HANKEL OPERATORS ON WEIGHTED BERGMAN SPACES AND NORM IDEALS

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Abstract. Consider Hankel operators H_f on the weighted Bergman space $L^2_a(\mathbf{B}, dv_\alpha)$. In this paper we characterize the membership of $(H_f^* H_f)^{s/2} = |H_f|^s$ in the norm ideal $\mathcal{C}_\Phi$, where $0 < s \leq 1$ and the symmetric gauge function Φ is allowed to be arbitrary.

1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. Write dv for the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. For each $-1 < \alpha < \infty$, we define the weighted measure

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

on $\mathbf{B}$, where the coefficient c_α is chosen so that $v_\alpha(\mathbf{B}) = 1$. Recall that the weighted Bergman space $L^2_a(\mathbf{B}, dv_\alpha)$ is defined to be the subspace

$$\{h \in L^2(\mathbf{B}, dv_\alpha) : h \text{ is analytic on } \mathbf{B}\}$$

of $L^2(\mathbf{B}, dv_\alpha)$. The orthogonal projection from $L^2(\mathbf{B}, dv_\alpha)$ onto $L^2_a(\mathbf{B}, dv_\alpha)$ is given by

$$(Pf)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}} dv_\alpha(w), \quad f \in L^2(\mathbf{B}, dv_\alpha).$$

Note that this integral formula defines Pf as a function even for $f \in L^1(\mathbf{B}, dv_\alpha)$. Although P is obviously α dependent, for the sake of simplicity we intentionally omit the weight of the space in the notation for this projection.

Given an appropriate symbol function f, the Hankel operator $H_f : L^2_a(\mathbf{B}, dv_\alpha) \to L^2(\mathbf{B}, dv_\alpha) \ominus L^2_a(\mathbf{B}, dv_\alpha)$ is defined by the formula

$$H_f h = fh - P(fh),$$

$h \in L^2_a(\mathbf{B}, dv_\alpha)$. A subject of intense research interest, the theory of Hankel operators can be conveniently divided into two natural components. Because of the relation

$$[M_f, P] = H_f - H^*_{\bar{f}},$$

the simultaneous study of the pair of Hankel operators H_f and $H_{\bar{f}}$ is equivalent to the study of the commutator $[M_f, P]$. Results that simultaneous concern the pair H_f, $H_{\bar{f}}$ are

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often called the "two-sided" theory of Hankel operators, of which we cite [1,8,11,17,20] as typical examples. In particular, [17] has attracted much recent attention.

By contrast, the study of H_f alone is often called the "one-sided" theory of Hankel operators, which presents its unique challenges. As examples of "one-sided" theory in the Bergman space case, let us cite [13-16]. Recall that in these papers, Li and Luecking characterized the boundedness, compactness and Schatten-class membership of H_f. Building on these results, in this paper we will take the logical next step. Namely, we will determine exactly when the operator $|H_f|^s = (H_f^*H_f)^{s/2}$ belongs to the norm ideal $\mathcal{C}_\Phi$, where $0 < s \leq 1$ and the symmetric gauge function Φ is allowed to be arbitrary.

Before going any further, a brief review of "symmetric gauge functions" and the associated "norm ideals" will be beneficial. Throughout the paper, [10] will be our standard reference in this connection. Following [10], let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A symmetric gauge function (also called *symmetric norming function*) is a map

$$\Phi : \hat{c} \to [0, \infty)$$

that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \dots, 0, \dots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [10,page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \dots, s_j(A), 0, \dots, 0, \dots\}) \tag{1.1}$$

for bounded operators. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{1.2}$$

is a norm ideal [10,page 68]. This term refers to the following properties of $\mathcal{C}_\Phi$:

- For any B, $C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \leq \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \leq \|A\|_\Phi$, and the equality holds when $\mathrm{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|.\|_\Phi$.

There are many familiar examples of symmetric gauge functions. For each $1 \leq p < \infty$, the formula $\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = (\sum_{j=1}^\infty |a_j|^p)^{1/p}$ defines a symmetric gauge function on $\hat{c}$, and the corresponding ideal $\mathcal{C}_{\Phi_p}$ defined by (1.2) is just the Schatten class $\mathcal{C}_p$. As another family of examples, let us mention the symmetric gauge function Φ_p^- defined by the formula

$$\Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|a_{\pi(j)}|}{j^{(p-1)/p}}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \cdots \geq |a_{\pi(j)}| \geq \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. In this case, the ideal $\mathcal{C}_{\Phi_p^-}$ defined by (1.2) is called a Lorentz ideal and often simply denoted by the symbol $\mathcal{C}_p^-$. When $p = 1$, $\mathcal{C}_1^-$ is just the trace class $\mathcal{C}_1$. But when $1 < p < \infty$, $\mathcal{C}_p^-$ is strictly smaller than the Schatten class $\mathcal{C}_p$. Moreover, when $1 < p < \infty$, the dual $\mathcal{C}^+_{p/(p-1)}$ of $\mathcal{C}_p^-$ is a norm ideal with interesting properties of its own [10].

Given a symmetric gauge Φ, it is a common practice to extend its domain of definition beyond the space $\hat{c}$. Suppose that $\{b_j\}_{j\in\mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., the set $\{j \in \mathbf{N} : b_j \neq 0\}$ is not necessarily finite. Then we define

$$\Phi(\{b_j\}_{j\in\mathbf{N}}) = \sup_{k\geq 1} \Phi(\{b_1, \dots, b_k, 0, \dots, 0, \dots\}). \tag{1.3}$$

Thus if A is a bounded operator, then $\|A\|_\Phi = \Phi(\{s_j(A)\}_{j\in\mathbf{N}})$. For each $0 < p < \infty$, the singular numbers of $|A|^p = (A^*A)^{p/2}$ are $\{(s_1(A))^p, \dots, (s_j(A))^p, \dots\}$, and therefore

$$\||A|^p\|_\Phi = \Phi(\{(s_j(A))^p\}_{j\in\mathbf{N}}). \tag{1.4}$$

For an unbounded operator X, it is consistent with [10,Theorem II.7.1] to interpret all its singular numbers as infinity. Therefore it is consistent with (1.4) to adopt the convention that $\||X|^p\|_\Phi = \infty$ for all $0 < p < \infty$ whenever the operator X is unbounded.

For our purpose we also need to deal with sequences indexed by sets other than $\mathbf{N}$. If W is a countable, infinite set, then we define

$$\Phi(\{b_\alpha\}_{\alpha\in W}) = \Phi(\{b_{\pi(j)}\}_{j\in\mathbf{N}}),$$

where $\pi : \mathbf{N} \to W$ is any bijection. The definition of symmetric gauge functions guarantees that the value of $\Phi(\{b_\alpha\}_{\alpha\in W})$ is independent of the choice of the bijection π. For a finite index set $F = \{x_1, \dots, x_\ell\}$, we simply define $\Phi(\{b_x\}_{x\in F}) = \Phi(\{b_{x_1}, \dots, b_{x_\ell}, 0, \dots, 0, \dots\})$.

Recall that the membership of the commutator $[M_f, P] = H_f - H^*_{\bar{f}}$ in $\mathcal{C}_\Phi$ was characterized in [20] for arbitrary symmetric gauge functions Φ, although in [20] the weight of the Bergman space was set at $\alpha = 0$. This paper deals with the corresponding "one-sided" problem for arbitrary weight $-1 < \alpha < \infty$, and we will go a little farther by introducing the power $0 < s \leq 1$ mentioned earlier.

The statement of our result involves modified kernel functions and the Bergman metric, which we will now review. First of all, the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{(n+1+\alpha)/2}}{(1 - \langle \zeta, z\rangle)^{n+1+\alpha}}, \quad z, \zeta \in \mathbf{B}, \tag{1.5}$$

gives us the normalized reproducing kernel for $L^2_a(\mathbf{B}, dv_\alpha)$. For each integer $i \geq 0$, we define the modified kernel function

$$\psi_{z,i}(\zeta) = \frac{(1 - |z|^2)^{\{(n+1+\alpha)/2\}+i}}{(1 - \langle \zeta, z\rangle)^{n+1+\alpha+i}}, \quad z, \zeta \in \mathbf{B}. \tag{1.6}$$

If we introduce the multiplier

$$m_z(\zeta) = \frac{1-|z|^2}{1-\langle \zeta, z\rangle} \tag{1.7}$$

for each $z \in \mathbf{B}$, then we have the relation $\psi_{z,i} = m_z^i k_z$. Similar to the analogous situations in the Hardy space and the Drury-Arveson space [5-7], this modification gives $\psi_{z,i}$ a faster "decaying rate" than k_z, which will allow us to establish certain crucial bounds.

Let β denote the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z, w \in \mathbf{B},$$

where φ_z is the Möbius transform of $\mathbf{B}$ [18,Section 2.2]. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z,a) = \{w \in \mathbf{B} : \beta(z,w) < a\}$.

Definition 1.1. [20,Definition 1.1] (i) Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) Let $0 < a < b < \infty$. A subset Γ of $\mathbf{B}$ is said to be an a,b-lattice if it is a-separated and has the property $\cup_{z\in\Gamma} D(z,b) = \mathbf{B}$.

Given an operator A, for example a Toeplitz operator or a Hankel operator, one is always interested in formulas for its set of singular numbers. But as a practical matter, a formula that is both explicit and exact, is usually not available. Thus one is frequently forced to search for alternatives: are there quantities given by simple formulas that are *equivalent* to $\{s_1(A), s_2(A), \dots, s_j(A), \dots\}$ in some clearly-defined sense?

In this general context, our investigation stems from the following intuition: if i is suitably large, i.e., if $\psi_{z,i}$ "decays fast enough", then for an a,b-lattice Γ in $\mathbf{B}$, the set of scalar quantities

$$\{\|H_f\psi_{z,i}\|\}_{z\in\Gamma}$$

should be equivalent to the set of singular numbers $\{s_1(H_f), s_2(H_f), \dots, s_j(H_f), \dots\}$ of the Hankel operator H_f. The main result of this paper confirms our intuition in a very specific way: if one allows a constant multiple, then the s-powers of these two sets of numbers are not distinguishable by the application of symmetric gauge functions.

Theorem 1.2. *Let $0 < s \le 1$ be given, and let $i \in \mathbf{Z}_+$ satisfy the condition $s(n+1+\alpha+2i) > 2n$. Let $0 < a < b < \infty$ be positive numbers such that $b \ge 2a$. Then there exist constants $0 < c \le C < \infty$ which depend only on the given s, i, a, b, the complex dimension n and the weight α such that the inequality*

$$c\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma}) \le \||H_f|^s\|_\Phi \le C\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma})$$

holds for every $f \in L^2(\mathbf{B}, dv_\alpha)$, every symmetric gauge function Φ and every a,b-lattice Γ in $\mathbf{B}$.

The reader may wonder, why does Theorem 1.2 only cover the powers $0 < s \le 1$? The simple answer is, we could consider all $0 < s < \infty$, but that would not add anything. The point is this: if Φ is a symmetric gauge function, then for each $1 < p < \infty$ the formula

$$\{a_j\}_{j\in\mathbf{N}} \mapsto \left(\Phi(\{|a_j|^p\}_{j\in\mathbf{N}})\right)^{1/p}$$

defines just another symmetric gauge function on $\hat{c}$, which Theorem 1.2 already covers. That is why we only *need to* consider $0 < s \le 1$.

The proof of Theorem 1.2 involves a somewhat complicated scheme. To conclude the Introduction, let us outline the main steps in the proof.

For both directions in Theorem 1.2, it is necessary to control the projection $1 - P$ by certain differential operators. This will be achieved in terms of the inequality

$$\|f - Pf\| \le C(\|\rho\bar{\partial}f\| + \|\rho^{1/2}\bar{\partial}f \wedge \bar{\partial}\rho\|) \tag{1.8}$$

for $f \in C^\infty(\mathbf{B}) \cap L^2(\mathbf{B}, dv_\alpha)$, which will be the main content of Section 2.

As one would expect, the proof of Theorem 1.2 uses properties of symmetric gauge functions and symmetric norms extensively. For that reason we begin Section 3 with a review of these properties. Another key ingredient in the proof is a workable decomposition system for the unit ball. For this we adopt the decomposition system from [20], which gives us the sets $T_{k,j}$ and $Q_{k,j}$, $(k, j) \in I$. Accordingly, we define the quantities $A(f; Q_{k,j})$, $(k, j) \in I$, for $f \in L^2(\mathbf{B}, dv_\alpha)$. With this decomposition system we have

$$\Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma}) \le C\Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}) \tag{1.9}$$

if Γ is a-separated for some $a > 0$. In (1.9), the integer $i \in \mathbf{Z}_+$ must satisfy the condition $s(n + 1 + \alpha + 2i) > 2n$, and that is why there is such a requirement in Theorem 1.2.

Section 4 is one of the two major steps, which shows that

$$\left\|\left|M_f \sum_{z\in\Gamma} \psi_{z,i'} \otimes e_z\right|^s\right\|_\Phi \le C\Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}), \tag{1.10}$$

where i' is appropriately large and $\{e_z : z \in \Gamma\}$ is an orthonormal set. Then, by using the atomic decomposition for $L^2_a(\mathbf{B}, dv_\alpha)$, in Section 5 we show that (1.10) implies

$$\||M_f P|^s\|_\Phi \le C\Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}). \tag{1.11}$$

In Section 6, we adopt ideas from [15,16] and introduce the local projections $P_{k,j}$, which have certain amazing properties. With the local projections $P_{k,j}$ we can define "analytic oscillations" $M(f; k, j)$ for a given symbol function f. Then, using Luecking's ideas in [16], we show that f admits a decomposition $f = f^{(1)} + f^{(2)}$ such that

$$\begin{cases} A(f^{(1)}; Q_{k,j}), \quad A(\rho|\bar{\partial}f^{(2)}|; Q_{k,j}), \quad A(\rho^{1/2}|\bar{\partial}f^{(2)} \wedge \bar{\partial}\rho|; Q_{k,j}) \\ \qquad\qquad \text{can be controlled by} \ \ \{M(f; k, j) : (k, j) \in I\} \end{cases} . \tag{1.12}$$

It is then easy to deduce from (1.8), (1.11) and (1.12) that

$$\||H_f|^s\|_\Phi \le C\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}).$$

This essentially proves the upper bound in Theorem 1.2, for it is routine to show that

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \le C\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma})$$

if Γ has the property that $\cup_{z\in\Gamma}D(z,b) = \mathbf{B}$ for some $0 < b < \infty$.

For the proof of the lower bound in Theorem 1.2, the most crucial step is Proposition 6.8, which establishes the inequality

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \le C\||H_f|^s\|_\Phi. \tag{1.13}$$

Then, from (1.12), (1.9) and (1.8) we deduce

$$\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma}) \le C\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}). \tag{1.14}$$

Obviously, the lower bound in Theorem 1.2 follows from (1.13) and (1.14).

To summarize, Sections 2-6 contain the technical steps outlined above, and the proof of Theorem 1.2 itself is formally completed in Section 7. Finally, the Appendix at the end of the paper contains technical proofs that are judged to be either similar to what can be found in the literature, or too elementary for the main text.

2. Projection and d-bar operators

We begin by recalling a particular integral estimate on $\mathbf{B}$. As in [3], define

$$\Delta(\zeta,z) = |1-\langle\zeta,z\rangle|^2 - (1-|\zeta|^2)(1-|z|^2), \quad \zeta,z\in\mathbf{B}.$$

Lemma 2.1. [3,Lemma 24] *Let $a,b,c,t\in\mathbf{R}$. If $c > -2n$ and $-2a < t+1 < 2b+2$, then the operator*

$$(Tf)(z) = \int \frac{(1-|z|^2)^a(1-|\zeta|^2)^b\Delta^{c/2}(\zeta,z)}{|1-\langle\zeta,z\rangle|^{n+1+a+b+c}}f(\zeta)dv(\zeta)$$

is bounded on $L^2(\mathbf{B},dv_t)$.

For any $f\in C^\infty(\mathbf{B})$, let $\bar\partial f$ denote the $(0,1)$-form $\sum_{j=1}^n(\bar\partial_jf)(\zeta)d\bar\zeta_j$ as usual. Write

$$|(\bar\partial f)(\zeta)| = \{|(\bar\partial_1f)(\zeta)|^2+\cdots+|(\bar\partial_nf)(\zeta)|^2\}^{1/2}$$

for $\zeta\in\mathbf{B}$. If φ is a scalar function on $\mathbf{B}$, then by $\|\varphi\bar\partial f\|$ we mean the norm of the scalar function $\varphi|\bar\partial f|$ in $L^2(\mathbf{B},dv_\alpha)$, allowing the possibility that $\|\varphi\bar\partial f\| = \infty$. For any (p,q)-form F on $\mathbf{B}$, $|F(\zeta)|$ and $\|\varphi F\|$ are similarly defined.

Let us write

$$\rho(\zeta) = 1 - |\zeta|^2 \quad \text{for} \quad \zeta \in \mathbf{B},$$

and this notation will be fixed for the rest of the paper.

Proposition 2.2. *There is a constant $C_{2.2}$ which depends only on n and α such that*

$$\|f - Pf\| \le C_{2.2}(\|\rho\bar{\partial}f\| + \|\rho^{1/2}\bar{\partial}f \wedge \bar{\partial}\rho\|) \tag{2.1}$$

for every $f \in C^\infty(\mathbf{B}) \cap L^2(\mathbf{B}, dv_\alpha)$.

Proof. Estimates of this type are more or less well known by now. But because of the importance of (2.1) in this paper, we will go through its proof anyway.

The standard proof is to solve a $\bar{\partial}$-problem using Charpentier's solution formula [2]. Pick an integer $k \ge \alpha + 3$. First we consider any $f \in C_c^\infty(\mathbf{B})$. Recall from [2] that the function

$$u(z) = \int (\bar{\partial}f)(\zeta) \wedge C_k(\zeta, z)$$

solves the equation $\bar{\partial}u = \bar{\partial}f$ on $\mathbf{B}$. Thus $f - u$ is analytic on $\mathbf{B}$ and $\|f - Pf\| = \|u - Pu\| \le \|u\|$. Hence it suffices to estimate $\|u\|$. For this we use the explicit decomposition of $C_k(\zeta, z)$ given on pages 136-138 in [2]:

$$C_k(\zeta, z) = C_k^{(1)}(\zeta, z) + C_k^{(2)}(\zeta, z),$$

where

$$C_k^{(1)}(\zeta, z) = \Psi_k(\zeta, z)\sum_{i=1}^{n}(-1)^{i-1}(\bar{\zeta}_i(1 - \langle\zeta, z\rangle) - \bar{z}_i(1 - |\zeta|^2)) \wedge_{j\ne i} d\bar{\zeta}_j \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$
$$C_k^{(2)}(\zeta, z) = \Psi_k(\zeta, z)\sum_{i<j}(-1)^{i+j}(\bar{\zeta}_i\bar{z}_j - \bar{\zeta}_j\bar{z}_i)(\bar{\partial}\rho)(\zeta) \wedge_{\ell\ne i,j} d\bar{\zeta}_\ell \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

The scalar function $\Psi_k(\zeta, z)$ above has the form

$$\Psi_k(\zeta, z) = c_n\psi_k(\zeta, z)\frac{(1 - \langle\zeta, z\rangle)^{n-1}}{\Delta^n(\zeta, z)} \quad \text{with} \quad |\psi_k(\zeta, z)| \le C\left(\frac{1 - |\zeta|^2}{|1 - \langle z, \zeta\rangle|}\right)^k.$$

To estimate $\|u\|$, we write $u = u^{(1)} + u^{(2)}$, where

$$u^{(\nu)}(z) = \int (\bar{\partial}f)(\zeta) \wedge C_k^{(\nu)}(\zeta, z),$$

$\nu = 1, 2$. To estimate $\|u^{(1)}\|$, note that

$$\bar{\zeta}_i(1 - \langle\zeta, z\rangle) - \bar{z}_i(1 - |\zeta|^2) = (\bar{\zeta}_i - \bar{z}_i)(1 - \langle\zeta, z\rangle) + \bar{z}_i\langle\zeta, \zeta - z\rangle.$$

It is obvious that $|\langle\zeta,\zeta-z\rangle| \le \Delta^{1/2}(\zeta,z)$ [3,page 508]. Since $1-\langle\zeta,z\rangle = 1-|\zeta|^2+\langle\zeta,\zeta-z\rangle$, the formulas given there also show that $|\zeta-z||1-\langle\zeta,z\rangle| \le C_1\Delta^{1/2}(\zeta,z)$. Hence

$$
\begin{aligned}
|u^{(1)}(z)| &\le C_2\int \frac{(1-|\zeta|^2)^k\Delta^{(-2n+1)/2}(\zeta,z)}{|1-\langle\zeta,z\rangle|^{k-n+1}}|(\bar\partial f)(\zeta)|dv(\zeta)\\
&= C_2\int \frac{(1-|\zeta|^2)^{k-1}\Delta^{(-2n+1)/2}(\zeta,z)}{|1-\langle\zeta,z\rangle|^{n+1+k-1+(-2n+1)}}\rho(\zeta)|(\bar\partial f)(\zeta)|dv(\zeta).
\end{aligned}
$$

Since $\alpha > -1$, we have $\alpha+1>0$. If we let $a=0$, $b=k-1$, $c=-2n+1$, and $t=\alpha$, then $-2a=0<t+1<2b+2$ and $c>-2n$. Applying Lemma 2.1 with these parameters, we find that

$$\int |u^{(1)}(z)|^2dv_\alpha(z) \le C_3\int \rho^2(\zeta)|(\bar\partial f)(\zeta)|^2dv_\alpha(\zeta) = C_3\|\rho\bar\partial f\|^2.$$

Thus we have $\|u^{(1)}\| \le C_3^{1/2}\|\rho\bar\partial f\|$.

To estimate $\|u^{(2)}\|$, note that

$$|\bar\zeta_i\bar z_j-\bar\zeta_j\bar z_i| = |(\bar\zeta_i-\bar z_i)\bar z_j-(\bar\zeta_j-\bar z_j)\bar z_i| \le 2|\zeta-z|. \tag{2.2}$$

From the formulas on page 508 in [3] we deduce that $|\zeta-z| \le C_4\Delta^{1/4}(\zeta,z)$. Hence

$$
\begin{aligned}
|u^{(2)}(z)| &\le C_5\int \frac{(1-|\zeta|^2)^k\Delta^{(-2n+(1/2))/2}(\zeta,z)}{|1-\langle\zeta,z\rangle|^{k-n+1}}|(\bar\partial f\wedge\bar\partial\rho)(\zeta)|dv(\zeta)\\
&= C_5\int \frac{(1-|\zeta|^2)^{k-(1/2)}\Delta^{(-2n+(1/2))/2}(\zeta,z)}{|1-\langle\zeta,z\rangle|^{n+1+k-(1/2)+(-2n+(1/2))}}\rho^{1/2}(\zeta)|(\bar\partial f\wedge\bar\partial\rho)(\zeta)|dv(\zeta).
\end{aligned}
$$

Now we set $a=0$, $b=k-(1/2)$, $c=-2n+(1/2)$, and $t=\alpha$. For these numbers we have $-2a=0<t+1<2b+2$ and $c>-2n$. Similar to the argument above, another application of Lemma 2.1 gives us $\|u^{(2)}\| \le C_6^{1/2}\|\rho^{1/2}\bar\partial f\wedge\bar\partial\rho\|$.

Thus we have proved (2.1) in the special case where $f\in C_c^\infty(\mathbf{B})$. To prove the general case, pick a C^∞ function η on $[0,\infty)$ such that $\eta=0$ on $[0,1]$, $\eta=1$ on $[2,\infty)$ and $0\le\eta\le1$ on $[1,2]$. For each $k\in\mathbf{N}$, define the function

$$h_k(\zeta)=\eta(k\rho(\zeta)),\quad \zeta\in\mathbf{B}.$$

Let an arbitrary $f\in C^\infty(\mathbf{B})\cap L^2(\mathbf{B},dv_\alpha)$ be given. Since $h_k(\zeta)=0$ when $1-|\zeta|^2\le1/k$, we have $h_kf\in C_c^\infty(\mathbf{B})$. Thus by the special case that we have already proved,

$$\|h_kf-P(h_kf)\| \le C_{2.2}(\|\rho\bar\partial(h_kf)\|+\|\rho^{1/2}\bar\partial(h_kf)\wedge\bar\partial\rho\|).$$

We have $\bar\partial(h_kf)=h_k\bar\partial f+f\bar\partial h_k$ and $\|h_k\|_\infty\le1$. It is obvious that $\bar\partial h_k\wedge\bar\partial\rho=0$. Hence

$$\|h_kf-P(h_kf)\| \le C_{2.2}(\|\rho\bar\partial f\|+\|f\rho\bar\partial h_k\|+\|\rho^{1/2}\bar\partial f\wedge\bar\partial\rho\|). \tag{2.3}$$

Moreover, $\rho\bar{\partial}h_k = k\rho\eta'(k\rho)\bar{\partial}\rho$ and $\eta'(k\rho(\zeta)) \neq 0$ only if $1 \leq k\rho(\zeta) \leq 2$. Therefore $\rho|\bar{\partial}h_k| \leq 2\|\eta'\|_\infty\chi_{H_k}$, where $H_k = \{\zeta : 1/k \leq 1-|\zeta|^2 \leq 2/k\}$. Thus $\|f\rho\bar{\partial}h_k\| \leq 2\|\eta'\|_\infty\|f\chi_{H_k}\| \to 0$ as $k \to \infty$. Taking the limit $k \to \infty$ in (2.3), the general case of (2.1) follows. □

Remark 2.3. Obviously, the above proof was meant for complex dimensions $n \geq 2$. When $n = 1$, one simply interprets $\|\rho^{1/2}\bar{\partial}f \wedge \bar{\partial}\rho\|$ as 0 and (2.1) still holds. In fact, the case $n = 1$ is much simpler, because $\bar{\partial}$-closedness is no longer an issue. To prove (2.1) in the case $n = 1$, one solves the $\bar{\partial}$-problem by the simple formula

$$u(z) = \frac{1}{2\pi i}\int \frac{1}{z-\zeta}\left(\frac{1-|\zeta|^2}{1-z\bar{\zeta}}\right)^k (\bar{\partial}f)(\zeta) \wedge d\zeta,$$

where the integration takes place on the unit disc in **C** [9,page 319]. For a sufficiently large $k \in \mathbf{N}$, one obtains the estimate $\|u\| \leq C\|\rho\bar{\partial}f\|$ as above.

Recall that for each pair of $i \neq j$ in $\{1, \dots, n\}$, one has the tangential derivatives

$$L_{i,j} = \bar{\zeta}_j\partial_i - \bar{\zeta}_i\partial_j \quad \text{and} \quad \bar{L}_{i,j} = \zeta_j\bar{\partial}_i - \zeta_i\bar{\partial}_j.$$

Thus $|(\bar{\partial}f \wedge \bar{\partial}\rho)(\zeta)|^2$ is simply the sum of all $|(\bar{L}_{i,j}f)(\zeta)|^2$, $i < j$. We end this section with an elementary estimate on derivatives that will be needed in Section 6.

Lemma 2.4. *There is a constant $C_{2.4}$ such that for every $z \in \mathbf{B}$, we have $\|\rho\partial_i\varphi_z\|_\infty \leq C_{2.4}$ for every $i \in \{1, \dots, n\}$ and $\|\rho^{1/2}L_{i,j}\varphi_z\|_\infty \leq C_{2.4}$ for all $i \neq j$ in $\{1, \dots, n\}$.*

Proof. Recall from [18,page 25] that

$$\varphi_z(\zeta) = \frac{1}{1-\langle\zeta, z\rangle}\left\{z - \frac{\langle\zeta, z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(\zeta - \frac{\langle\zeta, z\rangle}{|z|^2}z\right)\right\}.$$

Write $D_z(\zeta)$ for $1 - \langle\zeta, z\rangle$ and $N_z(\zeta)$ for the vector $\{\cdots\}$ above. In other words, we have $\varphi_z = D_z^{-1}N_z$. Note that $\|\rho/D_z\|_\infty \leq 2$ and that $\|\partial_iN_z\|_\infty \leq 3$. Since

$$(\partial_i\varphi_z)(\zeta) = \frac{\bar{z}_i}{D_z(\zeta)}\varphi_z(\zeta) + \frac{1}{D_z(\zeta)}(\partial_iN_z)(\zeta),$$

we have $\|\rho\partial_i\varphi_z\|_\infty \leq 2 + 2 \cdot 3 = 8$. For the tangential derivatives, we have

$$\begin{aligned}(L_{i,j}\varphi_z)(\zeta) &= \frac{\bar{\zeta}_j\bar{z}_i - \bar{\zeta}_i\bar{z}_j}{D_z(\zeta)}\varphi_z(\zeta) \\ &+ \frac{1}{D_z(\zeta)}\left\{((1-|z|^2)^{1/2}-1)\frac{\bar{\zeta}_j\bar{z}_i - \bar{\zeta}_i\bar{z}_j}{|z|^2}z - (1-|z|^2)^{1/2}L_{i,j}\zeta\right\}.\end{aligned} \tag{2.4}$$

By (2.2), $|\bar{\zeta}_j\bar{z}_i - \bar{\zeta}_i\bar{z}_j| \leq 2|\zeta - z|$. On the other hand, $|\zeta - z|^2 = |\zeta|^2 - 2\mathrm{Re}\langle\zeta, z\rangle + |z|^2 \leq 2(1 - \mathrm{Re}\langle\zeta, z\rangle)$. Therefore $|\bar{\zeta}_j\bar{z}_i - \bar{\zeta}_i\bar{z}_j| \leq 2\sqrt{2}|1 - \langle\zeta, z\rangle|^{1/2}$, which leads to

$$\rho^{1/2}(\zeta)\left|\frac{\bar{\zeta}_j\bar{z}_i - \bar{\zeta}_i\bar{z}_j}{D_z(\zeta)}\right| \leq 4. \tag{2.5}$$

Also, we have $\rho^{1/2}(\zeta)(1-|z|^2)^{1/2}/|D_z(\zeta)| = (1-|\varphi_z(\zeta)|^2)^{1/2} \le 1$ [18,Theorem 2.2.2]. Combining this with (2.4) and (2.5), we find that $\|\rho^{1/2}L_{i,j}\varphi_z\|_\infty \le 4+4+1=9$. □

3. Other preliminaries

The proof of Theorem 1.2 requires a familiarity with symmetric norms.

Lemma 3.1. [20,Lemma 2.2] *Suppose that X and Y are countable sets and that N is a natural number. Suppose that $T : X \to Y$ is a map that is at most N-to-1. That is, for every $y \in Y$, $\text{card}\{x \in X : T(x) = y\} \le N$. Then for every set of real numbers $\{b_y\}_{y\in Y}$ and every symmetric gauge function Φ, we have $\Phi(\{b_{T(x)}\}_{x\in X}) \le N\Phi(\{b_y\}_{y\in Y})$.*

Recall from [10,page 125] that given a symmetric gauge function Φ, the formula

$$\Phi^*(\{b_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{a_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi(\{a_j\}_{j\in\mathbf{N}}) \le 1\right\}, \quad \{b_j\}_{j\in\mathbf{N}} \in \hat{c},$$

defines the symmetric gauge function that is dual to Φ. Moreover, we have the relation $\Phi^{**} = \Phi$ [10,page 125]. This relation implies that for every $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, we have

$$\Phi(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi^*(\{b_j\}_{j\in\mathbf{N}}) \le 1\right\}. \tag{3.1}$$

Lemma 3.2. [20,Lemma 5.1] *Let $\{A_k\}$ be a sequence of bounded operators on a separable Hilbert space $\mathcal{H}$. If $\{A_k\}$ weakly converges to an operator A, then the inequality*

$$\|A\|_\Phi \le \sup_k \|A_k\|_\Phi$$

holds for every symmetric gauge function Φ.

Lemma 3.3. *Let A and B be two bounded operators. Then the inequalities*

$$\||AB|^s\|_\Phi \le \|B\|^s\||A|^s\|_\Phi \quad \text{and} \quad \||BA|^s\|_\Phi \le \|B\|^s\||A|^s\|_\Phi$$

hold for every symmetric gauge function Φ and every $0 < s \le 1$.

Proof. For the singular numbers of the operators involved, it is well known that

$$s_j(AB) \le s_j(A)\|B\| \quad \text{and} \quad s_j(BA) \le \|B\|s_j(A)$$

for every $j \in \mathbf{N}$ [10,page 61]. Therefore for any gauge function Φ and any $0 < s \le 1$,

$$\||AB|^s\|_\Phi = \Phi(\{(s_j(AB))^s\}_{j\in\mathbf{N}}) \le \|B\|^s\Phi(\{(s_j(A))^s\}_{j\in\mathbf{N}}) = \|B\|^s\||A|^s\|_\Phi.$$

The other inequality is similarly proved. □

Lemma 3.4. [20,Lemma 3.1] *Suppose that* $A_1, \dots, A_m$ *are finite-rank operators on a Hilbert space* $\mathcal{H}$ *and let* $A = A_1 + \cdots + A_m$. *Then for every symmetric gauge function* Φ *and every* $0 < s \le 1$, *we have*

$$\| |A|^s \|_\Phi \le 2^{1-s} (\| |A_1|^s \|_\Phi + \cdots + \| |A_m|^s \|_\Phi). \tag{3.2}$$

Remark 3.5. Although (3.2) was only proved for finite-rank operators $A_1, \dots, A_m$ in [20], it actually hold for all bounded operators $A_1, \dots, A_m$ and $A = A_1 + \cdots + A_m$ on any separable Hilbert space $\mathcal{H}$. Indeed let $A_1, \dots, A_m \in \mathcal{B}(\mathcal{H})$ and $A = A_1 + \cdots + A_m$, and let E and F be finite-rank orthogonal projections on $\mathcal{H}$. Then by (3.2) and Lemma 3.3,

$$\begin{aligned} \|E|FA|^s\|_\Phi \le \| |FA|^s \|_\Phi &\le 2^{1-s} (\| |FA_1|^s \|_\Phi + \cdots + \| |FA_m|^s \|_\Phi) \\ &\le 2^{1-s} (\| |A_1|^s \|_\Phi + \cdots + \| |A_m|^s \|_\Phi). \end{aligned}$$

Since $\mathrm{rank}(E) < \infty$, the supremum of $\|E|FA|^s\|_\Phi$ over all finite-rank orthogonal projections F dominates $\|E|A|^s\|_\Phi$. Then observe that, by (1.1), if we take the supremum of $\|E|A|^s\|_\Phi$ over all finite-rank orthogonal projections E, we obtain $\| |A|^s \|_\Phi$. Hence (3.2) holds for all $A_1, \dots, A_m \in \mathcal{B}(\mathcal{H})$ and $A = A_1 + \cdots + A_m$.

As one would expect, the proof of Theorem 1.2 also requires a suitable decomposition of the ball and the sphere. We will adopt the decomposition system in [20], for that paper showed that the system, however complicated it may appear, actually works. Next let us review the decomposition system in [20] and estimates related to it.

Let S denote the unit sphere $\{\xi \in \mathbf{C}^n : |\xi| = 1\}$. Recall that the formula

$$d(u, \xi) = |1 - \langle u, \xi \rangle|^{1/2}, \qquad u, \xi \in S,$$

defines a metric on S [18,page 66]. Throughout the paper, we denote

$$B(u, r) = \{\xi \in S : |1 - \langle u, \xi \rangle|^{1/2} < r\}$$

for $u \in S$ and $r > 0$. Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. We take the usual normalization $\sigma(S) = 1$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \le \sigma(B(u, r)) \le A_0 r^{2n} \tag{3.3}$$

for all $u \in S$ and $0 < r \le \sqrt{2}$ [18,Proposition 5.1.4]. Note that the upper bound actually holds when $r > \sqrt{2}$.

For each integer $k \ge 0$, let $\{u_{k,1}, \dots, u_{k,m(k)}\}$ be a subset of S which is *maximal* with respect to the property

$$B(u_{k,j}, 2^{-k-1}) \cap B(u_{k,j'}, 2^{-k-1}) = \emptyset \quad \text{for all } \ 1 \le j < j' \le m(k). \tag{3.4}$$

The maximality of $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ implies that

$$\cup_{j=1}^{m(k)} B(u_{k,j}, 2^{-k}) = S. \tag{3.5}$$

For each pair of $k \geq 0$ and $1 \leq j \leq m(k)$, define the subsets

$$T_{k,j} = \{ru : 1 - 2^{-2k} \leq r < 1 - 2^{-2(k+1)}, u \in B(u_{k,j}, 2^{-k})\} \quad \text{and} \tag{3.6}$$

$$Q_{k,j} = \{ru : 1 - 2^{-2k} \leq r < 1 - 2^{-2(k+2)}, u \in B(u_{k,j}, 9 \cdot 2^{-k})\} \tag{3.7}$$

of $\mathbf{B}$. Let us also introduce the index set

$$I = \{(k,j) : k \geq 0, 1 \leq j \leq m(k)\}. \tag{3.8}$$

Lemma 3.6. [20,Lemma 2.4] *Given any* $0 < a < \infty$, *there exists a natural number* K *which depends only on* a *and the complex dimension* n *such that the following holds true: Suppose that* Γ *is an* a*-separated subset of* $\mathbf{B}$. *Then there exist pairwise disjoint subsets* $\Gamma_1, \ldots, \Gamma_K$ *of* Γ *such that* $\cup_{\mu=1}^{K} \Gamma_\mu = \Gamma$ *and such that* $\mathrm{card}(\Gamma_\mu \cap T_{k,j}) \leq 1$ *for all* $\mu \in \{1, \ldots, K\}$ *and* $(k,j) \in I$.

Let E be a Borel set in $\mathbf{B}$ with $v_\alpha(E) > 0$. For any $f \in L^2(\mathbf{B}, dv_\alpha)$, we define

$$A(f;E) = \left(\frac{1}{v_\alpha(E)} \int_E |f|^2 dv_\alpha \right)^{1/2}.$$

Although we use the same decomposition system as that in [20], there is a major difference between [20] and this paper: Whereas most of the estimates in [20] were carried out in terms of the various mean oscillations introduced there, quantities of the form $A(f;E)$ and $\|f\psi_{z,i}\|$ will be much more prominent in this paper.

Proposition 3.7. *Let* $0 < s \leq 1$ *be given, and let* $i \in \mathbf{Z}_+$ *satisfy the condition* $s(n+1+\alpha+2i) > 2n$. *Let* $0 < a < \infty$ *also be given. Then there exists a constant* $0 < C_{3.7} < \infty$ *which depends only on* n, α, s, i *and* a *such that the inequality*

$$\Phi(\{\|f\psi_{z,i}\|^s\}_{z \in \Gamma}) \leq C_{3.7} \Phi(\{A^s(f;Q_{k,j})\}_{(k,j) \in I})$$

holds for every $f \in L^2(\mathbf{B}, dv_\alpha)$, *every symmetric gauge function* Φ, *and every* a*-separated subset* Γ *of* $\mathbf{B}$.

The proof of this proposition is essentially a combination of a part of the work for the proof of [20,Lemma 6.4] and a part of the work in [20,Section 2]. For this reason the proof of Proposition 3.7 is relegated to the Appendix at the end of the paper.

Next we recall some elementary facts related to the Bergman metric.

Lemma 3.8. [21,Lemma 2.3] *For all* $u, v, x, y \in \mathbf{B}$ *we have*

$$\frac{(1 - |\varphi_u(x)|^2)^{1/2}(1 - |\varphi_v(y)|^2)^{1/2}}{|1 - \langle \varphi_u(x), \varphi_v(y) \rangle|} \leq 2e^{\beta(x,0)+\beta(y,0)} \frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1 - \langle u, v \rangle|}.$$

Lemma 3.9. *The inequality* $1-|z|^2 \le 4e^{2\beta(z,w)}(1-|w|^2)$ *holds for all* $z, w \in \mathbf{B}$.

Proof. Applying [18,Theorem 2.2.2], we have

$$1-|\varphi_z(w)| \le 1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2} \le \frac{(1-|z|^2)(1-|w|^2)}{(1-|z|)^2} \le 4\frac{1-|w|^2}{1-|z|^2}.$$

Combining this with the fact that $2\beta(z,w) \ge \log(1-|\varphi_z(w)|)^{-1}$, the lemma follows. □

Lemma 3.10. [22,Lemma 1.24] *Given any* $r > 0$, *there are* $0 < c(r) \le C(r) < \infty$ *such that*

$$c(r)(1-|z|^2)^{n+1+\alpha} \le v_\alpha(D(z,r)) \le C(r)(1-|z|^2)^{n+1+\alpha}$$

for every $z \in \mathbf{B}$.

Lemma 3.11. *Given any* $r > 0$, *there is a* $\delta(r) > 0$ *such that* $|m_z(w)| \ge \delta(r)$ *for all* $z, w \in \mathbf{B}$ *satisfying the condition* $\beta(z,w) < r$.

Proof. For $z, w \in \mathbf{B}$ satisfying the condition $\beta(z,w) < r$, Lemma 3.9 gives us

$$|m_z(w)| = \frac{1-|z|^2}{|1-\langle w,z\rangle|} \ge \frac{1}{2e^r}\frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,z\rangle|} \ge \frac{1}{2e^r}\left(1-|\varphi_z(w)|\right)^{1/2}.$$

Since $\beta(z,w) < r$, we have $1-|\varphi_z(w)| > e^{-2r}$. Substituting this in the above, we see that the constant $\delta(r) = (2e^{2r})^{-1}$ will do for the lemma. □

The proof of Theorem 1.2 involves a familiar counting lemma:

Lemma 3.12. [19,Lemma 4.1] *Let* X *be a set and let* E *be a subset of* $X \times X$. *Suppose that* m *is a natural number such that*

$$\text{card}\{y \in X : (x,y) \in E\} \le m \quad \textit{and} \quad \text{card}\{y \in X : (y,x) \in E\} \le m$$

for every $x \in X$. *Then there exist pairwise disjoint subsets* E_1, E_2, ..., E_{2m} *of* E *such that*

$$E = E_1 \cup E_2 \cup ... \cup E_{2m}$$

and such that for each $1 \le j \le 2m$, *the conditions* $(x,y), (x',y') \in E_j$ *and* $(x,y) \ne (x',y')$ *imply both* $x \ne x'$ *and* $y \ne y'$.

We end the preliminaries with an elementary operator-theoretical fact.

Lemma 3.13. *Let* $A : \mathcal{H} \to \mathcal{H}'$ *and* $B : \mathcal{H} \to \mathcal{H}''$ *be bounded operators, where* $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}''$ *are Hilbert spaces. Suppose that there is a positive number* C *such that* $\|Ax\| \le C\|Bx\|$ *for every* $x \in \mathcal{H}$. *Then there is an operator* $T : \mathcal{H}'' \to \mathcal{H}'$ *with* $\|T\| \le C$ *such that* $A = TB$.

Proof. Let $\mathcal{R}_0$ denote the linear subspace $\{Bx : x \in \mathcal{H}\}$ of $\mathcal{H}''$, and let $\mathcal{R}$ be the closure of $\mathcal{R}_0$ in $\mathcal{H}''$. Since $\|Ax\| \le C\|Bx\|$ for every $x \in \mathcal{H}$, the formula

$$TBx = Ax, \qquad x \in \mathcal{H}, \tag{3.9}$$

gives us a well-defined linear operator T from $\mathcal{R}_0$ into $\mathcal{H}'$. Moreover, we have $\|Ty\| \le C\|y\|$ for every $y \in \mathcal{R}_0$. By the density of $\mathcal{R}_0$ in $\mathcal{R}$, T extends to a bounded operator $T : \mathcal{R} \to \mathcal{H}'$ with $\|T\| \le C$. It is then trivial to extend T to an operator on from $\mathcal{H}''$ to $\mathcal{H}'$ with the same norm. Finally, (3.9) implies the operator identity $A = TB$. □

4. Estimates involving the modified kernel

We begin with inner products involving $\psi_{z,i}$. First of all, there is a $\delta \in \mathbf{Z}_+$ such that

$$0 \le \delta - \alpha < 1. \tag{4.1}$$

Lemma 4.1. *Given any $i \in \mathbf{Z}_+$, there is a constant $C_{4.1}$ which depends only on n, α and i such that if $z = |z|\xi$ and $w = |w|\eta$ with $\xi, \eta \in S$, and if $0 \le |z| \le |w| < 1$, then*

$$|\langle f\psi_{z,3i+n+1+\delta}, f\psi_{w,3i+n+1+\delta}\rangle| \le C_{4.1}\left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1+\alpha)/2}\left(\frac{1-|z|^2}{d^2(\xi,\eta)}\right)^i \|f\psi_{w,i}\|^2$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$.

Proof. By (1.7), $\|m_z\|_\infty \le 1 + |z| < 2$ for every $z \in \mathbf{B}$. Thus

$$\begin{aligned}|\psi_{z,3i+n+1+\delta}\psi_{w,3i+n+1+\delta}| &= |\psi_{w,i}|^2\left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1+\alpha)/2}|m_w|^{i+\delta-\alpha}|m_z|^{3i+2n+2+\alpha+\delta}\\ &\le 2^{\delta-\alpha+2i+2n+2+\alpha+\delta}\left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1+\alpha)/2}|m_w m_z|^i\,|\psi_{w,i}|^2\end{aligned}$$

for all $z, w \in \mathbf{B}$. Thus if we write $C = 2^{2i+2n+2+2\delta}$, then

$$|\langle f\psi_{z,3i+n+1+\delta}, f\psi_{w,3i+n+1+\delta}\rangle| \le C\left(\frac{1-|w|^2}{1-|z|^2}\right)^{(n+1+\alpha)/2}\|(m_z m_w)^i\|_\infty \|f\psi_{w,i}\|^2 \tag{4.2}$$

for all $z, w \in \mathbf{B}$ and $f \in L^2(\mathbf{B}, dv_\alpha)$. Hence the proof will be complete if we can show that

$$\|m_z m_w\|_\infty \le 16\frac{1-|z|^2}{d^2(\xi,\eta)} \tag{4.3}$$

for all $z, w \in \mathbf{B}$ satisfying the conditions $z = |z|\xi$, $w = |w|\eta$, $\xi, \eta \in S$ and $|z| \le |w|$. For this, consider any $\zeta \in \mathbf{B}$. Then $\zeta = |\zeta|x$ for some $x \in S$. We have

$$2|1 - \langle\zeta, z\rangle| \ge |1 - \langle x, \xi\rangle| = d^2(x,\xi) \quad \text{and} \quad 2|1 - \langle\zeta, w\rangle| \ge |1 - \langle x, \eta\rangle| = d^2(x,\eta).$$

Hence we have either $|1 - \langle\zeta, z\rangle| \ge (1/8)d^2(\xi,\eta)$ or $|1 - \langle\zeta, w\rangle| \ge (1/8)d^2(\xi,\eta)$. Since $1 - |w|^2 \le 1 - |z|^2$, $\|m_z\|_\infty \le 2$ and $\|m_w\|_\infty \le 2$, (4.3) follows. □

Lemma 4.2. *Suppose that* $\{e_x : x \in X\}$ *is an orthonormal set in a Hilbert space* $\mathcal{H}$, *where* X *is a countable index set. Furthermore, suppose that* $\{g_x : x \in X\}$ *are vectors in* $\mathcal{H}$ *satisfying the following two conditions*:
(1) *There is an* $N \in \mathbf{N}$ *such that* $\mathrm{card}\{y \in X : \langle g_x, g_y\rangle \neq 0\} \le N$ *for every* $x \in X$.
(2) $g_x = 0$ *for all but a finite number of* $x \in X$.
Let $A = \sum_{x\in X} g_x \otimes e_x$. *Then for every symmetric gauge function* Φ *and every* $0 < s \le 1$, *we have* $\||A|^s\|_\Phi \le 2N\Phi(\{\|g_x\|^s\}_{x\in X})$.

Proof. By (1) and a standard maximality argument, there is a partition $X = X_1 \cup \cdots \cup X_N$ such that for every $r \in \{1, \dots, N\}$, the conditions $x, y \in X_r$ and $x \neq y$ imply $\langle g_x, g_y\rangle = 0$. Thus if we define $A_r = \sum_{x\in X_r} g_x \otimes e_x$, $r \in \{1, \dots, N\}$, then

$$A_r^* A_r = \sum_{x\in X_r} \|g_x\|^2 e_x \otimes e_x.$$

Thus for every $0 < s \le 1$ and every symmetric gauge function Φ,

$$\||A_r|^s\|_\Phi = \|(A_r^*A_r)^{s/2}\|_\Phi = \Phi(\{\|g_x\|^s\}_{x\in X_r}) \le \Phi(\{\|g_x\|^s\}_{x\in X}).$$

Since $A = A_1 + \cdots + A_N$, the conclusion of the lemma follows from this inequality and Lemma 3.4. $\square$

Lemma 4.3. *Let* $0 < s \le 1$ *be given, and let* $i \in \mathbf{N}$ *satisfy the condition* $si > 4n$. *Write* $i' = 3i+n+1+\delta$, *where* $\delta \in \mathbf{Z}_+$ *satisfies* (4.1). *Then there is a constant* $C_{4.3}$ *which depends only on* n, α, s *and* i *such that the following holds for every* $f \in L^2(\mathbf{B}, dv_\alpha)$ *and every symmetric gauge function* Φ: *Let* $\{e_{k,j} : (k,j) \in I\}$ *be an orthonormal set. Let* $z_{k,j} \in T_{k,j}$ *for every* $(k,j) \in I$. *For each* $(k,j) \in I$, *let* $c_{k,j}$ *be either* 1 or 0, *and suppose that* $c_{k,j} = 0$ *for all but a finite number of* $(k,j) \in I$. *Then the operator*

$$F = M_f \sum_{(k,j)\in I} c_{k,j}\psi_{z_{k,j},i'} \otimes e_{k,j} = \sum_{(k,j)\in I} c_{k,j}(f\psi_{z_{k,j},i'}) \otimes e_{k,j}$$

satisfies the estimate $\||F|^s\|_\Phi \le C_{4.3}\Phi(\{c_{k,j}\|f\psi_{z_{k,j},i}\|^s\}_{(k,j)\in I})$.

Proof. By (3.4) and (3.3), there is an $N \in \mathbf{N}$ such that for every $(k,j) \in I$,

$$\mathrm{card}\{j' \in \{1, \dots, m(k)\} : B(u_{k,j}, 2^{-k}) \cap B(u_{k,j'}, 2^{-k}) \neq \emptyset\} \le N. \tag{4.4}$$

This N will be fixed for the rest of the proof. To simplify the notation, let us write

$$\begin{cases} r(k,j) = c_{k,j}\|f\psi_{z_{k,j},i}\| & \text{for all} \quad (k,j) \in I \\ a(k,j;t,h) = c_{t,h}c_{k,j}\langle f\psi_{z_{k,j},i'}, f\psi_{z_{t,h},i'}\rangle & \text{for all} \quad (k,j),(t,h) \in I \end{cases}. \tag{4.5}$$

Then

$$F^*F = \sum_{(k,j),(t,h)\in I} a(k,j;t,h)e_{t,h} \otimes e_{k,j} = B_0 + \sum_{\ell=1}^{\infty}(B_\ell + B_\ell^*),$$

where

$$B_\ell = \sum_{(k,j),(k+\ell,h)\in I} a(k,j;k+\ell,h) e_{k+\ell,h} \otimes e_{k,j},$$

$\ell \ge 0$. It follows from Lemma 3.4 that

$$\| |F|^s \|_\Phi = \|(F^*F)^{s/2}\|_\Phi \le 2^{1-(s/2)} \| |B_0|^{s/2} \|_\Phi + 2^{2-(s/2)} \sum_{\ell=1}^{\infty} \| |B_\ell|^{s/2} \|_\Phi. \tag{4.6}$$

To estimate each $\| |B_\ell|^{s/2} \|_\Phi$, we need to group the terms in B_ℓ is a specific way.

By the assumption $z_{k,j} \in T_{k,j}$, $(k,j) \in I$, we can write each $z_{k,j}$ in the form $z_{k,j} = |z_{k,j}|\xi_{k,j}$, where $\xi_{k,j} \in B(u_{k,j}, 2^{-k})$. By (3.5), we can rewrite each B_ℓ in the form

$$B_\ell = \sum_{k=0}^{\infty} \sum_{1\le j,j'\le m(k)} \sum_{\xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})} \epsilon(k,j';k+\ell,h) a(k,j;k+\ell,h) e_{k+\ell,h} \otimes e_{k,j}, \tag{4.7}$$

where each $\epsilon(k,j';k+\ell,h)$ is either 1 or 0. Define the vector

$$g^{(\ell)}_{k,j;k,j'} = \sum_{\xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})} \epsilon(k,j';k+\ell,h) a(k,j;k+\ell,h) e_{k+\ell,h} \tag{4.8}$$

for such ℓ, k and j, j'. Note that for all $j, j', q, q' \in \{1, \dots, m(k)\}$, we have

$$\langle g^{(\ell)}_{k,j;k,j'}, g^{(\ell)}_{k,q;k,q'} \rangle = 0 \quad \text{whenever} \quad B(u_{k,j'}, 2^{-k}) \cap B(u_{k,q'}, 2^{-k}) = \emptyset. \tag{4.9}$$

Also, it is obvious that

$$\langle g^{(\ell)}_{k,j;k,j'}, g^{(\ell)}_{k',q;k',q'} \rangle = 0 \quad \text{whenever} \quad k \ne k'. \tag{4.10}$$

Let us introduce the index sets

$$\begin{aligned} E^{(0)} &= \{((k,j),(k,j')) : d(u_{k,j}, u_{k,j'}) < 2^{-k+2}\} \quad \text{and} \\ E^{(m)} &= \{((k,j),(k,j')) : 2^{-k+m+1} \le d(u_{k,j}, u_{k,j'}) < 2^{-k+m+2}\}, \quad m \ge 1. \end{aligned}$$

Then by (4.7) and (4.8), we have

$$\begin{aligned} B_\ell &= \sum_{k=0}^{\infty} \sum_{1\le j,j'\le m(k)} g^{(\ell)}_{k,j;k,j'} \otimes e_{k,j} = \sum_{m=0}^{\infty} B^{(m)}_\ell, \quad \text{where} \\ B^{(m)}_\ell &= \sum_{((k,j),(k,j'))\in E^{(m)}} g^{(\ell)}_{k,j;k,j'} \otimes e_{k,j} \quad \text{for each} \quad m \ge 0. \end{aligned}$$

But each $B^{(m)}_\ell$ needs to be further decomposed. By (3.4) and (3.3), there is a natural number C_1 such that for each $(k,j) \in I$ and each $m \ge 0$, we have

$$\text{card}\{j' \in \{1, \dots, m(k)\} : d(u_{k,j}, u_{k,j'}) < 2^{-k+m+2}\} \le C_1 2^{2nm}. \tag{4.11}$$

By (4.11) and Lemma 3.12, for each $m \geq 0$ we have a partition

$$E^{(m)} = E_1^{(m)} \cup \cdots \cup E_{2C_1 2^{2nm}}^{(m)} \tag{4.12}$$

such that for each $1 \leq \nu \leq 2C_1 2^{2nm}$, if $((k_1, j_1), (k_1, j_1'))$ and $((k_2, j_2), (k_2, j_2'))$ are two distinct elements in $E_\nu^{(m)}$, then we have both $(k_1, j_1) \neq (k_2, j_2)$ and $(k_1, j_1') \neq (k_2, j_2')$. Define

$$B_\ell^{(m,\nu)} = \sum_{((k,j),(k,j')) \in E_\nu^{(m)}} g_{k,j;k,j'}^{(\ell)} \otimes e_{k,j} \tag{4.13}$$

for $m \geq 0$ and $1 \leq \nu \leq 2C_1 2^{2nm}$. The above-mentioned property of $E_\nu^{(m)}$ implies that the projections $((k,j),(k,j')) \mapsto (k,j)$ and $((k,j),(k,j')) \mapsto (k,j')$ are both injective on $E_\nu^{(m)}$. It follows from the injectivity of this second projection and (4.9), (4.4) and (4.10) that for each $((k,j),(k,j')) \in E_\nu^{(m)}$, we have

$$\operatorname{card}\{((k',q),(k',q')) \in E_\nu^{(m)} : \langle g_{k,j;k,j'}^{(\ell)}, g_{k',q;k',q'}^{(\ell)} \rangle \neq 0\} \leq N.$$

Since $\{e_{k,j} : (k,j) \in I\}$ is an orthonormal set and since the projection $((k,j),(k,j')) \mapsto (k,j)$ is injective on $E_\nu^{(m)}$, we can now apply Lemma 4.2 to obtain

$$\| |B_\ell^{(m,\nu)}|^{s/2} \|_\Phi \leq 2N\Phi(\{\|g_{k,j;k,j'}^{(\ell)}\|^{s/2}\}_{((k,j),(k,j')) \in E_\nu^{(m)}}). \tag{4.14}$$

Next we estimate the right-hand side of (4.14).

For each triple of $\ell \geq 0$, $(k,j) \in I$ and $m \geq 0$, there is an $h(\ell; k, j; m) \in \{1, \ldots, m(k + \ell)\}$ such that $d(u_{k,j}, u_{k+\ell, h(\ell;k,j;m)}) < 2^{-k+m+3}$ and

$$r(k+\ell, h(\ell;k,j;m)) \geq r(k+\ell, h) \quad \text{whenever} \quad d(u_{k,j}, u_{k+\ell,h}) < 2^{-k+m+3}.$$

Claim: there is a C_0 such that if $((k,j),(k,j')) \in E^{(m)}$ and $\xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})$, then

$$|a(k,j;k+\ell,h)| \leq C_0 2^{-\ell(n+1+\alpha)} 2^{-2im} r^2(k+\ell, h(\ell;k,j;m)). \tag{4.15}$$

Using (4.5) and Lemma 4.1, let us verify it according to the following three cases.

(1) Suppose that $\ell = 0$ and that $m = 0$. Since $z_{k,h} = |z_{k,h}|\xi_{k,h}$ and $\xi_{k,h} \in B(u_{k,h}, 2^{-k})$, if $((k,j),(k,j')) \in E^{(0)}$ and $\xi_{k,h} \in B(u_{k,j'}, 2^{-k})$, then $d(u_{k,j}, u_{k,h}) \leq d(u_{k,j}, u_{k,j'}) + d(u_{k,j'}, u_{k,h}) < 2^{-k+2} + 2^{-k+1} < 2^{-k+3}$. In this case, recalling (4.5), it follows from (4.2) and the definition of $h(\cdot;\cdot,\cdot;\cdot)$ that $|a(k,j;k,h)| \leq 4^i C r^2(k, h(0;k,j;0))$.

(2) Suppose that $\ell = 0$ and that $m \geq 1$. If $((k,j),(k,j')) \in E^{(m)}$ and $\xi_{k,h} \in B(u_{k,j'}, 2^{-k})$, then $d(u_{k,j}, u_{k,h}) \leq d(u_{k,j}, u_{k,j'}) + d(u_{k,j'}, u_{k,h}) < 2^{-k+m+3}$ in this case. Hence, recalling (4.5), it follows from Lemma 4.1 and the definition of $h(\cdot;\cdot,\cdot;\cdot)$ that

$$|a(k,j;k,h)| \leq C_{4.1} \left(\frac{2^{-2k+1}}{d^2(\xi_{k,j}, \xi_{k,h})} \right)^i r^2(k, h(0;k,j;m)). \tag{4.16}$$

Since $((k,j),(k,j')) \in E^{(m)}$ and $m \geq 1$, it follows from the definition of $E^{(m)}$ that $d(u_{k,j}, u_{k,j'}) \geq 2^{-k+m+1} \geq 4d(u_{k,j}, \xi_{k,j})$. Similarly, $d(u_{k,j}, u_{k,j'}) \geq 4d(u_{k,j'}, \xi_{k,h})$ since $\xi_{k,h} \in B(u_{k,j'}, 2^{-k})$. By the triangle inequality, we have $d(\xi_{k,j}, \xi_{k,h}) \geq (1/2)d(u_{k,j}, u_{k,j'}) \geq 2^{-k+m}$. Substituting this in (4.16), we obtain

$$|a(k,j;k,h)| \leq 2^i C_{4.1} 2^{-2im} r^2(k, h(0;k,j;m)) \tag{4.17}$$

if $\xi_{k,h} \in B(u_{k,j'}, 2^{-k})$ and $((k,j),(k,j')) \in E^{(m)}$.

(3) Suppose that $\ell \geq 1$. Let $((k,j),(k,j')) \in E^{(m)}$ and $\xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})$. Then $d(u_{k,j}, u_{k+\ell,h}) < 2^{-k+m+2} + 2^{-k} + 2^{-k-\ell} < 2^{-k+m+3}$. Applying Lemma 4.1, we have

$$\begin{aligned} |a(k,j;k+\ell,h)| &\leq C_{4.1} \left(\frac{1-|z_{k+\ell,h}|^2}{1-|z_{k,j}|^2} \right)^{(n+1+\alpha)/2} \left(\frac{1-|z_{k,j}|^2}{d^2(\xi_{k,j}, \xi_{k+\ell,h})} \right)^i r^2(k+\ell,h) \\ &\leq C_{4.1} \left(\frac{2^{-2(k+\ell)+1}}{2^{-2(k+1)}} \right)^{(n+1+\alpha)/2} \left(\frac{2^{-2k+1}}{d^2(\xi_{k,j}, \xi_{k+\ell,h})} \right)^i r^2(k+\ell, h(\ell;k,j;m)). \end{aligned} \tag{4.18}$$

By (4.2), we can also replace the factor $(\cdots)^i$ above by 4^i, which covers the case $m = 0$. For the case $m \geq 1$, we can repeat the triangle inequality-argument between (4.16) and (4.17) to obtain $d(\xi_{k,j}, \xi_{k+\ell,h}) \geq (1/2)d(u_{k,j}, u_{k,j'}) \geq 2^{-k+m}$. Substituting this in (4.18), we see that (4.15) also holds in the case $\ell \geq 1$. This completes the verification of (4.15).

For each pair of $\ell \geq 0$ and $(k,j') \in I$, define

$$\mathcal{N}(\ell;k,j') = \text{card}\{h : \xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})\}.$$

Since $\xi_{k+\ell,h} \in B(u_{k+\ell,h}, 2^{-k-\ell})$, if $\xi_{k+\ell,h} \in B(u_{k,j'}, 2^{-k})$, then $d(u_{k,j'}, u_{k+\ell,h}) < 2^{-k+1}$. Hence it follows from (3.4) and (3.3) that there is a C_2 such that

$$\mathcal{N}(\ell;k,j') \leq C_2 2^{2n\ell}$$

for all $\ell \geq 0$ and $(k,j') \in I$. The fact that $\{e_{k,j} : (k,j) \in I\}$ is an orthonormal set now produces a quantitative effect: by (4.8), (4.15) and this orthonormality, we have

$$\begin{aligned} \|g^{(m)}_{k,j;k,j'}\| &\leq C_0 2^{-\ell(n+1+\alpha)} 2^{-2im} r^2(k+\ell, h(\ell;k,j;m)) \sqrt{\mathcal{N}(\ell;k,j')} \\ &\leq C_0 2^{-\ell(n+1+\alpha)} 2^{-2im} r^2(k+\ell, h(\ell;k,j;m)) \cdot C_2^{1/2} 2^{\ell n} \\ &= C_3 2^{-\ell(1+\alpha)} 2^{-2im} r^2(k+\ell, h(\ell;k,j;m)) \end{aligned} \tag{4.19}$$

for every $((k,j),(k,j')) \in E^{(m)}$, where $C_3 = C_0 C_2^{1/2}$. Thus

$$\|g^{(m)}_{k,j;k,j'}\|^{s/2} \leq C_3^{s/2} 2^{-\ell(1+\alpha)(s/2)} 2^{-sim} r^s(k+\ell, h(\ell;k,j;m)).$$

Since the projection $((k,j),(k,j')) \mapsto (k,j)$ is injective on $E^{(m)}_\nu$, (4.14) now leads to

$$\begin{aligned} \||B^{(m,\nu)}_\ell|^{s/2}\|_\Phi &\leq 2N\Phi(\{\|g^{(\ell)}_{k,j;k,j'}\|^{s/2}\}_{((k,j),(k,j')) \in E^{(m)}_\nu}) \\ &\leq C_4 2^{-\ell(1+\alpha)(s/2)} 2^{-sim} \Phi(\{r^s(k+\ell, h(\ell;k,j;m))\}_{(k,j) \in I}), \end{aligned} \tag{4.20}$$

where $C_4 = 2NC_3^{s/2}$. If $h(\ell;k,j;m) = h(\ell;k,j';m)$, then $d(u_{k,j}, u_{k,j'}) < 2^{-k+m+4}$. By (3.4) and (3.3), there is an $N_1 \in \mathbf{N}$ such that for every pair of $\ell \ge 0$ and $m \ge 0$, the map

$$(k,j) \mapsto (k+\ell, h(\ell;k,j;m))$$

is at most $N_1 2^{2nm}$-to-1 on I. Applying Lemma 3.1 in (4.20), we obtain

$$\||B_\ell^{(m,\nu)}|^{s/2}\|_\Phi \le N_1 C_4 2^{-\ell(1+\alpha)(s/2)} 2^{-(si-2n)m} \Phi(\{r^s(k,j)\}_{(k,j)\in I}).$$

By (4.12) and (4.13), $B_\ell^{(m)} = B_\ell^{(m,1)} + \cdots + B_\ell^{(m,2C_1 2^{2nm})}$. Thus Lemma 3.4 leads to

$$\begin{aligned}\||B_\ell^{(m)}|^{s/2}\|_\Phi &\le 2 \sum_{\nu=1}^{2C_1 2^{2nm}} \||B_\ell^{(m,\nu)}|^{s/2}\|_\Phi \\ &\le 4C_1 N_1 C_4 2^{-\ell(1+\alpha)(s/2)} 2^{-(si-4n)m} \Phi(\{r^s(k,j)\}_{(k,j)\in I}).\end{aligned}$$

Since $si > 4n$, another application of Lemma 3.4 gives us

$$\||B_\ell|^{s/2}\|_\Phi \le 2 \sum_{m=0}^{\infty} \||B_\ell^{(m)}|^{s/2}\|_\Phi \le C_5 2^{-\ell(1+\alpha)(s/2)} \Phi(\{c_{k,j}\|f\psi_{z_{k,j},i}\|^s\}_{(k,j)\in I}).$$

Finally, substituting this in (4.6), we see that the lemma holds for the constant

$$C_{4.3} = 2^{1-(s/2)} C_5 + 2^{2-(s/2)} C_5 \sum_{\ell=1}^{\infty} 2^{-\ell(1+\alpha)(s/2)},$$

which is finite because $\alpha > -1$. This completes the proof. □

Proposition 4.4. *Let $0 < s \le 1$ be given, and let $i \in \mathbf{N}$ satisfy the condition $si > 4n$. Set $i' = 3i + n + 1 + \delta$, where $\delta \in \mathbf{Z}_+$ satisfies (4.1). Let $a > 0$ also be given. Then there is a constant $C_{4.4}$ which depends only on n, α, s, i and a such that the following holds for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ: Let Γ be an a-separated set in $\mathbf{B}$, and let $\{e_z : z \in \Gamma\}$ be an orthonormal set. Then the operator*

$$Y = M_f \sum_{z\in\Gamma} \psi_{z,i'} \otimes e_z = \sum_{z\in\Gamma} (f\psi_{z,i'}) \otimes e_z$$

satisfies the estimate $\||Y|^s\|_\Phi \le C_{4.4} \Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I})$.

Proof. Given $a > 0$, let K denote the natural number provided by Lemma 3.6. According to that lemma, any a-separated set Γ admits a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_K$ such that for each $\mu \in \{1, \ldots, K\}$, we have $\mathrm{card}(\Gamma_\mu \cap T_{k,j}) \le 1$ for every $(k,j) \in I$. We can write Γ as the union of an increasing sequence of *finite* subsets $G_1 \subset G_2 \subset \cdots \subset G_m \subset \cdots$.

Consider any $f \in L^2(\mathbf{B}, dv_\alpha)$ and any symmetric gauge function Φ. The condition $si > 4n$ certainly implies $s(n+1+\alpha+2i) > 2n$. Thus by Proposition 3.7,

$$\Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma}) \le C_{3.7} \Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I}). \tag{4.21}$$

For every pair of $\mu \in \{1, \dots, K\}$, and $m \ge 1$, define

$$Y_\mu^{(m)} = M_f \sum_{z \in \Gamma_\mu \cap G_m} \psi_{z,i'} \otimes e_z = \sum_{z \in \Gamma_\mu \cap G_m} (f\psi_{z,i'}) \otimes e_z.$$

Since the finite set $\Gamma_\mu \cap G_m$ has the property $\mathrm{card}(\Gamma_\mu \cap G_m \cap T_{k,j}) \le 1$ for every $(k,j) \in I$, it follows from Lemma 4.3 and (4.21) that

$$\| |Y_\mu^{(m)}|^s \|_\Phi \le C_{4.3} \Phi(\{\|f\psi_{z,i}\|^s\}_{z \in \Gamma_\mu \cap G_m}) \le C_{4.3} C_{3.7} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}).$$

Set $C_{4.4} = 2^{1-s} K C_{4.3} C_{3.7}$. By the partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_K$ and Lemma 3.4, we have

$$\| |Y^{(m)}|^s \|_\Phi \le 2^{1-s} (\| |Y_1^{(m)}|^s \|_\Phi + \cdots + \| |Y_K^{(m)}|^s \|_\Phi) \le C_{4.4} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}),$$

where

$$Y^{(m)} = M_f \sum_{z \in G_m} \psi_{z,i'} \otimes e_z = \sum_{z \in G_m} (f\psi_{z,i'}) \otimes e_z,$$

$m \ge 1$. Thus for every $m \ge 1$ we have

$$\|(Y^{(m)} Y^{(m)*})^{s/2}\|_\Phi = \| |Y^{(m)*}|^s \|_\Phi = \| |Y^{(m)}|^s \|_\Phi \le C_{4.4} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}).$$

If $\Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}) < \infty$, then this bound guarantees that the increasing operator sequence $\{Y^{(m)} Y^{(m)*}\}$ converges to YY^* strongly. Hence the sequence $\{(Y^{(m)} Y^{(m)*})^{s/2}\}$ strongly converges to $(YY^*)^{s/2}$. Thus it follows from Lemma 3.2 that

$$\|(YY^*)^{s/2}\|_\Phi = \sup_{m \ge 1} \|(Y^{(m)} Y^{(m)*})^{s/2}\|_\Phi \le C_{4.4} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}).$$

But if $\Phi(\{A^s(f; Q_{k,j})\}_{(k,j) \in I}) = \infty$, then this inequality holds trivially. Finally, since $(YY^*)^{s/2} = |Y^*|^s$ and $\| |Y^*|^s \|_\Phi = \| |Y|^s \|_\Phi$, the proposition follows. □

Corollary 4.5. *Let $i \in \mathbf{N}$ satisfy the condition $i > 4n$. Set $i' = 3i + n + 1 + \delta$, where $\delta \in \mathbf{Z}_+$ satisfies* (4.1). *Let $a > 0$ also be given. Then there is a constant $C_{4.5}$ which depends only on n, α, i and a such that if Γ is an a-separated set in $\mathbf{B}$ and if $\{e_z : z \in \Gamma\}$ is an orthonormal set, then*

$$\left\| \sum_{z \in \Gamma} \psi_{z,i'} \otimes e_z \right\| \le C_{4.5}.$$

Proof. This follows from Proposition 4.4 by applying it to the specific symmetric gauge function

$$\Phi_\infty(\{a_j\}_{j \in \mathbf{N}}) = \sup\{|a_1|, \dots, |a_j|, \dots\}, \quad \{a_j\}_{j \in \mathbf{N}} \in \hat{c},$$

with $s = 1$ and f being the constant function 1 on $\mathbf{B}$. □

5. Discrete sums and the Bergman projection

Next we will show that operators of the form $M_f P$ can be dominated by the kind of discrete sums Y in Proposition 4.4. This will reduce the main estimate in the proof of the upper bound in Theorem 1.2 to the estimate provided by Proposition 4.4. What is involved here is the familiar atomic decomposition for the weighted Bergman space [4,22].

Lemma 5.1. [21,Lemma 2.2] *Let Γ be an a-separated set in $\mathbf{B}$ for some $a > 0$.*
(a) *For each $0 < R < \infty$, there is a natural number $N = N(\Gamma, R)$ such that* $\text{card}\{v \in \Gamma : \beta(u,v) \le R\} \le N$ *for every $u \in \Gamma$.*
(b) *For every pair of $z \in \mathbf{B}$ and $r > 0$, there is a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $\nu \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_\nu$ and $u \ne v$ imply $\beta(\varphi_u(z), \varphi_v(z)) > r$.*

Let Γ be an a-separated set in $\mathbf{B}$. For each pair of $i \in \mathbf{Z}_+$ and $z \in \mathbf{B}$, denote

$$E_{\Gamma,z,i} = \sum_{u\in\Gamma} \psi_{\varphi_u(z),i} \otimes \psi_{\varphi_u(z),i}.$$

Lemma 5.2. *Let Γ be an a-separated set in $\mathbf{B}$ for some $a > 0$. Given $0 < s \le 1$, let $i \in \mathbf{N}$ satisfy the condition $si > 4n$. Set $i' = 3i + n + 1 + \delta$, where $\delta \in \mathbf{Z}_+$ satisfies* (4.1)*. Then for every $z \in \mathbf{B}$, there is a constant $C_{5.2}(z)$ which depends only on n, α, Γ, s, i, and z such that*

$$\| |M_f E_{\Gamma,z,i'}|^s \|_\Phi \le C_{5.2}(z) \Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ.

Proof. For each $z \in \mathbf{B}$, Lemma 5.1(b) provides an $m = m(\Gamma, z) \in \mathbf{N}$ and a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for each $\nu \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_\nu$ and $u \ne v$ imply $\beta(\varphi_u(z), \varphi_v(z)) > 2$. In other words, each $\{\varphi_u(z) : u \in \Gamma_\nu\}$ is a 1-separated set. Thus we can pick an orthonormal set $\{e_u : u \in \Gamma\}$ and decompose $E_{\Gamma,z,i'}$ in the form

$$E_{\Gamma,z,i'} = F_1 F_1^* + \cdots + F_m F_m^*, \quad \text{where} \quad F_\nu = \sum_{u\in\Gamma_\nu} \psi_{\varphi_u(z),i'} \otimes e_u,$$

$1 \le \nu \le m$. Since each $\{\varphi_u(z) : u \in \Gamma_\nu\}$ is 1-separated, Corollary 4.5 guarantees that F_ν is bounded. For each $\nu \in \{1, \ldots, m\}$, we can apply Proposition 4.4 with $a = 1$ to obtain

$$\| |M_f F_\nu|^s \|_\Phi \le C_{4.4} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}) \tag{5.1}$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ. On the other hand, applying Lemma 3.4, Remark 3.5 and Lemma 3.3, we have

$$\begin{aligned} \| |M_f E_{\Gamma,z,i'}|^s \|_\Phi &\le 2^{1-s}(\| |M_f F_1 F_1^*|^s \|_\Phi + \cdots + \| |M_f F_m F_m^*|^s \|_\Phi) \\ &\le 2^{1-s}(\| |M_f F_1|^s \|_\Phi \|F_1^*\|^s + \cdots + \| |M_f F_m|^s \|_\Phi \|F_m^*\|^s). \end{aligned}$$

Combining this with (5.1), we see that the constant $C_{5.2}(z) = 2^{1-s} C_{4.4}(\|F_1\|^s + \cdots + \|F_m\|^s)$ will do for the lemma. □

Let us recall the well-known atomic decomposition for $L^2_a(\mathbf{B}, dv_\alpha)$:

Proposition 5.3. [22,pages 69-72] *Let $i \in \mathbf{Z}_+$ be given. Then there exist an a-separated set Γ in $\mathbf{B}$ for some $a > 0$ and a finite set $\{z_1, \dots, z_q\}$ in $\mathbf{B}$ such that every $h \in L^2_a(\mathbf{B}, dv_\alpha)$ admits the representation*

$$h = \sum_{u\in\Gamma} \sum_{1\leq j\leq q} c_{u,j} \psi_{\varphi_u(z_j),i},$$

where the coefficients $c_{u,j}$ satisfy the condition $\sum_{u\in\Gamma} \sum_{1\leq j\leq q} |c_{u,j}|^2 < \infty$.

Lemma 5.4. *Let $i \in \mathbf{N}$ satisfy the condition $i > 4n$. Set $i' = 3i + n + 1 + \delta$, where $\delta \in \mathbf{Z}_+$ satisfies* (4.1). *Then there exist an a-separated set Γ in $\mathbf{B}$ for some $a > 0$, a finite set $\{z_1, \dots, z_q\}$ in $\mathbf{B}$, and a bounded operator T on $L^2(\mathbf{B}, dv_\alpha)$ such that*

$$P = E_{\Gamma,z_1,i'} T + \cdots + E_{\Gamma,z_q,i'} T. \tag{5.2}$$

Proof. We apply Propositions 5.3 to this integer i': there is an a-separated set Γ for some $a > 0$ and $\{z_1, \dots, z_q\} \subset \mathbf{B}$ such that every $h \in L^2_a(\mathbf{B}, dv_\alpha)$ admits the representation

$$h = \sum_{u\in\Gamma} \sum_{1\leq j\leq q} c_{u,j} \psi_{\varphi_u(z_j),i'} \quad \text{with} \quad \sum_{u\in\Gamma} \sum_{1\leq j\leq q} |c_{u,j}|^2 < \infty. \tag{5.3}$$

Let $\{e_{u,j} : u \in \Gamma, 1 \leq j \leq q\}$ be an orthonormal set and define the operator

$$A = \sum_{u\in\Gamma} \sum_{1\leq j\leq q} \psi_{\varphi_u(z_j),i'} \otimes e_{u,j}.$$

By Lemma 5.1(b) and Corollary 4.5, A is a bounded operator. By (5.3), the range of A equals $L^2_a(\mathbf{B}, dv_\alpha)$. Thus a standard argument gives us a $c > 0$ such that $\|A^*h\| \geq c\|h\|$ for every $h \in L^2_a(\mathbf{B}, dv_\alpha)$. This lower bound implies that AA^*, which we regard as an operator on the whole of $L^2(\mathbf{B}, dv_\alpha)$, is invertible on the subspace $L^2_a(\mathbf{B}, dv_\alpha)$. In other words, there is a bounded operator X on $L^2_a(\mathbf{B}, dv_\alpha)$ such that $AA^*Xh = h$ for every $h \in L^2_a(\mathbf{B}, dv_\alpha)$. Now define the operator T by the formula $T(h + g) = Xh$ for $h \in L^2_a(\mathbf{B}, dv_\alpha)$ and $g \in L^2(\mathbf{B}, dv_\alpha) \ominus L^2_a(\mathbf{B}, dv_\alpha)$. Then $\|T\| = \|X\| < \infty$ and $P = AA^*T$. To complete the proof, simply observe that $AA^* = E_{\Gamma,z_1,i'} + \cdots + E_{\Gamma,z_q,i'}$. □

Proposition 5.5. *Let $0 < s \leq 1$ be given. Then there is a constant $C_{5.5}$ which depends only on n, α and s such that*

$$\||M_f P|^s\|_\Phi \leq C_{5.5} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and very symmetric gauge function Φ.

Proof. Given any $0 < s \leq 1$, we pick an $i \in \mathbf{N}$ such that $si > 4n$. Then set $i' = 3i+n+1+\delta$, where $\delta \in \mathbf{Z}_+$ satisfies (4.1). For this i', Lemma 5.4 provides an a-separated set Γ in $\mathbf{B}$ for some $a > 0$, a finite set $\{z_1, \dots, z_q\}$ in $\mathbf{B}$ and a bounded operator T such that (5.2) holds. Since $si > 4n$ and $i' = 3i + n + 1 + \delta$, by Lemma 5.2, for every $j \in \{1, \dots, q\}$ we have

$$\||M_f E_{\Gamma,z_j,i'}|^s\|_\Phi \leq C_{5.2}(z_j) \Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}) \tag{5.4}$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ, where $C_{5.2}(z_j)$ depends only on n, α, s, i, Γ and z_j. By (5.2), we have $M_f P = M_f E_{\Gamma,z_1,i'}T + \cdots + M_f E_{\Gamma,z_q,i'}T$. Applying Lemma 3.4, Remark 3.5 and Lemma 3.3 to this sum, we obtain

$$\begin{aligned}\||M_fP|^s\|_\Phi &\le 2(\||M_fE_{\Gamma,z_1,i'}T|^s\|_\Phi + \cdots + \||M_fE_{\Gamma,z_q,i'}T|^s\|_\Phi) \\ &\le 2\|T\|^s(\||M_fE_{\Gamma,z_1,i'}|^s\|_\Phi + \cdots + \||M_fE_{\Gamma,z_q,i'}|^s\|_\Phi).\end{aligned}$$

Combining this with (5.4), we have

$$\||M_fP|^s\|_\Phi \le 2\|T\|^s(C_{5.2}(z_1) + \cdots + C_{5.2}(z_q))\Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ. $\square$

6. Bergman balls and local projections

The cumbersome decomposition system adopted in Section 3 was designed to accommodate a disparity between the radial direction and the spherical direction of the ball. The best place to see this disparity is (4.19): the factor $2^{-\ell(1+\alpha)}$ is the best decaying rate that one can hope for in the radial direction. In contrast, the factor 2^{-2im} in (4.19), which is the decaying rate in the spherical direction, represents artificial improvement: one can pencil in as large an i as one pleases. But once we have proved Proposition 5.5, we no longer need to be concerned the disparity between the two directions. For the rest of the paper, it will simplify matters considerably if we adopt a new decomposition system in terms of balls in the Bergman metric.

For each $(k,j) \in I$, we fix the point

$$w_{k,j} = (1 - 2^{-2k-1})u_{k,j}$$

for the rest of the paper the paper. Recalling (3.6) and (3.7), we have $w_{k,j} \in T_{k,j} \subset Q_{k,j}$ for every $(k,j) \in I$, and we think of $w_{k,j}$ as the "center" for $T_{k,j}$.

Lemma 6.1. (1) *There is a $\tau_0 > 0$ such that $D(w_{k,j}, \tau_0) \cap D(w_{t,h}, \tau_0) = \emptyset$ for all $(k,j) \neq (t,h)$ in I.*
(2) *There is a $\tau_0 < \tau < \infty$ such that $D(w_{k,j}, \tau) \supset Q_{k,j}$ for every $(k,j) \in I$.*
(3) *There is an $N_0 \in \mathbf{N}$ such that* $\text{card}\{(t,h) \in I : D(w_{k,j}, \tau+1) \cap D(w_{t,h}, \tau+1) \neq \emptyset\} \le N_0$ *for every $(k,j) \in I$.*

Since the proof of Lemma 6.1 is completely elementary, it is best suited for the Appendix, and the reader can find it there.

Definition 6.2. For each $(k,j) \in I$, we denote

$$D_{k,j} = D(w_{k,j}, \tau), \quad G_{k,j} = D(w_{k,j}, \tau + 1), \quad U_{k,j} = D(w_{k,j}, 3\tau + 3)$$

and $I_{k,j} = \{(t,h) \in I : G_{k,j} \cap G_{t,h} \neq \emptyset\}$.

Note that

$$\text{(6.1)} \qquad \text{if} \;\; (t,h) \in I_{k,j}, \;\; \text{then} \;\; U_{t,h} \supset G_{k,j} \supset Q_{k,j}.$$

Also note that

$$\overline{D(0,\tau)} = \left\{ w \in \mathbf{B} : |w| \le \frac{e^{2\tau}-1}{e^{2\tau}+1} \right\} \quad \text{and} \quad D(0,\tau+1) = \left\{ w \in \mathbf{B} : |w| < \frac{e^{2\tau+2}-1}{e^{2\tau+2}+1} \right\}.$$

We now fix a C^∞ function η on $[0,\infty)$ with the following properties:

(i) $0 \le \eta(x) \le 1$ for every $x \in [0,\infty)$;

(ii) $\eta(x^2) = 1$ if $0 \le x \le (e^{2\tau}-1)/(e^{2\tau}+1)$;

(iii) $\eta(x^2) = 0$ if $x \ge (e^{2\tau+2}-1)/(e^{2\tau+2}+1)$.

For each $(k,j) \in I$, define

$$\eta_{k,j}(\zeta) = \eta(|\varphi_{w_{k,j}}(\zeta)|^2), \quad \zeta \in \mathbf{B}.$$

Then each $\eta_{k,j}$ is a C^∞ function on $\mathbf{B}$. Furthermore, because $\varphi_{w_{k,j}}(\overline{D_{k,j}}) = \overline{D(0,\tau)}$ and $\varphi_{w_{k,j}}(G_{k,j}) = D(0,\tau+1)$, we have

$$\eta_{k,j} = 1 \;\; \text{on} \;\; \overline{D_{k,j}} \;\; \text{and} \;\; \eta_{k,j} = 0 \;\; \text{on} \;\; \mathbf{B}\backslash G_{k,j}.$$

By Lemma 6.1(3), we have $\sum_{(k,j)\in I} \eta_{k,j} \le N_0$ on $\mathbf{B}$. On the other hand, since $\cup_{(k,j)\in I} T_{k,j} = \mathbf{B}$, we have $\sum_{(k,j)\in I} \eta_{k,j} \ge 1$ on $\mathbf{B}$. Now, for every $(k,j) \in I$ define

$$\gamma_{k,j} = \frac{\eta_{k,j}}{\sum_{(t,h)\in I} \eta_{t,h}}.$$

This gives us a family of C^∞-partition of unity on $\mathbf{B}$. More specifically, we have

(A) $\sum_{(k,j)\in I} \gamma_{k,j} = 1$ on $\mathbf{B}$;

(B) for each $(k,j) \in I$, $\gamma_{k,j} = 0$ on $\mathbf{B}\backslash G_{k,j}$.

Lemma 6.3. *There is a constant* $C_{6.3}$ *such that* $\|\rho\bar{\partial}_\nu\gamma_{k,j}\|_\infty \le C_{6.3}$ *and* $\|\rho^{1/2}\bar{L}_{\nu,\mu}\gamma_{k,j}\|_\infty \le C_{6.3}$ *for all* $(k,j) \in I$, $\nu \in \{1,\dots,n\}$ *and* $\mu \ne \nu$ *in* $\{1,\dots,n\}$.

Proof. Write $H = \sum_{(t,h)\in I} \eta_{t,h}$. Then $H \ge 1$ on $\mathbf{B}$. Straightforward differentiation yields

$$\begin{aligned} \bar{\partial}_\nu\gamma_{k,j} &= H^{-1}\bar{\partial}_\nu\eta_{k,j} - H^{-2}\eta_{k,j}\bar{\partial}_\nu H = H^{-1}\bar{\partial}_\nu\eta_{k,j} - H^{-2}\eta_{k,j}\sum_{(t,h)\in I_{k,j}} \bar{\partial}_\nu\eta_{t,h} \\ &= H^{-1}\eta'(|\varphi_{w_{k,j}}|^2)\langle\varphi_{w_{k,j}}, \partial_\nu\varphi_{w_{k,j}}\rangle - H^{-2}\eta_{k,j}\sum_{(t,h)\in I_{k,j}} \eta'(|\varphi_{w_{t,h}}|^2)\langle\varphi_{w_{t,h}}, \partial_\nu\varphi_{w_{t,h}}\rangle, \end{aligned}$$

where the $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbf{C}^n$. Similarly, for $\mu \ne \nu$ in $\{1,\dots,n\}$ we have

$$\bar{L}_{\nu,\mu}\gamma_{k,j} = H^{-1}\eta'(|\varphi_{w_{k,j}}|^2)\langle\varphi_{w_{k,j}}, L_{\nu,\mu}\varphi_{w_{k,j}}\rangle - \frac{\eta_{k,j}}{H^2}\sum_{(t,h)\in I_{k,j}} \eta'(|\varphi_{w_{t,h}}|^2)\langle\varphi_{w_{t,h}}, L_{\nu,\mu}\varphi_{w_{t,h}}\rangle.$$

Obviously, η' is bounded on $[0, \infty)$. Thus, combining the bounds provided by Lemma 2.4 with Lemma 6.1(3), the conclusion of the lemma follows. □

Let E be any Borel set in $\mathbf{B}$. Then by $L^2(E, dv_\alpha)$ we mean the collection of functions g in $L^2(\mathbf{B}, dv_\alpha)$ satisfying the condition $g = 0$ on $\mathbf{B}\backslash E$. The point is that we consider each element in $L^2(E, dv_\alpha)$ as a function on the whole of the unit ball $\mathbf{B}$.

For each $(k, j) \in I$, let $\mathcal{B}_{k,j}$ be the collection of functions h in $L^2(U_{k,j}, dv_\alpha)$ that are analytic on $U_{k,j}$. That is, $\mathcal{B}_{k,j}$ consists of functions h in $L^2(\mathbf{B}, dv_\alpha)$ that are analytic on $U_{k,j}$ and identically zero on $\mathbf{B}\backslash U_{k,j}$. Obviously, $\mathcal{B}_{k,j}$ is a closed linear subspace of $L^2(\mathbf{B}, dv_\alpha)$. One may think of $\mathcal{B}_{k,j}$ as a kind of "Bergman space", but keep in mind that the measure in question is the restriction of the weighted volume measure dv_α to $U_{k,j}$. For each $(k, j) \in I$, let

$$P_{k,j} : L^2(\mathbf{B}, dv_\alpha) \to \mathcal{B}_{k,j}$$

be the orthogonal projection. We consider each $P_{k,j}$ as a local projection, and it performs a little magic:

Lemma 6.4. *For all $f, g \in L^2(\mathbf{B}, dv_\alpha)$ and $(k, j) \in I$, we have*

$$\langle f - Pf, \chi_{U_{k,j}} g - P_{k,j} g\rangle = \langle \chi_{U_{k,j}} f - P_{k,j} f, \chi_{U_{k,j}} g - P_{k,j} g\rangle.$$

Proof. Note that $\langle h, \chi_{U_{k,j}} g - P_{k,j} g\rangle = 0$ for every $h \in L^2(\mathbf{B}, dv_\alpha)$ that is analytic on $U_{k,j}$. Therefore

$$\begin{aligned}\langle f - Pf, \chi_{U_{k,j}} g - P_{k,j} g\rangle = \langle f, \chi_{U_{k,j}} g - P_{k,j} g\rangle &= \langle \chi_{U_{k,j}} f, \chi_{U_{k,j}} g - P_{k,j} g\rangle \\ &= \langle \chi_{U_{k,j}} f - P_{k,j} f, \chi_{U_{k,j}} g - P_{k,j} g\rangle\end{aligned}$$

as promised. □

For all $f \in L^2(\mathbf{B}, dv_\alpha)$ and $(k, j) \in I$, we define

$$M(f; k, j) = \left(\frac{1}{v_\alpha(U_{k,j})} \int_{U_{k,j}} |f - P_{k,j} f|^2 dv_\alpha \right)^{1/2}.$$

Proposition 6.5. *There is a constant $C_{6.5}$ such that the following estimates hold: Every $f \in L^2(\mathbf{B}, dv_\alpha)$ admits a decomposition*

$$f = f^{(1)} + f^{(2)}$$

with $f^{(2)} \in C^\infty(\mathbf{B})$ such that for every $(k, j) \in I$, we have

$$A^2(f^{(1)}; Q_{k,j}) \le C_{6.5} \sum_{(t,h)\in I_{k,j}} M^2(f; t, h),$$

$$A^2(\rho|\bar\partial f^{(2)}|; Q_{k,j}) \le C_{6.5} \sum_{(t,h)\in I_{k,j}} M^2(f; t, h) \quad \text{and}$$

$$A^2(\rho^{1/2}|\bar\partial f^{(2)} \wedge \bar\partial\rho|; Q_{k,j}) \le C_{6.5} \sum_{(t,h)\in I_{k,j}} M^2(f; t, h).$$

Proof. If $(t,h) \in I_{k,j}$, then $U_{t,h} \subset D(w_{k,j}, 5\tau+5)$. By Lemma 3.10, there is a C_1 such that

$$v_\alpha(U_{t,h}) \le C_1 v_\alpha(Q_{k,j}) \quad \text{whenever} \quad (t,h) \in I_{k,j}. \tag{6.2}$$

Using the partition of unit $\{\gamma_{k,j} : (k,j) \in I\}$, for a given $f \in L^2(\mathbf{B}, dv_\alpha)$ we define

$$f^{(2)} = \sum_{(k,j)\in I} \gamma_{k,j} P_{k,j} f \quad \text{and} \quad f^{(1)} = f - f^{(2)} = \sum_{(k,j)\in I} (f - P_{k,j} f)\gamma_{k,j}.$$

If $(t,h) \notin I_{k,j}$, then $\gamma_{t,h} = 0$ on $G_{k,j} \supset Q_{k,j}$. Therefore for every $(k,j) \in I$ we have

$$\int_{Q_{k,j}} |f^{(1)}|^2 dv_\alpha = \int_{Q_{k,j}} \left| \sum_{(t,h)\in I_{k,j}} (f - P_{t,h} f)\gamma_{t,h} \right|^2 dv_\alpha \le N_0 \sum_{(t,h)\in I_{k,j}} \int_{Q_{k,j}} |f - P_{t,h} f|^2 dv_\alpha,$$

where the second $\le$ follows from the Cauchy-Schwarz inequality. Recalling (6.1), we have

$$\int_{Q_{k,j}} |f^{(1)}|^2 dv_\alpha \le N_0 \sum_{(t,h)\in I_{k,j}} \int_{U_{t,h}} |f - P_{t,h} f|^2 dv_\alpha.$$

Dividing both sides by $v_\alpha(Q_{k,j})$ and using (6.2), we find that

$$A^2(f^{(1)}; Q_{k,j}) \le N_0 C_1 \sum_{(t,h)\in I_{k,j}} M^2(f; t, h),$$

proving the first inequality.

Since each $\gamma_{k,j}$ vanishes on $\mathbf{B} \backslash G_{k,j}$, by Lemma 6.1(3) we have $f^{(2)} \in C^\infty(\mathbf{B})$. Moreover, since $P_{k,j} f$ is analytic on $U_{k,j}$, for each $\nu \in \{1, \dots, n\}$ we have

$$\bar\partial_\nu f^{(2)} = \sum_{(k,j)\in I} P_{k,j} f \cdot \bar\partial_\nu \gamma_{k,j}.$$

Thus if $\zeta \in G_{k,j}$, then

$$(\bar\partial_\nu f^{(2)})(\zeta) = \sum_{(t,h)\in I_{k,j}} (P_{t,h} f)(\zeta)(\bar\partial_\nu \gamma_{t,h})(\zeta) = \sum_{(t,h)\in I_{k,j}} \{(P_{t,h} f)(\zeta) - (P_{k,j} f)(\zeta)\}(\bar\partial_\nu \gamma_{t,h})(\zeta),$$

where the second $=$ is due to the fact that $\sum_{(t,h)\in I} \bar\partial_\nu \gamma_{t,h} = \bar\partial_\nu 1 = 0$. Combining this with Lemma 6.3, we obtain

$$\rho(\zeta) |(\bar\partial_\nu f^{(2)})(\zeta)| \le C_{6.3} \sum_{(t,h)\in I_{k,j}} |(P_{t,h} f)(\zeta) - (P_{k,j} f)(\zeta)| \quad \text{if} \quad \zeta \in G_{k,j}.$$

Using the Cauchy-Schwarz inequality, Lemma 6.1(3) and (6.1) again, we have

$$\begin{aligned}\int_{Q_{k,j}} |\rho\bar{\partial}_\nu f^{(2)}|^2 dv_\alpha &\le N_0 C_{6.3}^2 \sum_{(t,h)\in I_{k,j}} \int_{Q_{k,j}} |P_{t,h}f - P_{k,j}f|^2 dv_\alpha \\ &\le N_0 C_{6.3}^2 \sum_{(t,h)\in I_{k,j}} 2\left(\int_{U_{t,h}} |P_{t,h}f - f|^2 dv_\alpha + \int_{U_{k,j}} |f - P_{k,j}f|^2 dv_\alpha\right).\end{aligned}$$

Again, dividing both sides by $v_\alpha(Q_{k,j})$ and using (6.2), we have

$$A^2(\rho\bar{\partial}_\nu f^{(2)}; Q_{k,j}) \le 2(N_0 + N_0^2) C_{6.3}^2 C_1 \sum_{(t,h)\in I_{k,j}} M^2(f;t,h).$$

Since this holds for every $\nu \in \{1, \dots, n\}$, we obtain the second inequality. The proof of the third inequality is similar and will be omitted. □

Lemma 6.6. *Let $0 < s \le 1$, and suppose that $i \in \mathbf{N}$ satisfies the condition $si > n$. Then for any given $\epsilon > 0$, there is an $0 < R < \infty$ such that*

$$\sup_{(k,j)\in I} v_\alpha^{s/2}(U_{k,j}) \sum_{\substack{(t,h)\in I \\ \beta(w_{k,j}, w_{t,h})\ge R}} \sup_{\zeta\in U_{k,j}} |\psi_{w_{t,h},i}(\zeta)|^s \le \epsilon.$$

This lemma is in fact a discrete variant of the familiar Forelli-Rudin estimates [12,17, 18,21]. The interested reader can find its proof in the Appendix.

Lemma 6.7. *Let $0 < p < \infty$. Then for every pair of finite-rank operators A and B,*

$$\sum_{\nu=1}^{\infty} (s_\nu(AB))^p \le 2 \sum_{\nu=1}^{\infty} (s_\nu(A))^p (s_\nu(B))^p.$$

Proof. It is well known that $s_{\mu+\nu-1}(AB) \le s_\mu(A)s_\nu(B)$ for all $\mu, \nu \in \mathbf{N}$ [10,page 30]. In particular, we have $s_{2\nu-1}(AB) \le s_\nu(A)s_\nu(B)$ and $s_{2\nu}(AB) \le s_{\nu+1}(A)s_\nu(B) \le s_\nu(A)s_\nu(B)$ for every $\nu \in \mathbf{N}$. Hence for any $0 < p < \infty$, we have

$$(s_{2\nu-1}(AB))^p \le (s_\nu(A)s_\nu(B))^p \quad \text{and} \quad (s_{2\nu}(AB))^p \le (s_\nu(A)s_\nu(B))^p$$

for every $\nu \in \mathbf{N}$. The lemma obviously follows from these inequalities. □

Proposition 6.8. *Let $0 < s \le 1$ be given. Then there is a constant $C_{6.8}$ which depends only on n, α and s such that*

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \le C_{6.8} \||H_f|^s\|_\Phi$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ.

Proof. We begin by fixing certain constants. Given $0 < s \le 1$, pick an $i_0 \in \mathbf{N}$ such that $si_0 > 4n$. Then set $i = 3i_0 + n + 1 + \delta$, where $\delta \in \mathbf{Z}_+$ satisfies (4.1). Let $\{e_{k,j} : (k,j) \in I\}$ be an orthonormal set. By Lemma 6.1(1) and Corollary 4.5, there is a C_1 such that

$$\left\| \sum_{(k,j)\in J} \psi_{w_{k,j},i} \otimes e_{k,j} \right\| \le C_1 \tag{6.3}$$

for every subset J of I. Also, once this i is so fixed, by Lemmas 3.10 and 3.11, there is a $c > 0$ which depends only on n, α and i such that

$$v_\alpha^{1/2}(U_{k,j}) \inf_{\zeta\in U_{k,j}} |\psi_{w_{k,j},i}(\zeta)| \ge c \tag{6.4}$$

for every $(k,j) \in I$. For $R > 0$, write

$$\epsilon(R) = \sup_{(k,j)\in I} v_\alpha^{s/2}(U_{k,j}) \sum_{\substack{(t,h)\in I \\ \beta(w_{k,j},w_{t,h})\ge R}} \sup_{\zeta\in U_{k,j}} |\psi_{w_{t,h},i}(\zeta)|^s.$$

For this i, Lemma 6.6 allows us to pick an $R > 6\tau + 7$ such that

$$2\epsilon(R) \le c^s/2, \tag{6.5}$$

and this R is so fixed for the rest of the proof.

By Lemmas 6.1(1) and 5.1(a), there is an $M \in \mathbf{N}$ such that

$$\mathrm{card}\{(t,h) \in I : \beta(w_{k,j}, w_{t,h}) < R\} \le M$$

for every $(k,j) \in I$. By a standard maximality argument, there is a partition $I = E_1 \cup \cdots \cup E_M$ such that for every $m \in \{1, \dots, M\}$, we have $\beta(w_{k,j}, w_{t,h}) \ge R$ whenever $(k,j), (t,h) \in E_m$ and $(k,j) \ne (t,h)$. We will show that $C_{6.8} = 8M(C_1^s/c^s)$ suffices for the proposition.

Let a symmetric gauge function Φ be given, and let Φ^* be its dual. Fix an $m \in \{1, \dots, M\}$ for the moment. Given an $f \in L^2(\mathbf{B}, dv_\alpha)$, consider any

$$J_m \subset \{(k,j) \in E_m : M(f;k,j) \ne 0\} \quad \text{with} \quad \mathrm{card}(J_m) < \infty. \tag{6.6}$$

For each $(k,j) \in J_m$, define the unit vector

$$g_{k,j} = \frac{\chi_{U_{k,j}} f\psi_{w_{k,j},i} - P_{k,j}(f\psi_{w_{k,j},i})}{\|\chi_{U_{k,j}} f\psi_{w_{k,j},i} - P_{k,j}(f\psi_{w_{k,j},i})\|} \tag{6.7}$$

in $L^2(U_{k,j}, dv_\alpha)$. Let $\{b_{k,j} : (k,j) \in J_m\}$ be a family of non-negative numbers. We define the finite-rank operator

$$A = \sum_{(k,j)\in J_m} b_{k,j} e_{k,j} \otimes g_{k,j}.$$

Note that the choice $R > 6\tau+7$ ensures that for $(k,j) \neq (t,h)$ in E_m, we have $U_{k,j} \cap U_{t,h} = \emptyset$. Hence $\langle g_{k,j}, g_{t,h}\rangle = 0$ for $(k,j) \neq (t,h)$ in J_m. Consequently,

$$\Phi^*(\{(s_\nu(A))^s\}_{\nu\in\mathbf{N}}) = \Phi^*(\{b^s_{k,j}\}_{(k,j)\in J_m}). \tag{6.8}$$

Also, define the operator

$$T = \sum_{(k,j)\in J_m} \psi_{w_{k,j},i} \otimes e_{k,j}.$$

Then $\|T\| \leq C_1$ by (6.3).

By straightforward multiplication,

$$AH_fT = \sum_{(k,j),(t,h)\in J_m} b_{k,j}\langle H_f\psi_{w_{t,h},i}, g_{k,j}\rangle e_{k,j} \otimes e_{t,h} = Y + Z,$$

where

$$Y = \sum_{(k,j)\in J_m} b_{k,j}\langle H_f\psi_{w_{k,j},i}, g_{k,j}\rangle e_{k,j} \otimes e_{k,j} \quad \text{and}$$
$$Z = \sum_{(k,j)\in J_m} \sum_{\substack{(t,h)\neq(k,j)\\(t,h)\in J_m}} b_{k,j}\langle H_f\psi_{w_{t,h},i}, g_{k,j}\rangle e_{k,j} \otimes e_{t,h}.$$

Since $Y = AH_fT - Z$, an application of Lemma 3.4 to the symmetric gauge function for the trace class $\mathcal{C}_1$ yields

$$\||Y|^s\|_1 \leq 2\||AH_fT|^s\|_1 + 2\||Z|^s\|_1. \tag{6.9}$$

By (6.7) and Lemma 6.4, we have

$$\begin{aligned}\langle H_f\psi_{w_{k,j},i}, g_{k,j}\rangle &= \|\chi_{U_{k,j}} f\psi_{w_{k,j},i} - P_{k,j}(f\psi_{w_{k,j},i})\| \\ &= \|\chi_{U_{k,j}}\psi_{w_{k,j},i}(f - \psi^{-1}_{w_{k,j},i}P_{k,j}(f\psi_{w_{k,j},i}))\|.\end{aligned}$$

Recalling (6.4), we have

$$\langle H_f\psi_{w_{k,j},i}, g_{k,j}\rangle \geq c\frac{\|\chi_{U_{k,j}} f - \psi^{-1}_{w_{k,j},i}P_{k,j}(f\psi_{w_{k,j},i})\|}{v_\alpha^{1/2}(U_{k,j})} \geq c\frac{\|\chi_{U_{k,j}} f - P_{k,j}f\|}{v_\alpha^{1/2}(U_{k,j})} = cM(f;k,j),$$

where the second $\geq$ follows from the facts the that $\psi^{-1}_{w_{k,j},i}P_{k,j}(f\psi_{w_{k,j},i}) \in \mathcal{B}_{k,j}$ and that $P_{k,j}f$ is the element in $\mathcal{B}_{k,j}$ that minimizes the norm $\|\chi_{U_{k,j}} f - h\|$, $h \in \mathcal{B}_{k,j}$. Thus

$$\||Y|^s\|_1 = \sum_{(k,j)\in J_m} \{b_{k,j}\langle H_f\psi_{w_{k,j},i}, g_{k,j}\rangle\}^s \geq c^s \sum_{(k,j)\in J_m} b^s_{k,j}M^s(f;k,j). \tag{6.10}$$

On the other hand, since $0 < s \le 1$, the orthonormality of $\{e_{k,j} : (k,j) \in I\}$ leads to

$$(6.11) \quad \| |Z|^s \|_1 \le \sum_{(k,j),(t,h)\in J_m} |\langle Ze_{t,h}, e_{k,j}\rangle|^s = \sum_{(k,j)\in J_m} \sum_{\substack{(t,h)\neq(k,j)\\(t,h)\in J_m}} b_{k,j}^s |\langle H_f \psi_{w_{t,h},i}, g_{k,j}\rangle|^s.$$

Using Lemma 6.4 and the norm-minimizing property of $P_{k,j}$ again, we have

$$\begin{aligned}
|\langle H_f\psi_{w_{t,h},i}, g_{k,j}\rangle| &= \frac{|\langle \chi_{U_{k,j}} f\psi_{w_{t,h},i} - P_{k,j}(f\psi_{w_{t,h},i}), \chi_{U_{k,j}} f\psi_{w_{k,j},i} - P_{k,j}(f\psi_{w_{k,j},i})\rangle|}{\|\chi_{U_{k,j}} f\psi_{w_{k,j},i} - P_{k,j}(f\psi_{w_{k,j},i})\|}\\
&\le \|\chi_{U_{k,j}} f\psi_{w_{t,h},i} - P_{k,j}(f\psi_{w_{t,h},i})\| \le \|\chi_{U_{k,j}} f\psi_{w_{t,h},i} - \psi_{w_{t,h},i} P_{k,j} f\|\\
&\le v_\alpha^{1/2}(U_{k,j}) \sup_{\zeta\in U_{k,j}} |\psi_{w_{t,h},i}(\zeta)| M(f;k,j).
\end{aligned}$$

Substituting this in (6.11), since $\beta(w_{k,j}, w_{t,h}) \ge R$ for $(k,j) \neq (t,h)$ in E_m, we obtain

$$\begin{aligned}
\| |Z|^s \|_1 &\le \sum_{(k,j)\in J_m} b_{k,j}^s M^s(f;k,j) v_\alpha^{s/2}(U_{k,j}) \sum_{\substack{(t,h)\neq(k,j)\\(t,h)\in J_m}} \sup_{\zeta\in U_{k,j}} |\psi_{w_{t,h},i}(\zeta)|^s\\
&\le \epsilon(R) \sum_{(k,j)\in J_m} b_{k,j}^s M^s(f;k,j).
\end{aligned}$$

Combining this with (6.9) and (6.10), we find that

$$c^s \sum_{(k,j)\in J_m} b_{k,j}^s M^s(f;k,j) \le 2\| |AH_fT|^s \|_1 + 2\epsilon(R) \sum_{(k,j)\in J_m} b_{k,j}^s M^s(f;k,j).$$

Since J_m is a finite set, the sum $\sum_{(k,j)\in J_m} \cdots$ above is finite. By (6.5), $2\epsilon(R) \le c^s/2$. Thus the obvious cancellation leads to

$$(6.12) \qquad (c^s/2) \sum_{(k,j)\in J_m} b_{k,j}^s M^s(f;k,j) \le 2\| |AH_fT|^s \|_1.$$

To estimate $\| |AH_fT|^s \|_1$, we apply Lemma 6.7, which gives us

$$\| |AH_fT|^s \|_1 = \sum_{\nu=1}^\infty (s_\nu(AH_fT))^s \le 2 \sum_{\nu=1}^\infty (s_\nu(A))^s (s_\nu(H_fT))^s.$$

Applying (3.1) and (6.8) to the right-hand side, we obtain

$$\begin{aligned}
\| |AH_fT|^s \|_1 &\le 2\Phi^*(\{(s_\nu(A))^s\}_{\nu\in\mathbf{N}}) \Phi(\{(s_\nu(H_fT))^s\}_{\nu\in\mathbf{N}})\\
&= 2\Phi^*(\{b_{k,j}^s\}_{(k,j)\in J_m}) \| |H_fT|^s \|_\Phi\\
&\le 2C_1^s \Phi^*(\{b_{k,j}^s\}_{(k,j)\in J_m}) \| |H_f|^s \|_\Phi,
\end{aligned}$$

where the second $\leq$ follows from Lemma 3.3 and the fact that $\|T\| \leq C_1$. Substituting this in (6.12) and simplifying, we find that

$$\sum_{(k,j)\in J_m} b^s_{k,j} M^s(f;k,j) \leq 8(C_1^s/c^s)\Phi^*(\{b^s_{k,j}\}_{(k,j)\in J_m})\||H_f|^s\|_\Phi.$$

Since the non-negative numbers $\{b^s_{k,j} : (k,j) \in J_m\}$ are arbitrary, the duality between Φ and Φ^* (see (3.1)) implies

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in J_m}) \leq 8(C_1^s/c^s)\||H_f|^s\|_\Phi.$$

Since the above holds for every J_m given by (6.6), recalling (1.3), we conclude that

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in E_m}) \leq 8(C_1^s/c^s)\||H_f|^s\|_\Phi.$$

Finally, since this holds for every $m \in \{1, \dots, M\}$ and since $I = E_1 \cup \dots \cup E_M$, we have

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \leq \sum_{m=1}^{M} \Phi(\{M^s(f;k,j)\}_{(k,j)\in E_m}) \leq 8M(C_1^s/c^s)\||H_f|^s\|_\Phi.$$

This completes the proof. □

Lemma 6.9. *There is a constant $C_{6.9}$ such that*

$$\Phi(\{\sum_{(t,h)\in I_{k,j}} a_{t,h}\}_{(k,j)\in I}) \leq C_{6.9}\Phi(\{a_{k,j}\}_{(k,j)\in I})$$

for every set of non-negative numbers $\{a_{k,j}\}_{(k,j)\in I}$ and every symmetric gauge function Φ.

Proof. First of all, by Lemmas 6.1(1) and 5.1(a), there is an $N_1 \in \mathbf{N}$ such that

$$\text{card}\{(t,h) \in I : \beta(w_{k,j}, w_{t,h}) < 4\tau + 4\} \leq N_1 \tag{6.13}$$

for every $(k,j) \in I$. Let non-negative numbers $\{a_{k,j}\}_{(k,j)\in I}$ be given. For every $(k,j) \in I$, there is a $\pi(k,j) \in I_{k,j}$ such that $a_{\pi(k,j)} \geq a_{t,h}$ for every $(t,h) \in I_{k,j}$. Thus $\sum_{t,h} a_{t,h} \leq \text{card}(I_{k,j})a_{\pi(k,j)} \leq N_0 a_{\pi(k,j)}$, where the second $\leq$ follows from Lemma 6.1(3). Hence

$$\Phi(\{\sum_{(t,h)\in I_{k,j}} a_{t,h}\}_{(k,j)\in I}) \leq N_0\Phi(\{a_{\pi(k,j)}\}_{(k,j)\in I}). \tag{6.14}$$

Obviously, $\beta(w_{k,j}, w_{\pi(k,j)}) < 2\tau+2$ for every $(k,j) \in I$. Thus for any pair of $(k,j), (k',j') \in I$, if $\pi(k,j) = \pi(k',j')$, then $\beta(w_{k,j}, w_{k',j'}) < 4\tau + 4$ by the triangle inequality. By (6.13), the map $\pi : I \mapsto I$ is at most N_1-to-1. Applying Lemma 3.1, we obtain $\Phi(\{a_{\pi(k,j)}\}_{(k,j)\in I}) \leq N_1\Phi(\{a_{k,j}\}_{(k,j)\in I})$. Recalling (6.14), the lemma holds for the constant $C_{6.9} = N_0N_1$. □

Proposition 6.10. *Let $0 < s \leq 1$ be given, and let $i \in \mathbf{Z}_+$ satisfy the condition $s(n+1+\alpha+2i) > 2n$. Let $a > 0$ also be given. Then there is a constant $C_{6.10}$ which depends only on n, α, s, i and a such that*

$$\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma}) \leq C_{6.10}\Phi(\{M^s(f;k,j)\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$, every symmetric gauge function Φ, and every a-separated set Γ in $\mathbf{B}$.

Proof. Given any $f \in L^2(\mathbf{B}, dv_\alpha)$, let $f = f^{(1)} + f^{(2)}$ be the decomposition provided by Proposition 6.5. Applying Proposition 2.2 to $f^{(2)}\psi_{z,i} - P(f^{(2)}\psi_{z,i})$, $z \in \mathbf{B}$, we have

$$\begin{aligned}\|H_f\psi_{z,i}\| &\leq \|H_{f^{(1)}}\psi_{z,i}\| + \|H_{f^{(2)}}\psi_{z,i}\| \leq \|f^{(1)}\psi_{z,i}\| + \|H_{f^{(2)}}\psi_{z,i}\| \\ &\leq \|f^{(1)}\psi_{z,i}\| + C_{2.2}\|\rho\bar\partial(f^{(2)}\psi_{z,i})\| + C_{2.2}\|\rho^{1/2}\bar\partial(f^{(2)}\psi_{z,i}) \wedge \bar\partial\rho\| \\ &= \|f^{(1)}\psi_{z,i}\| + C_{2.2}\|\rho\psi_{z,i}\bar\partial f^{(2)}\| + C_{2.2}\|\rho^{1/2}\psi_{z,i}\bar\partial f^{(2)} \wedge \bar\partial\rho\|.\end{aligned}$$

For $0 < s \leq 1$, the above implies

$$\|H_f\psi_{z,i}\|^s \leq \|f^{(1)}\psi_{z,i}\|^s + C_{2.2}^s\|\rho\psi_{z,i}\bar\partial f^{(2)}\|^s + C_{2.2}^s\|\rho^{1/2}\psi_{z,i}\bar\partial f^{(2)} \wedge \bar\partial\rho\|^s.$$

Thus it suffices to find a C that depends only on n, α, s, i and a such that

$$\begin{aligned}\Phi(\{\|f^{(1)}\psi_{z,i}\|^s\}_{z\in\Gamma}) &\leq C\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}), \\ \Phi(\{\|\rho\psi_{z,i}\bar\partial f^{(2)}\|^s\}_{z\in\Gamma}) &\leq C\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \quad \text{and} \\ \Phi(\{\|\rho^{1/2}\psi_{z,i}\bar\partial f^{(2)} \wedge \bar\partial\rho\|^s\}_{z\in\Gamma}) &\leq C\Phi(\{M^s(f;k,j)\}_{(k,j)\in I})\end{aligned}$$

for every symmetric gauge function Φ and every a-separated set Γ in $\mathbf{B}$.

Since $s(n+1+\alpha+2i) > 2n$ and Γ is a-separated, by Propositions 3.7 and 6.5,

$$\begin{aligned}\Phi(\{\|f^{(1)}\psi_{z,i}\|^s\}_{z\in\Gamma}) &\leq C_{3.7}\Phi(\{A^s(f^{(1)};Q_{k,j})\}_{(k,j)\in I}) \\ &\leq C_{3.7}C_{6.5}^{s/2}\Phi(\{\sum_{(t,h)\in I_{k,j}} M^s(f;t,h)\}_{(k,j)\in I}).\end{aligned}$$

Applying Lemma 6.9, we obtain

$$\Phi(\{\|f^{(1)}\psi_{z,i}\|^s\}_{z\in\Gamma}) \leq C_{3.7}C_{6.5}^{s/2}C_{6.9}\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}).$$

That is, the first inequality holds for the constant $C = C_{3.7}C_{6.5}^{s/2}C_{6.9}$. By the same argument, the other two inequalities also hold for the same C. □

Lemma 6.11. *Let $i \in \mathbf{Z}_+$ and $b > 0$ be given. Then there is a constant $C_{6.11}$ which depends only on n, α, i and b such that*

$$M(f;k,j) \leq C_{6.11}\|H_f\psi_{z,i}\|$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every pair of $(k,j) \in I$ and $z \in \mathbf{B}$ satisfying the condition $\beta(w_{k,j}, z) < b$.

Proof. Let $b > 0$ be given. By Lemma 3.10, there is a C_1 such that

$$v_\alpha(D(w, 2b + 3\tau + 3)) \le C_1 v_\alpha(D(w, 3\tau + 3)) \quad \text{for every} \ \ w \in \mathbf{B}. \tag{6.15}$$

Let $i \in \mathbf{Z}_+$. By Lemmas 3.10 and 3.11, there is a $c_0 > 0$ such that for every $z \in \mathbf{B}$,

$$|\psi_{z,i}(\zeta)| \ge c_0 v_\alpha^{-1/2}(D(z, b + 3\tau + 3)) \quad \text{whenever} \ \ \zeta \in D(z, b + 3\tau + 3). \tag{6.16}$$

Let $(k,j) \in I$ and $z \in \mathbf{B}$ be such that $\beta(w_{k,j}, z) < b$. Then $D(z, b+3\tau+3) \subset D(w_{k,j}, 2b+3\tau+3)$. By (6.15), we have $v_\alpha(D(z, b+3\tau+3)) \le C_1 v_\alpha(D(w_{k,j}, 3\tau+3))$, and consequently

$$v_\alpha^{-1/2}(D(z, b + 3\tau + 3)) \ge C_1^{-1/2} v_\alpha^{-1/2}(D(w_{k,j}, 3\tau + 3)) = C_1^{-1/2} v_\alpha^{-1/2}(U_{k,j}). \tag{6.17}$$

Since $U_{k,j} = D(w_{k,j}, 3\tau + 3)$, we have $U_{k,j} \subset D(z, b + 3\tau + 3)$. Writing $c_1 = c_0 C_1^{-1/2}$, from (6.16) and (6.17) we obtain

$$|\psi_{z,i}(\zeta)| \ge c_1 v_\alpha^{-1/2}(U_{k,j}) \quad \text{for every} \ \ \zeta \in U_{k,j}.$$

Hence

$$\begin{aligned} \|H_f \psi_{z,i}\| &= \|f\psi_{z,i} - P(f\psi_{z,i})\| \ge \|\chi_{U_{k,j}} \psi_{z,i}(f - \psi_{z,i}^{-1} P(f\psi_{z,i}))\| \\ &\ge c_1 v_\alpha^{-1/2}(U_{k,j}) \|\chi_{U_{k,j}} f - \chi_{U_{k,j}} \psi_{z,i}^{-1} P(f\psi_{z,i})\| \\ &\ge c_1 v_\alpha^{-1/2}(U_{k,j}) \|\chi_{U_{k,j}} f - P_{k,j} f\| = c_1 M(f; k, j), \end{aligned}$$

where the last $\ge$ again follows from the norm-minimizing property of $P_{k,j} f$. $\square$

Proposition 6.12. *Let $i \in \mathbf{Z}_+$ and $b > 0$ be given. Then there is a constant $C_{6.12}$ which depends only on n, α, i and b such that*

$$\Phi(\{M^s(f; k, j)\}_{(k,j) \in I}) \le C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z \in \Gamma})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$, every $0 < s \le 1$, every symmetric gauge function Φ, and every countable subset Γ of **B** *with the property $\cup_{z \in \Gamma} D(z, b) = \mathbf{B}$.*

Proof. Let $b > 0$ be given. Then by Lemmas 6.1 and 5.1, there is an $N \in \mathbf{N}$ such that

$$\text{card}\{(k', j') \in I : \beta(w_{k,j}, w_{k',j'}) < 2b\} \le N \quad \text{for every} \ \ (k,j) \in I. \tag{6.18}$$

Let Γ be a countable subset of $\mathbf{B}$ with the property $\cup_{z \in \Gamma} D(z, b) = \mathbf{B}$. Then for every $(k,j) \in I$, there is a $z_{k,j} \in \Gamma$ such that $\beta(w_{k,j}, z_{k,j}) < b$. Let $i \in \mathbf{Z}_+$ also be given. By Lemma 6.11, we have

$$M(f; k, j) \le C_{6.11} \|H_f \psi_{z_{k,j}, i}\|$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every $(k,j) \in I$, where $C_{6.11}$ depends only on n, α, i and b. Hence for every $0 < s \le 1$ and every symmetric gauge function Φ we have

$$\Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \le \max\{C_{6.11}, 1\}\Phi(\{\|H_f\psi_{z_{k,j},i}\|^s\}_{(k,j)\in I}). \tag{6.19}$$

If $(k,j), (k',j') \in I$ are such that $z_{k,j} = z_{k',j'}$, then

$$\beta(w_{k,j}, w_{k',j'}) \le \beta(w_{k,j}, z_{k,j}) + \beta(z_{k,j}, w_{k',j'}) = \beta(w_{k,j}, z_{k,j}) + \beta(z_{k',j'}, w_{k',j'}) < 2b.$$

Thus, by (6.18), the map $(k,j) \mapsto z_{k,j}$ is at most N-to-1. Applying Lemma 3.1, we have

$$\Phi(\{\|H_f\psi_{z_{k,j},i}\|^s\}_{(k,j)\in I}) \le N\Phi(\{\|H_f\psi_{z,i}\|^s\}_{z\in\Gamma}).$$

The combination of this with (6.19) proves the proposition. $\square$

7. Proof of Theorem 1.2

We need one more proposition for the proof of the upper bound in Theorem 1.2.

Proposition 7.1. *Set $C_{7.1} = 2(1+\sqrt{2}C_{2.2})$, where $C_{2.2}$ is the constant in Proposition* 2.2. *Then for every $f \in C^\infty(\mathbf{B}) \cap L^2(\mathbf{B}, dv_\alpha)$, every $0 < s \le 1$ and every symmetric symmetric gauge function Φ we have*

$$\||H_f|^s\|_\Phi \le C_{7.1}(\||M_{\rho|\bar\partial f|}P|^s\|_\Phi + \||M_{\rho^{1/2}|\bar\partial f\wedge\bar\partial\rho|}P|^s\|_\Phi). \tag{7.1}$$

Proof. Given f, s and Φ as above, it suffices to consider the case where the right-hand side of (7.1) is finite, for otherwise the inequality holds trivially. This finiteness implies that every $M_{\rho\bar\partial_i f}P$ and every $M_{\rho^{1/2}\bar L_{i,j}f}P$ is a bounded operator on $L^2(\mathbf{B}, dv_\alpha)$. Let $\mathcal{H}$ be the orthogonal sum of $n + (1/2)n(n-1)$ copies of $L^2(\mathbf{B}, dv_\alpha)$. We now define an operator

$$X : L^2_a(\mathbf{B}, dv_\alpha) \to \mathcal{H}$$

as follows: for each $h \in L^2_a(\mathbf{B}, dv_\alpha)$, the first n components of Xh are $(\rho\bar\partial_1 f)h, \dots, (\rho\bar\partial_n f)h$, while the other $(1/2)n(n-1)$ components of Xh are $(\rho^{1/2}\bar L_{i,j}f)h$, arranged according to a fixed enumeration of the pairs $i < j$ in $\{1, \dots, n\}$. Then obviously we have

$$\|Xh\|^2 = \langle X^*Xh, h\rangle = \|M_{\rho|\bar\partial f|}h\|^2 + \|M_{\rho^{1/2}|\bar\partial f\wedge\bar\partial\rho|}h\|^2,$$

$h \in L^2_a(\mathbf{B}, dv_\alpha)$. For $h \in L^2_a(\mathbf{B}, dv_\alpha)$, its analyticity leads to $\bar\partial(fh) = h\bar\partial f$. Hence

$$\|Xh\|^2 = \|\rho\bar\partial(fh)\|^2 + \|\rho^{1/2}\bar\partial(fh)\wedge\bar\partial\rho\|^2 \ge \frac{1}{2}(\|\rho\bar\partial(fh)\| + \|\rho^{1/2}\bar\partial(fh)\wedge\bar\partial\rho\|)^2$$

for every $h \in L^2_a(\mathbf{B}, dv_\alpha)$. Applying Proposition 2.2, for every $g \in H^\infty(\mathbf{B})$ we have

$$\|H_f g\| = \|fg - P(fg)\| \le C_{2.2}(\|\rho\bar\partial(fg)\| + \|\rho^{1/2}\bar\partial(fg)\wedge\bar\partial\rho\|) \le \sqrt{2}C_{2.2}\|Xg\|.$$

For $h \in L^2_a(\mathbf{B}, dv_\alpha)$ and $0 < r < 1$, the function h_r defined by the formula $h_r(z) = h(rz)$ belongs to $H^\infty(\mathbf{B})$. Thus an obvious application of Fatou's lemma in the above gives us

$$\|H_f h\| = \|fh - P(fh)\| \le \sqrt{2}C_{2.2}\|Xh\| \quad \text{for every} \quad h \in L^2_a(\mathbf{B}, dv_\alpha).$$

By Lemma 3.13, there is an operator $T : \mathcal{H} \to L^2(\mathbf{B}, dv_\alpha)$ with $\|T\| \le \sqrt{2}C_{2.2}$ such that

$$H_f = TX.$$

Thus it follows from Lemma 3.3 that

$$\| |H_f|^s \|_\Phi \le \|T\|^s \| |X|^s \|_\Phi \le (\sqrt{2}C_{2.2})^s \| |X|^s \|_\Phi \le (1 + \sqrt{2}C_{2.2}) \| |X|^s \|_\Phi. \tag{7.2}$$

To estimate $\| |X|^s \|_\Phi$, write $F = \rho|\bar{\partial} f|$ and $G = \rho^{1/2}|\bar{\partial} f \wedge \bar{\partial}\rho|$. Then note that

$$X^*X = PM_{F^2}P + PM_{G^2}P = (M_F P)^* M_F P + (M_G P)^* M_G P.$$

By Lemma 3.4 and Remark 3.5, we have

$$\begin{aligned} \| |X|^s \|_\Phi &= \|(X^*X)^{s/2}\|_\Phi \le 2\|((M_F P)^* M_F P)^{s/2}\|_\Phi + 2\|((M_G P)^* M_G P)^{s/2}\|_\Phi \\ &= 2\| |M_F P|^s \|_\Phi + 2\| |M_G P|^s \|_\Phi = 2(\| |M_{\rho|\bar{\partial} f|} P|^s \|_\Phi + \| |M_{\rho^{1/2}|\bar{\partial} f \wedge \bar{\partial}\rho|} P|^s \|_\Phi). \end{aligned}$$

Combining this with (7.2), the proposition follows. □

At this point, we are finally ready to assemble the previous steps and present

Proof of Theorem 1.2. Let s, i, Γ, f and Φ be given as in the statement of the theorem. Applying Propositions 6.10 and 6.8, we obtain

$$\Phi(\{\|H_f \psi_{z,i}\|^s\}_{z \in \Gamma}) \le C_{6.10}\Phi(\{M^s(f; k, j)\}_{(k,j) \in I}) \le C_{6.10}C_{6.8}\| |H_f|^s \|_\Phi,$$

which establishes the lower bound in Theorem 1.2.

To prove the upper bound, let $f = f^{(1)} + f^{(2)}$ be the decomposition provided by Proposition 6.5. Then by Lemma 3.4 and Remark 3.5, we have

$$\| |H_f|^s \|_\Phi \le 2^{1-s}(\| |H_{f^{(1)}}|^s \|_\Phi + \| |H_{f^{(2)}}|^s \|_\Phi). \tag{7.3}$$

Since $H_{f^{(1)}} = (1 - P)M_{f^{(1)}}P$, it follows from Lemma 3.3 and Proposition 5.5 that

$$\| |H_{f^{(1)}}|^s \|_\Phi \le \| |M_{f^{(1)}} P|^s \|_\Phi \le C_{5.5}\Phi(\{A^s(f^{(1)}; Q_{k,j})\}_{(k,j) \in I}). \tag{7.4}$$

Since $0 < s/2 < 1$, it follows from Propositions 6.5 that

$$A^s(f^{(1)}; Q_{k,j}) \le C_{6.5}^{s/2} \sum_{(t,h) \in I_{k,j}} M^s(f; t, h)$$

for every $(k,j) \in I$. Substituting this in (7.4) and then applying Lemma 6.9 and Proposition 6.12, we obtain

$$\begin{aligned} \| |H_{f^{(1)}}|^s \|_\Phi \le \| |M_{f^{(1)}} P|^s \|_\Phi &\le C_{5.5} C_{6.5}^{s/2} C_{6.9} \Phi(\{M^s(f;k,j)\}_{(k,j)\in I}) \\ &\le C_{5.5} C_{6.5}^{s/2} C_{6.9} C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z\in\Gamma}). \end{aligned} \tag{7.5}$$

To bound $\| |H_{f^{(2)}}|^s \|_\Phi$, we first apply Proposition 7.1, which gives us

$$\| |H_{f^{(2)}}|^s \|_\Phi \le C_{7.1} (\| |M_{\rho|\bar\partial f^{(2)}|} P|^s \|_\Phi + \| |M_{\rho^{1/2}|\bar\partial f^{(2)} \wedge \bar\partial \rho|} P|^s \|_\Phi).$$

Then, applying Propositions 5.5 and 6.5, Lemma 6.9 and Proposition 6.12 in the same manner as above, we obtain

$$\| |M_{\rho|\bar\partial f^{(2)}|} P|^s \|_\Phi \le C_{5.5} C_{6.5}^{s/2} C_{6.9} C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z\in\Gamma}) \quad \text{and}$$
$$\| |M_{\rho^{1/2}|\bar\partial f^{(2)} \wedge \bar\partial \rho|} P|^s \|_\Phi \le C_{5.5} C_{6.5}^{s/2} C_{6.9} C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z\in\Gamma}).$$

That is,

$$\| |H_{f^{(2)}}|^s \|_\Phi \le 2 C_{7.1} C_{5.5} C_{6.5}^{s/2} C_{6.9} C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z\in\Gamma}).$$

Finally, combining this with (7.5) and (7.3), we find that

$$\| |H_f|^s \|_\Phi \le 2^{1-s} (1 + 2C_{7.1}) C_{5.5} C_{6.5}^{s/2} C_{6.9} C_{6.12} \Phi(\{\|H_f \psi_{z,i}\|^s\}_{z\in\Gamma}).$$

This proves the upper bound in Theorem 1.2 and completes the proof. □

Appendix

Several propositions in the previous sections are similar to their respective counter parts in previous papers. For that reason their proofs were left out of the main text of the paper. We present these proofs here in this Appendix, both for the completeness of the paper and for those readers who are careful with details.

For each $(k,j) \in I$, we define the subset

$$F_{k,j} = \{(\ell,i) : \ell > k, 1 \le i \le m(\ell), B(u_{\ell,i}, 2^{-\ell}) \cap B(u_{k,j}, 3 \cdot 2^{-k}) \ne \emptyset\}$$

of I. We then define

$$W_{k,j} = Q_{k,j} \cup \{\cup_{(\ell,i)\in F_{k,j}} Q_{\ell,i}\},$$

$(k,j) \in I$. By (3.6) and (3.7), we have $W_{k,j} \supset \{ru : 1 - 2^{-2k} \le r < 1, u \in B(u_{k,j}, 3 \cdot 2^{-k})\}$.

Lemma A.1. *There is a constant $C_{\mathrm{A}.1}$ which depends only on n and α such that*

$$\Phi(\{A^s(f; W_{k,j})\}_{(k,j)\in I}) \le \frac{C_{\mathrm{A}.1}}{1 - 2^{-(1+\alpha)s}} \Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$, every symmetric gauge function Φ, and every $0 < s \le 1$.

Proof. It is clear from the above that for every $(k,j) \in I$, we have

$$A^2(f;W_{k,j}) \le \frac{v_\alpha(Q_{k,j})}{v_\alpha(W_{k,j})} A^2(f;Q_{k,j}) + \sum_{(\ell,i)\in F_{k,j}} \frac{v_\alpha(Q_{\ell,i})}{v_\alpha(W_{k,j})} A^2(f;Q_{\ell,i}).$$

Since $v_\alpha(Q_{\ell,i}) \le C_1 2^{-2(n+1+\alpha)\ell}$ and $v_\alpha(W_{k,j}) \ge C_2 2^{-2(n+1+\alpha)k}$, it follows that

$$A^2(f;W_{k,j}) \le A^2(f;Q_{k,j}) + C_3 \sum_{(\ell,i)\in F_{k,j}} 2^{-2(n+1+\alpha)(\ell-k)} A^2(f;Q_{\ell,i}). \tag{A.1}$$

For each integer $\ell \ge 0$, define the set

$$G_{k,j}^{(\ell)} = \{(k+\ell,h) : 1 \le h \le m(k+\ell), B(u_{k+\ell,h}, 2^{-k-\ell}) \cap B(u_{k,j}, 3\cdot 2^{-k}) \ne \emptyset\}$$

as in [20]. By (3.4) and (3.3), there is a natural number M such that

$$\mathrm{card}(G_{k,j}^{(\ell)}) \le M 2^{2n\ell} \tag{A.2}$$

for all $(k,j) \in I$ and $\ell \ge 0$. Similarly, there is an $N \in \mathbf{N}$ such that

$$\mathrm{card}\{j' \in \{1,\dots,m(k)\} : G_{k,j'}^{(\ell)} \cap G_{k,j}^{(\ell)} \ne \emptyset\} \le N \tag{A.3}$$

for all $(k,j) \in I$ and $\ell \ge 0$.

Setting $C_4 = \max\{1, C_3\}$, (A.1) gives us

$$A^2(f;W_{k,j}) \le C_4 \sum_{\ell=0}^{\infty} S_{k,j}^{(\ell)}, \tag{A.4}$$

where

$$S_{k,j}^{(\ell)} = 2^{-2(n+1+\alpha)\ell} \sum_{(k+\ell,h)\in G_{k,j}^{(\ell)}} A^2(f;Q_{k+\ell,h}).$$

Given any $(k,j) \in I$ and $\ell \ge 0$, there is a $(k+\ell, h(k,j;\ell)) \in G_{k,j}^{(\ell)}$ such that

$$A(f;Q_{k+\ell,h(k,j;\ell)}) \ge A(f;Q_{k+\ell,h}) \quad \text{for every} \quad (k+\ell,h) \in G_{k,j}^{(\ell)}.$$

Combining this with (A.2), we have

$$S_{k,j}^{(\ell)} \le M 2^{-2(1+\alpha)\ell} A^2(f;Q_{k+\ell,h(k,j;\ell)}).$$

For each $\ell \ge 0$, define the map $F_\ell : I \to I$ by the formula

$$F_\ell(k,j) = (k+\ell, h(k,j;\ell)).$$

If $k \neq k_1$, then $F_\ell(k,j) \neq F_\ell(k_1,j_1)$ for all possible j and j_1. Since $(k+\ell, h(k,j;\ell)) \in G^{(\ell)}_{k,j}$, (A.3) tells us that for each ℓ, the map F_ℓ is at most N-to-1. Hence, by Lemma 3.1, for each $0 < s \leq 1$ and each symmetric gauge function Φ, we have

$$\begin{aligned}\Phi(\{(S^{(\ell)}_{k,j})^{s/2}\}_{(k,j)\in I}) &\leq M^{s/2}2^{-(1+\alpha)s\ell}\Phi(\{A^s(f;Q_{k+\ell,h(k,j;\ell)})\}_{(k,j)\in I}) \\ &= M^{s/2}2^{-(1+\alpha)s\ell}\Phi(\{A^s(f;Q_{F_\ell(k,j)})\}_{(k,j)\in I}) \\ &\leq NM2^{-(1+\alpha)s\ell}\Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

Since $C_4 \geq 1$ and $0 < s/2 < 1$, (A.4) implies

$$A^s(f;W_{k,j}) \leq C_4\sum_{\ell=0}^{\infty}(S^{(\ell)}_{k,j})^{s/2}$$

for every $(k,j) \in I$. Therefore

$$\begin{aligned}\Phi(\{A^s(f;W_{k,j})\}_{(k,j)\in I}) &\leq C_4\sum_{\ell=0}^{\infty}\Phi(\{(S^{(\ell)}_{k,j})^{s/2}\}_{(k,j)\in I}) \\ &\leq C_4NM\sum_{\ell=0}^{\infty}2^{-(1+\alpha)s\ell}\Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I}).\end{aligned}$$

Thus the constant $C_{\text{A.1}} = C_4NM$ will do for the lemma. □

As in [20], for each $(k,j) \in I$ we define

$$H_{k,j} = \{(t,h) \in I : 0 \leq t \leq k, 1 \leq h \leq m(t), B(u_{t,h}, 2^{-t}) \cap B(u_{k,j}, 2^{-k}) \neq \emptyset\}. \tag{A.5}$$

Lemma A.2. *Given any $i \in \mathbf{Z}_+$, there is a constant $C_{\text{A.2}}$ which depends only on n, α and i such that the following estimate holds: Let $(k,j) \in I$ and $z \in T_{k,j}$. Then there exist $(\ell, \nu(\ell)) \in H_{k,j}$ for $\ell = 0, \dots, k$ such that for every $f \in L^2(\mathbf{B}, dv_\alpha)$, we have*

$$\|f\psi_{z,i}\| \leq C_{\text{A.2}}\sum_{\ell=0}^{k}2^{-(n+1+\alpha+2i)(k-\ell)}A(f;W_{\ell,\nu(\ell)}).$$

Proof. Let $(k,j) \in I$ and $z \in T_{k,j}$ be given. Then $z = |z|\xi$ for some $\xi \in B(u_{k,j}, 2^{-k})$. Set $\nu(k) = j$. If $0 \leq \ell < k$, by (3.5), there is a $\nu(\ell) \in \{1, \dots, m(\ell)\}$ such that $\xi \in B(u_{\ell,\nu(\ell)}, 2^{-\ell})$. Let us show that the inequality

$$|\psi_{z,i}|^2 \leq C_1\sum_{\ell=0}^{k}2^{-2(n+1+\alpha+2i)(k-\ell)}\frac{1}{v_\alpha(W_{\ell,\nu(\ell)})}\chi_{W_{\ell,\nu(\ell)}} \tag{A.6}$$

holds on $\mathbf{B}$, where C_1 depends only on n, α and i.

First of all, $W_{0,\nu(0)} = \mathbf{B}$. Suppose that $w \in W_{\ell-1,\nu(\ell-1)} \backslash W_{\ell,\nu(\ell)}$ and let us estimate the value of $|1 - \langle w, z\rangle|$. Since $w \notin W_{\ell,\nu(\ell)}$, there are two possibilities. Either $|w| \leq 1 - 2^{-2\ell}$, in which case we have $|1 - \langle w, z\rangle| \geq 1 - |w| \geq 2^{-2\ell}$. Or $w/|w| \notin B(u_{\ell,\nu(\ell)}, 3 \cdot 2^{-\ell})$, in which case we have $d(w/|w|, \xi) > 2 \cdot 2^{-\ell}$ since $\xi \in B(u_{\ell,\nu(\ell)}, 2^{-\ell})$ by the choice of $\nu(\ell)$. In the latter case, $|1 - \langle w, z\rangle| \geq (1/2)|1 - \langle w/|w|, \xi\rangle| \geq 2 \cdot 2^{-2\ell}$. Thus we have shown that if $w \in W_{\ell-1,\nu(\ell-1)} \backslash W_{\ell,\nu(\ell)}$, then $|1-\langle w, z\rangle|^{-1} \leq 4 \cdot 2^{2(\ell-1)}$. On the other hand, the definition of $T_{k,j}$ gives us $1 - |z| \leq 2^{-2k}$. It is elementary that $v_\alpha(W_{\ell-1,\nu(\ell-1)}) \leq C2^{-2(n+1+\alpha)(\ell-1)}$. Combining these three inequalities, we see that (A.6) holds on $\mathbf{B}\backslash W_{k,\nu(k)} = \mathbf{B}\backslash W_{k,j}$. But on the set $W_{k,j}$, (A.6) follows from the simple fact that $|1 - \langle w, z\rangle| \geq 1 - |z| \geq 2^{-2(k+1)} = (1/4)2^{-2k}$ since $z \in T_{k,j}$. Thus (A.6) holds on $\mathbf{B}$.

Obviously, (A.6) implies that

$$\|f\psi_{z,i}\|^2 \leq C_1 \sum_{\ell=0}^{k} 2^{-2(n+1+\alpha+2i)(k-\ell)} A^2(f; W_{\ell,\nu(\ell)}),$$

$f \in L^2(\mathbf{B}, dv_\alpha)$. For every $0 \leq \ell \leq k$, since $\xi \in B(u_{\ell,\nu(\ell)}, 2^{-\ell}) \cap B(u_{k,j}, 2^{-k})$, we have $(\ell, \nu(\ell)) \in H_{k,j}$. Thus, taking square-roots in the above, the lemma follows. □

Lemma A.3. *Let $0 < s \leq 1$ be given, and let $i \in \mathbf{Z}_+$ satisfy the condition $s(n+1+\alpha+2i) > 2n$. Then there exists a constant $0 < C_{\mathrm{A.3}} < \infty$ which depends only on n, α, s and i such that the following estimate holds: Let $z(k,j) \in T_{k,j}$ for each $(k,j) \in I$. Then for each $f \in L^2(\mathbf{B}, dv_\alpha)$ and each symmetric gauge function Φ, we have*

$$\Phi(\{\|f\psi_{z(k,j),i}\|^s\}_{(k,j)\in I}) \leq C_{\mathrm{A.3}}\Phi(\{A^s(f; Q_{k,j})\}_{(k,j)\in I}).$$

Proof. By Lemma A.1, it suffices to show that

$$\Phi(\{\|f\psi_{z(k,j),i}\|^s\}_{(k,j)\in I}) \leq C\Phi(\{A^s(f; W_{k,j})\}_{(k,j)\in I}), \tag{A.7}$$

where C depends only on n, α, s and i. To prove this, in addition to the $H_{k,j}$ given by (A.5), we further defined the set

$$H_{k,j}^{(\ell)} = \{(\ell, h) : (\ell, h) \in H_{k,j}\}$$

for each integer $0 \leq \ell \leq k$.

Let $f \in L^2(\mathbf{B}, dv_\alpha)$ be given. For each triple of integers $0 \leq \ell \leq k$ and $1 \leq j \leq m(k)$, there is an element $(\ell, h(k, j; \ell)) \in H_{k,j}^{(\ell)}$ such that

$$A(f; W_{\ell,h(k,j;\ell)}) \geq A(f; W_{\ell,h}) \quad \text{for every} \quad (\ell, h) \in H_{k,j}^{(\ell)}.$$

Let $0 < s \leq 1$ be given. Let $z(k,j) \in T_{k,j}$, $(k,j) \in I$. Applying Lemma A.2, we have

$$\begin{aligned}\|f\psi_{z(k,j),i}\|^s &\leq C_{\mathrm{A.2}}^s \sum_{\ell=0}^{k} A^s(f; W_{\ell,h(k,j;\ell)})2^{-s(n+1+\alpha+2i)(k-\ell)} \\ &= C_1 \sum_{\nu=0}^{k} A^s(f; W_{k-\nu,h(k,j;k-\nu)})2^{-s(n+1+\alpha+2i)\nu}\end{aligned}$$

for each $(k,j) \in I$, where $C_1 = C^s_{\mathrm{A}.2}$. Define

$$a_{k,j;\nu} = \begin{cases} A(f; W_{k-\nu,h(k,j;k-\nu)}) & \text{if} \quad \nu \le k \\ 0 & \text{if} \quad \nu > k \end{cases}$$

for all $(k,j) \in I$ and all $\nu \ge 0$. Then

$$\|f\psi_{z(k,j),i}\|^s \le C_1 \sum_{\nu=0}^{\infty} a^s_{k,j;\nu} 2^{-s(n+1+\alpha+2i)\nu}.$$

Consequently, for each symmetric gauge function Φ we have

$$\text{(A.8)} \qquad \Phi(\{\|f\psi_{z(k,j),i}\|^s\}_{(k,j)\in I}) \le C_1 \sum_{\nu=0}^{\infty} 2^{-s(n+1+\alpha+2i)\nu} \Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}).$$

Since $a_{k,j;\nu} = 0$ whenever $k < \nu$, for each $\nu \ge 0$ we have

$$\Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}) = \Phi(\{A^s(f; W_{k-\nu,h(k,j;k-\nu)})\}_{(k,j)\in I^{(\nu)}}),$$

where $I^{(\nu)} = \{(k,j) : k \ge \nu, 1 \le j \le m(k)\}$.

For each $\nu \ge 0$, consider the map $G_\nu : I^{(\nu)} \to I$ defined by the formula

$$G_\nu(k,j) = (k-\nu, h(k,j;k-\nu)), \quad (k,j) \in I^{(\nu)}.$$

If $k \ne k'$, then, of course, $G_\nu(k,j) \ne G_\nu(k',j')$ for all possible j and j'. Now suppose that integers j and j' are in the set $\{1,\dots,m(k)\}$ such that $G_\nu(k,j) = G_\nu(k,j')$. Then $h(k,j;k-\nu) = h(k,j';k-\nu)$. A chase of definitions gives us

$$B(u_{k-\nu,h(k,j;k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j}, 2^{-k}) \ne \emptyset \quad \text{and}$$
$$B(u_{k-\nu,h(k,j';k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j'}, 2^{-k}) \ne \emptyset.$$

Since $h(k,j;k-\nu) = h(k,j';k-\nu)$, we have $d(u_{k,j}, u_{k,j'}) \le 4 \cdot 2^{-(k-\nu)}$. Thus we conclude from (3.4) and (3.3) that there is a $C_2 \in \mathbf{N}$ which depends only on n such that for all $\nu \le k$ and all $1 \le j \le m(k)$,

$$\mathrm{card}\{j' \in \{1,\dots,m(k)\} : G_\nu(k,j') = G_\nu(k,j)\} \le C_2 2^{2n\nu}.$$

That is, the map $G_\nu : I^{(\nu)} \to I$ is at most $C_2 2^{2n\nu}$-to-1. Applying Lemma 3.1, we have

$$\Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}) = \Phi(\{A^s(f; W_{G_\nu(k,j)})\}_{(k,j)\in I^{(\nu)}}) \le C_2 2^{2n\nu} \Phi(\{A^s(f; W_{k,j})\}_{(k,j)\in I}).$$

Substituting this in (A.8), we find that

$$\Phi(\{\|f\psi_{z(k,j),i}\|^s\}_{(k,j)\in I}) \le C_1 C_2 \sum_{\nu=0}^{\infty} 2^{-\{s(n+1+\alpha+2i)-2n\}\nu} \Phi(\{A^s(f; W_{k,j})\}_{(k,j)\in I}).$$

Since we assume $s(n+1+\alpha+2i) > 2n$, (A.7) follows. This completes the proof. $\square$

Proof of Proposition 3.7. Let $0 < s \le 1$, and let $i \in \mathbf{Z}_+$ satisfy the condition $s(n+1+\alpha+2i) > 2n$. Given $0 < a < \infty$, let K be the natural number provided by Lemma 3.6. According to that lemma, each a-separated set Γ is the union of pairwise disjoint subsets $\Gamma_1, \dots, \Gamma_K$ such that $\mathrm{card}(\Gamma_\mu \cap T_{k,j}) \le 1$ for all $\mu \in \{1, \dots, K\}$ and $(k,j) \in I$. Thus for each $\mu \in \{1, \dots, K\}$, it follows from Lemma A.3 that

$$\Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma_\mu}) \le C_{\mathrm{A.3}}\Phi(\{A^s(f;Q_{k,j})\}_{(k,j)\in I})$$

for every $f \in L^2(\mathbf{B}, dv_\alpha)$ and every symmetric gauge function Φ. Since $\Gamma_1 \cup \cdots \cup \Gamma_K = \Gamma$, we have

$$\Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma}) \le \Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma_1}) + \cdots + \Phi(\{\|f\psi_{z,i}\|^s\}_{z\in\Gamma_K}).$$

Hence Proposition 3.7 holds for the constant $C_{3.7} = KC_{\mathrm{A.3}}$. $\square$

Proof of Lemma 6.1. (1) It is elementary that for $z, w \in \mathbf{B}$ satisfying the condition $|z| \le |w|$, we have

$$\beta(w,z) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} \ge \frac{1}{2}\log\frac{(1+|w|)(1-|z|)}{(1-|w|)(1+|z|)}.$$

Thus if $k' > k \ge 0$, then $\beta(w_{k',j'}, w_{k,j}) \ge (1/2)\log 4$. Hence it suffices to find a $c > 0$ such that $\beta(w_{k,j'}, w_{k,j}) \ge c$ for all $k \in \mathbf{Z}_+$ and $j' \ne j$ in $\{1, \dots, m(k)\}$. To find such a c, note that for any given pair of $\xi', \xi \in S$, if we write $1 - \langle \xi, \xi' \rangle = a + ib$ with $a, b \in \mathbf{R}$, then $a \ge 0$. Using this positivity, for every $0 \le r < 1$ we have

$$\begin{aligned}1-|\varphi_{r\xi'}(r\xi)|^2 &= \frac{(1-r^2)^2}{|1-r^2\langle\xi,\xi'\rangle|^2} = \frac{(1-r^2)^2}{|1-r^2+r^2(a+ib)|^2} = \left|1+\frac{r^2}{1-r^2}(a+ib)\right|^{-2} \\ &\le \left(1+\frac{r^4}{(1-r^2)^2}(a^2+b^2)\right)^{-1} = \left(1+\frac{r^4}{(1-r^2)^2}|1-\langle\xi,\xi'\rangle|^2\right)^{-1}.\end{aligned}$$

Hence, recalling (3.4), for all $k \in \mathbf{Z}_+$ and $j' \ne j$ in $\{1, \dots, m(k)\}$ we have

$$1-|\varphi_{w_{k,j'}}(w_{k,j})| \le 1-|\varphi_{w_{k,j'}}(w_{k,j})|^2 \le (1+2^{-4}\cdot 2^{-2}\cdot 2^{-2})^{-1},$$

which leads to $\beta(w_{k,j'}, w_{k,j}) \ge (1/2)\log(1+2^{-8})$, as was to be proved.

(2) Denote $r_k = |w_{k,j}| = 1 - 2^{-2k-1}$. Consider any $\xi \in B(u_{k,j}, 9\cdot 2^{-k})$. We have

$$1-|\varphi_{w_{k,j}}(r_k\xi)|^2 = \frac{(1-r_k^2)^2}{|1-r_k^2\langle\xi,u_{k,j}\rangle|^2} \ge \frac{(1-r_k^2)^2}{(1-r_k^2+|1-\langle\xi,u_{k,j}\rangle|)^2} \ge \frac{1}{\left(1+\frac{|1-\langle\xi,u_{k,j}\rangle|}{1-r_k}\right)^2}.$$

Thus $1-|\varphi_{w_{k,j}}(r_k\xi)| \ge (1/2)(1-|\varphi_{w_{k,j}}(r_k\xi)|^2) \ge 2^{-1}\cdot(1+2\cdot 9^2)^{-2}$. Consequently,

$$\beta(r_k\xi, w_{k,j}) \le \frac{1}{2}\log\frac{2}{1-|\varphi_{w_{k,j}}(r_k\xi)|} \le \log 326 \tag{A.9}$$

for every $\xi \in B(u_{k,j}, 9 \cdot 2^{-k})$. On the other hand, for $\xi \in S$ and $0 \le r < 1$, we have

$$\beta(r_k\xi, r\xi) = \frac{1}{2}\left|\log \frac{(1+r)(1-r_k)}{(1-r)(1+r_k)}\right|. \tag{A.10}$$

Recalling (3.7), the definition of $Q_{k,j}$, the conclusion of (2) follows from (A.9) and (A.10).

(3) This follows from (1) and Lemma 5.1(a). This completes the proof of Lemma 6.1. □

We now turn to the proof of Lemma 6.6, the Forelli-Rudin estimate.

Lemma A.4. *Let $i \in \mathbf{Z}_+$ be given. Then there is a constant $C_{A.4}$ which depends only on n, α and i such that*

$$v_\alpha^{1/2}(U_{k,j}) \sup_{\zeta \in U_{k,j}} |\psi_{w_{t,h},i}(\zeta)| \le C_{A.4}\left(\frac{(1-|w_{k,j}|^2)^{1/2}(1-|w_{t,h}|^2)^{1/2}}{|1-\langle w_{k,j}, w_{t,h}\rangle|}\right)^{n+1+\alpha} |m_{w_{t,h}}(w_{k,j})|^i$$

for every pair of $(k,j), (t,h) \in I$.

Proof. By Lemma 3.10, we have $v_\alpha(U_{k,j}) \le C_1(1-|w_{k,j}|^2)^{n+1+\alpha}$, $(k,j) \in I$. Hence

$$v_\alpha^{1/2}(U_{k,j})|\psi_{w_{t,h},i}(\zeta)| \le C_1^{1/2}\left(\frac{(1-|w_{k,j}|^2)^{1/2}(1-|w_{t,h}|^2)^{1/2}}{|1-\langle \zeta, w_{t,h}\rangle|}\right)^{n+1+\alpha} |m_{w_{t,h}}(\zeta)|^i. \tag{A.11}$$

If $\zeta \in U_{k,j} = D(w_{k,j}, 3\tau+3)$, then $\varphi_{w_{k,j}}(\zeta) \in D(0, 3\tau+3)$. Since $\zeta = \varphi_{w_{k,j}}(\varphi_{w_{k,j}}(\zeta))$ and $w_{t,h} = \varphi_{w_{t,h}}(0)$, we can apply Lemma 3.8 and 3.9 to obtain

$$\begin{aligned}\frac{(1-|w_{k,j}|^2)^{1/2}(1-|w_{t,h}|^2)^{1/2}}{|1-\langle \zeta, w_{t,h}\rangle|} &= \frac{(1-|w_{k,j}|^2)^{1/2}}{(1-|\zeta|^2)^{1/2}} \cdot \frac{(1-|\zeta|^2)^{1/2}(1-|w_{t,h}|^2)^{1/2}}{|1-\langle \zeta, w_{t,h}\rangle|} \\ &\le 2e^{3\tau+3} \cdot 2e^{3\tau+3}\frac{(1-|w_{k,j}|^2)^{1/2}(1-|w_{t,h}|^2)^{1/2}}{|1-\langle w_{k,j}, w_{t,h}\rangle|}\end{aligned} \tag{A.12}$$

for every $\zeta \in U_{k,j}$. Combining this with (A.11), we see that the lemma holds for constant $C_{A.4} = C_1^{1/2}(4e^{6\tau+6})^{n+1+\alpha+i}$. □

As usual, let $d\lambda$ denote the standard Möbius-invariant measure on $\mathbf{B}$. That is,

$$d\lambda(\zeta) = \frac{dv(\zeta)}{(1-|\zeta|^2)^{n+1}}.$$

Let $t > 0$. Then on the unit disc $\{u \in \mathbf{C} : |u| < 1\}$ we have the power-series expansion

$$\frac{1}{(1-u)^t} = \sum_{m=0}^{\infty} a_m u^m, \quad \text{where} \quad a_m = \frac{1}{m!}\prod_{\nu=0}^{m-1}(t+\nu) \quad \text{for} \quad m \ge 1.$$

By Stirling's asymptotic formula, we have $a_m \approx (1+m)^{t-1}$.

Proof of Lemma 6.6. Let $0 < s \le 1$ and $i \in \mathbf{N}$ be such that $si > n$. By Lemma A.4, it suffices to show that

$$\lim_{R\to\infty} \sup_{(k,j)\in I} \mathcal{S}_{k,j}(R) = 0, \tag{A.13}$$

where

$$\mathcal{S}_{k,j}(R) = \sum_{\substack{(t,h)\in I \\ \beta(w_{k,j},w_{t,h})\ge R}} \left(1 - |\varphi_{w_{k,j}}(w_{t,h})|^2\right)^{(s/2)(n+1+\alpha)} |m_{w_{t,h}}(w_{k,j})|^{si}$$

for $(k,j) \in I$ and $R > 0$. Using Lemma 3.8 and 3.9 in a manner similar to the proof of (A.12), we see that there is a C_1 such that if $\zeta_{t,h} \in D(w_{t,h},\tau_0)$ for every $(t,h) \in I$, then

$$\mathcal{S}_{k,j}(R) \le C_1 \sum_{\substack{(t,h)\in I \\ \beta(w_{k,j},w_{t,h})\ge R}} (1 - |\varphi_{w_{k,j}}\left(\zeta_{t,h})|^2\right)^{(s/2)(n+1+\alpha)} |m_{\zeta_{t,h}}(w_{k,j})|^{si}$$

for all $(k,j) \in I$ and $R > 0$. Since $D(w_{t,h},\tau_0) \cap D(w_{t',h'},\tau_0) = \emptyset$ for all $(t,h) \ne (t',h')$ in I, the above implies

$$\begin{aligned} \mathcal{S}_{k,j}(R) &\le C_1 \sum_{\substack{(t,h)\in I \\ \beta(w_{k,j},w_{t,h})\ge R}} \int_{D(w_{t,h},\tau_0)} \frac{(1 - |\varphi_{w_{k,j}}\left(\zeta)|^2\right)^{(s/2)(n+1+\alpha)}}{\lambda(D(w_{t,h},\tau_0))} |m_\zeta(w_{k,j})|^{si} d\lambda(\zeta) \\ &\le \frac{C_1}{\lambda(D(0,\tau_0))} \int_{\beta(w_{k,j},\zeta)\ge R-\tau_0} (1 - |\varphi_{w_{k,j}}\left(\zeta)|^2\right)^{(s/2)(n+1+\alpha)} |m_\zeta(w_{k,j})|^{si} d\lambda(\zeta). \end{aligned}$$

Note that $|m_{\varphi_w(z)}(w)| = |m_z(w)|$. Thus, making the substitution $\zeta = \varphi_{w_{k,j}}(z)$ in the above and using the Möbius invariance of $d\lambda$, we obtain

$$\mathcal{S}_{k,j}(R) \le C_2 \int_{\beta(0,z)\ge R'} (1 - |z|^2)^{(s/2)(n+1+\alpha)} |m_z(w_{k,j})|^{si} d\lambda(z),$$

where $C_2 = C_1/\lambda(D(0,\tau_0))$ and $R' = R - \tau_0$. That is,

$$\mathcal{S}_{k,j}(R) \le C_2 \int_{\beta(0,z)\ge R'} \frac{(1 - |z|^2)^{(s/2)(n+1+\alpha)+si-n-1}}{|1 - \langle w_{k,j}, z\rangle|^{si}} dv(z).$$

For convenience, let us write $\kappa = (s/2)(n+1+\alpha) + si - n - 1$, $t = si/2$, and $r_{k,j} = |w_{k,j}|$. Then $\kappa > -1$, $t > 0$, and $\kappa - 2t + n > -1$. By the rotation invariance of dv, we have

$$\mathcal{S}_{k,j}(R) \le C_2 \int_{R''\le|z|<1} \frac{(1 - |z|^2)^\kappa}{|1 - r_{k,j}z_1|^{2t}} dv(z),$$

where z_1 denotes the first component of z and $R'' = (e^{2R'} - 1)/(e^{2R'} + 1)$. We have $R'' \uparrow 1$ as $R \to \infty$. Using the power series expansion for $(1 - r_{k,j}z_1)^{-t}$ and the spherical symmetry of the region $R'' \leq |z| < 1$, we have

$$\begin{aligned}\mathcal{S}_{k,j}(R) &\leq C_3 \sum_{m=0}^{\infty} (1+m)^{2t-2} \int_{R'' \leq |z| < 1} (1 - |z|^2)^{\kappa} |z_1|^{2m} dv(z) \\ &= C_3 \sum_{m=0}^{\infty} (1+m)^{2t-2} \int |\xi_1|^{2m} d\sigma(\xi) \int_{R''}^{1} 2n(1-r^2)^{\kappa} r^{2m+2n-1} dr.\end{aligned}$$

By [18,Proposition 1.4.9], the $d\sigma$-integral above equals $\frac{(n-1)!m!}{(n-1+m)!} \approx (1+m)^{-n+1}$. Hence

$$\mathcal{S}_{k,j}(R) \leq C_4 \sum_{m=0}^{\infty} (1+m)^{2t-n-1} \int_{R''}^{1} (1-r^2)^{\kappa} r^{2m+2n-1} dr.$$

Since $2t = si > n$, we have $2t - n > 0$, and from the above we obtain

$$\mathcal{S}_{k,j}(R) \leq C_5 \int_{R''}^{1} \frac{(1-r^2)^{\kappa}}{(1-r^2)^{2t-n}} r^{2n-1} dr.$$

Since $\kappa - 2t + n > -1$, this proves (A.13) and completes the proof of Lemma 6.6. $\square$

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