^\infty(\mathbf{C}^n)$.

**Proposition 5.1.** [14, Corollary 9.3] *There are constants* $0 < C_{5.1} < \infty$ *and* $0 < C'_{5.1} < \infty$ *such that the following bounds hold: Given an* $f \in \mathcal{T}(\mathbf{C}^n)$, *let*

$$f = f^{(1)} + f^{(2)}$$

*be the decomposition defined by* (5.1). *Then for every symmetric gauge function* $\Phi$,

$$\Phi(\{A(f^{(1)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C_{5.1}\|H_f\|_\Phi \quad \textit{and}$$
$$\Phi(\{A(\bar{\partial}_j f^{(2)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C'_{5.1}\|H_f\|_\Phi, \quad j = 1, \dots, n.$$

**Lemma 5.2.** [14, Lemma 9.4] *Suppose that* $f \in L^\infty(\mathbf{C}^n)$. *Then the functions* $f^{(1)}$, $f^{(2)}$ *defined by* (5.1) *also belong to* $L^\infty(\mathbf{C}^n)$.

**Lemma 5.3.** [14, Lemma 5.1] *Let* $f \in C^2(\mathbf{C}^n) \cap L^\infty(\mathbf{C}^n)$ *be a function which has the property that* $\bar{\partial}_j f \in \mathcal{L}_n^{2,p} = \mathcal{L}_n^{2,\Phi_p}$ *for some* $j \in \{1, \dots, n\}$ *and* $1 < p < \infty$. *Then*

$$\partial_j f = -\pi^{-1} T_j(\bar{\partial}_j f),$$

*where* $T_j$ *is the operator defined by* (3.5).

**Proof of Theorem 1.5.** Let $\Phi$ be a symmetric gauge function such that $1 < p_\Phi \le q_\Phi < \infty$. Let $f \in L^\infty(\mathbf{C}^n)$. To prove (1.2), it suffices to consider the case where $\|H_f\|_\Phi < \infty$. We apply decomposition (5.1) to this $f$:

$$f = f^{(1)} + f^{(2)} \quad \text{with} \quad f^{(2)} \in C^\infty(\mathbf{C}^n).$$

By Lemma 5.2, $f^{(1)}, f^{(2)} \in L^\infty(\mathbf{C}^n)$. Applying Propositions 4.5 and 5.1, we have

$$\|H_{\bar{f}^{(1)}}\|_\Phi \le \|[M_{f^{(1)}}, P]\|_\Phi \le C_{4.5}\Phi(\{A(f^{(1)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C_{4.5}C_{5.1}\|H_f\|_\Phi. \tag{5.2}$$

Next we consider $H_{\bar{f}^{(2)}}$.

By Proposition 5.1, (4.3) and (3.3), the condition $\|H_f\|_\Phi < \infty$ implies $\bar{\partial}_j f^{(2)} \in \mathcal{L}_n^{2,\Phi}$ for $j = 1, \dots, n$. By Lemma 2.3, we have $\mathcal{L}_n^{2,\Phi} \subset \mathcal{L}_n^{2,q} = \mathcal{L}_n^{2,\Phi_q}$ for every $q \in (q_\Phi, \infty)$. Since $f^{(2)} \in L^\infty(\mathbf{C}^n)$, Lemma 5.3 is applicable to $f^{(2)}$. By Lemma 5.3,

$$\partial_j f^{(2)} = -\pi^{-1} T_j(\bar{\partial}_j f^{(2)}),$$

$j = 1, \dots, n$. Thus it follows from Proposition 3.3 that

$$\|\partial_j f^{(2)}\|_{2,\Phi} \le C \|\bar{\partial}_j f^{(2)}\|_{2,\Phi},$$

$j = 1, \dots, n$. Recalling (3.3) and (4.3), this means

$$\Phi(\{A(\partial_j f^{(2)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C \Phi(\{A(\bar{\partial}_j f^{(2)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}), \tag{5.3}$$

$j = 1, \dots, n$. By Proposition 5.1, we have

$$\Phi(\{A(\bar{\partial}_j f^{(2)}; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C'_{5.1} \|H_f\|_\Phi.$$

$j = 1, \dots, n$. Combining this with (5.3), we find that

$$\Phi(\{A(|\nabla f^{(2)}|; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C_1 \|H_f\|_\Phi.$$

Applying Proposition 4.4, we now have

$$\|H_{\bar{f}^{(2)}}\|_\Phi \le \|[M_{f^{(2)}}, P]\|_\Phi \le C_{4.4} \Phi(\{A(|\nabla f^{(2)}|; \alpha)\}_{\alpha \in \mathbf{Z}^{2n}}) \le C_{4.4} C_1 \|H_f\|_\Phi. \tag{5.4}$$

Since $\bar{f} = \bar{f}^{(1)} + \bar{f}^{(2)}$, (1.2) follows from (5.2) and (5.4). This completes the proof. □

## 6. More on Hankel operators

Having proved Theorem 1.5, next we turn to the proof of Theorem 1.6, which requires quite a bit of preparation.

For any pair of $a \in \mathbf{C}^n$ and $r > 0$, denote

$$B(a, r) = \{z \in \mathbf{C}^n : |a - z| < r\}.$$

Let $\mathcal{M}$ denote the collection of $\varphi \in L^\infty(\mathbf{C}^n)$ for which there is some $0 < \rho = \rho(\varphi) < \infty$ such that $\varphi = 0$ on $\mathbf{C}^n \backslash B(0, \rho)$. If $\varphi \in \mathcal{M}$, then the Hankel operator $H_\varphi$ is in the trace class $\mathcal{C}_1$. This fact can be easily verified by hand, but it certainly is a consequence of Proposition 4.1. Furthermore, we have

**Lemma 6.1.** *If $\varphi \in \mathcal{M}$, then*

$$\lim_{r \to \infty} \|M_\varphi P M_{\chi_{\mathbf{C}^n \backslash B(0,r)}}\|_1 = 0.$$

*Proof.* For any $\varphi \in \mathcal{M}$, by definition there is some $0 < \rho = \rho(\varphi) < \infty$ such that $\varphi = 0$ on $\mathbf{C}^n \backslash B(0, \rho)$, and $\varphi \in L^\infty(\mathbf{C}^n)$. Using these properties, it is elementary to verify that $M_\varphi P \in \mathcal{C}_1$. Now it suffices to observe that $M_{\chi_{\mathbf{C}^n \backslash B(0,r)}} \to 0$ strongly as $r \to \infty$. □

For each $a \in \mathbf{C}^n$, we have the translation

$$\tau_a(z) = z - a, \quad z \in \mathbf{C}^n.$$

It is well known that for each $a \in \mathbf{C}^n$, the formula

$$V_a f = f \circ \tau_a \cdot k_a, \quad f \in L^2(\mathbf{C}^n, d\mu),$$

defines a unitary operator on $L^2(\mathbf{C}^n, d\mu)$, where $k_a(z) = e^{\langle z,a \rangle} e^{-|a|^2/2}$. The restriction of $V_a$ to $H^2(\mathbf{C}^n, d\mu)$ is also a unitary operator that maps the Fock space onto itself.

For any $f \in L^\infty(\mathbf{C}^n)$, we will identify the Hankel operator $H_f$ with the operator $(1 - P)M_f P$ on the space $L^2(\mathbf{C}^n, d\mu)$. Thus for $f, \varphi, \psi \in L^\infty(\mathbf{C}^n)$, $M_\varphi H_f M_\psi$ means the operator $M_\varphi(1 - P)M_f P M_\psi$ on $L^2(\mathbf{C}^n, d\mu)$.

Our next lemma simplifies the proof of Theorem 1.6:

**Lemma 6.2.** *Let $\Phi$ be a symmetric gauge function. Suppose that there exists a set of functions $\{f_k : k \in \mathbf{N}\} \subset \mathcal{M}$ such that $\sup_{k\in\mathbf{N}} \|f_k\|_\infty < \infty$ and such that*

$$\bigoplus_{k=1}^{\infty} H_{f_k} \in \mathcal{C}_\Phi^{(0)} \quad \textit{while} \quad \bigoplus_{k=1}^{\infty} H_{\bar{f}_k} \notin \mathcal{C}_\Phi. \tag{6.1}$$

*Then there is an $f \in L^\infty(\mathbf{C}^n)$ such that $H_f \in \mathcal{C}_\Phi^{(0)}$ while $H_{\bar{f}} \notin \mathcal{C}_\Phi$.*

*Proof.* First of all, it is a basic fact about symmetrically normed ideals that $\mathcal{C}_\Phi^{(0)} \supset \mathcal{C}_1$.

Since $\{f_k : k \in \mathbf{N}\} \subset \mathcal{M}$, there is a sequence $\{\rho_k\}$ in $(0, \infty)$ such that $f_k = 0$ on $\mathbf{C}^n \backslash B(0, \rho_k)$ for every $k \in \mathbf{N}$. For each $k \in \mathbf{N}$, Lemma 6.1 allows us to pick an $r_k \in (\rho_k, \infty)$ such that

$$\begin{cases} \|H_{f_k} - M_{\chi_{B(0,r_k)}} H_{f_k} M_{\chi_{B(0,r_k)}}\|_1 \le 2^{-k} & \text{and} \\ \|H_{\bar{f}_k} - M_{\chi_{B(0,r_k)}} H_{\bar{f}_k} M_{\chi_{B(0,r_k)}}\|_1 \le 2^{-k} & \end{cases}. \tag{6.2}$$

Thus the operators

$$\bigoplus_{k=1}^{\infty} H_{f_k} - \bigoplus_{k=1}^{\infty} M_{\chi_{B(0,r_k)}} H_{f_k} M_{\chi_{B(0,r_k)}} \quad \text{and} \quad \bigoplus_{k=1}^{\infty} H_{\bar{f}_k} - \bigoplus_{k=1}^{\infty} M_{\chi_{B(0,r_k)}} H_{\bar{f}_k} M_{\chi_{B(0,r_k)}}$$

are in the trace class. Applying (6.1), we have

$$\bigoplus_{k=1}^{\infty} M_{\chi_{B(0,r_k)}} H_{f_k} M_{\chi_{B(0,r_k)}} \in \mathcal{C}_\Phi^{(0)} \quad \text{while} \quad \bigoplus_{k=1}^{\infty} M_{\chi_{B(0,r_k)}} H_{\bar{f}_k} M_{\chi_{B(0,r_k)}} \notin \mathcal{C}_\Phi. \tag{6.3}$$

We can inductively select a sequence $\{a_k\}$ in $\mathbf{C}^n$ such that $B(a_k, r_k) \cap B(a_j, r_j) = \emptyset$ for all $j \neq k$. We have

$$V_{a_k} M_{\chi_{B(0,r_k)}} H_\varphi M_{\chi_{B(0,r_k)}} V_{a_k}^* = M_{\chi_{B(a_k,r_k)}} H_{\varphi\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}}$$

for every $\varphi \in L^\infty(\mathbf{C}^n)$. Combining this unitary equivalence with (6.3), we see that

$$\bigoplus_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{f_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}} \in \mathcal{C}_\Phi^{(0)} \quad \text{while} \quad \bigoplus_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{\bar{f}_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}} \notin \mathcal{C}_\Phi.$$

Since $B(a_k, r_k) \cap B(a_j, r_j) = \emptyset$ for all $j \neq k$, the above implies that as operators on $L^2(\mathbf{C}^n, d\mu)$, we have

(6.4)
$$\sum_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{f_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}} \in \mathcal{C}_\Phi^{(0)} \quad \text{while} \quad \sum_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{\bar{f}_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}} \notin \mathcal{C}_\Phi.$$

Using the unitary operator $V_{a_k}$ again, from (6.2) we obtain

$$\begin{cases} \|H_{f_k\circ\tau_{a_k}} - M_{\chi_{B(a_k,r_k)}} H_{f_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}}\|_1 \le 2^{-k} & \text{and} \\ \|H_{\bar{f}_k\circ\tau_{a_k}} - M_{\chi_{B(a_k,r_k)}} H_{\bar{f}_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}}\|_1 \le 2^{-k} & \end{cases},$$

$k \in \mathbf{N}$. Thus the operators

$$\sum_{k=1}^\infty H_{f_k\circ\tau_{a_k}} - \sum_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{f_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}} \quad \text{and}$$
$$\sum_{k=1}^\infty H_{\bar{f}_k\circ\tau_{a_k}} - \sum_{k=1}^\infty M_{\chi_{B(a_k,r_k)}} H_{\bar{f}_k\circ\tau_{a_k}} M_{\chi_{B(a_k,r_k)}}$$

are in the trace class. Combining this fact with (6.4), we see that

$$\sum_{k=1}^\infty H_{f_k\circ\tau_{a_k}} \in \mathcal{C}_\Phi^{(0)} \quad \text{while} \quad \sum_{k=1}^\infty H_{\bar{f}_k\circ\tau_{a_k}} \notin \mathcal{C}_\Phi. \tag{6.5}$$

The property that $f_k = 0$ on $\mathbf{C}^n \backslash B(0, r_k)$ implies that $f_k \circ \tau_{a_k} = 0$ on $\mathbf{C}^n \backslash B(a_k, r_k)$. Since $B(a_k, r_k) \cap B(a_j, r_j) = \emptyset$ for $j \neq k$ and $\sup_{k\in\mathbf{N}} \|f_k\|_\infty < \infty$, the function

$$f = \sum_{k=1}^\infty f_k \circ \tau_{a_k}$$

is in $L^\infty(\mathbf{C}^n)$. On $L^2(\mathbf{C}^n, d\mu)$, we have the obvious strong convergence $\sum_{k=1}^\ell M_{f_k\circ\tau_{a_k}} \to M_f$ and $\sum_{k=1}^\ell M_{\bar{f}_k\circ\tau_{a_k}} \to M_{\bar{f}}$ as $\ell \to \infty$. Therefore

$$\sum_{k=1}^\infty H_{f_k\circ\tau_{a_k}} = H_f \quad \text{and} \quad \sum_{k=1}^\infty H_{\bar{f}_k\circ\tau_{a_k}} = H_{\bar{f}}.$$

Thus (6.5) tells us that $H_f \in \mathcal{C}_\Phi^{(0)}$ while $H_{\bar f} \notin \mathcal{C}_\Phi$. This completes the proof. □

**Proposition 6.3.** *For each $f \in C^\infty(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, d\mu)$ we have*

$$\|(1-P)f\| \le \|\bar\partial_1 f\| + \cdots + \|\bar\partial_n f\|.$$

*Proof.* For each $j \in \{1, \dots, n\}$, define the operator $P_j$ by the formula

$$(P_j\psi)(\zeta_1, \dots, \zeta_n) = \frac{1}{\pi}\int_{\mathbf{C}} \psi(\zeta_1, \dots, \zeta_{j-1}, z, \zeta_{j+1}, \dots, \zeta_n) e^{\zeta_j \bar z} e^{-|z|^2} dA(z),$$

$\psi \in L^2(\mathbf{C}^n, d\mu)$. Then it is easy to see that $P_jP_k = P_kP_j$ for all $j, k \in \{1, \dots, n\}$, and that $P = P_1 \cdots P_n$. Thus

$$\begin{aligned} 1 - P &= 1 - P_1 \cdots P_{n-1} + P_1 \cdots P_{n-1}(1 - P_n) \\ &= 1 - P_1 \cdots P_{n-2} + P_1 \cdots P_{n-2}(1 - P_{n-1}) + P_1 \cdots P_{n-1}(1 - P_n) \\ &= \cdots . \end{aligned}$$

Since each $P_j$ is an orthogonal projection, from the above we see that

$$\|(1-P)\psi\| \le \|(1-P_1)\psi\| + \cdots + \|(1-P_n)\psi\| \tag{6.6}$$

for every $\psi \in L^2(\mathbf{C}^n, d\mu)$. If $f \in C^\infty(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, d\mu)$, then by [14, Proposition 12.1] we have $\|(1-P_j)f\| \le \|\bar\partial_j f\|$ for $j = 1, \dots, n$. Combining this with (6.6), the proposition is proved. □

Recall that we used the function

$$g(z) = \begin{cases} z^{-1} & \text{if} \quad |z| \ge 1 \\ \\ 0 & \text{if} \quad |z| < 1 \end{cases}$$

in [14] in the case $n = 1$. That is, when $n = 1$, we have $H_g \in \mathcal{C}_1$ while $H_{\bar g} \notin \mathcal{C}_1$ [14, Theorem 1.5]. We will modify this $g$ for use in the general case $n \ge 1$.

To do this, we first introduce appropriate cutoff functions. We begin with a $\xi \in C^\infty(\mathbf{R})$ satisfying the following three conditions:

(a) $0 \le \xi \le 1$ on $\mathbf{R}$.

(b) $\xi = 0$ on $(-\infty, 2]$.

(c) $\xi = 1$ on $[3, \infty)$.

Next, for each $R > 6$ we define the function

$$\eta_R(x) = \xi(x)\xi(R + 3 - x), \quad x \in \mathbf{R}.$$

Then $0 \le \eta_R \le 1$ on $\mathbf{R}$, $\eta_R = 0$ on $(-\infty, 2] \cup [R + 1, \infty)$, and $\eta_R = 1$ on $[3, R]$. Note that $\|\eta_R'\|_\infty \le \|\xi'\|_\infty$ for every $R > 6$. With $\eta_R$ so defined, for each $R > 6$ we now define the function $\gamma_R$ on $\mathbf{C}^n$ by the formula

$$\gamma_R(\zeta_1, \dots, \zeta_n) = \prod_{j=1}^n \eta_R(|\zeta_j|)g(\zeta_j), \quad (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n. \tag{6.7}$$

**Lemma 6.4.** *There is a* $0 < C_{6.4} < \infty$ *such that* $\|H_{\gamma_R}\|_1 \leq C_{6.4}R^{n-1}$ *for every* $R > 6$.

*Proof.* We have

$$\|H_{\gamma_R}\|_1 = \mathrm{tr}((H^*_{\gamma_R}H_{\gamma_R})^{1/2}) = \frac{1}{\pi^n}\int \langle (H^*_{\gamma_R}H_{\gamma_R})^{1/2}k_z, k_z\rangle dV(z) \leq \frac{1}{\pi^n}\int \|H_{\gamma_R}k_z\| dV(z),$$

where the $\leq$ follows from the spectral decomposition of $H^*_{\gamma_R}H_{\gamma_R}$ and the Cauchy-Schwarz inequality. By Proposition 6.3, we have

$$\|H_{\gamma_R}k_z\| \leq \sum_{j=1}^n \|\bar\partial_j(\gamma_R k_z)\| = \sum_{j=1}^n \|k_z\bar\partial_j\gamma_R\|.$$

Thus

$$\|H_{\gamma_R}\|_1 \leq \frac{1}{\pi^n}\sum_{j=1}^n \int \|k_z\bar\partial_j\gamma_R\| dV(z) = \frac{n}{\pi^n}\int \|k_z\bar\partial_1\gamma_R\| dV(z),$$

where the second step involves the fact that $\gamma_R$ is invariant under any permutation of the variable $\zeta_1, \dots, \zeta_n$. Therefore

$$\text{(6.8)}\quad \|H_{\gamma_R}\|_1 \leq \frac{n}{\pi^n}\int \langle |\bar\partial_1\gamma_R|^2 k_z, k_z\rangle^{1/2} dV(z) \leq C_1 \sum_{\alpha\in\mathbf{Z}^{2n}} \left(\int_{Q_\alpha} |(\bar\partial_1\gamma_R)(\zeta)|^2 dV(\zeta)\right)^{1/2},$$

where the second $\leq$ is a well-known fact that is easy to prove (see [16, Lemma 6.34] for the case $n = 1$, and the general case $n \geq 1$ is proved by the same kind of estimates).

We have $\bar\partial|z| = (2|z|)^{-1}z$ for $z \in \mathbf{C}\backslash\{0\}$. Therefore

$$(\bar\partial_1\gamma_R)(\zeta_1, \dots, \zeta_n) = \frac{1}{2|\zeta_1|}\eta'_R(|\zeta_1|)\prod_{j=2}^n \eta_R(|\zeta_j|)g(\zeta_j) \quad \text{if} \quad |\zeta_1| \geq 2.$$

(In the case $n = 1$, the product $\prod_{j=2}^n \cdots$ is interpreted to be 1. The same convention applies in similar situations.) Also, by the definition of $\eta_R$ we have

$$(\bar\partial_1\gamma_R)(\zeta_1, \dots, \zeta_n) = 0 \quad \text{if} \quad |\zeta_1| < 2.$$

Recalling (3.2), if $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbf{Z}^{2n}$ is such that $\int_{Q_\alpha} |(\bar\partial_1\gamma_R)(\zeta)|^2 dV(\zeta) \neq 0$, then then the function $\zeta_1 \mapsto \eta'_R(|\zeta_1|)$ must not identically vanish on $I_{(\alpha_1,\alpha_2)}$. There are two types of $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ for which this is possible. The first type are those $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ satisfying the condition

$$I_{(\alpha_1,\alpha_2)} \cap \{\zeta_1 \in \mathbf{C} : |\zeta_1| \leq 3\} \neq \emptyset.$$

Thus if $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ is of this type, then we have $|\alpha_1| \leq 4$ and $|\alpha_2| \leq 4$. Hence the total number of this first type of $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ does not exceed 81.

The second type of $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ with the property that the function $\zeta_1 \mapsto \eta'_R(|\zeta_1|)$ does not identically vanish on $I_{(\alpha_1,\alpha_2)}$ are those satisfying the condition

$$I_{(\alpha_1,\alpha_2)} \cap \{\zeta_1 \in \mathbf{C} : R \le |\zeta_1| \le R+1\} \ne \emptyset.$$

It is obvious that the total area of such $I_{(\alpha_1,\alpha_2)}$ does not exceed $C_2R$. Moreover, if $(\alpha_1, \alpha_2) \in \mathbf{Z}^2$ is of this type, then we have $1/|\zeta_1| \le C_3/R$ for $\zeta_1 \in I_{(\alpha_1,\alpha_2)}$.

Combining the analysis in the last two paragraphs with (6.8), we find that

$$\|H_{\gamma_R}\|_1 \le C_4 \sum_{(\alpha_3,\dots,\alpha_{2n})\in\mathbf{Z}^{2n-2}} \left(\prod_{j=2}^{n} \int_{I_{(\alpha_{2j-1},\alpha_{2j})}} |\eta_R(|\zeta_j|)g(\zeta_j)|^2 dA(\zeta_j)\right)^{1/2}$$

(see (3.2) and (3.1)). For $z \in \mathbf{C}$, if $\eta_R(|z|) \ne 0$, then $2 \le |z| \le R+1$. Thus if we define $Z_R$ to be the collection of all $(\alpha_3, \dots, \alpha_{2n}) \in \mathbf{Z}^{2n-2}$ satisfying the condition that

$$I_{(\alpha_{2j-1},\alpha_{2j})} \cap \{\zeta_j \in \mathbf{C} : 2 \le |\zeta_j| \le R+1\} \ne \emptyset \quad \text{for every} \ \ 2 \le j \le n,$$

then

$$\begin{aligned}\|H_{\gamma_R}\|_1 &\le C_5 \sum_{(\alpha_3,\dots,\alpha_{2n})\in Z_R} \prod_{j=2}^{n} \frac{1}{1+(\alpha_{2j-1}^2+\alpha_{2j}^2)^{1/2}} \\ &\le C_6\left(\int_{|z|\le R+1+\sqrt{2}} \frac{1}{1+|z|} dA(z)\right)^{n-1} \le C_7R^{n-1}.\end{aligned}$$

This completes the proof. □

**Proposition 6.5.** [6, Lemma 6.3] *Let $\Phi$ be an arbitrary symmetric gauge function. Then there is a constant $0 < C < \infty$ such that*

$$\Phi(\{J(f;u)\}_{u\in\mathbf{Z}^{2n}}) \le C\|[M_f, P]\|_\Phi$$

*for every $f \in \mathcal{T}(\mathbf{C}^n)$.*

**Lemma 6.6.** *There exist positive numbers $6 < M_{6.6} < \infty$ and $0 < c_{6.6} < \infty$ such that if $R \ge M_{6.6}$, then*

$$\|H_{\bar{\gamma}_R}\|_1 \ge c_{6.6}R^{n-1}\log R.$$

*Proof.* Define $T = \{(z_1, \dots, z_n) \in \mathbf{C}^n : |z_j| < 1 \text{ for } 1 \le j \le n\}$. Also, write $D = \{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{Z}^2 = \mathbf{Z} + i\mathbf{Z}$. Consider any $u = (u_1, \dots, u_n) \in (\mathbf{Z}^2)^n$ such that $|u_j| \ge 6$ for every $1 \le j \le n$. For such a $u$, we have

$$\begin{aligned}
&\iint_{(T+u)\times(T+u)} \left| \frac{1}{z_1\cdots z_n} - \frac{1}{w_1\cdots w_n}\right|^2 dV(z_1,\dots,z_n)dV(w_1,\dots,w_n)\\
&\ge \frac{c_1}{|u_1|^4\cdots|u_n|^4}\iint_{(T+u)\times(T+u)} |z_1\cdots z_n - w_1\cdots w_n|^2 dV(z_1,\dots,z_n)dV(w_1,\dots,w_n)\\
&= \frac{2\pi^n c_1}{|u_1|^4\cdots|u_n|^4}\left(\int_{T+u}|z_1\cdots z_n|^2dV(z_1,\dots,z_n) - \pi^n|u_1|^2\cdots|u_n|^2\right)\\
&= \frac{2\pi^n c_1}{|u_1|^4\cdots|u_n|^4}\left(\prod_{j=1}^n\int_D|z+u_j|^2dA(z) - \pi^n|u_1|^2\cdots|u_n|^2\right)\\
&= \frac{2\pi^{2n} c_1}{|u_1|^4\cdots|u_n|^4}\left(\prod_{j=1}^n\left(\frac{1}{2}+|u_j|^2\right) - |u_1|^2\cdots|u_n|^2\right)\\
&\ge \frac{\pi^{2n} c_1}{|u_1|^4\cdots|u_n|^4}\sum_{j=1}^n\prod_{\nu\ne j}|u_\nu|^2.
\end{aligned}$$

Let $U_R$ be the collection of $u = (u_1,\dots,u_n) \in (\mathbf{Z}^2)^n$ satisfying the conditions that $|u_j| \ge 6$ for every $1 \le j \le n$ and that $\prod_{j=1}^n \eta_R(|z_j|) = 1$ for every $(z_1,\dots,z_n) \in W + u$. Since $T \subset W$, for each $u \in U_R$ it follows from the above that

$$J(\gamma_R;u) \ge \frac{\pi^n c_1^{1/2}}{n^{1/2}|u_1|^2\cdots|u_n|^2}\sum_{j=1}^n\prod_{\nu\ne j}|u_\nu| = c_2\sum_{j=1}^n\frac{1}{|u_j|^2}\prod_{\nu\ne j}\frac{1}{|u_\nu|}.$$

Recall that we have $\eta_R = 1$ on $[3,R]$. Thus it is obvious that there is a $10 < C < \infty$ such that if $R \ge 3C$, then for $u = (u_1,\dots,u_n) \in (\mathbf{Z}^2)^n$, the condition that $C \le |u_j| \le R - C$ for $j = 1,\dots,n$ implies that $u \in U_R$. Applying Proposition 6.5 to the symmetric gauge function $\Phi_1$, for $R \ge 3C$ we have

$$\|[M_{\gamma_R},P]\|_1 \ge c_3\sum_{u\in U_R}J(\gamma_R;u) \ge c_4\sum_{(u_1,\dots,u_n)\in\tilde{U}_R}\sum_{j=1}^n\frac{1}{|u_j|^2}\prod_{\nu\ne j}\frac{1}{|u_\nu|}, \tag{6.9}$$

where $\tilde{U}_R$ is the collection of $(u_1,\dots,u_n) \in (\mathbf{Z}^2)^n$ such that $C \le |u_j| \le R - C$ for $j = 1,\dots,n$.

It is easy to see that there are $C_1 \in [10C,\infty)$ and $c_5 > 0$ such that for $R \ge C_1$,

$$\begin{aligned}
\sum_{(u_1\dots,u_n)\in\tilde{U}_R}\frac{1}{|u_1|^2}\prod_{\nu=2}^n\frac{1}{|u_\nu|} &\ge c_5\int_{2C\le|z|\le R-2C}\frac{1}{|z|^2}dA(z)\left(\int_{2C\le|z|\le R-2C}\frac{1}{|z|}dA(z)\right)^{n-1}\\
&\ge c_6(\log R)R^{n-1}.
\end{aligned}$$

Combining this with (6.9), we find that

$$\|[M_{\gamma_R},P]\|_1 \ge c_7R^{n-1}\log R$$

when $R \geq C_1$. Note that $[M_f, P] = H_f - H^*_{\bar{f}}$. Hence if $R \geq C_1$, then

$$\|H_{\gamma_R}\|_1 + \|H_{\bar{\gamma}_R}\|_1 \geq c_7 R^{n-1} \log R.$$

On the other hand, Lemma 6.4 tells us that $\|H_{\gamma_R}\|_1 \leq C_{6.4} R^{n-1}$. Now let $C_1 \leq M < \infty$ be such that $(c_7/2)\log R \geq C_{6.4}$ for $R \geq M$. Then for $R \geq M$, from the above we obtain

$$\|H_{\bar{\gamma}_R}\|_1 \geq (c_7/2) R^{n-1} \log R.$$

This completes the proof. □

**Lemma 6.7.** *There exists a* $0 < C_{6.7} < \infty$ *such that the following holds true: Let* $h : [0, \infty) \to [0, 1]$ *be any measurable function satisfying the conditions*

(a) $h = 1$ *on* $[0, C_{6.7}]$;

(b) $h = 0$ *on* $[\rho, \infty)$ *for some* $C_{6.7} < \rho < \infty$.

*Then the function*

$$\eta(z_1, \ldots, z_n) = h(|z_1|) z_1, \quad (z_1, \ldots, z_n) \in \mathbf{C}^n, \tag{6.10}$$

*has the property that* $\|H_{\bar{\eta}}\| \geq 1/2$.

*Proof.* We have $\|z_1\| = 1$ in $H^2(\mathbf{C}^n, d\mu)$. Thus for such an $\eta$,

$$\begin{aligned}
\|H_{\bar{\eta}}\|^2 &\geq \langle H^*_{\bar{\eta}} H_{\bar{\eta}} z_1, z_1 \rangle = \langle M_{|\eta|^2} z_1, z_1 \rangle - \|P M_{\bar{\eta}} z_1\|^2 \\
&= \frac{1}{\pi} \int_{\mathbf{C}} h^2(|z|)|z|^4 e^{-|z|^2} dA(z) - \|P M_{\bar{\eta}} z_1\|^2 \\
&\geq \frac{1}{\pi} \int_{|z| < C_{6.7}} |z|^4 e^{-|z|^2} dA(z) - \|P M_{\bar{\eta}} z_1\|^2.
\end{aligned}$$

Since $\pi^{-1} \int_{\mathbf{C}} |z|^4 e^{-|z|^2} dA(z) = 2$, we see that for a sufficiently large $C_{6.7}$ we have

$$\|H_{\bar{\eta}}\|^2 \geq (5/4) - \|P M_{\bar{\eta}} z_1\|^2.$$

Note that $\bar{\eta}(z_1, \ldots, z_n) z_1 = h(|z_1|)|z_1|^2 \perp z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ whenever there is a $j \in \{1, \ldots, n\}$ such that $k_j \geq 1$. Therefore

$$\|P M_{\bar{\eta}} z_1\| = |\langle \bar{\eta} z_1, 1 \rangle| = \frac{1}{\pi} \int_{\mathbf{C}} h(|z|)|z|^2 e^{-|z|^2} dA(z) \leq \frac{1}{\pi} \int_{\mathbf{C}} |z|^2 e^{-|z|^2} dA(z) = 1.$$

Consequently, $\|H_{\bar{\eta}}\|^2 \geq (5/4) - 1 = 1/4$. This completes the proof. □

**Lemma 6.8.** *Given an* $\eta$ *defined by* (6.10), *where* $h$ *satisfies the conditions in Lemma* 6.7, *there is a* $0 < C_{6.8} = C_{6.8}(\eta) < \infty$ *which has the following property: Let* $\psi \in L^\infty(\mathbf{C}^n)$ *be such that* $\psi = 1$ *on* $B(0, C_{6.8})$ *and* $\|\psi\|_\infty = 1$. *Then* $\|H_{\psi\bar{\eta}}\| \geq 1/3$.

*Proof.* If $\{A_k\}$ is a sequence of bounded operators strongly convergent to an operator $A$, then $\|A\| \leq \liminf_{k\to\infty} \|A_k\|$. The conclusion of the lemma follows from this fact and Lemma 6.7. □

We now generalize [14, Proposition 13.2] to arbitrary complex dimensions $n \geq 1$:

**Proposition 6.9.** *There does not exist any constant* $0 < C < \infty$ *such that the inequality*

$$\|H_{\bar{\varphi}}\| \leq C\|H_\varphi\|$$

*holds for every* $\varphi \in \mathcal{M}$.

*Proof.* Let $\epsilon > 0$ be given. By the argument on pages 43 and 44 in [14], there is a $C^\infty$ function $h : [0, \infty) \to [0, 1]$ satisfying conditions (a) and (b) in Lemma 6.7 such that the inequality $\|\bar{\partial}_1\eta\|_\infty \leq \epsilon$ holds for the function

$$\eta(z_1, \ldots, z_n) = h(|z_1|)z_1, \quad (z_1, \ldots, z_n) \in \mathbf{C}^n.$$

To be more precise about condition (b), there is a $T \in (C_{6.7}, \infty)$ such that $h = 0$ on $[T, \infty)$. We have, of course, $\bar{\partial}_j\eta = 0$ for $2 \leq j \leq n$.

For this $\eta$, Lemma 6.8 provides a $0 < C_{6.8} = C_{6.8}(\eta) < \infty$. There is a $C^\infty$ function $\beta : \mathbf{R} \to [0, 1]$ satisfying the following four conditions:
(1) $0 \leq \beta \leq 1$ on $\mathbf{R}$.
(2) $\beta(x) = 1$ if $x \leq C_{6.8}$.
(3) $\|\beta'\|_\infty \leq \epsilon/T$.
(4) There is an $r \in (C_{6.8}, \infty)$ such that $\beta = 0$ on $[r, \infty)$.
With this $\beta$ we define the function

$$\psi(z_1, \ldots, z_n) = \beta(|z_2|) \cdots \beta(|z_n|), \quad (z_1, \ldots, z_n) \in \mathbf{C}^n.$$

We obviously have $\psi\eta \in \mathcal{M}$. By conditions (1) and (2) above and Lemma 6.8, we have $\|H_{\overline{\psi\eta}}\| = \|H_{\psi\bar{\eta}}\| \geq 1/3$. Next we show that $\|H_{\psi\eta}\| \leq n\epsilon$. Since $\epsilon > 0$ is arbitrary, this will complete the proof of the proposition.

For each $f \in H^2(\mathbf{C}^n, d\mu)$, Proposition 6.3 tells us that

$$\|H_{\psi\eta}f\| \leq \|\bar{\partial}_1(\psi\eta f)\| + \cdots + \|\bar{\partial}_n(\psi\eta f)\| = \|f\bar{\partial}_1(\psi\eta)\| + \cdots + \|f\bar{\partial}_n(\psi\eta)\|.$$

Thus the desired conclusion will follow if we can show that $\|\bar{\partial}_j(\psi\eta)\|_\infty \leq \epsilon$ for every $1 \leq j \leq n$.

In the case $j = 1$, we have $\bar{\partial}_1(\psi\eta) = \psi\bar{\partial}_1\eta$. Since $\|\psi\|_\infty = 1$, from the condition $\|\bar{\partial}_1\eta\|_\infty \leq \epsilon$ we deduce $\|\bar{\partial}_1(\psi\eta)\|_\infty \leq \epsilon$. Now consider any $2 \leq j \leq n$. Then $\bar{\partial}_j(\psi\eta) = \eta\bar{\partial}_j\psi$. By condition (3), we have $\|\bar{\partial}_j\psi\|_\infty \leq \epsilon/T$. On the other hand, since $h = 0$ on $[T, \infty)$, we have $\|\eta\|_\infty \leq T$. Consequently, $\|\bar{\partial}_j(\psi\eta)\|_\infty \leq T \cdot (\epsilon/T) = \epsilon$. This completes the proof. □

## 7. Proof of Theorem 1.6

In addition to the material about Hankel operators in Section 6, to prove Theorem 1.6, we also need the following three lemmas concerning symmetric gauge functions.

**Lemma 7.1.** *Let* $\Phi$ *be a symmetric gauge function such that* $p_\Phi = 1$. *Then* $q_{\Phi^*} = \infty$.

*Proof.* Suppose that $q_{\Phi^*} < \infty$. We will show that this contradicts the condition $p_\Phi = 1$. Let $q < \infty$ be such that $q_{\Phi^*} < q$. Then by Lemma 2.2, there is a $c > 0$ such that

$$cm^{1/q}\Phi^*(b) \le \Phi^*(b^{[m]}) \quad \text{for all} \quad b \in \hat{c} \quad \text{and} \quad m \in \mathbf{N}. \tag{7.1}$$

Since $\Phi = \Phi^{**}$, by [13, Proposition 3.2], (7.1) implies that there are $0 < t < 1$ and $0 < C < \infty$ such that

$$\Phi(a^{[m]}) \le Cm^t\Phi(a) \quad \text{for all} \quad a \in \hat{c} \quad \text{and} \quad m \in \mathbf{N}.$$

This implies that $p_\Phi \ge (t + \epsilon)^{-1}$ for every $\epsilon > 0$. Since $0 < t < 1$, this contradicts the condition $p_\Phi = 1$. □

**Lemma 7.2.** *Let $\Phi$ be a symmetric gauge function such that $q_\Phi = \infty$. Then for every $m \in \mathbf{N}$, there exists an $x_m \in \hat{c}$, $x_m \ne 0$, such that $\Phi(x_m^{[m]}) \le 2\Phi(x_m)$.*

*Proof.* Given any $m \in \mathbf{N}$, we pick a $k \in \mathbf{N}$ such that $2^k \ge m$. Since $q_\Phi > k$, for a sufficiently large $\ell \in \mathbf{N}$ there is an $a \in \hat{c}$, $a \ne 0$, such that

$$\frac{\log 2^{k\ell}}{\log\left(\Phi(a^{[2^{k\ell}]})/\Phi(a)\right)} > k.$$

That is,

$$\frac{\Phi(a^{[2^{k\ell}]})}{\Phi(a)} < 2^\ell. \tag{7.2}$$

On the other hand, we have

$$\frac{\Phi(a^{[2^{k\ell}]})}{\Phi(a)} = \prod_{j=1}^{\ell} \frac{\Phi(a^{[2^{jk}]})}{\Phi(a^{[2^{(j-1)k}]})}. \tag{7.3}$$

By a comparison of (7.2) and (7.3), there is an $i \in \{1, \dots, \ell\}$ such that

$$\frac{\Phi(a^{[2^{ik}]})}{\Phi(a^{[2^{(i-1)k}]})} < 2.$$

With this $i$ we define $x_m = a^{[2^{(i-1)k}]}$. Then

$$\Phi(x_m^{[m]}) \le \Phi(x_m^{[2^k]}) = \Phi(a^{[2^{ik}]}) \le 2\Phi(a^{[2^{(i-1)k}]}) = 2\Phi(x_m).$$

This completes the proof. □

**Lemma 7.3.** *For each $A \in \mathcal{C}_1$, there is a $\nu = \nu(A) \in \mathbf{N}$ such that the inequality*

$$\left\| \bigoplus_{j=1}^{\infty} \alpha_j A \right\|_\Phi \le \|A\|\Phi(\alpha^{[\nu]}) + \Phi(\alpha)$$

*holds for every symmetric gauge function* $\Phi$ *and every* $\alpha = \{\alpha_1, \dots, \alpha_j, \dots\} \in \hat{c}$.

*Proof.* Since $A \in \mathcal{C}_1$, there is a $\nu = \nu(A) \in \mathbf{N}$ such that

$$\sum_{k=\nu+1}^{\infty} s_k(A) \le 1.$$

There are orthonormal sets $\{u_k : k \in \mathbf{N}\}$ and $\{v_k : k \in \mathbf{N}\}$ such that

$$A = \sum_{k=1}^{\infty} s_k(A) u_k \otimes v_k.$$

Since

$$\bigoplus_{j=1}^{\infty} \alpha_j A = \left(\bigoplus_{j=1}^{\infty} \alpha_j \sum_{k=1}^{\nu} s_k(A) u_k \otimes v_k\right) + \left(\bigoplus_{j=1}^{\infty} \alpha_j \sum_{k=\nu+1}^{\infty} s_k(A) u_k \otimes v_k\right),$$

the desired conclusion is now obvious. □

**Proof of Theorem 1.6.** (1) First, we consider the case where $p_\Phi = 1$. In this case we need the functions defined by (6.7). Given a $d \in \mathbf{N}$, we take an $R_d \ge M_{6.6}$ such that $\log R_d \ge 1 + d^3$. Consider the Hankel operator $H_{\bar{\gamma}_{R_d}}$. Since $\bar{\gamma}_{R_d} \in \mathcal{M}$, which implies $H_{\bar{\gamma}_{R_d}} \in \mathcal{C}_1$, we have

$$H_{\bar{\gamma}_{R_d}} = \sum_{\ell=1}^{\infty} s_\ell(H_{\bar{\gamma}_{R_d}}) u_{d,\ell} \otimes v_{d,\ell},$$

where $\{u_{d,\ell} : \ell \in \mathbf{N}\}$ and $\{v_{d,\ell} : \ell \in \mathbf{N}\}$ are orthonormal set. By Lemma 6.6, we have

$$\|H_{\bar{\gamma}_{R_d}}\|_1 \ge c_{6.6} R_d^{n-1}(1 + d^3).$$

Thus there is a $\nu_d \in \mathbf{N}$ such that

$$\sum_{\ell=1}^{\nu_d} s_\ell(H_{\bar{\gamma}_{R_d}}) \ge c_{6.6} R_d^{n-1} d^3. \tag{7.4}$$

By Lemma 7.1, the condition $p_\Phi = 1$ implies $q_{\Phi^*} = \infty$. Thus by Lemma 7.2, there is an

$$\eta_d = \{\eta_{d,1}, \eta_{d,2}, \dots, \eta_{d,j}, \dots\} \in \hat{c}$$

such that

$$\Phi^*(\eta_d) = 1 \quad \text{and} \quad \Phi^*(\eta_d^{[\nu_d]}) \le 2. \tag{7.5}$$

Since $\Phi^{**} = \Phi$, there is an

$$\alpha_d = \{\alpha_{d,1}, \alpha_{d,2}, \dots, \alpha_{d,j}, \dots\} \in \hat{c}$$

such that

$$\Phi(\alpha_d) = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} \alpha_{d,j}\eta_{d,j} = \Phi^*(\eta_d) = 1. \tag{7.6}$$

Thus we obtain the $R_d$, $\nu_d$, $\eta_d$ and $\alpha_d$ described above for every $d \in \mathbf{N}$.

We know from (6.7) that $\|\gamma_R\|_\infty \le 1$ for every $R > 6$. Hence $\|\gamma_{R_d}\|_\infty \le 1$ for every $d \in \mathbf{N}$. Moreover, the condition $\Phi(\alpha_d) = 1$ implies that $|\alpha_{d,j}| \le 1$ for every $j \in \mathbf{N}$. Thus, by Lemma 6.2, the case $p_\Phi = 1$ of the theorem will be proved if we can show that

$$\bigoplus_{d=1}^{\infty} \frac{1}{d^2 R_d^{n-1}} \bigoplus_{j=1}^{\infty} \alpha_{d,j} H_{\gamma_{R_d}} \in \mathcal{C}_\Phi^{(0)} \quad \text{while} \tag{7.7}$$

$$\bigoplus_{d=1}^{\infty} \frac{1}{d^2 R_d^{n-1}} \bigoplus_{j=1}^{\infty} \alpha_{d,j} H_{\bar{\gamma}_{R_d}} \notin \mathcal{C}_\Phi. \tag{7.8}$$

It is obvious that for every $d \in \mathbf{N}$,

$$\left\| \bigoplus_{j=1}^{\infty} \alpha_{d,j} H_{\gamma_{R_d}} \right\|_\Phi \le \Phi(\alpha_d)\|H_{\gamma_{R_d}}\|_1 \le C_{6.4} R_d^{n-1}, \tag{7.9}$$

where for the second $\le$ we apply Lemma 6.4 and the condition $\Phi(\alpha_d) = 1$. Since $H_{\gamma_{R_d}} \in \mathcal{C}_1$ and $\alpha_d \in \hat{c}$, we have $\oplus_{j=1}^{\infty} \alpha_{d,j} H_{\gamma_{R_d}} \in \mathcal{C}_1 \subset \mathcal{C}_\Phi^{(0)}$, $d \in \mathbf{N}$. Hence (7.7) follows from (7.9).

To prove (7.8), for each $k \in \mathbf{N}$ we define the finite-rank operator

$$T_k = \bigoplus_{d=1}^{\infty} A_{k,d},$$

where $A_{k,d} = 0$ for $d \ne k$ and

$$A_{k,k} = \bigoplus_{j=1}^{\infty} \eta_{k,j} \sum_{\ell=1}^{\nu_k} v_{k,\ell} \otimes u_{k,\ell}.$$

For every $k \in \mathbf{N}$,

$$\|T_k\|_{\Phi^*} = \|A_{k,k}\|_{\Phi^*} = \Phi^*(\eta_k^{[\nu_k]}) \le 2,$$

where for the $\le$ we recall (7.5). For each $k \in \mathbf{N}$,

$$\begin{aligned} \mathrm{tr}\left( T_k \bigoplus_{d=1}^{\infty} \frac{1}{d^2 R_d^{n-1}} \bigoplus_{j=1}^{\infty} \alpha_{d,j} H_{\bar{\gamma}_{R_d}} \right) &= \frac{1}{k^2 R_k^{n-1}} \sum_{j=1}^{\infty} \alpha_{k,j}\eta_{k,j} \sum_{\ell=1}^{\nu_k} \langle H_{\bar{\gamma}_{R_k}} v_{k,\ell}, u_{k,\ell} \rangle \\ &= \frac{1}{k^2 R_k^{n-1}} \sum_{j=1}^{\infty} \alpha_{k,j}\eta_{k,j} \sum_{\ell=1}^{\nu_k} s_\ell(H_{\bar{\gamma}_{R_k}}) \\ &\ge \frac{1}{k^2 R_k^{n-1}} \cdot c_{6.6} R_k^{n-1} k^3 = c_{6.6} k, \end{aligned} \tag{7.10}$$

where the $\geq$ is obtained from (7.6) and (7.4). Since $\|T_k\|_{\Phi^*} \leq 2$, it follows from (2.1) and (7.10) that

$$\left\| \bigoplus_{d=1}^{\infty} \frac{1}{d^2 R_d^{n-1}} \bigoplus_{j=1}^{\infty} \alpha_{d,j} H_{\bar{\gamma}_{R_d}} \right\|_{\Phi} \geq c_{6.6} k/2.$$

Since $k \in \mathbf{N}$ is arbitrary, this proves (7.8). Thus the theorem holds in the case $p_\Phi = 1$.

(2) Suppose now that $q_\Phi = \infty$. In this case we need the condition that $\Phi$ is not equivalent to $\Phi_\infty$. This simply means that

$$\lim_{k\to\infty} \Phi(\{\overbrace{1,\dots,1}^{k},0,\dots,0,\dots\}) = \infty. \tag{7.11}$$

By Proposition 6.9, for each $d \in \mathbf{N}$ there is a $\varphi_d \in \mathcal{M}$ such that

$$\|H_{\varphi_d}\| = 1 \quad \text{while} \quad \|H_{\bar{\varphi}_d}\| \geq d.$$

Since the membership $\varphi_d \in \mathcal{M}$ implies $H_{\varphi_d} \in \mathcal{C}_1$, let $\nu(d)$ be the natural number for $H_{\varphi_d}$ provided by Lemma 7.3. That is,

$$\left\| \bigoplus_{j=1}^{\infty} \beta_j H_{\varphi_d} \right\|_{\Phi} \leq \Phi(\beta^{[\nu(d)]}) + \Phi(\beta). \tag{7.12}$$

for every $\beta = (\beta_1, \dots, \beta_j, \dots) \in \hat{c}$.

By the condition $q_\Phi = \infty$ and Lemma 7.2, for each $m \in \mathbf{N}$ there is a

$$\xi_m = \{\xi_{m,1}, \xi_{m,2}, \dots, \xi_{m,j}, \dots\} \in \hat{c}$$

satisfying the following conditions:

(1) $\xi_{m,j} \geq 0$ for every $j \in \mathbf{N}$.
(2) $\xi_{m,j} \geq \xi_{m,j+1}$ for every $j \in \mathbf{N}$.
(3) $\xi_{m,1} = 1$.
(4) $\Phi(\xi_m^{[m]}) \leq 2\Phi(\xi_m)$.

By (3) and (7.11), we have $\Phi(\xi_m^{[m]}) \to \infty$ as $m \to \infty$. Hence $\Phi(\xi_m) \to \infty$ as $m \to \infty$. Thus for each $d \in \mathbf{N}$, we can pick an $m(d) \in \mathbf{N}$ such that

$$m(d) \geq \nu(d) \quad \text{and} \quad \Phi(\xi_{m(d)}) \geq \|\varphi_d\|_\infty.$$

We now define

$$X = \bigoplus_{d=1}^{\infty} \frac{1}{d^2 \Phi(\xi_{m(d^3)})} \bigoplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\varphi_{d^3}} \quad \text{and} \quad Y = \bigoplus_{d=1}^{\infty} \frac{1}{d^2 \Phi(\xi_{m(d^3)})} \bigoplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\bar{\varphi}_{d^3}}.$$

We have $\|\varphi_{d^3}\|_\infty / \Phi(\xi_{m(d^3)}) \leq 1$ for every $d \in \mathbf{N}$. Thus, by Lemma 6.2, to complete the proof in the case $q_\Phi = \infty$, it suffices to show that $X \in \mathcal{C}_\Phi^{(0)}$ while $Y \notin \mathcal{C}_\Phi$.

Since $H_{\varphi_{d^3}} \in \mathcal{C}_1$ and $\xi_{m(d^3)} \in \hat{c}$, we have $\oplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\varphi_{d^3}} \in \mathcal{C}_1 \subset \mathcal{C}_\Phi^{(0)}$ for every $d \in \mathbf{N}$. By (7.12) and condition (4), for every $d \in \mathbf{N}$ we have

$$\left\| \bigoplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\varphi_{d^3}} \right\|_\Phi \le \Phi(\xi_{m(d^3)}^{[\nu(d^3)]}) + \Phi(\xi_{m(d^3)})$$

$$\le \Phi(\xi_{m(d^3)}^{[m(d^3)]}) + \Phi(\xi_{m(d^3)}) \le 3\Phi(\xi_{m(d^3)}).$$

Therefore $X \in \mathcal{C}_\Phi^{(0)}$.

On the other hand, since $\|H_{\bar{\varphi}_{d^3}}\| \ge d^3$, we have

$$\left\| \bigoplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\bar{\varphi}_{d^3}} \right\|_\Phi \ge d^3 \Phi(\xi_{m(d^3)})$$

for every $d \in \mathbf{N}$. Therefore

$$\|Y\|_\Phi \ge \frac{1}{d^2 \Phi(\xi_{m(d^3)})} \left\| \bigoplus_{j=1}^{\infty} \xi_{m(d^3),j} H_{\bar{\varphi}_{d^3}} \right\|_\Phi \ge \frac{d^3 \Phi(\xi_{m(d^3)})}{d^2 \Phi(\xi_{m(d^3)})} = d$$

for every $d \in \mathbf{N}$. Clearly, this means $Y \notin \mathcal{C}_\Phi$. This completes the proof. □

## 8. More on Boyd indices

We will now take a closer look at the condition $1 < p_\Phi \le q_\Phi < \infty$. We begin with a proposition which is essentially a known fact. The reason for presenting this proposition is that it sets the stage for the main result of the section, Theorem 8.2.

**Proposition 8.1.** *Let $\Phi$ be a symmetric gauge function such that $1 < p_\Phi \le q_\Phi < \infty$. Then for any $1 < r < s < \infty$ satisfying the condition $r < p_\Phi \le q_\Phi < s$, we have*

(8.1) $$\mathcal{C}_r \subset \mathcal{C}_\Phi \subset \mathcal{C}_s.$$

*Proof.* (1) We first prove the inclusion $\mathcal{C}_\Phi \subset \mathcal{C}_s$, which is essentially a repeat of the proof of Lemma 2.3. Indeed we pick a $q$ such that $q_\Phi < q < s$. By Lemma 2.2, there is a $c > 0$ such that $\Phi(\xi^{[m]}) \ge c m^{1/q} \Phi(\xi)$ for all $m \in \mathbf{N}$ and $\xi \in \hat{c}$. Since $s > q$, by [13, Lemma 3.1], there is a $0 < B < \infty$ such that

$$\Phi_s(\alpha) \le B\Phi(\alpha) \quad \text{for every } \alpha \in \hat{c}.$$

This obviously implies that $\mathcal{C}_\Phi \subset \mathcal{C}_s$.

(2) To prove the inclusion $\mathcal{C}_r \subset \mathcal{C}_\Phi$, we pick a $p$ such that $r < p < p_\Phi$. By the definition of $p_\Phi$, there is an $N \in \mathbf{N}$ such that

$$\frac{\log m}{\log\{\Phi(a^{[m]})/\Phi(a)\}} > p \quad \text{for all} \;\; m \ge N \;\; \text{and} \;\; a \in \hat{c}\backslash\{0\}.$$

Thus $\Phi(a^{[m]}) \le m^{1/p}\Phi(a)$ for all $m \ge N$ and $a \in \hat{c}$. Consequently,

$$\Phi(a^{[m]}) \le N m^{1/p}\Phi(a) \qquad \text{for all} \;\; m \in \mathbf{N} \;\; \text{and} \;\; a \in \hat{c}. \tag{8.2}$$

Using this fact, next we show that

$$N^{-1} m^{(p-1)/p}\Phi^*(b) \le \Phi^*(b^{[m]}) \qquad \text{for all} \;\; m \in \mathbf{N} \;\; \text{and} \;\; b \in \hat{c}. \tag{8.3}$$

Indeed given a $b = \{b_1, \dots, b_j, \dots\} \in \hat{c}$, since $\Phi^{**} = \Phi$, there is an $a = \{a_1, \dots, a_j, \dots\} \in \hat{c}$ with $\Phi(a) = 1$ such that $\Phi^*(b) = \sum_{j=1}^{\infty} b_j a_j$. Thus for any $m \in \mathbf{N}$, (8.2) gives us

$$m\Phi^*(b) = m\sum_{j=1}^{\infty} b_j a_j \le \Phi^*(b^{[m]})\Phi(a^{[m]}) \le N m^{1/p}\Phi^*(b^{[m]})\Phi(a) = N m^{1/p}\Phi^*(b^{[m]}).$$

Thus (8.3) holds. Note that (8.3) implies that $q_{\Phi^*} \le p/(p-1)$. Since $p/(p-1) < r/(r-1)$, by the argument in (1), there is a $0 < C < \infty$ such that

$$\Phi_{r/(r-1)}(a) \le C\Phi^*(a) \quad \text{for every} \;\; a \in \hat{c}. \tag{8.4}$$

Given any $x = \{x_1, \dots, x_j, \dots\} \in \hat{c}$, there is a $y = \{y_1, \dots, y_j, \dots\} \in \hat{c}$ with $\Phi^*(y) = 1$ such that $\Phi(x) = \sum_{j=1}^{\infty} x_j y_j$. Applying Hölder's inequality and (8.4), we have

$$\Phi(x) = \sum_{j=1}^{\infty} x_j y_j \le \Phi_r(x)\Phi_{r/(r-1)}(y) \le C\Phi_r(x)\Phi^*(y) = C\Phi_r(x).$$

That is, $\Phi(x) \le C\Phi_r(x)$ for every $x \in \hat{c}$. This implies that $\mathcal{C}_r \subset \mathcal{C}_\Phi$. □

Once we have Proposition 8.1, an obvious question asserts itself. Namely, if (8.1) holds for some $1 < r < s < \infty$, does it follow that $1 < p_\Phi \le q_\Phi < \infty$? Note that (8.1) gives both an upper bound and a lower bound on the size of the ideal $\mathcal{C}_\Phi$. Thus an affirmative answer to this question would say that the condition $1 < p_\Phi \le q_\Phi < \infty$ is solely determined by the size of $\mathcal{C}_\Phi$. But the truth is quite the opposite:

**Theorem 8.2.** *Given any $1 < r < s < \infty$, there is a symmetric gauge function $\Phi$ satisfying the conditions that*

$$\mathcal{C}_r \subset \mathcal{C}_\Phi \subset \mathcal{C}_s,$$

*that $q_\Phi = \infty$ and that $p_\Phi = 1$.*

*Proof.* Given any $1 < r < s < \infty$, we pick $p, q$ satisfying the condition

$$r < p < q < s.$$

It is elementary that $\mathcal{C}_r \subset \mathcal{C}_p^+$ and that $C_q^+ \subset \mathcal{C}_s$. Thus it suffices to find a symmetric gauge function $\Phi$ which satisfies the conditions that

$$\mathcal{C}_p^+ \subset \mathcal{C}_\Phi \subset \mathcal{C}_q^+, \tag{8.5}$$

that $q_\Phi = \infty$ and that $p_\Phi = 1$. We prefer to use the Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_q^+$ because they are easier to handle than Schatten classes.

To construct the desired $\Phi$, we begin with the fact that $q/p > 1$. This allows us to pick an $\alpha \in (0, \infty)$ such that $(q/p)^\alpha\{(q/p) - 1\} > 2$. With $\alpha$ so chosen, for each $j \in \mathbf{N}$ we let $k_j$ be the unique natural number satisfying the inequality

$$(q/p)^{\alpha+j} \le k_j < (q/p)^{\alpha+j} + 1.$$

The choice of $\alpha$ ensures that $k_{j+1} - k_j > 1$ for every $j \in \mathbf{N}$. For each $j \in \mathbf{N}$, let $N_j$ be the unique natural number satisfying the inequality

$$2^{qk_j} \le N_j < 2^{qk_j} + 1.$$

We now enumerate the sequence

$$1, \overbrace{\frac{1}{2^{k_1}}, \dots, \frac{1}{2^{k_1}}}^{N_1}, \overbrace{\frac{1}{2^{k_2}}, \dots, \frac{1}{2^{k_2}}}^{N_2}, \dots, \overbrace{\frac{1}{2^{k_j}}, \dots, \frac{1}{2^{k_j}}}^{N_j}, \dots,$$

in the descending order, as $\gamma_1, \gamma_2, \dots, \gamma_\nu, \dots$.

Obviously, we have $\gamma_1 = 1$, $\lim_{\nu\to\infty} \gamma_\nu = 0$ and $\sum_{\nu=1}^\infty \gamma_\nu = \infty$. That is, the sequence $\{\gamma_\nu\}$ is "binormalizing" [7, page 141]. Thus, according to [7, Section III.14], the formula

$$\Phi(a) = \sup_{\nu\in\mathbf{N}} \frac{|a_{\pi(1)}| + \cdots + |a_\pi(\nu)|}{\gamma_1 + \cdots + \gamma_\nu}, \quad a = \{a_1, \dots, a_\nu, \dots\} \in \hat{c}, \tag{8.6}$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \ge |a_{\pi(2)}| \ge \cdots \ge |a_{\pi(\nu)}| \ge \cdots$, defines a symmetric gauge function. Let us verify that this $\Phi$ satisfies (8.5) and has the properties that $q_\Phi = \infty$ and that $p_\Phi = 1$.

Consider any $\nu \in \mathbf{N}$ such that

$$1 + N_1 + \cdots + N_j < \nu \le 1 + N_1 + \cdots + N_j + N_{j+1} \tag{8.7}$$

for some $j \in \mathbf{N}$. Then by definition we have $\gamma_\nu = 2^{-k_{j+1}}$. Since $k_{i+1} - k_i > 1$ for every $i \in \mathbf{N}$, we have

$$\nu \le 1 + N_1 + \cdots + N_j + N_{j+1} \le 1 + 2(2^{qk_1} + \cdots + 2^{qk_j} + 2^{qk_{j+1}}) \le C2^{qk_{j+1}}.$$

Thus $\gamma_\nu \le (C/\nu)^{1/q}$ for such a $\nu$. This obviously implies that $\mathcal{C}_\Phi \subset \mathcal{C}_q^+$. For any $\nu \in \mathbf{N}$ satisfying (8.7), we have $\nu \ge N_j \ge 2^{qk_j}$. Thus

$$(1/\nu)^{1/p} \le 2^{-(q/p)k_j} = 2^{-k_{j+1}}2^{k_{j+1}-(q/p)k_j} \le \gamma_\nu 2^{k_{j+1}-(q/p)^{\alpha+j+1}} \le 2\gamma_\nu$$

for any $\nu \in \mathbf{N}$ satisfying (8.7), $j \in \mathbf{N}$. This obviously implies that $\mathcal{C}_p^+ \subset \mathcal{C}_\Phi$. Thus we have verified (8.5) for the symmetric gauge function $\Phi$ defined by (8.6).

To prove that $q_\Phi = \infty$, we observe that there is an $L \in \mathbf{N}$ such that

$$1 + N_1 + N_2 + \cdots + N_j \le 2^{qk_j + L}$$

for every $j \in \mathbf{N}$. Then note that

$$k_{j+1} - k_j \ge (q/p)^{\alpha+j+1} - (q/p)^{\alpha+j} - 1 = ((q/p) - 1)(q/p)^{\alpha+j} - 1. \tag{8.8}$$

Thus there is a $J \in \mathbf{N}$ such that if $j \ge J$, then $(q-1)(k_{j+1} - k_j) \ge L + 3$. For each $j \ge J$, we define $\ell_j$ to be the largest natural number satisfying the condition

$$\ell_j + L + 2 \le (q-1)(k_{j+1} - k_j). \tag{8.9}$$

Since $\ell_j$ is the largest of such natural number, it follows from (8.8) that $\ell_j \to \infty$ as $j \to \infty$. Obviously, (8.9) implies $1 + k_{j+1} + (q-1)k_j \le qk_{j+1} - \ell_j - L - 1$, $j \ge J$. Therefore for each $j \ge J$, there is a natural number $d_j \in \mathbf{N}$ such that

$$k_{j+1} + (q-1)k_j \le d_j \le qk_{j+1} - \ell_j - L - 1. \tag{8.10}$$

With this $d_j$ we now define the element

$$u_j = \{ \overbrace{1, \ldots\ldots\ldots, 1}^{1+N_1+\cdots+N_j+2^{d_j}}, 0, \ldots, 0, \ldots \}$$

in $\hat{c}$, $j \ge J$. We will show that there is a $1 < C_1 < \infty$ such that

$$\Phi(u_j^{[2^{\ell_j}]}) \le C_1 \Phi(u_j) \quad \text{for every} \quad j \ge J + 1. \tag{8.11}$$

Since $\ell_j \to \infty$ as $j \to \infty$, this obviously implies that $q_\Phi = \infty$. Note that (8.11) will follow if we can find constants $0 < c < \infty$ and $0 < C_2 < \infty$ such that

$$\Phi(u_j) \ge c 2^{k_{j+1}} \quad \text{while} \tag{8.12}$$

$$\Phi(u_j^{[2^{\ell_j}]}) \le C_2 2^{k_{j+1}} \tag{8.13}$$

for every $j \ge J + 1$.

For each $j \ge J$ we have

$$\Phi(u_j) \ge \frac{2^{d_j}}{\sum_{\nu=1}^{1+N_1+\cdots N_j+2^{d_j}} \gamma_\nu} \ge \frac{2^{d_j}}{1 + 2\sum_{i=1}^{j} 2^{(q-1)k_i} + 2^{d_j - k_{j+1}}}. \tag{8.14}$$

Since $k_{i+1} - k_i > 1$ for every $i \in \mathbf{N}$, we have $1 + 2\sum_{i=1}^{j} 2^{(q-1)k_i} \le C_3 2^{(q-1)k_j}$, $j \ge J$. By (8.10), we have $d_j - k_{j+1} \ge (q-1)k_j$, $j \ge J$. Hence (8.12) follows from (8.14).

To prove (8.13), denote $M_j = 1 + N_1 + \cdots + N_j + 2^{d_j}$ for each $j \geq J+1$. Then

$$u_j^{[2^{\ell_j}]} = \{\overbrace{1, \ldots, 1}^{2^{\ell_j} M_j}, 0, \ldots, 0, \ldots\},$$

and consequently

$$\Phi(u_j^{[2^{\ell_j}]}) = \max_{1 \leq \nu \leq 2^{\ell_j} M_j} \frac{\nu}{\gamma_1 + \cdots + \gamma_\nu}, \tag{8.15}$$

$j \geq J+1$. First, consider $\nu \in \mathbf{N}$ satisfying the condition

$$1 + N_1 + \cdots + N_i < \nu \leq 1 + N_1 + \cdots + N_i + N_{i+1},$$

where $1 \leq i \leq j-1$, $j \geq J+1$. In this case, we have $\nu = 1 + N_1 + \cdots + N_i + \nu_0$ for some $1 \leq \nu_0 \leq N_{i+1}$. Thus

$$\begin{aligned} \frac{\nu}{\gamma_1 + \cdots + \gamma_\nu} &\leq \frac{1 + N_1 + \cdots + N_i + \nu_0}{1 + \sum_{\mu=1}^{i} 2^{(q-1)k_\mu} + \frac{\nu_0}{2^{k_{i+1}}}} \leq \frac{2^{qk_i + L} + \nu_0}{2^{(q-1)k_i} + \frac{\nu_0}{2^{k_{i+1}}}} \\ &= \frac{2^{qk_i+L}}{2^{(q-1)k_i} + \frac{\nu_0}{2^{k_{i+1}}}} + \frac{\nu_0}{2^{(q-1)k_i} + \frac{\nu_0}{2^{k_{i+1}}}} \leq 2^{k_i + L} + 2^{k_{i+1}} \leq C_4 2^{k_j}. \end{aligned} \tag{8.16}$$

Now consider $\nu$ satisfying the condition

$$1 + N_1 + \cdots + N_j < \nu \leq 2^{\ell_j} M_j. \tag{8.17}$$

First of all, note that since $d_j \geq (q-1)k_j + k_{j+1} > qk_j$, we have

$$2^{\ell_j} M_j \leq 2^{\ell_j}(2^{qk_j + L} + 2^{d_j}) \leq 2^{\ell_j + 1} 2^{d_j + L} = 2^{d_j + \ell_j + L + 1} \leq 2^{qk_{j+1}} \leq N_{j+1}.$$

Thus if $1 + N_1 + \cdots + N_j < i \leq 2^{\ell_j} M_j$, then $\gamma_i = 2^{-k_{j+1}}$. Hence if $\nu$ satisfies (8.17), then $\nu = 1 + N_1 + \cdots + N_j + \nu_0$ for some $1 \leq \nu_0 \leq 2^{\ell_j} M_j - (1 + N_1 + \cdots + N_j)$, and

$$\begin{aligned} \frac{\nu}{\gamma_1 + \cdots + \gamma_\nu} &\leq \frac{1 + N_1 + \cdots + N_j + \nu_0}{1 + \sum_{\mu=1}^{j} 2^{(q-1)k_\mu} + \frac{\nu_0}{2^{k_{j+1}}}} \leq \frac{2^{qk_j + L} + \nu_0}{2^{(q-1)k_j} + \frac{\nu_0}{2^{k_{j+1}}}} \\ &= \frac{2^{qk_j+L}}{2^{(q-1)k_j} + \frac{\nu_0}{2^{k_{j+1}}}} + \frac{\nu_0}{2^{(q-1)k_j} + \frac{\nu_0}{2^{k_{j+1}}}} \leq 2^{k_j + L} + 2^{k_{j+1}} \leq C_4 2^{k_{j+1}}. \end{aligned} \tag{8.18}$$

Combining (8.15), (8.16) and (8.18), we obtain (8.13). Hence $q_\Phi = \infty$.

Last but not least, we need to verify that $p_\Phi = 1$. To do that, we define the element

$$v_j = \{\overbrace{1, \ldots\ldots, 1}^{1 + N_1 + \cdots + N_j}, 0, \ldots, 0, \ldots\}$$

in $\hat{c}$ for each $j \geq J$. Then

$$\Phi(v_j) = \max_{1 \leq \nu \leq K_j} \frac{\nu}{\gamma_1 + \cdots + \gamma_\nu}$$

for each $j \geq J$, where we write $K_j = 1 + N_1 + \cdots + N_j$. Thus it follows from (8.16) that

$$\Phi(v_j) \leq C_5 2^{k_j} \quad \text{for every} \quad j \geq J+1. \tag{8.19}$$

For each $j \geq J$, let $m_j$ be the smallest natural number satisfying the inequality

$$(q-1)k_j + k_{j+1} \leq m_j.$$

By (8.10), we have $m_j \leq d_j$ for every $j \geq J$. Thus

$$2^{k_{j+1}-k_j} K_j \geq 2^{k_{j+1}-k_j} 2^{qk_j} = 2^{k_{j+1}+(q-1)k_j} \geq 2^{m_j-1}$$

for every $j \geq J$. Also,

$$2^{k_{j+1}-k_j} K_j \leq 2^{k_{j+1}-k_j} 2^{qk_j+L} = 2^{k_{j+1}+(q-1)k_j+L} \leq 2^{m_j+L} \leq 2^{qk_{j+1}} \leq N_{j+1}$$

for every $j \geq J$, where the third $\leq$ follows from the inequality $m_j \leq d_j$ and (8.10). It follows from the above two inequalities that

$$\begin{aligned}
\Phi(v_j^{[2^{k_{j+1}-k_j}]}) &\geq \frac{2^{k_{j+1}-k_j} K_j}{\sum_{\nu=1}^{2^{k_{j+1}-k_j} K_j} \gamma_\nu} \geq \frac{2^{m_j-1}}{1 + 2\sum_{i=1}^{j} 2^{(q-1)k_i} + \frac{2^{k_{j+1}-k_j} K_j - K_j}{2^{k_{j+1}}}} \\
&\geq \frac{2^{m_j-1}}{C_6 2^{(q-1)k_j} + \frac{2^{m_j+L}}{2^{k_{j+1}}}} \geq \frac{2^{m_j-1}}{C_6 2^{(q-1)k_j} + \frac{2^{(q-1)k_j+k_{j+1}+1+L}}{2^{k_{j+1}}}} \\
&\geq c_1 \frac{2^{m_j}}{2^{(q-1)k_j}} \geq c_1 2^{k_{j+1}} = c_1 2^{k_{j+1}-k_j} \cdot 2^{k_j} \geq (c_1/C_5) 2^{k_{j+1}-k_j} \Phi(v_j)
\end{aligned} \tag{8.20}$$

for every $j \geq J+1$, where for the last step we apply (8.19). By (8.8), we have $k_{j+1} - k_j \to \infty$ as $j \to \infty$. It is now straightforward to deduce from (8.20) that $p_\Phi = 1$. □

## Data availability

No data was used for the research described in the article.

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