THE BERGER-COBURN PHENOMENON FOR HANKEL OPERATORS ON THE FOCK SPACE

Jingbo Xia

Abstract. The Berger-Coburn phenomenon of Hankel operators was recently reported by Hu and Virtanen in [14] for the Schatten classes $\mathcal{C}_p$, $1 < p < \infty$. But a careful reading of [14] finds that in the case $1 < p < 2$, there is a technical problem in their proof. In this paper we first fix this problem. We then establish the Berger-Coburn phenomenon for the Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$, $1 < p < \infty$. Last but not least, we show that there is no Berger-Coburn phenomenon for the trace class $\mathcal{C}_1$, for $\mathcal{C}_1^+$, and for the Macaev ideal $\mathcal{C}_\infty^-$.

1. Introduction

Let $d\mu$ denote the Gaussian measure on $\mathbf{C}^n$. More precisely, we write

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z),$$

where dV is the standard volume measure on $\mathbf{C}^n$. Recall that the Fock space $H^2(\mathbf{C}^n, d\mu)$ is the norm closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(\mathbf{C}^n, d\mu)$. Let $P : L^2(\mathbf{C}^n, d\mu) \to H^2(\mathbf{C}^n, d\mu)$ be the orthogonal projection. Given an appropriate symbol function f, the Hankel operator $H_f : H^2(\mathbf{C}^n, d\mu) \to L^2(\mathbf{C}^n, d\mu) \ominus H^2(\mathbf{C}^n, d\mu)$ is defined by the formula

$$H_f h = (1 - P)(fh),$$

$h \in H^2(\mathbf{C}^n, d\mu)$.

For an arbitrary symbol function f, in general very little about $H_{\bar{f}}$ can be inferred from the properties of H_f. Therefore it was all the more remarkable that Berger and Coburn proved the following result in [4]: for $f \in L^\infty(\mathbf{C}^n)$, H_f is compact if and only if $H_{\bar{f}}$ is compact. From the author's conversations with Lew Coburn about this result in the late 1990s and early 2000s arose a natural question, which was reported in [20]:

Question 1.1. *For $f \in L^\infty(\mathbf{C}^n)$ and $1 \le p < \infty$, does the membership $H_f \in \mathcal{C}_p$ imply $H_{\bar{f}} \in \mathcal{C}_p$?*

Here, $\mathcal{C}_p$ denotes the Schatten p-class. That is, $\mathcal{C}_p$ is the collection of operators A satisfying the condition $\|A\|_p < \infty$, where $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$.

In [3], Bauer answered Question 1.1 in the affirmative for the Hilbert-Schmidt class $\mathcal{C}_2$: for $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_2$ if and only if $H_{\bar{f}} \in \mathcal{C}_2$.

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After Bauer's paper, no progress was made on Question 1.1 for the next sixteen years. Then in December, 2020, Hu and Virtanen posted [14] (arXiv:2012.13768), which reported an affirmative answer to Question 1.1 for all $1 < p < \infty$:

Theorem 1.2. [14, Theorem 1.2] *Suppose that* $1 < p < \infty$. *For* $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p$ *if and only if* $H_{\bar{f}} \in \mathcal{C}_p$.

Actually, the Hankel operators in [14] are on the weighted Segal-Bargman space $F^2(\varphi)$, which is more general than the $H^2(\mathbf{C}^n, d\mu)$ considered in this paper. Moreover, Hu and Virtanen gave the norm bound $\|H_{\bar{f}}\|_p \le C\|H_f\|_p$ for $f \in L^\infty(\mathbf{C}^n)$, where C depends only on the $p \in (1, \infty)$.

Hu and Virtanen referred to Theorem 1.2 as the Berger-Coburn phenomenon for the Schatten classes $\mathcal{C}_p$, $1 < p < \infty$. We will follow their terminology. In [21,13], the root cause of the Berger-Coburn phenomenon was ascribed to the absence of bounded, non-constant analytic functions on $\mathbf{C}^n$.

But unfortunately, for $1 < p < 2$, there is a technical problem in the proof of Theorem 1.2 presented in [14], which, for convenience of discussion, will be referred to as the Original Proof. For $f \in L^\infty(\mathbf{C}^n)$ such that $H_f \in \mathcal{C}_p$, the Original Proof decomposes f in a specific form $f = f_1 + f_2$ with $f_1 \in C^2(\mathbf{C}^n)$. Then the goal is to show that $H_{\bar{f}_1} \in \mathcal{C}_p$ and $H_{\bar{f}_2} \in \mathcal{C}_p$. There is no problem with the argument for $H_{\bar{f}_2} \in \mathcal{C}_p$. The problem occurs when it comes to the membership $H_{\bar{f}_1} \in \mathcal{C}_p$. In the Original Proof, the membership $H_{\bar{f}_1} \in \mathcal{C}_p$ is justified by the condition $\|\bar{\partial}\bar{f}_1\|_{L^p} < \infty$, which is incorrect. To ensure $H_{\bar{f}_1} \in \mathcal{C}_p$, what one needs (in the notation of [14]) is the condition

$$\|M_{2,r}(\bar{\partial}\bar{f}_1)\|_{L^p} < \infty, \tag{1.1}$$

where

$$M_{2,r}(\varphi)(z) = \left\{\frac{1}{|B(z,r)|}\int_{B(z,r)} |\varphi|^2 dV\right\}^{1/2}. \tag{1.2}$$

If $2 \le p < \infty$, then by Hölder's inequality the condition $\|\bar{\partial}\bar{f}_1\|_{L^p} < \infty$ implies (1.1). Therefore the Original Proof works in the case $2 \le p < \infty$. But in the case $1 < p < 2$, there is no Hölder's inequality to apply, and it is not a priori clear why the condition $\|\bar{\partial}\bar{f}_1\|_{L^p} < \infty$ implies (1.1).

The point is that in the definition of $M_{2,r}(\varphi)(z)$, $|\varphi|$ has to be squared before it is integrated over $B(z, r)$. This actually corresponds to the fact that one proves the Schatten class membership of $M_\varphi P$ by considering $\{(M_\varphi P)^* M_\varphi P\}^{1/2} = T^{1/2}_{|\varphi|^2}$. In other words, the $|\varphi|^2$ in (1.2) is dictated by the underlying operator theory.

For each $1 < p < 2$, one can certainly produce examples of φ such that $\|\varphi\|_{L^p} < \infty$ while $\|M_{2,r}(\varphi)\|_{L^p} = \infty$. In fact, the condition $\|M_{2,r}(\varphi)\|_{L^p} < \infty$ is structurally different from the condition $\|\varphi\|_{L^p} < \infty$.

Note added in revision, November 3, 2022. After the initial submission of this paper, an updated version of [14], arXiv.2012.13768v3, was posted. In this updated version, a correct proof was given for the case $1 < p < 2$ in Theorem 1.2. The main part of this new proof is that (1.1) indeed holds in the case $1 < p < 2$.

Now let us discuss what we will do in this paper. This paper has a threefold purpose. First, we will present a correct proof for the case $1 < p < 2$ in Theorem 1.2. Specifically, we will show that (1.1) indeed holds for the particular f_1, which takes quite a bit of work.

Even though the original proof of Theorem 1.2 had a technical problem, the idea in [14] of decomposing f for the purpose of proving the Berger-Coburn phenomenon is a really good one. This idea can be further exploited. One can use the same idea to prove the Berger-Coburn phenomenon for operator ideals other than the Schatten classes. The second purpose of this paper is to establish the Berger-Coburn phenomenon for two classes of Lorentz ideals. Before going any further, let us introduce these ideals.

Let $\mathcal{H}$ be a Hilbert space. For any given $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{j\ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

defines a norm for bounded operators on $\mathcal{H}$. Here and in what follows, we write $s_1(A)$, $s_2(A)$, ..., $s_j(A)$, ... for the s-numbers [12] of the operator A. It is well known that the collection of operators

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

form a norm ideal, for which we cite [12] as our primary reference.

For each $1 \le p < \infty$, the formula

$$\|A\|_p^- = \sum_{j=1}^\infty \frac{s_j(A)}{j^{(p-1)/p}}$$

also defines a norm for bounded operators on $\mathcal{H}$. Denote

$$\mathcal{C}_p^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^- < \infty\},$$

which is also a norm ideal of operators on $\mathcal{H}$ [12].

It is well known that $\mathcal{C}_p^+$ is not separable with respect to the norm $\|\cdot\|_p^+$ [12]. For $1 \le p < p' < \infty$, the inclusion relation

$$\mathcal{C}_p^- \subset \mathcal{C}_p \subset \mathcal{C}_p^+ \subset \mathcal{C}_{p'}^-$$

is also well known. When $p > 1$, the above inclusions are all proper. For $p = 1$, it is easy to see that $\mathcal{C}_1^- = \mathcal{C}_1$ while $\mathcal{C}_1^+ \ne \mathcal{C}_1$. The ideal $\mathcal{C}_1^+$ commands special interest in that it is the domain of every Dixmier trace [6,5].

For these ideals, we will prove the following two theorems:

Theorem 1.3. *Let $1 < p < \infty$. For $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p^+$, if and only if $H_{\bar{f}} \in \mathcal{C}_p^+$.*

Theorem 1.4. *Let $1 < p < \infty$. For $f \in L^\infty(\mathbf{C}^n)$, $H_f \in \mathcal{C}_p^-$, if and only if $H_{\bar{f}} \in \mathcal{C}_p^-$.*

The third purpose of this paper is to report the absence of Berger-Coburn phenomenon for a number of ideals. First of all, there is no Berger-Coburn phenomenon for the trace class $\mathcal{C}_1$ and for the ideal $\mathcal{C}_1^+$. The case of trace class $\mathcal{C}_1$ is settled by a simple example, while the case of $\mathcal{C}_1^+$ requires a rather elaborate construction.

Theorem 1.5. *Consider the case where $n = 1$. On $\mathbf{C}$, define the function*

$$g(z) = \begin{cases} z^{-1} & \text{if} \quad |z| \geq 1 \\ 0 & \text{if} \quad |z| < 1 \end{cases}. \tag{1.3}$$

Then H_g is in the trace class while $H_{\bar{g}}$ is not in the trace class.

Theorem 1.6. *Consider the case where $n = 1$. There is a $\psi \in L^\infty(\mathbf{C})$ such that $H_\psi \in \mathcal{C}_1^+$ while $H_{\bar{\psi}} \notin \mathcal{C}_1^+$.*

Then there is the matter of the Macaev ideal $\mathcal{C}_\infty^-$. Recall that on any Hilbert space $\mathcal{H}$, we have $\mathcal{C}_\infty^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\infty^- < \infty\}$, where the norm $\|\cdot\|_\infty^-$ is defined by the formula

$$\|A\|_\infty^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j}.$$

It is well known that $\mathcal{C}_\infty^-$ is the pre-dual of $\mathcal{C}_1^+$ [12]. In contrast to $\mathcal{C}_1$ and $\mathcal{C}_1^+$, the Macaev ideal $\mathcal{C}_\infty^-$ is at the other end of the scale. That is, $\mathcal{C}_\infty^-$ is a large ideal; in fact, $\mathcal{C}_\infty^-$ is not much smaller than $\mathcal{K}$, the ideal of compact operators. Since the original Berger-Coburn phenomenon in [4] was about $\mathcal{K}$, one is obligated to ask, is there Berger-Coburn phenomenon for $\mathcal{C}_\infty^-$? Notwithstanding the size of $\mathcal{C}_\infty^-$, the answer is negative:

Theorem 1.7. *Consider the case where $n = 1$. There is a $q \in L^\infty(\mathbf{C})$ such that $H_q \in \mathcal{C}_\infty^-$ while $H_{\bar{q}} \notin \mathcal{C}_\infty^-$.*

Taking the results in this paper as a whole, we now have a much better understanding of the Berger-Coburn phenomenon.

The rest of the paper is organized as follows. Sections 2, 3 and 4 are the technical foundation for proving the case $1 < p < 2$ in Theorem 1.2 and for proving Theorems 1.3 and 1.4. For this part, the space that matters is $\mathcal{L}_n^{2,p}$, which will be introduced in Section 4. Locally $\mathcal{L}_n^{2,p}$ behaves like L^2, but at long range this space is more like ℓ^p. The culmination of these three sections is Proposition 4.5, which says that the operator

$$(T_j\varphi)(\zeta_1, \ldots \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_n) = \text{p.v.} \int_{\mathbf{C}} \frac{\varphi(\zeta_1, \ldots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n)}{(\zeta_j - z)^2} dA(z)$$

is bounded on $\mathcal{L}_n^{2,p}$, $j = 1, \ldots, n$. The proof of the boundedness of T_j requires treatment of the discrete Hilbert transform in Section 2 and other steps in Section 3.

In Section 5, we give a correct proof of the case $1 < p < 2$ in Theorem 1.2.

We then begin our preparation to prove Theorems 1.3 and 1.4. This preparation consists of two parts. First, we need to show that each T_j is bounded on the spaces $\mathcal{L}_n^{2,p,+}$ and $\mathcal{L}_n^{2,p,-}$, $1 < p < \infty$. This requires Proposition 4.5 and the unconventional interpolation in Sections 6 and 7. The second part of this preparation consists of Sections 8 and 9, which deal with the membership of Hankel operators in general norm ideals. After this extensive preparation, we prove Theorems 1.3 and 1.4 in Section 10.

We then prove Theorems 1.5, 1.6 and 1.7 in Sections 11, 12 and 13 respectively. Finally, in Section 14 we present a generalization of Theorem 1.7.

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2. Discrete Hilbert transform

Let $\mathcal{I}$ denote the collection of half-open, half-closed intervals $[a, b)$ in $\mathbf{R}$, $-\infty < a < b < \infty$. For each interval I in $\mathbf{R}$, we write $|I|$ for its length.

Lemma 2.1. *Given any $m \in \mathbf{N}$, there are pairwise disjoint $I_1, \dots, I_k \in \mathcal{I}$ satisfying the following four conditions*:

(a) $I_1 \cup \cdots \cup I_k \supset \{1, \dots, m\}$.
(b) $|I_j| = \mathrm{dist}(I_j, \mathbf{R}\backslash(0, m+1))$ *for every* $1 \le j \le k$.
(c) $\cup_{j=1}^k I_j \subset (0, m+1)$.
(d) $3|I_j| \ge \mathrm{card}(I_j \cap \{1, \dots, m\}) > 0$ *for every* $1 \le j \le k$.

Proof. It is elementary that there are pairwise disjoint $J_1, \dots, J_\nu, \dots$ in $\mathcal{I}$ such that $\cup_{\nu=1}^\infty J_\nu = (0, m+1)$ and such that $|J_\nu| = \mathrm{dist}(J_\nu, \mathbf{R}\backslash(0, m+1))$ for every $\nu \in \mathbf{N}$. Let $I_1, \dots, I_k$ be the intervals J_ν satisfying the condition $J_\nu \cap \{1, \dots, m\} \neq 0$. Then (a), (b) and (c) follow from the properties of $J_1, \dots, J_\nu, \dots$. For each $1 \le j \le k$, since $I_j \cap \{1, \dots, m\} \neq \emptyset$ by choice, (b) implies $|I_j| \ge 1/2$. Therefore

$$3|I_j| \ge 1 + |I_j| \ge \mathrm{card}(I_j \cap \mathbf{Z}) \ge \mathrm{card}(I_j \cap \{1, \dots, m\}).$$

That is, (d) also holds. $\square$

Let $\mathcal{H}$ be a Hilbert space and consider $\ell^p(\mathbf{Z}, \mathcal{H})$, $1 \le p \le \infty$. For $1 \le p < \infty$, $\ell^p(\mathbf{Z}, \mathcal{H})$ consists of sequences $a = \{a_k\}$ such that $a_k \in \mathcal{H}$ for every $k \in \mathbf{Z}$ and such that

$$\|a\|_p = \left(\sum_{k \in \mathbf{Z}} \|a_k\|^p \right)^{1/p} < \infty.$$

We define the maximal operator M on $\ell^p(\mathbf{Z}, \mathcal{H})$ as follows. For each $a = \{a_k\} \in \ell^p(\mathbf{Z}, \mathcal{H})$, Ma is the scalar sequence $\{(Ma)_k\}$, where each $(Ma)_k$ is defined by the formula

$$(Ma)_k = \sup \left\{ \frac{1}{|I|} \sum_{j \in I} \|a_j\| : k \in I,\ I \in \mathcal{I} \text{ and } |I| \ge 1 \right\}. \tag{2.1}$$

Lemma 2.2. *The maximal operator M satisfies the weak-type* 1-1 *estimate on* $\ell^1(\mathbf{Z}, \mathcal{H})$.

Proof. Let $a = \{a_k\} \in \ell^1(\mathbf{Z}, \mathcal{H})$. Given any $\lambda > 0$, define $\Lambda = \{k \in \mathbf{Z} : (Ma)_k > \lambda\}$. Since $a \in \ell^1(\mathbf{Z}, \mathcal{H})$, from (2.1) we see that there is a $K \in \mathbf{N}$ such that $\Lambda \subset [-K, K]$. That is, Λ is a finite set. For each $k \in \Lambda$, there is an $I_k \in \mathcal{I}$ such that $k \in I_k$, $|I_k| \geq 1$, and

$$\frac{1}{|I_k|} \sum_{j \in I_k} \|a_j\| \geq \lambda/2. \tag{2.2}$$

By arranging the finite collection of intervals $\{I_k : k \in \Lambda\}$ in the descending order of $|I_k|$, we obtain a subset Λ' of Λ which has the following two properties:

(1) $I_j \cap I_k = \emptyset$ for all $j \neq k$ in Λ'.

(2) If $\nu \in \Lambda \backslash \Lambda'$, then there is a $k(\nu) \in \Lambda'$ such that $I_\nu \cap I_{k(\nu)} \neq \emptyset$ and $|I_{k(\nu)}| \geq |I_\nu|$.

For each $k \in \Lambda'$, let $\tilde{I}_k$ be the interval in $\mathcal{I}$ that has the same center as I_k but 3 times the length. If $\nu \in \Lambda \backslash \Lambda'$ and $k \in \Lambda'$ are such that $I_\nu \cap I_k \neq \emptyset$ and $|I_k| \geq |I_\nu|$, then $\tilde{I}_k \supset I_\nu$. That is, $\cup_{k \in \Lambda'} \tilde{I}_k \supset \Lambda$. Thus from (2.2) and property (1) above we obtain

$$\|a\|_1 \geq \sum_{k \in \Lambda'} \sum_{j \in I_k} \|a_j\| \geq \frac{\lambda}{2} \sum_{k \in \Lambda'} |I_k| = \frac{\lambda}{6} \sum_{k \in \Lambda'} |\tilde{I}_k|.$$

For each $k \in \Lambda'$, since $|\tilde{I}_k| \geq 3$, we have $2|\tilde{I}_k| > 1 + |\tilde{I}_k| \geq \mathrm{card}(\tilde{I}_k \cap \mathbf{Z}) \geq \mathrm{card}(\tilde{I}_k \cap \Lambda)$. Therefore the above implies

$$\mathrm{card}(\Lambda) \leq 12\lambda^{-1}\|a\|_1.$$

That is, M satisfies the weak-type 1-1 estimate on $\ell^1(\mathbf{Z}, \mathcal{H})$ as promised. □

Obviously, the maximal operator $M : \ell^\infty(\mathbf{Z}, \mathcal{H}) \to \ell^\infty(\mathbf{Z})$ is bounded. Therefore, by the usual interpolation, from Lemma 2.2 we obtain

Corollary 2.3. *For each $1 < p < \infty$, the maximal operator $M : \ell^p(\mathbf{Z}, \mathcal{H}) \to \ell^p(\mathbf{Z})$ is bounded.*

Next we consider the discrete Hilbert transform. For any $a = \{a_k\} \in \ell^p(\mathbf{Z}, \mathcal{H})$, $1 \leq p < \infty$, we define the sequence $Da = \{(Da)_k\}$ by the formula

$$(Da)_k = \sum_{j \in \mathbf{Z} \backslash \{k\}} \frac{1}{k - j} a_j, \quad k \in \mathbf{Z}.$$

Lemma 2.4. *The discrete Hilbert transform D maps $\ell^2(\mathbf{Z}, \mathcal{H})$ into itself. Moreover, the norm of the operator $D : \ell^2(\mathbf{Z}, \mathcal{H}) \to \ell^2(\mathbf{Z}, \mathcal{H})$ does not exceed π.*

Proof. It is straightforward to verify that when acting on $\ell^2(\mathbf{Z}, \mathcal{H})$, D is unitarily equivalent to the operator of multiplication by the scalar function

$$\sum_{k=1}^{\infty} \frac{1}{k}(e^{ikx} - e^{-ikx}) = 2i \sum_{k=1}^{\infty} \frac{\sin kx}{k} = i(\pi - x)$$

on $L^2([0, 2\pi), \mathcal{H})$. □

Lemma 2.5. *The discrete Hilbert transform D satisfies the weak-type* 1-1 *estimate on* $\ell^1(\mathbf{Z},\mathcal{H})$.

Proof. This is an adaptation of a classic argument. See, e.g., [11, pages 129-131]. Let $a = \{a_k\} \in \ell^1(\mathbf{Z},\mathcal{H})$. Given any $\lambda > 0$, define $\Lambda = \{k \in \mathbf{Z} : (Ma)_k > \lambda\}$ and $F = \mathbf{Z}\backslash\Lambda$. By Lemma 2.2, we have $\mathrm{card}(\Lambda) < \infty$. Thus if $\Lambda \neq \emptyset$, then Λ is the union of a finite number of segments $\nu+1,\dots,\nu+m$, where $\nu \in \mathbf{Z}$ and $m \in \mathbf{N}$ have the properties that $\nu \in F$ and $\nu+m+1 \in F$. Applying Lemma 2.1 to each such segment, we obtain a finite number of pairwise disjoint intervals $I_1,\dots,I_r \in \mathcal{I}$ satisfying the following conditions:

(1) $I_1 \cup \cdots \cup I_r \supset \Lambda$.
(2) $|I_j| = \mathrm{dist}(I_j, F)$ for every $1 \le j \le r$.
(3) $I_j \cap \mathbf{Z} = I_j \cap \Lambda$ for every $1 \le j \le r$.
(4) $3|I_j| \ge \mathrm{card}(I_j \cap \Lambda) > 0$ for every $1 \le j \le r$.

Denote $N_j = \mathrm{card}(I_j \cap \Lambda)$, $1 \le j \le r$. From (3) and (4) we obtain $2N_j \ge N_j + 1 \ge |I_j|$, $1 \le j \le r$. For each $k \in \mathbf{Z}$, we define

$$g_k = a_k\chi_F(k) + \sum_{j=1}^r \left(\frac{1}{N_j}\sum_{i \in I_j\cap\mathbf{Z}} a_i\right)\chi_{I_j}(k) \quad \text{and}$$

$$b_k = a_k - g_k = \sum_{j=1}^r \left(a_k - \frac{1}{N_j}\sum_{i \in I_j\cap\mathbf{Z}} a_i\right)\chi_{I_j}(k).$$

Then $a = g + b$, where $g = \{g_k\}$ and $b = \{b_k\}$. Obviously, we have $\|b\|_1 \le 2\|a\|_1$.

To complete the proof, it suffices to consider $E = \{k \in \mathbf{Z} : \|(Da)_k\| > 2\lambda\}$. The above gives us $E \subset G \cup B$, where $G = \{k \in \mathbf{Z} : \|(Dg)_k\| > \lambda\}$ and $B = \{k \in \mathbf{Z} : \|(Db)_k\| > \lambda\}$. We will estimate $\mathrm{card}(G)$ and $\mathrm{card}(B)$ separately.

It follows from Lemma 2.4 that

$$\mathrm{card}(G) \le \pi^2\lambda^{-2}\|g\|_2^2. \tag{2.3}$$

Thus we need to estimate $\|g\|_2^2$. For each $1 \le j \le r$, if we let J_j be the interval in $\mathcal{I}$ that has the same center as I_j but 5 times the length, then (2) tells us that $J_j \cap F \neq \emptyset$. That is, for each $1 \le j \le r$, there is a $k_j \in J_j \cap F$. Thus for each $1 \le j \le r$,

$$\left\|\frac{1}{N_j}\sum_{i \in I_j\cap\mathbf{Z}} a_i\right\| \le \frac{10}{|J_j|}\sum_{i \in J_j\cap\mathbf{Z}} \|a_i\| \le 10(Ma)_{k_j} \le 10\lambda,$$

where the last $\le$ follows from the definition of F. Consequently,

$$\|g\|_2^2 = \sum_{k \in F}\|a_k\|^2 + \sum_{j=1}^r \left\|\frac{1}{N_j}\sum_{i \in I_j\cap\mathbf{Z}} a_i\right\|^2 N_j \le \lambda\sum_{k \in F}\|a_k\| + 10\lambda\sum_{j=1}^r\sum_{i \in I_j\cap\mathbf{Z}}\|a_i\| \le 10\lambda\|a\|_1.$$

Combining this with (2.3), we find that

$$\mathrm{card}(G) \le 10\pi^2\lambda^{-1}\|a\|_1.$$

Next we consider $\text{card}(B)$.

By Lemma 2.2, we have $\text{card}(\Lambda) \le C_1\lambda^{-1}\|a\|_1$. We will show that

$$\text{card}(B\backslash\Lambda) \le 16\lambda^{-1}\|a\|_1, \tag{2.4}$$

which will complete the proof. For each $1 \le j \le r$, define $b^{(j)} = \{b_k^{(j)}\}$, where

$$b_k^{(j)} = \Big(a_k - \frac{1}{N_j}\sum_{i\in I_j\cap\mathbf{Z}} a_i\Big)\chi_{I_j}(k), \quad k \in \mathbf{Z}.$$

Then $b = b^{(1)} + \cdots + b^{(r)}$. Moreover $\sum_{i\in\mathbf{Z}} b_i^{(j)} = \sum_{i\in I_j\cap\mathbf{Z}} b_i^{(j)} = 0$ for each j. Let t_j denote the center of I_j, $1 \le j \le r$. For $k \in F$,

$$\begin{aligned}(Db)_k &= \sum_{j=1}^{r}\sum_{i\in\mathbf{Z}\backslash\{k\}}\frac{b_i^{(j)}}{k-i} = \sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\frac{b_i^{(j)}}{k-i}\\ &= \sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\Big(\frac{1}{k-i} - \frac{1}{k-t_j}\Big)b_i^{(j)} = \sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\frac{i-t_j}{(k-i)(k-t_j)}b_i^{(j)}.\end{aligned}$$

Since $k \in F$, for each j we have $|k-t_j| \ge \text{dist}(I_j, F) = |I_j|$. If $i \in I_j\cap\mathbf{Z}$, then $|k-t_j|+|I_j| \ge |k-i|$. That is, $2|k-t_j| \ge |k-i|$ for every pair of $1 \le j \le r$ and $i \in I_j \cap \mathbf{Z}$. Hence

$$\begin{aligned}\sum_{k\in F}\|(Db)_k\| &\le \sum_{k\in F}\sum_{j=1}^{r}2|I_j|\sum_{i\in I_j\cap\mathbf{Z}}\frac{\|b_i^{(j)}\|}{(k-i)^2} = \sum_{j=1}^{r}2|I_j|\sum_{i\in I_j\cap\mathbf{Z}}\sum_{k\in F}\frac{\|b_i^{(j)}\|}{(k-i)^2}\\ &\le 4\sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\|b_i^{(j)}\||I_j|\sum_{\nu\ge\text{dist}(I_j,F)}\frac{1}{\nu^2} = 4\sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\|b_i^{(j)}\||I_j|\sum_{\nu\ge|I_j|}\frac{1}{\nu^2},\end{aligned}$$

where the last step follows from (2). Consequently,

$$\sum_{k\in F}\|(Db)_k\| \le 8\sum_{j=1}^{r}\sum_{i\in I_j\cap\mathbf{Z}}\|b_i^{(j)}\| = 8\|b\|_1 \le 16\|a\|_1.$$

Since $B\backslash\Lambda = \{k \in F : \|(Db)_k\| > \lambda\}$, (2.4) follows from this. This completes the proof. □

Proposition 2.6. *For each $1 < p < \infty$, the discrete Hilbert transform D is bounded on $\ell^p(\mathbf{Z}, \mathcal{H})$.*

Proof. Applying Lemmas 2.4 and 2.5, when $1 < p < 2$, the boundedness of D on $\ell^p(\mathbf{Z}, \mathcal{H})$ is obtained by standard interpolation. The boundedness of D in the case $2 < p < \infty$ is then obtained from its boundedness in the case $1 < p < 2$ and the duality between $\ell^p(\mathbf{Z}, \mathcal{H})$ and $\ell^{p/(p-1)}(\mathbf{Z}, \mathcal{H})$. □

3. The space $L^{2,p}$

Let $1 < p < \infty$ and let $\mathcal{G}$ be a Hilbert space. For a $\mathcal{G}$-valued measurable function f on $\mathbf{R}$, we define

$$\|f\|_{2,p} = \left\{ \sum_{k \in \mathbf{Z}} \left(\int_0^1 \|f(k+x)\|^2 dx \right)^{p/2} \right\}^{1/p}.$$

Further, we define $L^{2,p}(\mathcal{G})$ to be the collection of $\mathcal{G}$-valued measurable functions f on $\mathbf{R}$ satisfying the condition $\|f\|_{2,p} < \infty$.

Proposition 3.1. *For each $1 < p < \infty$, the Hilbert transform*

$$(Hf)(x) = \text{p.v.} \int \frac{f(t)}{x-t} dt, \quad x \in \mathbf{R},$$

is a bounded operator on $L^{2,p}(\mathcal{G})$.

Proof. Obviously, $H = H_1 + H_2$, where

$$(H_1 f)(x) = \int_{|x-t|>1} \frac{f(t)}{x-t} dt \quad \text{and} \quad (H_2 f)(x) = \text{p.v.} \int_{|x-t|\le 1} \frac{f(t)}{x-t} dt,$$

$f \in L^{2,p}(\mathcal{G})$. It suffices to show that both H_1 and H_2 are bounded on $L^{2,p}(\mathcal{G})$.

In the case of H_1, note that for any $k \in \mathbf{Z}$ and $x \in [0,1)$, we have

$$(H_1 f)(k+x) = \int_{|k+x-t|>1} \frac{f(t)}{k+x-t} dt = g_k(x) + h_k(x),$$

where

$$g_k(x) = \sum_{j \in \mathbf{Z} \setminus \{k-1,k,k+1\}} \int_0^1 \frac{f(j+t)}{k-j+x-t} dt \quad \text{and}$$

$$h_k(x) = \int_{[k-1,k+2) \setminus [k+x-1,k+x+1]} \frac{f(t)}{k+x-t} dt.$$

We have

$$g_k(x) = \sum_{|j-k|\ge 2} \int_0^1 \frac{f(j+t)}{k-j+x-t} dt = u_k - v_k(x) + w_k, \tag{3.1}$$

$x \in [0,1)$, where

$$u_k = \sum_{j \in \mathbf{Z} \setminus \{k\}} \frac{1}{k-j} \int_0^1 f(j+t) dt,$$

$$v_k(x) = \sum_{|j-k|\ge 2} \int_0^1 \frac{(x-t) f(j+t)}{(k-j+x-t)(k-j)} dt \quad \text{and}$$

$$w_k = \int_0^1 \{f(k+1+t) - f(k-1+t)\} dt.$$

For each $j \in \mathbf{Z}$, we define

$$a_j = \int_0^1 f(j+t)dt, \quad b_j = \int_0^1 \|f(j+t)\|dt \quad \text{and} \quad c_j = \left(\int_0^1 \|f(j+t)\|^2 dt\right)^{1/2}.$$

Obviously, we have $\|a_j\| \le b_j \le c_j$ and $\sum_{j\in\mathbf{Z}} c_j^p = \|f\|_{2,p}^p$. It is also easy to see that

$$\|v_k(x)\| \le C \sum_{|j-k|\ge 2} \frac{b_j}{(k-j)^2}$$

for all $k \in \mathbf{Z}$ and $x \in [0,1)$. Thus

$$\begin{aligned}\left(\int_0^1 \|v_k(x)\|^2 dx\right)^{p/2} &\le \left(C \sum_{|j-k|\ge 2} \frac{b_j}{(k-j)^2}\right)^p \\ &\le C^p \sum_{|j-k|\ge 2} \frac{c_j^p}{|k-j|^p} \left(\sum_{|j-k|\ge 2} \frac{1}{|k-j|^{p/(p-1)}}\right)^{p-1}.\end{aligned}$$

Therefore

$$\sum_{k\in\mathbf{Z}} \left(\int_0^1 \|v_k(x)\|^2 dx\right)^{p/2} \le C_1 \sum_{k\in\mathbf{Z}} \sum_{|j-k|\ge 2} \frac{c_j^p}{|k-j|^p} \le C_2 \|f\|_{2,p}^p. \tag{3.2}$$

Since $w_k = a_{k+1} - a_{k-1}$, we have

$$\sum_{k\in\mathbf{Z}} \|w_k\|^p \le 2^{p-1} \sum_{k\in\mathbf{Z}} (\|a_{k+1}\|^p + \|a_{k-1}\|^p) \le 2^p \|f\|_{2,p}^p. \tag{3.3}$$

If we set $u = \{u_k\}$ and $a = \{a_k\}$, then $u = Da$, where D is the discrete Hilbert transform introduced in Section 2. Applying Proposition 2.6, we have

$$\sum_{k\in\mathbf{Z}} \|u_k\|^p \le C_3 \sum_{k\in\mathbf{Z}} \|a_k\|^p \le C_3 \|f\|_{2,p}^p. \tag{3.4}$$

Combining (3.1)-(3.4), we conclude that

$$\sum_{k\in\mathbf{Z}} \left(\int_0^1 \|g_k(x)\|^2 dx\right)^{p/2} \le 3^{p-1}(C_2 + 2^p + C_3)\|f\|_{2,p}^p. \tag{3.5}$$

It is obvious that $\|h_k(x)\| \le b_{k-1} + b_k + b_{k+1}$ for all $k \in \mathbf{Z}$ and $x \in [0,1)$. Hence

$$\sum_{k\in\mathbf{Z}} \left(\int_0^1 \|h_k(x)\|^2 dx\right)^{p/2} \le 3^p \|f\|_{2,p}^p. \tag{3.6}$$

Combining (3.5), (3.6) with the relation $(H_1 f)(k + x) = g_k(x) + h_k(x)$, we obtain

$$\|H_1 f\|_{2,p}^p = \sum_{k \in \mathbf{Z}} \left(\int_0^1 \|(H_1 f)(k + x)\|^2 dx \right)^{p/2} \leq C_5 \|f\|_{2,p}^p.$$

That is, H_1 is bounded on $L^{2,p}(\mathcal{G})$. Next we consider H_2.

First of all, H_2 is bounded on $L^2(\mathbf{R}, \mathcal{G})$. To see this, observe that $L^{2,2}(\mathcal{G}) = L^2(\mathbf{R}, \mathcal{G})$. Thus by what we proved in the above, H_1 is bounded on $L^2(\mathbf{R}, \mathcal{G})$. Since we know that H is bounded on $L^2(\mathbf{R}, \mathcal{G})$, $H_2 = H - H_1$ is bounded on $L^2(\mathbf{R}, \mathcal{G})$.

Given any $f \in L^{2,p}(\mathcal{G})$, we define

$$\varphi_k(x) = \chi_{[k,k+1)}(x) f(x)$$

for all $k \in \mathbf{Z}$ and $x \in \mathbf{R}$. For $k \in \mathbf{Z}$ and $x \in [0, 1)$, it is easy to see that

$$(H_2 f)(k + x) = (H_2 \varphi_{k-1})(k + x) + (H_2 \varphi_k)(k + x) + (H_2 \varphi_{k+1})(k + x).$$

Therefore by the preceding paragraph, we have

$$\begin{aligned} \int_0^1 \|(H_2 f)(k + x)\|^2 dx &\leq 3 \sum_{j=k-1}^{k+1} \int_0^1 \|(H_2 \varphi_j)(k + x)\|^2 dx \\ &\leq C_6 \sum_{j=k-1}^{k+1} \int_{-\infty}^{\infty} \|\varphi_j(x)\|^2 dx = C_6 \sum_{j=k-1}^{k+1} \int_0^1 \|f(j + x)\|^2 dx. \end{aligned}$$

Consequently,

$$\begin{aligned} \|H_2 f\|_{2,p}^p &= \sum_{k \in \mathbf{Z}} \left(\int_0^1 \|(H_2 f)(k + x)\|^2 dx \right)^{p/2} \\ &\leq C_7 \sum_{k \in \mathbf{Z}} \sum_{j=k-1}^{k+1} \left(\int_0^1 \|f(j + x)\|^2 dx \right)^{p/2} = 3C_7 \|f\|_{2,p}^p. \end{aligned}$$

This proves the boundedness of H_2 and completes the proof of the proposition. □

Proposition 3.2. *Let $-\infty < a < b < \infty$ and let $\mathcal{X}$ be any Hilbert space. For each $1 < p < \infty$, let $L^{2,p}([a, b), \mathcal{X})$ be the collection of $\mathcal{X}$-valued measurable functions f on $\mathbf{R} \times [a, b)$ satisfying the condition $\|f\|_{2,p;a,b} < \infty$, where*

$$\|f\|_{2,p;a,b}^p = \sum_{k \in \mathbf{Z}} \left(\int_0^1 \int_a^b \|f(k + x, y)\|^2 dy dx \right)^{p/2}.$$

Define the operator $H_{a,b}$ *by the formula*

$$(H_{a,b}f)(x,y) = \text{p.v.} \int \frac{f(t,y)}{x-t}dt, \quad x \in \mathbf{R} \;\; and \;\; y \in [a,b),$$

$f \in L^{2,p}([a,b),\mathcal{X})$. *Then* $H_{a,b}$ *is bounded on* $L^{2,p}([a,b),\mathcal{X})$ *with a norm independent of* a, b.

Proof. Observe that if we set $\mathcal{G} = L^2([a,b),\mathcal{X})$, then $L^{2,p}(\mathcal{G}) = L^{2,p}([a,b),\mathcal{X})$. Thus $H_{a,b}$ has the same norm as that of the H in Proposition 3.1. □

4. The operator T

Let dA denote the natural area measure on $\mathbf{C}$. For each $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}^2$, we define the square

$$I_\alpha = \{\alpha_1 + x + i(\alpha_2 + y) : x, y \in [0,1)\} \tag{4.1}$$

in $\mathbf{C}$. Let $\mathcal{X}$ be a Hilbert space. For each $1 < p < \infty$, we define $\mathcal{L}^{2,p}(\mathcal{X})$ to be the collection of $\mathcal{X}$-valued measurable functions φ on $\mathbf{C}$ satisfying the condition $\|\varphi\|_{2,p} < \infty$, where

$$\|\varphi\|_{2,p}^p = \sum_{\alpha \in \mathbf{Z}^2} \left(\int_{I_\alpha} \|\varphi(z)\|^2 dA(z) \right)^{p/2}.$$

Furthermore, we define

$$(\tilde{H}\varphi)(x+iy) = \text{p.v.} \int \frac{\varphi(t+iy)}{x-t}dt, \quad x, y \in \mathbf{R},$$

Proposition 4.1. *For each* $1 < p < \infty$, *the operator* $\tilde{H}$ *is bounded on* $\mathcal{L}^{2,p}(\mathcal{X})$.

Proof. For each $j \in \mathbf{Z}$, define the subspace

$$\mathcal{L}_j = \{\varphi \in \mathcal{L}^{2,p}(\mathcal{X}) : \varphi(x+iy) = 0 \text{ for all } y \in \mathbf{R}\backslash[j, j+1) \text{ and } x \in \mathbf{R}\}$$

of $\mathcal{L}^{2,p}(\mathcal{X})$. By Proposition 3.2, $\tilde{H}$ is bounded on each $\mathcal{L}_j$, with a norm that is independent of j. Obviously, $\tilde{H}$ maps each $\mathcal{L}_j$ into itself. Therefore $\tilde{H}$ is bounded on $\mathcal{L}^{2,p}(\mathcal{X})$. □

Let $\theta \in \mathbf{R}$. For $\varphi \in \mathcal{L}^{2,p}(\mathcal{X})$, we define

$$(R_\theta\varphi)(\zeta) = \varphi(e^{i\theta}\zeta), \quad \zeta \in \mathbf{C}.$$

Proposition 4.2. *For every pair of* $1 < p < \infty$ *and* $\theta \in \mathbf{R}$, *the operator* R_θ *is bounded on* $\mathcal{L}^{2,p}(\mathcal{X})$. *Moreover, given any* $1 < p < \infty$, *there is a constant* $0 < C_{4.2} = C_{4.2}(p) < \infty$ *such that for every* $\theta \in \mathbf{R}$, *the norm of the operator* $R_\theta : \mathcal{L}^{2,p}(\mathcal{X}) \to \mathcal{L}^{2,p}(\mathcal{X})$ *is bounded by* $C_{4.2}$.

Proof. Given any $\theta \in \mathbf{R}$, we define $I_{\alpha;\theta} = e^{i\theta} I_\alpha$, $\alpha \in \mathbf{Z}^2$. Given any pair of $\theta \in \mathbf{R}$ and $\alpha \in \mathbf{Z}^2$, define $B_{\alpha;\theta} = \{\beta \in \mathbf{Z}^2 : I_\beta \cap I_{\alpha;\theta} \neq \emptyset\}$. For $\alpha = (\alpha_1, \alpha_2)$, if $I_\beta \cap I_{\alpha;\theta} \neq \emptyset$, then we have $|e^{i\theta}(\alpha_1 + i\alpha_2) - z| \leq 2\sqrt{2}$ for every $z \in I_\beta$. Therefore $\mathrm{card}(B_{\alpha;\theta}) < 32$. For any $\varphi \in \mathcal{L}^{2,p}(\mathcal{X})$, by the rotation invariance of dA, we have

$$\|R_\theta \varphi\|_{2,p}^p = \sum_{\alpha \in \mathbf{Z}^2} \left(\int_{I_{\alpha;\theta}} \|\varphi(z)\|^2 dA(z) \right)^{p/2} \leq 32^{p-1} \sum_{\alpha \in \mathbf{Z}^2} \sum_{\beta \in B_{\alpha;\theta}} \left(\int_{I_\beta} \|\varphi(z)\|^2 dA(z) \right)^{p/2}.$$

Hence

$$\|R_\theta \varphi\|_{2,p}^p \leq 32^{p-1} \sum_{\beta \in \mathbf{Z}^2} \mathrm{card}\{\alpha \in \mathbf{Z}^2 : I_\beta \cap I_{\alpha;\theta} \neq \emptyset\} \left(\int_{I_\beta} \|\varphi(z)\|^2 dA(z) \right)^{p/2}. \tag{4.2}$$

For $\beta = (\beta_1, \beta_2)$, if $I_\beta \cap I_{\alpha;\theta} \neq \emptyset$, then we have $|\beta_1 + i\beta_2 - z| \leq 2\sqrt{2}$ for every $z \in I_{\alpha;\theta}$. Hence $\mathrm{card}\{\alpha \in \mathbf{Z}^2 : I_\beta \cap I_{\alpha;\theta} \neq \emptyset\} < 32$ for every $\beta \in \mathbf{Z}^2$. Substituting this bound in (4.2), the proof is complete. □

Proposition 4.3. *Let $1 < p < \infty$. For $\varphi \in \mathcal{L}^{2,p}(\mathcal{X})$ we define*

$$(S\varphi)(\zeta) = \frac{1}{2\pi} \mathrm{p.v.} \int \frac{\varphi(\zeta + z)}{z|z|} dA(z), \quad \zeta \in \mathbf{C}.$$

Then S is a bounded operator on $\mathcal{L}^{2,p}(\mathcal{X})$.

Proof. Integrating in the polar coordinates, for any $\epsilon > 0$ we have

$$\begin{aligned} \frac{1}{2\pi} \int_{|z| \geq \epsilon} \frac{\varphi(\zeta + z)}{z|z|} dA(z) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \int_\epsilon^\infty \frac{\varphi(\zeta + re^{i\theta})}{r} dr d\theta \\ &= \frac{1}{2\pi} \int_0^{\pi} e^{-i\theta} \int_\epsilon^\infty \frac{\varphi(\zeta + re^{i\theta}) - \varphi(\zeta - re^{i\theta})}{r} dr d\theta \\ &= \frac{-1}{2\pi} \int_0^{\pi} e^{-i\theta} (R_{-\theta} \tilde{H}_\epsilon R_\theta \varphi)(\zeta) d\theta, \end{aligned}$$

where

$$(\tilde{H}_\epsilon \psi)(x + iy) = \int_\epsilon^\infty \frac{\psi(x + iy - r) - \psi(x + iy + r)}{r} dr = \int_{|x-t| \geq \epsilon} \frac{\psi(t + iy)}{x - t} dt.$$

Letting ϵ descend to 0 in the above, we obtain the operator identity

$$S = \frac{-1}{2\pi} \int_0^\pi e^{-i\theta} R_{-\theta} \tilde{H} R_\theta d\theta.$$

Thus the boundedness of $S : \mathcal{L}^{2,p}(\mathcal{X}) \to \mathcal{L}^{2,p}(\mathcal{X})$ follows from Propositions 4.1 and 4.2. □

Proposition 4.4. *On each $\mathcal{L}^{2,p}(\mathcal{X})$, $1 < p < \infty$, the formula*

$$(T\varphi)(\zeta) = \text{p.v.} \int \frac{\varphi(z)}{(\zeta - z)^2} dA(z), \quad \zeta \in \mathbf{C},$$

defines a bounded operator.

Proof. It was shown on page 65 in [1] that $T = \pi S^2$. (Note that our T is $-\pi$ times the T defined on page 62 of [1].) Hence the boundedness of T follows from Proposition 4.3. □

We now consider the standard partition of $\mathbf{C}^n$ by cubes of the size $1 \times 1 \times \cdots \times 1 \times 1$. That is, for each $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbf{Z}^{2n}$, we introduce the cube

$$Q_\alpha = I_{(\alpha_1,\alpha_2)} \times \cdots \times I_{(\alpha_{2n-1},\alpha_{2n})}, \tag{4.3}$$

where, for each $1 \le j \le n$, $I_{(\alpha_{2j-1},\alpha_{2j})}$ is defined by (4.1). For each $1 < p < \infty$, we define $\mathcal{L}_n^{2,p}$ to be the collection of complex-valued measurable functions φ on $\mathbf{C}^n$ satisfying the condition $\|\varphi\|_{2,p} < \infty$, where

$$\|\varphi\|_{2,p}^p = \sum_{\alpha \in \mathbf{Z}^{2n}} \left(\int_{Q_\alpha} |\varphi(z)|^2 dV(z) \right)^{p/2}.$$

Proposition 4.5. *On each $\mathcal{L}_n^{2,p}$, $1 < p < \infty$, the operators*

$$(T_j\varphi)(\zeta_1, \dots, \zeta_n) = \text{p.v.} \int \frac{\varphi(\zeta_1, \cdots, \zeta_{j-1}, z, \zeta_{j+1}, \dots, \zeta_n)}{(\zeta_j - z)^2} dA(z),$$

$(\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$, $1 \le j \le n$, *are bounded.*

Proof. In the case $n = 1$, $\mathcal{L}_1^{2,p} = \mathcal{L}^{2,p}(\mathbf{C})$ and T_1 is just the T in Proposition 4.4. Hence T_1 is bounded if $n = 1$.

Suppose that $n \ge 2$. Obviously, in this case we also only need to consider T_1. For $\tau = (\tau_1, \dots, \tau_{2n-2}) \in \mathbf{Z}^{2n-2}$, we define

$$\Delta_\tau = I_{(\tau_1,\tau_2)} \times \cdots \times I_{(\tau_{2n-3},\tau_{2n-2})},$$

where $I_{(\tau_{2j-1},\tau_{2j})}$ is given by (4.1), $1 \le j \le n-1$. We then define, for each $\tau \in \mathbf{Z}^{2n-2}$, the subspace

$$\mathcal{L}_{n;\tau}^{2,p} = \{\varphi \in \mathcal{L}_n^{2,p} : \varphi = 0 \text{ on } \mathbf{C}^n \backslash (\mathbf{C} \times \Delta_\tau)\}$$

of $\mathcal{L}_n^{2,p}$. If we set $\mathcal{X}_\tau = L^2(\Delta_\tau)$, then $\mathcal{L}_{n;\tau}^{2,p} = \mathcal{L}^{2,p}(\mathcal{X}_\tau)$. Therefore, by Proposition 4.4,

$$T_1 : \mathcal{L}_{n;\tau}^{2,p} \to \mathcal{L}_{n;\tau}^{2,p}$$

is bounded, and its norm is independent of $\tau \in \mathbf{Z}^{2n-2}$, because all the $\mathcal{X}_\tau$'s are isometric images of each other. For each $\varphi \in \mathcal{L}_n^{2,p}$, we have

$$\|\varphi\|_{2,p}^p = \sum_{\tau \in \mathbf{Z}^{2n-2}} \sum_{\beta \in \mathbf{Z}^2} \left(\int_{I_\beta \times \Delta_\tau} |\varphi(\zeta)|^2 dV(\zeta) \right)^{p/2}.$$

From these facts we conclude that T_1 is bounded on $\mathcal{L}^{2,p}_n$. This completes the proof. □

5. Proof of Theorem 1.2 in the case $1 < p < 2$

We begin with a slightly different version of [14, Lemma 7.1].

Lemma 5.1. *Let $f \in C^2(\mathbf{C}^n) \cap L^\infty(\mathbf{C}^n)$ be a function which has the property that $\bar{\partial}_j f \in \mathcal{L}^{2,p}_n$ for some $j \in \{1, \dots, n\}$ and $1 < p < \infty$. Then*

$$\partial_j f = -\pi^{-1} T_j(\bar{\partial}_j f), \tag{5.1}$$

where T_j is the operator in Proposition 4.5.

Proof. As in [14], we pick a $\gamma \in C^\infty(\mathbf{R})$ which has the properties that $0 \le \gamma \le 1$ on $\mathbf{R}$, that $\gamma = 1$ on $(-\infty, 0]$, and that $\gamma = 0$ on $[1, \infty)$. For each $w \in \mathbf{C}$, we have

$$\lim_{R\to\infty} \int \frac{|\gamma'(|z| - R)|}{|z - w|^2} dA(z) \le \lim_{R\to\infty} \int_{0 \le |z| - R \le 1} \frac{\|\gamma'\|_\infty}{|z - w|^2} dA(z) = 0. \tag{5.2}$$

For each $R > 0$, define $\gamma_R(\zeta_1, \dots, \zeta_n) = \gamma(|\zeta_j| - R)$. Given an f as in the statement of the lemma, we define the function $f_R = \gamma_R f$ for each $R > 0$. Then, as in the proof of [14, Lemma 7.1], we have

$$f_R(\zeta_1, \dots \zeta_{j-1}, \zeta_j \zeta_{j+1}, \dots, \zeta_n) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{(\bar{\partial}_j f_R)(\zeta_1, \dots, \zeta_{j-1}, z, \zeta_{j+1}, \dots, \zeta_n)}{z - \zeta_j} dz \wedge d\bar{z}$$

and

$$-\pi \partial_j f_R = T_j(\bar{\partial}_j f_R) = T_j(f \bar{\partial}_j \gamma_R) + T_j(\gamma_R \bar{\partial}_j f). \tag{5.3}$$

Also see [2, pages 94-95]. Since $f \in L^\infty(\mathbf{C}^n)$, it follows from (5.2) that

$$\lim_{R\to\infty} T_j(f \bar{\partial}_j \gamma_R)(\zeta) = 0 \tag{5.4}$$

for every $\zeta \in \mathbf{C}^n$. It is obvious that, as $R \to \infty$, we have $\|\gamma_R \bar{\partial}_j f - \bar{\partial}_j f\|_{2,p} \to 0$. Applying Proposition 4.5, we have $\|T_j(\gamma_R \bar{\partial}_j f) - T_j(\bar{\partial}_j f)\|_{2,p} \to 0$ as $R \to \infty$. Combining this $\mathcal{L}^{2,p}_n$ convergence with (5.4), if we take the limit $R \to \infty$ in (5.3), we obtain (5.1). □

Proof of Theorem 1.2, *the case* $1 < p < 2$. Let $f \in L^\infty(\mathbf{C}^n)$ be such that $H_f \in \mathcal{C}_p$. Take the decomposition $f = f_1 + f_2$ with $f_1 \in C^2(\mathbf{C}^n)$ as in the original proof in [14]. As was correctly shown in [14], $H_{\bar{f}_2} \in \mathcal{C}_p$. The problem in [14] was the inequality

$$\|H_{\bar{f}_1}\|_p \le C \|\bar{\partial} \bar{f}_1\|_{L^p},$$

which is incorrect in the case $1 < p < 2$. The correct inequality is, in the notation of [14],

$$\|H_{\bar{f}_1}\|_p \le C \|M_{2,r}(|\bar{\partial} \bar{f}_1|)\|_{L^p}.$$

Thus we need to show that $\|M_{2,r}(|\bar\partial \bar f_1|)\|_{L^p} < \infty$. By the definition of $M_{2,r}$ in [14], the condition $\|M_{2,r}(|\bar\partial \bar f_1|)\|_{L^p} < \infty$ is equivalent to the membership $\bar\partial_j \bar f_1 \in \mathcal{L}_n^{2,p}$ for every $1 \le j \le n$. Since $\bar\partial_j \bar f_1 = \overline{\partial_j f_1}$, it suffices to show that $\partial_j f_1 \in \mathcal{L}_n^{2,p}$ for $j = 1, \dots, n$.

It was shown in [14] that the membership $H_f \in \mathcal{C}_p$ implies $\|M_{2,r}(|\bar\partial f_1|)\|_{L^p} < \infty$. Thus $\bar\partial_j f_1 \in \mathcal{L}_n^{2,p}$ for every $1 \le j \le n$. Further, it is known that the membership $f \in L^\infty(\mathbf{C}^n)$ implies the membership $f_1 \in L^\infty(\mathbf{C}^n)$. Therefore, by Lemma 5.1,

$$\partial_j f_1 = -\pi^{-1} T_j(\bar\partial_j f_1)$$

for each $j \in \{1, \dots, n\}$. Thus by Proposition 4.5, the membership $\bar\partial_j f_1 \in \mathcal{L}_n^{2,p}$ implies the desired membership $\partial_j f_1 \in \mathcal{L}_n^{2,p}$, $j = 1, \dots, n$. This completes the proof. □

6. Symmetric gauge functions and associated ideals

As preparation for the proofs of Theorems 1.3 and 1.4, we now introduce general operator ideals. Our main reference for this discussion will be [12]. Following [12], let $\hat c$ denote the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \ne 0\}$ is finite. A symmetric gauge function (also called *symmetric norming function*) is a map

$$\Phi : \hat c \to [0, \infty)$$

that has the following properties:

(a) Φ is a norm on $\hat c$.

(b) $\Phi(\{1, 0, \dots, 0, \dots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [12, page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\ge1} \Phi(\{s_1(A), \dots, s_j(A), 0, \dots, 0, \dots\}) \tag{6.1}$$

for operators. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{6.2}$$

is a norm ideal [12, page 68].

For our purpose, we need to extend the domain of definition of a symmetric gauge Φ beyond the space $\hat c$. Suppose that $\{b_j\}_{j\in\mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., the set $\{j \in \mathbf{N} : b_j \ne 0\}$ is not necessarily finite. Then we define

$$\Phi(\{b_j\}_{j\in\mathbf{N}}) = \sup_{k\ge1} \Phi(\{b_1, \dots, b_k, 0, \dots, 0, \dots\}). \tag{6.3}$$

More generally, for any countable, infinite index set A, we define

$$\Phi(\{b_\alpha\}_{\alpha\in A}) = \Phi(\{b_{h(j)}\}_{j\in\mathbf{N}}), \tag{6.4}$$

where $h : \mathbf{N} \to A$ is a bijection. Property (c) above ensures that the value of $\Phi(\{b_\alpha\}_{\alpha\in A})$ is independent of the choice of the bijection $h : \mathbf{N} \to A$.

Let us recall some familiar examples. For each $1 \le p < \infty$, the formula $\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = (\sum_{j=1}^\infty |a_j|^p)^{1/p}$ defines a symmetric gauge function on $\hat{c}$, and the corresponding ideal $\mathcal{C}_{\Phi_p}$ defined by (6.2) is just the Schatten class $\mathcal{C}_p$. For each $1 \le p < \infty$, we define the symmetric gauge functions Φ_p^+ and Φ_p^- defined by the formulas

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\ge 1} \frac{|a_{\pi(1)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + \cdots + j^{-1/p}} \quad \text{and} \quad \Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|a_{\pi(j)}|}{j^{(p-1)/p}},$$

$\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \ge |a_{\pi(2)}| \ge \cdots \ge |a_{\pi(j)}| \ge \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. Then the ideals $\mathcal{C}_{\Phi_p^+}$ and $\mathcal{C}_{\Phi_p^-}$ defined by (6.2) using Φ_p^+ and Φ_p^- are none other than the Lorentz ideals $\mathcal{C}_p^+$ and $\mathcal{C}_p^-$ introduced earlier.

Let $a = \{a_1, \ldots, a_j, \ldots\}$ be a sequence of non-negative numbers. For each $s > 0$, we denote

$$N(a; s) = \operatorname{card}\{j \in \mathbf{N} : a_j > s\}.$$

Lemma 6.1. [8, Lemma 2.1] *Let $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$ we have*

$$\int_0^\infty \{N(a; s)\}^{1/p} ds \le \Phi_p^-(a) \le p \int_0^\infty \{N(a; s)\}^{1/p} ds.$$

Proposition 6.2. [8, Proposition 2.2] *For every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$ and every $s > 0$, define the sequence $a^\vee(s) = \{a_1^\vee(s), \ldots, a_j^\vee(s), \ldots\}$, where*

$$a_j^\vee(s) = \begin{cases} 0 & \text{if} \quad a_j > s \\ a_j & \text{if} \quad a_j \le s \end{cases}, \quad j \in \mathbf{N}.$$

Then given any $1 < p < r < \infty$, there exists a constant $0 < C_{6.2} < \infty$ such that

$$\int_0^\infty \left(\frac{1}{s} \Phi_r^+ (a^\vee(s)) \right)^{r/p} ds \le C_{6.2} \Phi_p^-(a) \tag{6.5}$$

for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$.

Proposition 6.3. [8, Proposition 2.3] *For every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$ and every $s > 0$, define the sequence $a^\wedge(s) = \{a_1^\wedge(s), \ldots, a_j^\wedge(s), \ldots\}$, where*

$$a_j^\wedge(s) = \begin{cases} a_j & \text{if} \quad a_j > s \\ 0 & \text{if} \quad a_j \le s \end{cases}, \quad j \in \mathbf{N}.$$

Then given any $1 < r' < p < \infty$, there exists a constant $0 < C_{6.3} < \infty$ such that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r'}^+(a^\wedge(s))\right)^{r'/p} ds \le C_{6.3}\Phi_p^-(a) \tag{6.6}$$

for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$.

The significance of Propositions 6.2 and 6.3 is this: if we want to dominate a quantity Q by $\Phi_p^-(a)$, it suffices to dominate parts of Q by the integrals in (6.5) and (6.6). This in turn highlights the importance of the symmetric gauge functions Φ_r^+, $1 < r < \infty$. Below are two important facts about this family of symmetric gauge functions.

Lemma 6.4. [7, Lemma 5.6] *Suppose that $1 < p < \infty$. Let $\alpha = \{\alpha_1, \ldots, \alpha_k, \ldots\}$ be a non-increasing sequence of non-negative numbers. Define*

$$F_p(\alpha) = \sup_{k\ge 1} k^{1/p}\alpha_k.$$

Then

$$\frac{p-1}{p}F_p(\alpha) \le \Phi_p^+(\alpha) \le F_p(\alpha).$$

Given a sequence of non-negative numbers $a = \{a_1, \ldots, a_k, \ldots\}$, the conventional weak-type inequality states

$$N(a; s) \le (\Phi_p(a)/s)^p \tag{6.7}$$

for $s > 0$ and $1 < p < \infty$, where Φ_p is the symmetric gauge function for the Schatten class $\mathcal{C}_p$. Below is an improved version of (6.7):

Lemma 6.5. [8, Lemma 2.6] *Suppose that $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \ldots, a_k, \ldots\}$ and every $s > 0$ we have*

$$N(a; s) \le \left(\frac{p}{p-1}\right)^p \left(\frac{1}{s}\Phi_p^+(a)\right)^p.$$

Lemma 6.6. *Let $a = \{a_1, \ldots, a_k, \ldots\}$ be a sequence of non-negative numbers. Let $1 < p < \infty$, $0 < M < \infty$ and $0 < \tau < \infty$. If the inequality*

$$N(a; s) \le M(\tau/s)^p \tag{6.8}$$

holds for every $s > 0$, then $\Phi_p^+(a) \le 2M^{1/p}\tau$.

Proof. There is an injection $\pi : \mathbf{N} \to \mathbf{N}$ such that $a_{\pi(i)} \ge a_{\pi(i+1)}$ for every $i \in \mathbf{N}$ and such that $a_k = 0$ for every $k \in \mathbf{N}\backslash\{\pi(i) : i \in \mathbf{N}\}$. Consider any $i \in \mathbf{N}$ such that $a_{\pi(i)} \ne 0$. Set $s = a_{\pi(i)}/2$. Then by (6.8),

$$i \le N(a; s) \le M(\tau/s)^p = M(2\tau)^p a_{\pi(i)}^{-p}.$$

Solving this, we find that

$$a_{\pi(i)} \le 2M^{1/p}\tau i^{-1/p}$$

if $a_{\pi(i)} \neq 0$. This inequality, of course, also holds in the case $a_{\pi(i)} = 0$. Obviously, this inequality implies $\Phi_p^+(a) \le 2M^{1/p}\tau$. $\square$

7. Interpolation

Let X be a Banach space. We now define three families of spaces.

The first family consists of the familiar spaces $\ell^p(\mathbf{N}, X)$, $1 \le p < \infty$. That is, $\ell^p(\mathbf{N}, X)$ is the collection of the sequences $a = \{a_j\}$ satisfying the condition $\|a\|_p < \infty$, where $a_j \in X$ for every $j \in \mathbf{N}$ and

$$\|a\|_p = \left(\sum_{j=1}^{\infty} \|a_j\|^p \right)^{1/p} = \Phi_p(\{\|a_j\|\}).$$

For each $1 \le p < \infty$, let $\ell^p_+(\mathbf{N}, X)$ be the collection of the sequences $a = \{a_j\}$ satisfying the condition $\|a\|_p^+ < \infty$, where $a_j \in X$ for every $j \in \mathbf{N}$ and

$$\|a\|_p^+ = \Phi_p^+(\{\|a_j\|\}).$$

This gives us the second family of spaces. The third family consists of the spaces $\ell^p_-(\mathbf{N}, X)$, $1 \le p < \infty$. For each $1 \le p < \infty$, let $\ell^p_-(\mathbf{N}, X)$ be the collection of the sequences $a = \{a_j\}$ satisfying the condition $\|a\|_p^- < \infty$, where $a_j \in X$ for every $j \in \mathbf{N}$ and

$$\|a\|_p^- = \Phi_p^-(\{\|a_j\|\}).$$

Furthermore, we define $\ell_{00}(\mathbf{N}, X)$ to be the collection of $a = \{a_j\}$ satisfying the conditions that $a_j \in X$ for every $j \in \mathbf{N}$ and that

$$\mathrm{card}\{j \in \mathbf{N} : a_j \neq 0\} < \infty.$$

That is, if $a = \{a_j\} \in \ell_{00}(\mathbf{N}, X)$, then the sequence $\{a_j\}$ has at most a finite number of nonzero terms.

For any $a = \{a_j\}$, where $a_j \in X$ for every $j \in \mathbf{N}$, and any $s > 0$, we denote

$$N(a; s) = \mathrm{card}\{j \in \mathbf{N} : \|a_j\| > s\},$$

which is consistent with the corresponding notion in Section 6.

We now prove two interpolation results. The first result tells us that boundedness with respect to $\|\cdot\|_p^+$ can be obtained through boundedness with respect to $\|\cdot\|_{r'}$ and $\|\cdot\|_r$, $r' < p < r$.

Proposition 7.1. *Let $1 < r' < r < \infty$. Suppose that $A : \ell^r(\mathbf{N}, X) \to \ell^r(\mathbf{N}, X)$ is a bounded operator. Furthermore, suppose that there is a $0 < B_{r'} < \infty$ such that*

$$\|Ax\|_{r'} \le B_{r'} \|x\|_{r'} \tag{7.1}$$

for every $x \in \ell_{00}(\mathbf{N}, X)$. *Then for each* $r' < p < r$, A *maps* $\ell^p_+(\mathbf{N}, X)$ *into itself, and there is a* $0 < C(p) < \infty$ *such that*

$$\|Aa\|_p^+ \le C(p)\|a\|_p^+$$

for every $a \in \ell^p_+(\mathbf{N}, X)$.

Proof. For each $r' < p < r$, since $\ell^p_+(\mathbf{N}, X) \subset \ell^r(\mathbf{N}, X)$, A is uniquely defined on $\ell^p_+(\mathbf{N}, X)$. What we need to show is that there is a constant $0 < C(p) < \infty$ promised above.

Given an $a \in \ell_{00}(\mathbf{N}, X)$, denote

$$R = \frac{p}{p-1}\|a\|_p^+.$$

By Lemma 6.4, there is a bijection $\pi : \mathbf{N} \to \mathbf{N}$ such that

$$\|a_{\pi(i)}\| \le R/i^{1/p} \quad \text{for every } i \in \mathbf{N}. \tag{7.2}$$

For each $s > 0$, we define the sequences $b(s) = \{b_j(s)\}$ and $c(s) = \{c_j(s)\}$, where the terms are given by the formulas

$$b_{\pi(i)}(s) = \begin{cases} a_{\pi(i)} & \text{if} \quad 1 \le i < (R/s)^p \\ 0 & \text{if} \quad i \ge (R/s)^p \end{cases} \quad \text{and} \quad c_{\pi(i)}(s) = \begin{cases} 0 & \text{if} \quad 1 \le i < (R/s)^p \\ a_{\pi(i)} & \text{if} \quad i \ge (R/s)^p \end{cases},$$

$i \in \mathbf{N}$. Applying (7.1), (7.2) and using the fact that $r'/p < 1$, we have

$$N(Ab(s); s) \le s^{-r'}\|Ab(s)\|_{r'}^{r'} \le B_{r'}^{r'} s^{-r'}\|b(s)\|_{r'}^{r'} = B_{r'}^{r'} s^{-r'} \sum_{1 \le i < (R/s)^p} \|a_{\pi(i)}\|^{r'}$$

$$\le B_{r'}^{r'} s^{-r'} \sum_{1 \le i < (R/s)^p} (R/i^{1/p})^{r'} \le C_1 s^{-r'} R^{r'} \{(R/s)^p\}^{1-(r'/p)} = C_1 (R/s)^p. \tag{7.3}$$

Write B_r for the norm of the operator $A : \ell^r(\mathbf{N}, X) \to \ell^r(\mathbf{N}, X)$, which is finite by assumption. Applying (7.2) and using the fact that $r/p > 1$, we also have

$$N(Ac(s); s) \le s^{-r}\|Ac(s)\|_r^r \le B_r^r s^{-r}\|c(s)\|_r^r = B_r^r s^{-r} \sum_{i \ge (R/s)^p} \|a_{\pi(i)}\|^r$$

$$\le B_r^r s^{-r} \sum_{i \ge (R/s)^p} (R/i^{1/p})^r \le C_2 s^{-r} R^r \{(R/s)^p\}^{1-(r/p)} = C_2 (R/s)^p. \tag{7.4}$$

Since $a = b(s) + c(s)$, we have $Aa = Ab(s) + Ac(s)$. Thus (7.3) and (7.4) together give us

$$N(Aa; 2s) \le N(Ab(s); s) + N(Ac(s); s) \le C_3 (R/s)^p$$

for every $s > 0$, where $C_3 = C_1 + C_2$. A simple rescaling gives us the inequality

$$N(Aa; s) \le 2^p C_3 (R/s)^p = C_4 (\|a\|_p^+/s)^p$$

for all $a \in \ell_{00}(\mathbf{N}, X)$ and $s > 0$, where $C_4 = C_3\{2p/(p-1)\}^p$.

Now consider an arbitrary $a = \{a_k\} \in \ell^p_+(\mathbf{N}, X)$. For each $m \in \mathbf{N}$, if we define the truncated sequence

$$a^{(m)} = \{a_1, \ldots, a_m, 0, \ldots, 0, \ldots\},$$

then obviously $a^{(m)} \in \ell_{00}(\mathbf{N}, X)$ and $\|a^{(m)}\|^+_p \le \|a\|^+_p$. Moreover, $\|Aa - Aa^{(m)}\|_r \to 0$ as $m \to \infty$. Thus it follows from the above inequality that

$$N(Aa; s) \le C_4(\|a\|^+_p/s)^p$$

for all $a \in \ell^p_+(\mathbf{N}, X)$ and $s > 0$. By Lemma 6.6, this means $\|Aa\|^+_p \le 2C_4^{1/p}\|a\|^+_p$ for every $a \in \ell^p_+(\mathbf{N}, X)$. This completes the proof. □

Proposition 7.2. *Let $1 < r' < r < \infty$. Suppose that $A : \ell_{00}(\mathbf{N}, X) \to \ell^{r'}_+(\mathbf{N}, X)$ is a linear transformation. Furthermore, suppose that there are $0 < C(r') < \infty$ and $0 < C(r) < \infty$ such that*

$$\|Ax\|^+_{r'} \le C(r')\|x\|^+_{r'} \quad \text{and} \quad \|Ax\|^+_r \le C(r)\|x\|^+_r \tag{7.5}$$

for every $x \in \ell_{00}(\mathbf{N}, X)$. Then for each $r' < p < r$, there is a $0 < D(p) < \infty$ such that

$$\|Aa\|^-_p \le D(p)\|a\|^-_p$$

for every $a \in \ell_{00}(\mathbf{N}, X)$. Consequently, A naturally and uniquely extends to a bounded operator on $\ell^p_-(\mathbf{N}, X)$.

Proof. Let $a = \{a_j\} \in \ell_{00}(\mathbf{N}, X)$ and $s > 0$. We again decompose a in the form $a = b(s) + c(s)$, but this time the sequences $b(s) = \{b_j(s)\}$ and $c(s) = \{c_j(s)\}$ are defined according to the following rules:

$$b_j(s) = \begin{cases} a_j & \text{if} \quad \|a_j\| > s \\ \\ 0 & \text{if} \quad \|a_j\| \le s \end{cases} \quad \text{and} \quad c_j(s) = \begin{cases} 0 & \text{if} \quad \|a_j\| > s \\ \\ a_j & \text{if} \quad \|a_j\| \le s \end{cases},$$

$j \in \mathbf{N}$. Writing $C_1 = \{r'/(r'-1)\}^{r'}$, it follows from Lemma 6.5 and (7.5) that

$$N(Ab(s); s) \le C_1(\|Ab(s)\|^+_{r'}/s)^{r'} \le C_2(\|b(s)\|^+_{r'}/s)^{r'}.$$

Since $r' < p$, we can apply Proposition 6.3 to obtain

$$\int_0^\infty \{N(Ab(s); s)\}^{1/p} ds \le C_2^{1/p} \int_0^\infty (\|b(s)\|^+_{r'}/s)^{r'/p} ds \le C_3\|a\|^-_p. \tag{7.6}$$

It also follows from Lemma 6.5 and (7.5) that

$$N(Ac(s); s) \le C_4(\|Ac(s)\|^+_r/s)^r \le C_5(\|c(s)\|^+_r/s)^r.$$

Since $p < r$, we can apply Proposition 6.2 to obtain

$$\int_0^\infty \{N(Ac(s); s)\}^{1/p} ds \le C_5^{1/p} \int_0^\infty (\|c(s)\|_r^+/s)^{r/p} ds \le C_6 \|a\|_p^- . \tag{7.7}$$

The relation $a = b(s) + c(s)$ means that $N(Aa; 2s) \le N(Ab(s); s) + N(Ac(s); s)$ for every $s > 0$. Hence from (7.6) and (7.7) we obtain

$$\int_0^\infty \{N(Aa; 2s)\}^{1/p} ds \le \int_0^\infty (\{N(Ab(s); s)\}^{1/p} + \{N(Ac(s); s)\}^{1/p}) ds \le C_7 \|a\|_p^- ,$$

where $C_7 = C_3 + C_6$. Applying Lemma 6.1, we find that

$$\|Aa\|_p^- \le p \int_0^\infty \{N(Aa; s)\}^{1/p} ds = 2p \int_0^\infty \{N(Aa; 2s)\}^{1/p} ds \le 2pC_7 \|a\|_p^- ,$$

which proves the proposition. □

We recall the cubes $Q_\alpha \subset \mathbf{C}^n$ defined by (4.3), $\alpha \in \mathbf{Z}^{2n}$. In addition to the spaces $\mathcal{L}_n^{2,p}$ defined in Section 4, $1 < p < \infty$, we now define two more families of spaces.

For each $1 < p < \infty$, we define $\mathcal{L}_n^{2,p,+}$ to be the collection of complex-valued measurable functions φ on $\mathbf{C}^n$ satisfying the condition $\|\varphi\|_{2,p}^+ < \infty$, where

$$\|\varphi\|_{2,p}^+ = \Phi_p^+ \left(\left\{ \left(\int_{Q_\alpha} |\varphi(z)|^2 dV(z) \right)^{1/2} \right\}_{\alpha \in \mathbf{Z}^{2n}} \right). \tag{7.8}$$

Similarly, for each $1 < p < \infty$, we define $\mathcal{L}_n^{2,p,-}$ to be the collection of complex-valued measurable functions φ on $\mathbf{C}^n$ satisfying the condition $\|\varphi\|_{2,p}^- < \infty$, where

$$\|\varphi\|_{2,p}^- = \Phi_p^- \left(\left\{ \left(\int_{Q_\alpha} |\varphi(z)|^2 dV(z) \right)^{1/2} \right\}_{\alpha \in \mathbf{Z}^{2n}} \right). \tag{7.9}$$

The main conclusion of the section is that the analogue of Proposition 4.5 holds for these two families of spaces.

Proposition 7.3. *On each* $\mathcal{L}_n^{2,p,+}$, $1 < p < \infty$, *the operators*

$$(T_j\varphi)(\zeta_1, \dots, \zeta_n) = \text{p.v.} \int \frac{\varphi(\zeta_1, \cdots, \zeta_{j-1}, z, \zeta_{j+1}, \dots, \zeta_n)}{(\zeta_j - z)^2} dA(z),$$

$(\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$, $1 \le j \le n$, *are bounded.*

Proof. From (4.3) we clearly see that $Q_\alpha = Q_0 + \alpha$ for every $\alpha \in \mathbf{Z}^{2n}$. Let $X = L^2(Q_0)$. Let $b : \mathbf{N} \to \mathbf{Z}^{2n}$ be a bijection. Then any function φ on $\mathbf{C}^n$ is naturally identified with the sequence of functions $\{\varphi_j\}$ on Q_0, where

$$\varphi_j(z) = \varphi(b(j) + z), \quad z \in Q_0,$$

$j \in \mathbf{N}$. This naturally identifies $\mathcal{L}_n^{2,p}$ with $\ell^p(\mathbf{N}, X)$ and $\mathcal{L}_n^{2,p,+}$ with $\ell_+^p(\mathbf{N}, X)$. Under this identification, Proposition 4.5 tells us that $T_j : \ell^p(\mathbf{N}, X) \to \ell^p(\mathbf{N}, X)$ is bounded for every $1 < p < \infty$. Thus it follows from Proposition 7.1 that $T_j : \ell_+^p(\mathbf{N}, X) \to \ell_+^p(\mathbf{N}, X)$ is also bounded for every $1 < p < \infty$. □

Proposition 7.4. *On each* $\mathcal{L}_n^{2,p,-}$, $1 < p < \infty$, *the operators*

$$(T_j\varphi)(\zeta_1, \ldots, \zeta_n) = \text{p.v.} \int \frac{\varphi(\zeta_1, \cdots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n)}{(\zeta_j - z)^2} dA(z),$$

$(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$, $1 \le j \le n$, *are bounded.*

Proof. Again, if we let $X = L^2(Q_0)$, then $\mathcal{L}_n^{2,p,-}$ is naturally identified with $\ell_-^p(\mathbf{N}, X)$. Proposition 7.3 tells us that under this identification, $T_j : \ell_+^p(\mathbf{N}, X) \to \ell_+^p(\mathbf{N}, X)$ is bounded for every $1 < p < \infty$. Applying Proposition 7.2, we see that $T_j : \ell_-^p(\mathbf{N}, X) \to \ell_-^p(\mathbf{N}, X)$ is also bounded for every $1 < p < \infty$. □

8. Hankel operators in norm ideals

The purpose of this section is to derive two convenient sufficient conditions for a Hankel operator on $H^2(\mathbf{C}^n, d\mu)$ to belong to a general ideal, and these sufficient conditions will be applied in the proofs of Theorems 1.3 and 1.4.

Let k_z denote the normalized reproducing kernel for $H^2(\mathbf{C}^n, d\mu)$. That is,

$$k_z(\zeta) = e^{\langle \zeta, z \rangle} e^{-|z|^2/2}, \qquad z, \zeta \in \mathbf{C}^n.$$

For each $z \in \mathbf{C}^n$, let τ_z be the translation $\tau_z(\zeta) = \zeta + z$, $\zeta \in \mathbf{C}^n$. As in [20,3,10], we define

$$\mathcal{T}(\mathbf{C}^n) = \{f \in L^2(\mathbf{C}^n, d\mu) : f \circ \tau_z \in L^2(\mathbf{C}^n, d\mu) \ \text{ for every } \ z \in \mathbf{C}^n\}.$$

Obviously, $f \in \mathcal{T}(\mathbf{C}^n)$ if and only if $fk_z \in L^2(\mathbf{C}^n, d\mu)$ for every $z \in \mathbf{C}^n$. Thus if $f \in \mathcal{T}(\mathbf{C}^n)$, then the Hankel operator H_f at least has a dense domain in $H^2(\mathbf{C}^n, d\mu)$.

From now on it will be convenient to identify $\mathbf{Z}^{2n}$ with the standard lattice in $\mathbf{C}^n$. That is, for $j_1, k_1, \ldots, j_n, k_n \in \mathbf{Z}$, we will identify

$$(j_1, k_1, \ldots, j_n, k_n) \quad \text{with} \quad (j_1 + ik_1, \ldots, j_n + ik_n).$$

Thus for $E \subset \mathbf{C}^n$ and $u \in \mathbf{Z}^{2n}$, the translation $E + u$ makes sense.

We define the open cube

$$W = \{(x_1 + iy_1, \ldots, x_n + iy_n) : x_1, y_1, \ldots, x_n, y_n \in (-1, 2)\} \tag{8.1}$$

in $\mathbf{C}^n$. For $f \in \mathcal{T}(\mathbf{C}^n)$ and $u \in \mathbf{Z}^{2n}$, we define the quantity

$$J(f; u) = \left\{ \int_{W+u} \int_{W+u} |f(z) - f(w)|^2 dV(w) dV(z) \right\}^{1/2}.$$

We need the following result:

Theorem 8.1. [10, Theorem 1.2] *Let $f \in \mathcal{T}(\mathbf{C}^n)$ and let Φ be an arbitrary symmetric gauge function. Then we have the simultaneous membership of $H_f \in \mathcal{C}_\Phi$ and $H_{\bar{f}} \in \mathcal{C}_\Phi$ if and only if*

$$\Phi(\{J(f;u)\}_{u\in\mathbf{Z}^{2n}}) < \infty. \tag{8.2}$$

Recall that for each $\alpha \in \mathbf{Z}^{2n}$, the cube Q_α was defined by (4.3). Now for $f \in \mathcal{T}(\mathbf{C}^n)$ and $\alpha \in \mathbf{Z}^{2n}$, we define the quantities

$$A(f;\alpha) = \left\{ \int_{Q_\alpha} |f(z)|^2 dV(z) \right\}^{1/2} \quad \text{and} \quad B(f;\alpha) = \left\{ \int_{W+\alpha} |f(z)|^2 dV(z) \right\}^{1/2}. \tag{8.3}$$

We further define the set

$$\mathcal{E} = \{(j_1 + ik_1, \dots, j_n + ik_n) : j_1, k_1, \dots j_n, k_n \in \{-1, 0, 1\}\}. \tag{8.4}$$

Lemma 8.2. *For any set of non-negative numbers $\{x_\alpha\}_{\alpha\in\mathbf{Z}^{2n}}$ and any symmetric gauge function Φ, we have*

$$\Phi\left(\left\{ \sum_{\epsilon\in\mathcal{E}} x_{\alpha+\epsilon} \right\}_{\alpha\in\mathbf{Z}^{2n}}\right) \le 3^{2n}\Phi(\{x_\alpha\}_{\alpha\in\mathbf{Z}^{2n}}).$$

Proof. By the properties of symmetric gauge functions, we have

$$\Phi\left(\left\{ \sum_{\epsilon\in\mathcal{E}} x_{\alpha+\epsilon} \right\}_{\alpha\in\mathbf{Z}^{2n}}\right) \le \sum_{\epsilon\in\mathcal{E}} \Phi(\{x_{\alpha+\epsilon}\}_{\alpha\in\mathbf{Z}^{2n}})$$

and $\Phi(\{x_{\alpha+\epsilon}\}_{\alpha\in\mathbf{Z}^{2n}}) = \Phi(\{x_\alpha\}_{\alpha\in\mathbf{Z}^{2n}})$ for every $\epsilon \in \mathcal{E}$. □

The main goal of the section is to derive Propositions 8.3 and 8.5 below.

Proposition 8.3. *Let $f \in \mathcal{T}(\mathbf{C}^n) \cap C^1(\mathbf{C}^n)$ and let Φ be an arbitrary symmetric gauge function. If*

$$\Phi(\{A(|\nabla f|;u)\}_{u\in\mathbf{Z}^{2n}}) < \infty, \tag{8.5}$$

then we have $H_f \in \mathcal{C}_\Phi$ and $H_{\bar{f}} \in \mathcal{C}_\Phi$.

To prove Proposition 8.3, let us first establish the following elementary fact:

Lemma 8.4. *Let φ be any non-negative, measurable function on W. Then*

$$\int_W \int_W \int_0^1 \varphi(tz + (1-t)w)dtdV(w)dV(z) \le 6^{2n} \int_W \varphi(x)dV(x). \tag{8.6}$$

Proof. Observe that, by the limit theorems in Lebesgue integral, it suffices to prove (8.6) for the simple case where φ is the characteristic function χ_E of a measurable subset $E \subset W$. For χ_E, we have

$$\int_W \int_W \int_0^1 \chi_E(tz + (1-t)w)dtdV(w)dV(z) = I_1 + I_2,$$

where

$$I_1 = \int_{1/2}^1 \left\{ \int_W \int_W \chi_E(tz + (1-t)w)dV(w)dV(z) \right\} dt \quad \text{and}$$
$$I_2 = \int_0^{1/2} \left\{ \int_W \int_W \chi_E(tz + (1-t)w)dV(w)dV(z) \right\} dt.$$

For each pair of $t \in [1/2, 1]$ and $w \in W$, let $E_{t,w} = \{z \in W : tz + (1-t)w \in E\}$. Then we have $E_{t,w} \subset t^{-1}E - t^{-1}(1-t)w$. Thus by the translation and scaling properties of the volume measure,

$$V(E_{t,w}) \le V(t^{-1}E) = t^{-2n}V(E) \le 2^{2n}V(E).$$

Consequently,

$$I_1 = \int_{1/2}^1 \int_W V(E_{t,w})dV(w)dt \le (1/2)V(W)2^{2n}V(E) = (1/2)6^{2n}V(E).$$

Similarly, for each pair of $t \in [0, 1/2]$ and $z \in W$, if we define $F_{t,z} = \{w \in W : tz+(1-t)w \in E\}$, then $V(F_{t,z}) \le 2^{2n}V(E)$. Consequently,

$$I_2 = \int_0^{1/2} \int_W V(F_{t,z})dV(z)dt \le (1/2)V(W)2^{2n}V(E) = (1/2)6^{2n}V(E).$$

Thus we see that (8.6) holds in the case $\varphi = \chi_E$. As we explained above, this means that (8.6) holds for all non-negative, measurable functions φ on W. $\square$

Proof of Proposition 8.3. For each $\alpha \in \mathbf{Z}^{2n}$, $W + \alpha \subset \cup_{\epsilon \in \mathcal{E}} Q_{\alpha+\epsilon}$. Hence by (8.3) and Lemma 8.2, condition (8.5) implies

$$\Phi(\{B(|\nabla f|; u)\}_{u \in \mathbf{Z}^{2n}}) < \infty. \tag{8.7}$$

Thus, by Theorem 8.1, to complete the proof, it suffices to show that (8.7) implies (8.2). To show that (8.2) holds, it will be convenient to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the natural way. Since our f is in C^1, for any $u \in \mathbf{Z}^{2n}$ and any $z, w \in W + u$, we have

$$f(z) - f(w) = \int_0^1 \frac{d}{dt} f(tz + (1-t)w)dt = \int_0^1 \langle (\nabla f)(tz + (1-t)w), z - w \rangle dt,$$

where the $\langle \cdot, \cdot \rangle$ is taken in the sense of the inner product on $\mathbf{R}^{2n}$. Since $z, w \in W + u$, we have $|z - w| \le 3\sqrt{2n}$. Hence the above implies

$$|f(z) - f(w)|^2 \le 18n \int_0^1 |(\nabla f)(tz + (1-t)w)|^2 dt.$$

Applying Lemma 8.4, we have

$$\begin{aligned} J^2(f; u) &\le 18n \int_{W+u} \int_{W+u} \int_0^1 |(\nabla f)(tz + (1-t)w)|^2 dt dV(w) dV(z) \\ &\le 18n 6^{2n} \int_{W+u} |(\nabla f)(x)|^2 dV(x) = 18n 6^{2n} B^2(|\nabla f|; u). \end{aligned}$$

Therefore (8.7) implies (8.2). This completes the proof. □

Proposition 8.5. *Let $f \in \mathcal{T}(\mathbf{C}^n)$ and let Φ be an arbitrary symmetric gauge function. If*

$$\Phi(\{A(f; u)\}_{u \in \mathbf{Z}^{2n}}) < \infty, \tag{8.8}$$

then we have $H_f \in \mathcal{C}_\Phi$ and $H_{\bar{f}} \in \mathcal{C}_\Phi$.

Proof. Again, by Lemma 8.2, (8.8) implies

$$\Phi(\{B(f; u)\}_{u \in \mathbf{Z}^{2n}}) < \infty.$$

It is obvious that for any $u \in \mathbf{Z}^{2n}$,

$$J(f; u) \le 2\{V(W)\}^{1/2} B(f; u) = 2 \cdot 3^n B(f; u).$$

Hence (8.8) implies (8.2). Applying Theorem 8.1, we obtain the memberships $H_f \in \mathcal{C}_\Phi$ and $H_{\bar{f}} \in \mathcal{C}_\Phi$. □

9. Local projections

We now introduce a partition by smooth functions, which is well known in the case of Bergman space. See [17,15,16,9]. The following is its Fock-space adaptation. Denote

$$\begin{aligned} Q &= \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, \dots, x_n, y_1, \dots, y_n \in [0, 1)\} \quad \text{and} \\ S &= \{(x_1 + iy_1, \dots, x_n + iy_n) : x_1, \dots, x_n, y_1, \dots, y_n \in (-1/2, 3/2)\}. \end{aligned}$$

Thus $Q = Q_0$ (see (4.3)). Fix an $\eta \in C^\infty(\mathbf{C}^n)$ satisfying the following three conditions:

(1) $0 \le \eta \le 1$ on $\mathbf{C}^n$.

(2) $\eta = 1$ on Q.

(3) $\eta = 0$ on $\mathbf{C}^n \backslash S$.

For each $z \in \mathbf{C}^n$, we define the function $\eta_z(\zeta) = \eta(\zeta - z)$ in $\mathbf{C}^n$. By (3), for $\zeta \in \mathbf{C}^n$ and $u \in \mathbf{Z}^{2n}$, if $\eta_u(\zeta) \ne 0$, then $\zeta - u \in S$, i.e., $u \in \zeta - S$. This ensures that the function

$$\varphi = \sum_{u \in \mathbf{Z}^{2n}} \eta_u$$

belongs to $C^\infty(\mathbf{C}^n)$. Also, by (1)-(3), the inequality $1 \le \varphi \le 3^{2n}$ holds on $\mathbf{C}^n$. Note that the identity $\varphi(\zeta) = \varphi(\zeta - u)$ holds for all $u \in \mathbf{Z}^{2n}$ and $\zeta \in \mathbf{C}^n$. Now we define

$$\gamma_z = \varphi^{-1}\eta_z$$

for every $z \in \mathbf{Z}^{2n}$. Then $\{\gamma_z : z \in \mathbf{Z}^{2n}\}$ is a set of C^∞-partition of the unity on $\mathbf{C}^n$. Moreover, for every $z \in \mathbf{Z}^{2n}$, we have $\gamma_z = 0$ on the set $\mathbf{C}^n \backslash \{S + z\}$.

Since φ is invariant under translations by $z \in \mathbf{Z}^{2n}$, we have $(\bar{\partial}_j\gamma_z)(\zeta) = (\bar{\partial}_j\gamma_0)(\zeta - z)$ for all $z \in \mathbf{Z}^{2n}$, $\zeta \in \mathbf{C}^n$ and $j \in \{1, \dots, n\}$. Thus if we set

$$C = \max_{1 \le k \le n} \|\bar{\partial}_k\gamma_0\|_\infty,$$

then

$$\|\bar{\partial}_j\gamma_z\|_\infty \le C \tag{9.1}$$

for all $z \in \mathbf{Z}^{2n}$ and $j \in \{1, \dots, n\}$.

If U is an open set in $\mathbf{C}^n$, we write $\mathrm{Hol}(U)$ for the collection of analytic functions on U. Recall that the open cube W was defined by (8.1). For $f \in \mathcal{T}(\mathbf{C}^n)$ and $z \in \mathbf{Z}^{2n}$, we define

$$M(f; z) = \inf_{h \in \mathrm{Hol}(W+z)} \left(\int_{W+z} |f(\zeta) - h(\zeta)|^2 dV(\zeta) \right)^{1/2}. \tag{9.2}$$

We also recall the set $\mathcal{E}$ defined by (8.4).

Proposition 9.1. *Set $C_{9.1} = 3^n\sqrt{2}$ and $C'_{9.1} = 2(1 + 3^{2n})^{1/2}3^nC$, where C is the constant that appears in* (9.1). *Then every $f \in \mathcal{T}(\mathbf{C}^n)$ admits a decomposition*

$$f = f^{(1)} + f^{(2)} \quad \textit{with} \quad f^{(2)} \in C^\infty(\mathbf{C}^n)$$

such that for every $\alpha \in \mathbf{Z}^{2n}$, we have

$$A(f^{(1)}; \alpha) \le C_{9.1} \sum_{\epsilon \in \mathcal{E}} M(f; \alpha - \epsilon) \quad \textit{and} \tag{9.3}$$

$$A(\bar{\partial}_j f^{(2)}; \alpha) \le C'_{9.1} \sum_{\epsilon \in \mathcal{E}} M(f; \alpha - \epsilon), \quad j = 1, \dots, n. \tag{9.4}$$

Proof. Let $f \in \mathcal{T}(\mathbf{C}^n)$. For each $z \in \mathbf{Z}^{2n}$, there is an $h_{f,z} \in \mathrm{Hol}(W + z)$ such that

$$\int_{W+z} |f(\zeta) - h_{f,z}(\zeta)|^2 dV(\zeta) \le 2M^2(f; z). \tag{9.5}$$

Note that this is true even if $M(f;z)=0$. We extend the definition of $h_{f,z}$ to the entire $\mathbf{C}^n$ by setting $h_{f,z}=0$ on $\mathbf{C}^n\backslash\{W+z\}$. Now define the functions

$$f^{(1)}=\sum_{z\in\mathbf{Z}^{2n}}(f-h_{f,z})\gamma_z \quad\text{and}\quad f^{(2)}=\sum_{z\in\mathbf{Z}^{2n}}h_{f,z}\gamma_z. \tag{9.6}$$

We have $f=f^{(1)}+f^{(2)}$ because $\{\gamma_z : z\in\mathbf{Z}^{2n}\}$ is a partition of the unity on $\mathbf{C}^n$. For $\alpha,z\in\mathbf{Z}^{2n}$, if γ_z is not identically zero on Q_α, then $Q_\alpha\cap\{W+z\}\neq\emptyset$, which is equivalent to $Q_{\alpha-z}\cap W\neq\emptyset$. That is, if γ_z is not identically zero on Q_α, then $\alpha-z\in\mathcal{E}$, i.e., $z=\alpha-\epsilon$ for some $\epsilon\in\mathcal{E}$. If $\epsilon\in\mathcal{E}$, then $V(Q_\alpha\backslash\{W+\alpha-\epsilon\})=0$. Hence for every $\alpha\in\mathbf{Z}^{2n}$,

$$\int_{Q_\alpha}|f^{(1)}|^2dV=\int_{Q_\alpha}\left|\sum_{\epsilon\in\mathcal{E}}(f-h_{f,\alpha-\epsilon})\gamma_{\alpha-\epsilon}\right|^2dV\leq 3^{2n}\sum_{\epsilon\in\mathcal{E}}\int_{W+\alpha-\epsilon}|f-h_{f,\alpha-\epsilon}|^2dV.$$

Recalling (8.3), (9.3) follows from this inequality and (9.5).

Obviously, $h_{f,z}\gamma_z\in C^\infty(\mathbf{C}^n)$ for every $z\in\mathbf{Z}^{2n}$. Since the condition $\gamma_z(\zeta)\neq 0$ implies $z\in\zeta-S$, it follows that $f^{(2)}\in C^\infty(\mathbf{C}^n)$. Moreover, since $h_{f,z}$ is analytic on $W+z$, for each $j\in\{1,\dots,n\}$ we have

$$\bar\partial_jf^{(2)}=\sum_{z\in\mathbf{Z}^{2n}}h_{f,z}\bar\partial_j\gamma_z. \tag{9.7}$$

To prove (9.4), consider any $\alpha\in\mathbf{Z}^{2n}$. As before, if $\zeta\in Q_\alpha$, then (9.7) gives us

$$(\bar\partial_jf^{(2)})(\zeta)=\sum_{\epsilon\in\mathcal{E}}h_{f,\alpha-\epsilon}(\zeta)(\bar\partial_j\gamma_{\alpha-\epsilon})(\zeta)=\sum_{\epsilon\in\mathcal{E}}(h_{f,\alpha-\epsilon}(\zeta)-h_{f,\alpha}(\zeta))(\bar\partial_j\gamma_{\alpha-\epsilon})(\zeta),$$

where the second $=$ is due to the fact that $\bar\partial_j\sum_{z\in\mathbf{Z}^{2n}}\gamma_z=\bar\partial_j1=0$. By (9.1), we have

$$|(\bar\partial_jf^{(2)})(\zeta)|\leq C\sum_{\epsilon\in\mathcal{E}}|h_{f,\alpha-\epsilon}(\zeta)-h_{f,\alpha}(\zeta)|$$

for $\zeta\in Q_\alpha$, $\alpha\in\mathbf{Z}^{2n}$. Therefore

$$\begin{aligned}\int_{Q_\alpha}|\bar\partial_jf^{(2)}|^2dV&\leq 3^{2n}C^2\sum_{\epsilon\in\mathcal{E}}\int_{Q_\alpha}|h_{f,\alpha-\epsilon}-h_{f,\alpha}|^2dV\\&\leq 3^{2n}C^2\sum_{\epsilon\in\mathcal{E}}2\left(\int_{W+\alpha-\epsilon}|h_{f,\alpha-\epsilon}-f|^2dV+\int_{W+\alpha}|f-h_{f,\alpha}|^2dV\right).\end{aligned}$$

Thus for every $\alpha\in\mathbf{Z}^{2n}$ we have

$$\int_{Q_\alpha}|\bar\partial_jf^{(2)}|^2dV\leq 2(1+3^{2n})3^{2n}C^2\sum_{\epsilon\in\mathcal{E}}\int_{W+\alpha-\epsilon}|h_{f,\alpha-\epsilon}-f|^2dV.$$

Obviously, (9.4) follows from this inequality and (9.5). □

Proposition 9.2. *Let* $0 < s \le 1$. *Then there is a constant* $0 < C_{9.2} < \infty$ *that depends only on* s *and the complex dimension* n *such that*

$$\Phi(\{M^s(f;z)\}_{z\in\mathbf{Z}^{2n}}) \le C_{9.2}\||H_f|^s\|_\Phi$$

for every $f \in \mathcal{T}(\mathbf{C}^n)$ *and every symmetric gauge function* Φ.

Proposition 9.2 is the Fock-space analogue of [9, Proposition 6.8]. In other words, Proposition 9.2 is essentially known. Moreover, the proof in the Fock-space case is easier than the proof in the Bergman-space case in [9]. For these reasons the proof of Proposition 9.2 is relegated to Appendix 1.

Corollary 9.3. *There are constants* $0 < C_{9.3} < \infty$ *and* $0 < C'_{9.3} < \infty$ *such that the following bounds hold: Given an* $f \in \mathcal{T}(\mathbf{C}^n)$, *let*

$$f = f^{(1)} + f^{(2)} \quad \textit{with} \;\; f^{(2)} \in C^\infty(\mathbf{C}^n)$$

be the decomposition defined by (9.6). *Then for every symmetric gauge function* Φ,

$$\Phi(\{A(f^{(1)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) \le C_{9.3}\|H_f\|_\Phi \quad \textit{and} \tag{9.8}$$

$$\Phi(\{A(\bar\partial_j f^{(2)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) \le C'_{9.3}\|H_f\|_\Phi, \quad j = 1,\dots,n. \tag{9.9}$$

Proof. Applying (9.4), Lemma 8.2 and Proposition 9.2, we have

$$\begin{aligned}\Phi(\{A(\bar\partial_j f^{(2)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) &\le C'_{9.1}\Phi\left(\left\{\sum_{\epsilon\in\mathcal{E}} M(f;\alpha-\epsilon)\right\}_{\alpha\in\mathbf{Z}^{2n}}\right)\\ &\le 3^{2n}C'_{9.1}\Phi(\{M(f;\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) \le 3^{2n}C'_{9.1}C_{9.2}\|H_f\|_\Phi,\end{aligned}$$

proving (9.9). Similarly, (9.8) follows from (9.3), Lemma 8.2 and Proposition 9.2. □

Lemma 9.4. *Suppose that* $f \in L^\infty(\mathbf{C}^n)$. *Then the functions* $f^{(1)}$, $f^{(2)}$ *defined by* (9.6) *also belong to* $L^\infty(\mathbf{C}^n)$.

Proof. It suffices to consider $f^{(2)}$. By (9.5), there is a $0 < C_1 < \infty$ such that

$$\int_{W+z} |h_{f,z}(\zeta)|^2 dV(\zeta) \le C_1\|f\|_\infty^2$$

for every $z \in \mathbf{Z}^{2n}$. Since the closure of S is a compact subset of W and since $h_{f,z}(\zeta)$ is the average of $h_{f,z}$ on an appropriate ball centered at ζ, the above implies that there is a $0 < C_2 < \infty$ such that

$$\sup_{\zeta\in S+z} |h_{f,z}(\zeta)| \le C_2\|f\|_\infty \quad \text{for every} \;\; z \in \mathbf{Z}^{2n}.$$

Combining this with (9.6), since $\gamma_z = 0$ on $\mathbf{C}^n\backslash\{S+z\}$, we see that $f^{(2)} \in L^\infty(\mathbf{C}^n)$. $\square$

10. Proofs of Theorems 1.3 and 1.4

First, let us prove Theorem 1.3. Suppose that $1 < p < \infty$. Let $f \in L^\infty(\mathbf{C}^n)$ be such that $H_f \in \mathcal{C}_p^+$. We decompose f in the form

$$f = f^{(1)} + f^{(2)} \quad \text{with} \quad f^{(2)} \in C^\infty(\mathbf{C}^n), \tag{10.1}$$

where $f^{(1)}$ and $f^{(2)}$ are defined by (9.6). By Lemma 9.4, we have $f^{(1)}, f^{(2)} \in L^\infty(\mathbf{C}^n)$. It suffices to show that $H_{\bar{f}^{(1)}} \in \mathcal{C}_p^+$ and that $H_{\bar{f}^{(2)}} \in \mathcal{C}_p^+$. Since $A(f^{(1)};\alpha) = A(\bar{f}^{(1)};\alpha)$, an application of Corollary 9.3 gives us

$$\Phi_p^+(\{A(\bar{f}^{(1)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) = \Phi_p^+(\{A(f^{(1)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) \le C_{9.3}\|H_f\|_p^+ < \infty.$$

Hence it follows from Proposition 8.5 that $H_{\bar{f}^{(1)}} \in \mathcal{C}_p^+$.

To prove that $H_{\bar{f}^{(2)}} \in \mathcal{C}_p^+$, we note that by Corollary 9.3 and (7.8), the membership $H_f \in \mathcal{C}_p^+$ implies $\bar{\partial}_j f^{(2)} \in \mathcal{L}_n^{2,p,+}$ for $j = 1, \ldots, n$. By Lemma 6.4, we have $\mathcal{L}_n^{2,p,+} \subset \mathcal{L}_n^{2,t}$ for $p < t < \infty$. Since $f^{(2)} \in L^\infty(\mathbf{C}^n)$, Lemma 5.1 is applicable to $f^{(2)}$. By Lemma 5.1,

$$\partial_j f^{(2)} = -\pi^{-1}T_j(\bar{\partial}_j f^{(2)}), \tag{10.2}$$

$j = 1, \ldots, n$. It now follows from Proposition 7.3 that $\partial_j f^{(2)} \in \mathcal{L}_n^{2,p,+}$ for $j = 1, \ldots, n$. Thus we conclude that $|\nabla f^{(2)}| \in \mathcal{L}_n^{2,p,+}$. By (7.8) and Proposition 8.3, the membership $|\nabla f^{(2)}| \in \mathcal{L}_n^{2,p,+}$ implies $H_{\bar{f}^{(2)}} \in \mathcal{C}_p^+$. This completes the proof of Theorem 1.3.

The proof of Theorem 1.4 proceeds along a similar line, so we will be more brief. Let $f \in L^\infty(\mathbf{C}^n)$ be such that $H_f \in \mathcal{C}_p^-$ for some $1 < p < \infty$. We again take the decomposition (10.1), and we only need to show that $H_{\bar{f}^{(1)}} \in \mathcal{C}_p^-$ and that $H_{\bar{f}^{(2)}} \in \mathcal{C}_p^-$. This time, Corollary 9.3 gives us

$$\Phi_p^-(\{A(\bar{f}^{(1)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) = \Phi_p^-(\{A(f^{(1)};\alpha)\}_{\alpha\in\mathbf{Z}^{2n}}) \le C_{9.3}\|H_f\|_p^- < \infty.$$

Hence it follows from Proposition 8.5 that $H_{\bar{f}^{(1)}} \in \mathcal{C}_p^-$.

By Corollary 9.3 and (7.9), the membership $H_f \in \mathcal{C}_p^-$ implies $\bar{\partial}_j f^{(2)} \in \mathcal{L}_n^{2,p,-}$ for $j = 1, \ldots, n$. Since we know that $f^{(2)} \in L^\infty(\mathbf{C}^n)$, by Lemma 5.1, (10.2) again holds. This time, it follows from (10.2) and Proposition 7.4 that $\partial_j f^{(2)} \in \mathcal{L}_n^{2,p,-}$ for $j = 1, \ldots, n$. Hence $|\nabla f^{(2)}| \in \mathcal{L}_n^{2,p,-}$. By (7.9) and Proposition 8.3, the membership $|\nabla f^{(2)}| \in \mathcal{L}_n^{2,p,-}$ implies $H_{\bar{f}^{(2)}} \in \mathcal{C}_p^-$. This completes the proof of Theorem 1.4.

11. No Berger-Coburn phenomenon for the trace class

We now prove Theorem 1.5. Recall that for $\varphi \in L^\infty(\mathbf{C})$, the Toeplitz operator T_φ is defined by the formula

$$T_\varphi h = P(\varphi h), \quad h \in H^2(\mathbf{C}, d\mu).$$

Also recall that the standard orthonormal basis $\{e_k : k \in \mathbf{Z}_+\}$ for $H^2(\mathbf{C}, d\mu)$ is given by the formula

$$e_k(z) = (k!)^{-1/2} z^k,$$

$k \geq 0$. For the function g defined by (1.3), it is easy to see that for any $j, k \in \mathbf{Z}_+$, if $\langle T_g e_k, e_j \rangle \neq 0$, then $j = k - 1$, which also forces $k \geq 1$. Integrating in the polar coordinates and making the obvious substitution, for each $k \geq 1$ we have

$$\begin{aligned}\langle T_g e_k, e_{k-1} \rangle &= \frac{1}{(k!(k-1)!)^{1/2}\pi} \int_{|z|\geq 1} |z^{k-1}|^2 e^{-|z|^2} dA(z) \\ &= \frac{1}{(k!(k-1)!)^{1/2}} \int_1^\infty t^{k-1} e^{-t} dt = \frac{1}{\sqrt{k}}(1 - c_k),\end{aligned}$$

where

$$c_k = \frac{1}{(k-1)!} \int_0^1 t^{k-1} e^{-t} dt. \tag{11.1}$$

Thus

$$T_g = \sum_{k=1}^\infty \langle T_g e_k, e_{k-1} \rangle e_{k-1} \otimes e_k = \sum_{k=1}^\infty \frac{1}{\sqrt{k}}(1 - c_k) e_{k-1} \otimes e_k. \tag{11.2}$$

Similarly, if $\langle T_{|g|^2} e_k, e_j \rangle \neq 0$, then $j = k$, and we have

$$\langle T_{|g|^2} e_k, e_k \rangle = \frac{1}{k!\pi} \int_{|z|\geq 1} |z^{k-1}|^2 e^{-|z|^2} dA(z) = \frac{1}{k}(1 - c_k)$$

when $k \geq 1$. Thus

$$T_{|g|^2} = \sum_{k=0}^\infty \langle T_{|g|^2} e_k, e_k \rangle e_k \otimes e_k = E_0 + \sum_{k=1}^\infty \frac{1}{k}(1 - c_k) e_k \otimes e_k, \tag{11.3}$$

where we denote $E_0 = \langle T_{|g|^2} e_0, e_0 \rangle e_0 \otimes e_0$.

Combining (11.2) and (11.3), we have

$$\begin{aligned}H_g^* H_g &= P M_{\bar{g}} (1 - P) M_g P = T_{|g|^2} - T_g^* T_g \\ &= E_0 + \sum_{k=1}^\infty \frac{1}{k}(1 - c_k) e_k \otimes e_k - \sum_{k=1}^\infty \frac{1}{k}(1 - c_k)^2 e_k \otimes e_k \\ &= E_0 + \sum_{k=1}^\infty \frac{1}{k} c_k (1 - c_k) e_k \otimes e_k.\end{aligned}$$

Consequently,

$$|H_g| = (H_g^* H_g)^{1/2} = E_0^{1/2} + \sum_{k=1}^\infty \frac{1}{\sqrt{k}} c_k^{1/2} (1 - c_k)^{1/2} e_k \otimes e_k.$$

By (11.1), we have $c_k^{1/2} \le \{(k-1)!\}^{-1/2}$. Hence $|H_g| \in \mathcal{C}_1$. That is, the Hankel operator H_g is in the trace class.

On the other hand, it also follows from (11.2) and (11.3) that

$$\begin{aligned} H_{\bar g}^* H_{\bar g} &= PM_g(1-P)M_{\bar g}P = T_{|g|^2} - T_g T_g^* \\ &= E_0 + \sum_{k=1}^{\infty} \frac{1}{k}(1-c_k)e_k \otimes e_k - \sum_{k=1}^{\infty} \frac{1}{k}(1-c_k)^2 e_{k-1} \otimes e_{k-1} \\ &= E_0 - F_0 + \sum_{k=1}^{\infty} \frac{1}{k}(1-c_k)e_k \otimes e_k - \sum_{k=1}^{\infty} \frac{1}{k+1}(1-c_{k+1})^2 e_k \otimes e_k \\ &= E_0 - F_0 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left(1-(k+1)c_k + 2kc_{k+1} - kc_{k+1}^2\right) e_k \otimes e_k, \end{aligned}$$

where $F_0 = (1-c_1)^2 e_0 \otimes e_0$. Consequently,

$$|H_{\bar g}| = (H_{\bar g}^* H_{\bar g})^{1/2}$$

(11.4)

$$= (E_0 - F_0)^{1/2} + \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}} \left(1-(k+1)c_k + 2kc_{k+1} - kc_{k+1}^2\right)^{1/2} e_k \otimes e_k.$$

From (11.1) we see that

$$\lim_{k\to\infty} \left(1-(k+1)c_k + 2kc_{k+1} - kc_{k+1}^2\right)^{1/2} = 1. \tag{11.5}$$

Combining (11.4) and (11.5), we conclude that $|H_{\bar g}| \notin \mathcal{C}_1$. That is, the Hankel operator $H_{\bar g}$ is not in the trace class. This completes the proof of Theorem 1.5.

12. No Berger-Coburn phenomenon for the ideal $\mathcal{C}_1^+$

We need a general fact about the orthogonal projection $P : L^2(\mathbf{C}, d\mu) \to H^2(\mathbf{C}, d\mu)$, which may be of independent interest:

Proposition 12.1. *For each $f \in C^\infty(\mathbf{C}) \cap L^2(\mathbf{C}, d\mu)$ we have $\|f - Pf\| \le \|\bar\partial f\|$.*

The proof of Proposition 12.1 is essentially an exercise in CCR, the canonical commutation relation. In order not to distract from the main line of our construction here, we leave the proof of Proposition 12.1 to Appendix 2 at the end of the paper.

We now turn to the proof of Theorem 1.6. The ψ promised in Theorem 1.6 will be constructed from "modified pieces" of the function g given by (1.3). This takes quite a few steps. We begin with a C^∞-function ξ on $\mathbf{R}$ satisfying the following three conditions:
(1) $0 \le \xi \le 1$ on $\mathbf{R}$.
(2) $\xi = 0$ on $(-\infty, 1]$.
(3) $\xi = 1$ on $[2, \infty)$.

Now, for each $j \in \mathbf{N}$, we define

$$\xi_j(z) = \xi(|z| - j), \quad z \in \mathbf{C}.$$

For each $j \in \mathbf{N}$, we then define the function g_j on $\mathbf{C}$ by the formula

$$g_j(z) = \xi_j(z)g(z), \quad z \in \mathbf{C},$$

where g is given by (1.3). Obviously, we have $\xi_j, g_j \in C^\infty(\mathbf{C})$ and $\|g_j\|_\infty \le (j+1)^{-1}$ for every $j \in \mathbf{N}$.

Lemma 12.2. *There is a $0 < C < \infty$ such that $\|H_{g_j}\|_1 \le C$ for every $j \in \mathbf{N}$.*

Proof. Applying Proposition 3.3 and Lemma 3.4 in [21], we have

$$\|H_{g_j}\|_1 = \mathrm{tr}((H_{g_j}^* H_{g_j})^{1/2}) = \frac{1}{\pi}\int \langle (H_{g_j}^* H_{g_j})^{1/2}k_z, k_z\rangle dA(z) \le \frac{1}{\pi}\int \|H_{g_j}k_z\| dA(z).$$

By Proposition 12.1, we have $\|H_{g_j}k_z\| \le \|\bar\partial(g_jk_z)\| = \|k_z\bar\partial g_j\|$. Thus

$$\begin{aligned}\|H_{g_j}\|_1 &\le \frac{1}{\pi}\int \|k_z\bar\partial g_j\| dA(z) = \frac{1}{\pi}\int \langle |\bar\partial g_j|^2 k_z, k_z\rangle^{1/2} dA(z)\\ &\le C_1 \sum_{\alpha\in\mathbf{Z}^2}\left(\int_{|z-\alpha|<1} |(\bar\partial g_j)(z)|^2 dA(z)\right)^{1/2},\end{aligned}$$

where the last $\le$ follows from the estimates on page 249 in [21]. Since g is analytic on $\{z \in \mathbf{C} : |z| > 1\}$, we have $\bar\partial g_j = g\bar\partial\xi_j$. Thus

$$\|H_{g_j}\|_1 \le C_1 \sum_{\alpha\in\mathbf{Z}^2}\left(\int_{|z-\alpha|<1} |g(z)(\bar\partial\xi_j)(z)|^2 dA(z)\right)^{1/2}. \tag{12.1}$$

Let E_j denote the collection of $\alpha \in \mathbf{Z}^2$ such that

$$\int_{|z-\alpha|<1} |g(z)(\bar\partial\xi_j)(z)|^2 dA(z) \ne 0.$$

By the choice of the function ξ, we have $(\bar\partial\xi_j)(z) \ne 0$ only if $1 \le |z| - j \le 2$, i.e., only if $j+1 \le |z| \le j+2$. Thus $\alpha \in E_j$ only if $j \le |\alpha| < j+3$. Consequently, there is a constant C_2 such that $\mathrm{card}(E_j) \le C_2 j$ for every $j \in \mathbf{N}$. Obviously, we have $\|\bar\partial\xi_j\|_\infty \le \|\xi'\|_\infty$ for every $j \in \mathbf{N}$. Furthermore, the condition $j+1 \le |z| \le j+2$ implies $|g(z)| \le (j+1)^{-1}$. Substituting these bounds in (12.1), we find that

$$\|H_{g_j}\|_1 \le C_1\mathrm{card}(E_j)\sqrt{\pi}(j+1)^{-1}\|\xi'\|_\infty \le C_1C_2 j\sqrt{\pi}(j+1)^{-1}\|\xi'\|_\infty \le C_1C_2\sqrt{\pi}\|\xi'\|_\infty$$

for every $j \in \mathbf{N}$. This completes the proof. $\square$

Lemma 12.3. *Let $\{Y_j\}$ be a sequence of operators satisfying the following two conditions*:

(a) $\|Y_j\| \to 0$ *as* $j \to \infty$.

(b) *There is a* $0 < C < \infty$ *such that* $\|Y_j\|_1 \le C$ *for every* $j \in \mathbf{N}$.

Then $\|Y_j\|_1^+ \to 0$ *as* $j \to \infty$.

Proof. For any pair of j and k, by the condition $\|Y_j\|_1 \le C$, we have

$$\frac{s_1(Y_j) + \cdots + s_k(Y_j)}{1^{-1} + \cdots + k^{-1}} \le \min\left\{k\|Y_j\|, \frac{C}{1^{-1} + \cdots + k^{-1}}\right\}.$$

Given any $\epsilon > 0$, we first pick a $K \in \mathbf{N}$ such that $C(1^{-1} + \cdots + K^{-1})^{-1} \le \epsilon$. We then pick a $J \in \mathbf{N}$ such that $K\|Y_j\| \le \epsilon$ for all $j \ge J$. By the above inequality, if $j \ge J$, then $\|Y_j\|_1^+ \le \epsilon$. This completes the proof. $\square$

Lemma 12.4. (1) *We have* $\|g_j\|_\infty \le (j+1)^{-1}$ *for every* $j \in \mathbf{N}$.
(2) *We have* $\|H_{g_j}\|_1^+ \to 0$ *as* $j \to \infty$.

Proof. (1) follows from the fact that $\xi_j(z) = 0$ when $|z| - j \le 1$ and (1.3). Since $\|g_j\|_\infty \le (j+1)^{-1}$, we have $\|H_{g_j}\| \to 0$ as $j \to \infty$. Thus (2) follows Lemmas 12.2 and 12.3. $\square$

Lemma 12.5. *We have* $H_{\bar{g}_j} \in \mathcal{C}_1^+$ *for every* $j \in \mathbf{N}$. *Moreover, there is a positive number* $c > 0$ *such that* $\|H_{\bar{g}_j}\|_1^+ \ge c$ *for every* $j \in \mathbf{N}$.

Proof. This is a modified version of the calculation in Section 11. Let $j \in \mathbf{N}$ be given. For any $i, k \in \mathbf{Z}_+$, if $\langle T_{g_j} e_k, e_i\rangle \ne 0$, then $i = k - 1$, which also forces $k \ge 1$. Moreover, for each $k \ge 1$ we have

$$\begin{aligned}\langle T_{g_j} e_k, e_{k-1}\rangle &= \frac{1}{(k!(k-1)!)^{1/2}\pi}\int \xi(|z| - j)|z^{k-1}|^2 e^{-|z|^2} dA(z) \\ &= \frac{1}{(k!(k-1)!)^{1/2}}\int_0^\infty \xi(\sqrt{t} - j)t^{k-1}e^{-t}dt = \frac{1}{\sqrt{k}}(1 - c_{j;k}),\end{aligned}$$

where

$$c_{j;k} = \frac{1}{(k-1)!}\int_0^\infty \{1 - \xi(\sqrt{t} - j)\}t^{k-1}e^{-t}dt. \tag{12.2}$$

Thus

$$T_{g_j} = \sum_{k=1}^\infty \langle T_{g_j} e_k, e_{k-1}\rangle e_{k-1} \otimes e_k = \sum_{k=1}^\infty \frac{1}{\sqrt{k}}(1 - c_{j;k})e_{k-1} \otimes e_k. \tag{12.3}$$

Similarly, if $\langle T_{|g_j|^2} e_k, e_i\rangle \ne 0$, then $i = k$, and for $k \ge 1$ we have

$$\langle T_{|g_j|^2} e_k, e_k\rangle = \frac{1}{k!\pi}\int \xi^2(|z| - j)|z^{k-1}|^2 e^{-|z|^2} dA(z) = \frac{1}{k}(1 - \tilde{c}_{j;k}),$$

where

$$\tilde{c}_{j;k} = \frac{1}{(k-1)!}\int_0^\infty \{1 - \xi^2(\sqrt{t} - j)\} t^{k-1} e^{-t} dt.$$

Thus

$$T_{|g_j|^2} = \sum_{k=0}^\infty \langle T_{|g_j|^2} e_k, e_k\rangle e_k \otimes e_k = E_j + \sum_{k=1}^\infty \frac{1}{k}(1 - \tilde{c}_{j;k}) e_k \otimes e_k, \tag{12.4}$$

where we denote $E_j = \langle T_{|g_j|^2} e_0, e_0\rangle e_0 \otimes e_0$.

Combining (12.3) and (12.4), we have

$$\begin{aligned} H^*_{\bar{g}_j} H_{\bar{g}_j} &= P M_{g_j}(1-P) M_{\bar{g}_j} P = T_{|g_j|^2} - T_{g_j} T^*_{g_j} \\ &= E_j + \sum_{k=1}^\infty \frac{1}{k}(1-\tilde{c}_{j;k}) e_k \otimes e_k - \sum_{k=1}^\infty \frac{1}{k}(1 - c_{j;k})^2 e_{k-1} \otimes e_{k-1} \\ &= E_j - F_j + \sum_{k=1}^\infty \frac{1}{k}(1-\tilde{c}_{j;k}) e_k \otimes e_k - \sum_{k=1}^\infty \frac{1}{k+1}(1 - c_{j;k+1})^2 e_k \otimes e_k \\ &= E_j - F_j + \sum_{k=1}^\infty \frac{1 - (k+1)\tilde{c}_{j;k} + 2k c_{j;k+1} - k c^2_{j;k+1}}{k(k+1)} e_k \otimes e_k, \end{aligned}$$

where $F_j = (1 - c_{j;1})^2 e_0 \otimes e_0$. Consequently,

$$|H_{\bar{g}_j}| = (H^*_{\bar{g}_j} H_{\bar{g}_j})^{1/2} = (E_j - F_j)^{1/2} + \sum_{k=1}^\infty \frac{d_{j;k}}{\sqrt{k(k+1)}} e_k \otimes e_k, \tag{12.5}$$

where

$$d_{j;k} = (1 - (k+1)\tilde{c}_{j;k} + 2k c_{j;k+1} - k c^2_{j;k+1})^{1/2}.$$

Recall that $0 \le \xi(x) \le 1$ for every $x \in \mathbf{R}$ and that $\xi(x) = 1$ when $x \ge 2$. Applying these facts in (12.2), we obtain

$$c_{j;k} \le \frac{1}{(k-1)!}\int_0^{(j+2)^2} t^{k-1} e^{-t} dt \le \frac{(j+2)^{2(k-1)}}{(k-1)!}$$

for every $k \ge 1$. A similar bound holds for $\tilde{c}_{j;k}$. Hence

$$\lim_{k\to\infty} d_{j;k} = 1. \tag{12.6}$$

This means that there is a $K_j \in \mathbf{N}$ such that $d_{j;k} \ge 1/2$ for every $k \ge K_j$. Combining this with (12.5), we see that for every $m \in \mathbf{N}$,

$$\frac{s_1(H_{\bar{g}_j}) + \cdots + s_m(H_{\bar{g}_j})}{1^{-1} + \cdots + m^{-1}} \ge \frac{1}{2} \cdot \frac{(K_j + 1)^{-1} + \cdots + (K_j + m)^{-1}}{1^{-1} + \cdots + m^{-1}}.$$

Taking supremum over $m \in \mathbf{N}$, we obtain the lower bound $\|H_{\bar{g}_j}\|_1^+ \geq 1/2$. Also, it is obvious from (12.5) and (12.6) that $H_{\bar{g}_j} \in \mathcal{C}_1^+$. This completes the proof. □

The construction of the desired ψ involves the duality between $\mathcal{C}_1^+$ and the Macaev ideal $\mathcal{C}_\infty^-$ [12]. This duality is better explained in terms of the symmetric gauge function Φ_∞^-, which is defined by the formula

$$\Phi_\infty^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^{\infty} \frac{|a_{\pi(j)}|}{j}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \cdots \geq |a_{\pi(j)}| \geq \cdots$. Thus

$$\|A\|_\infty^- = \|A\|_{\Phi_\infty^-} \quad \text{and} \quad \mathcal{C}_\infty^- = \mathcal{C}_{\Phi_\infty^-}$$

(see (6.1) and (6.2)). The symmetric gauge functions Φ_1^+ and Φ_∞^- are dual to each other in the sense that for every $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, we have

$$\Phi_\infty^-(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi_1^+(\{b_j\}_{j\in\mathbf{N}}) \leq 1\right\} \quad \text{and}$$
$$\Phi_1^+(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi_\infty^-(\{b_j\}_{j\in\mathbf{N}}) \leq 1\right\}.$$

See [12, pages 148, 149 and 125]. Thus it follows that for every $T \in \mathcal{B}(\mathcal{H})$,

$$\|T\|_1^+ = \sup\{|\mathrm{tr}(TF)| : \|F\|_\infty^- \leq 1 \text{ and } \mathrm{rank}(F) < \infty\}. \tag{12.7}$$

With the above extensive preparation, we can now accomplish the main goal of the section:

Proof of Theorem 1.6. By Lemma 12.4, we can pick a sequence of positive numbers $\{r_j\}$ with the following properties:

(i) $r_j\|g_j\|_\infty \leq 2(j+1)^{-1/2}$ for every $j \in \mathbf{N}$.
(ii) $r_j\|H_{g_j}\|_1^+ \to 0$ as $j \to \infty$.
(iii) $r_j \to \infty$ as $j \to \infty$.

Our desired function ψ will be the sum of a subsequence of the sequence $\{r_j g_j\}$, chosen as follows.

We begin by choosing a $j_1 \in \mathbf{N}$ such that

$$r_{j_1}\|g_{j_1}\|_\infty \leq 2^{-1} \quad \text{and} \quad r_{j_1}\|H_{g_{j_1}}\|_1^+ \leq 2^{-1}.$$

Lemma 12.5 says that $\|H_{\bar{g}_{j_1}}\|_1^+ \geq c$. Therefore, by (12.7), there is a finite-rank operator F_1 such that

$$\|F_1\|_\infty^- \leq 1 \quad \text{and} \quad |\mathrm{tr}(H_{\bar{g}_{j_1}} F_1)| \geq c/2.$$

Let $\nu \geq 1$ and suppose that we have chosen $j_1, \ldots, j_\nu \in \mathbf{N}$ and finite-rank operators $F_1, \ldots, F_\nu$. By (i), (ii) and (iii) above, there is a natural number $j_{\nu+1} > j_\nu$ such that

$$r_{j_{\nu+1}} \|g_{j_{\nu+1}}\|_\infty \leq 2^{-(\nu+1)}, \quad r_{j_{\nu+1}} \|H_{g_{j_{\nu+1}}}\|_1^+ \leq 2^{-(\nu+1)}, \tag{12.8}$$

$$r_{j_{\nu+1}} \|H_{g_{j_{\nu+1}}} F_i\|_1 \leq 2^{-(\nu+1-i)} c \quad \text{for every } 1 \leq i \leq \nu \tag{12.9}$$

and

$$r_{j_{\nu+1}} c \geq 4 \sum_{i=1}^{\nu} r_{j_i} \|H_{\bar{g}_{j_i}}\|_1^+. \tag{12.10}$$

Note that (12.10) requires the part of Lemma 12.5 which says that $\|H_{\bar{g}_j}\|_1^+ < \infty$ for every j. Lemma 12.5 also tells us that $\|H_{\bar{g}_{j_{\nu+1}}}\|_1^+ \geq c$. Therefore, by (12.7), there is a finite-rank operator $F_{\nu+1}$ such that

$$\|F_{\nu+1}\|_\infty^- \leq 1 \quad \text{and} \quad |\mathrm{tr}(H_{\bar{g}_{j_{\nu+1}}} F_{\nu+1})| \geq c/2. \tag{12.11}$$

Thus, inductively, we obtain a sequence of natural numbers $j_1 < \cdots < j_\nu < \cdots$ along with a sequence of finite-rank operators $\{F_\nu\}$ such that (12.8-11) hold for every $\nu \geq 1$.

With the sequences chosen above, we now define

$$\psi = \sum_{\nu=1}^{\infty} r_{j_\nu} g_{j_\nu}.$$

Let us verify that ψ has all the promised properties. First of all, it follows from (12.8) that

$$\|\psi\|_\infty \leq 1 \quad \text{and} \quad \|H_\psi\|_1^+ \leq 1.$$

That is, ψ is bounded on $\mathbf{C}$ and $H_\psi \in \mathcal{C}_1^+$. To show that $H_{\bar{\psi}} \notin \mathcal{C}_1^+$, take any $\nu \geq 1$. Then

$$\mathrm{tr}(H_{\bar{\psi}} F_{\nu+1}) = A + B + C,$$

where

$$A = \sum_{i=1}^{\nu} r_{j_i} \mathrm{tr}(H_{\bar{g}_{j_i}} F_{\nu+1}), \quad B = r_{j_{\nu+1}} \mathrm{tr}(H_{\bar{g}_{j_{\nu+1}}} F_{\nu+1}) \quad \text{and} \quad C = \sum_{i=\nu+2}^{\infty} r_{j_i} \mathrm{tr}(H_{\bar{g}_{j_i}} F_{\nu+1}).$$

By (12.11) and (12.10), we have

$$|B| \geq r_{j_{\nu+1}}(c/2) = r_{j_{\nu+1}}(c/4) + r_{j_{\nu+1}}(c/4) \geq r_{j_{\nu+1}}(c/4) + \sum_{i=1}^{\nu} r_{j_i} \|H_{\bar{g}_{j_i}}\|_1^+.$$

Since $\|F_{\nu+1}\|_\infty^- \le 1$, by (12.7), this means

$$|B| \ge r_{j_{\nu+1}}(c/4) + |A|.$$

On the other hand, it follows from (12.9) that

$$|C| \le \sum_{i=\nu+2}^{\infty} r_{j_i}\|H_{\bar{g}_{j_i}}F_{\nu+1}\|_1 \le \sum_{i=\nu+2}^{\infty} 2^{-(i-\nu-1)}c = c.$$

Using the fact $\|F_{\nu+1}\|_\infty^- \le 1$ and (12.7) again, we now have

$$\|H_{\bar\psi}\|_1^+ \ge |\mathrm{tr}(H_{\bar\psi}F_{\nu+1})| \ge |B| - |A| - |C| \ge r_{j_{\nu+1}}(c/4) - c.$$

Since $c > 0$ and since $r_{j_{\nu+1}} \to \infty$ as $\nu \to \infty$, this inequality implies that $H_{\bar\psi} \notin \mathcal{C}_1^+$. This completes the proof. □

13. No Berger-Coburn phenomenon for the Macaev ideal

The proof of Theorem 1.7 begins with an elementary lower bound:

Lemma 13.1. *There is a positive number $0 < R < \infty$ such that the following holds true: Let $h : [0,\infty) \to [0,1]$ be any measurable function that satisfies the conditions that $h = 1$ on $[0,R]$ and that $h = 0$ on $[R',\infty)$ for some $R < R' < \infty$. Then the function*

$$\eta(z) = h(|z|)z, \quad z \in \mathbf{C}, \tag{13.1}$$

has the property that $\|H_{\bar\eta}\| \ge 1/2$.

Proof. We have $\|z\| = 1$ in $H^2(\mathbf{C}, d\mu)$. Therefore

$$\|H_{\bar\eta}\|^2 \ge \langle H_{\bar\eta}^* H_{\bar\eta} z, z\rangle = \langle M_{|\eta|^2} z, z\rangle - \|PM_{\bar\eta}z\|^2 = \frac{1}{\pi}\int_{\mathbf{C}} h^2(|z|)|z|^4 d\mu(z) - \|PM_{\bar\eta}z\|^2$$
$$\ge \frac{1}{\pi}\int_{|z|<R} |z|^4 d\mu(z) - \|PM_{\bar\eta}z\|^2.$$

Since $\pi^{-1}\int_{\mathbf{C}} |z|^4 d\mu(z) = 2$, we see that for a sufficiently large R we have

$$\|H_{\bar\eta}\|^2 \ge (5/4) - \|PM_{\bar\eta}z\|^2.$$

Since $\bar\eta(z)z = h(|z|)|z|^2$, we have $\bar\eta z \perp z^k$ for all $k \ge 1$. Therefore

$$\|PM_{\bar\eta}z\| = |\langle \bar\eta z, 1\rangle| = \frac{1}{\pi}\int_{\mathbf{C}} h(|z|)|z|^2 d\mu(z) \le \frac{1}{\pi}\int_{\mathbf{C}} |z|^2 d\mu(z) = 1.$$

Consequently, $\|H_{\bar\eta}\|^2 \ge (5/4) - 1 = 1/4$. This completes the proof. □

For any pair of $a \in \mathbf{C}$ and $r > 0$, denote

$$D(a,r) = \{z \in \mathbf{C} : |a - z| < r\}.$$

Let $\mathcal{L}$ denote the collection of $\varphi \in L^\infty(\mathbf{C})$ for which there is some $0 < r = r(\varphi) < \infty$ such that $\varphi = 0$ on $\mathbf{C}\backslash D(0,r)$. Obviously, $\mathcal{L}$ is closed under complex conjugation. If $\varphi \in \mathcal{L}$, then the Hankel operator H_φ is in the trace class $\mathcal{C}_1$; this fact is well known, but it certainly follows from Theorem 8.1.

Our proof of Theorem 1.7 is based on the following fact, which, in view of the original theorem of Berger and Coburn in [4], may be of independent interest:

Proposition 13.2. *There does not exist any constant $0 < C < \infty$ such that the inequality*

$$\|H_{\bar\varphi}\| \le C\|H_\varphi\| \tag{13.2}$$

holds for every $\varphi \in \mathcal{L}$.

Proof. Let a small $\epsilon > 0$ be given. We will show that there is an $h : [0,\infty) \to [0,1]$ satisfying the two conditions in Lemma 13.1 such that the corresponding η defined by (13.1) has the property that $\|H_\eta\| \le \epsilon$. Combining this upper bound with the lower bound in Lemma 13.1, we see that no $0 < C < \infty$ exists such that (13.2) holds for every $\varphi \in \mathcal{L}$.

Our desired h will be a C^∞ function on $[0,\infty)$. Thus the corresponding η will be a C^∞ function on $\mathbf{C}$. By Proposition 12.1, for any $f \in H^2(\mathbf{C}, d\mu)$ we have

$$\|H_\eta f\| = \|\eta f - P(\eta f)\| \le \|\bar\partial(\eta f)\| = \|f\bar\partial\eta\|.$$

Thus it suffices to find a C^∞ function $h : [0,\infty) \to [0,1]$ which satisfies the two conditions in Lemma 13.1 and for which the corresponding η has the property that $\|\bar\partial\eta\|_\infty \le \epsilon$.

By straightforward differentiation, we have

$$(\bar\partial\eta)(z) = zh'(|z|)\bar\partial|z|.$$

Moreover, $\|\bar\partial|z|\|_\infty \le 1$. Thus our task is reduced to the construction of a C^∞ function $h : [0,\infty) \to [0,1]$ which satisfies the condition

$$x|h'(x)| \le \epsilon \quad \text{for every} \quad x \in [0,\infty), \tag{13.3}$$

as well as the two conditions in Lemma 13.1.

To construct such an h, we begin with the function u on $\mathbf{R}$ defined by the formula

$$u(x) = \begin{cases} \frac{1}{x\log x} & \text{if} \quad x \ge 2 \\ 0 & \text{if} \quad x < 2 \end{cases}.$$

Let R be the same as in Lemma 13.1. We pick a $T \geq R + 30$ such that

$$\frac{1}{\log(T-1)} \leq \epsilon.$$

With this T, we let v be a C^∞ function on $\mathbf{R}$ satisfying the conditions that $0 \leq v \leq 1$ on $\mathbf{R}$, that $v = 1$ on $[T, T^3]$, and that $v = 0$ on $\mathbf{R}\backslash(T-1, T^3+1)$. We further define

$$a(x) = v(x)u(x), \quad x \in \mathbf{R}.$$

Then a is a non-negative C^∞ function on $\mathbf{R}$, and $a = 0$ on $\mathbf{R}\backslash(T-1, T^3+1)$. Write

$$A = \int_0^\infty a(x)dx.$$

Then

$$A = \int_{T-1}^{T^3+1} a(x)dx \geq \int_T^{T^3} u(x)dx = \int_T^{T^3} \frac{1}{x\log x}dx = \log 3 > 1.$$

Define

$$b(x) = \int_0^x a(t)dt, \quad x \in [0, \infty),$$

which is a C^∞ function. Obviously, b is non-negative and non-decreasing on $[0, \infty)$. Moreover, we have $b = 0$ on $[0, T-1]$ and $b = A$ on $[T^3+1, \infty)$. Finally, we define

$$h(x) = A^{-1}(A - b(x)), \quad x \in [0, \infty).$$

Obviously, h is a C^∞ function and satisfies the condition $0 \leq h \leq 1$ on $[0, \infty)$. Let us verify that this h has the other desired properties.

First of, the properties of b imply that $h = 1$ on $[0, T-1]$ and $h = 0$ on $[T^3+1, \infty)$. Since $T - 1 > R$, this h satisfies the two conditions in Lemma 13.1. What remains is to show that this h satisfies (13.3). Obviously, $h'(x) = -A^{-1}a(x)$ for every $x \in [0, \infty)$. Thus we have $x|h'(x)| = 0$ for $x \in [0, \infty)\backslash(T-1, T^3+1)$. If $x \in (T-1, T^3+1)$, then

$$x|h'(x)| = A^{-1}xa(x) = A^{-1}xv(x)u(x) \leq xu(x) = \frac{1}{\log x} \leq \frac{1}{\log(T-1)} \leq \epsilon$$

by the choice of T. This verifies (13.3) and completes the proof. □

Lemma 13.3. *If $\varphi \in \mathcal{L}$, then*

$$\lim_{r\to\infty} \|M_\varphi P M_{\chi_{\mathbf{C}\backslash D(0,r)}}\|_1 = 0.$$

Proof. We can write the orthogonal projection $P : L^2(\mathbf{C}, d\mu) \to H^2(\mathbf{C}, d\mu)$ in the form $P = \sum_{k=0}^\infty e_k \otimes e_k$, where

$$e_k(z) = (k!)^{-1/2}z^k, \quad k \geq 0.$$

If $\varphi \in \mathcal{L}$, then there is a $0 < \rho < \infty$ such that $\varphi = 0$ on $\mathbf{C}\backslash D(0,\rho)$. Thus for each $k \geq 0$,

$$\|\varphi e_k\|^2 = \frac{1}{k!\pi}\int_{\mathbf{C}} |\varphi(z)z^k|^2 d\mu(z) \leq \frac{\|\varphi\|_\infty^2}{k!\pi}\int_{|z|\leq\rho} |z^k|^2 e^{-|z|^2} dA(z) \leq \|\varphi\|_\infty^2 \frac{\rho^{2k}}{k!}.$$

Therefore

$$\|M_\varphi P M_{\chi_{\mathbf{C}\backslash D(0,r)}}\|_1 \leq \sum_{k=0}^{\infty} \|\varphi e_k\|\|\chi_{\mathbf{C}\backslash D(0,r)} e_k\| \leq \|\varphi\|_\infty \sum_{k=0}^{\infty} \frac{\rho^k}{\sqrt{k!}}\|\chi_{\mathbf{C}\backslash D(0,r)} e_k\|. \tag{13.4}$$

It is obvious that for each $k \geq 0$, we have $\|\chi_{\mathbf{C}\backslash D(0,r)} e_k\| \leq 1$ and

$$\lim_{r\to\infty} \|\chi_{\mathbf{C}\backslash D(0,r)} e_k\| = 0.$$

Applying these facts in (13.4), the lemma follows. □

For any operator A and any $\nu \in \mathbf{N}$, we denote

$$A^{[\nu]} = \overbrace{A \oplus \cdots \oplus A}^{\nu \text{ copies}}.$$

Lemma 13.4. *If $A \in \mathcal{C}_1$, then $\|A^{[\nu]}\|_\infty^- \leq (1 + \log\nu)\|A\|_1$ for every $\nu \in \mathbf{N}$.*

Proof. It is obvious that if x and y are unit vectors, then for every $\nu \in \mathbf{N}$,

$$\|(x \otimes y)^{[\nu]}\|_\infty^- = \sum_{j=1}^{\nu} \frac{1}{j} \leq 1 + \log\nu.$$

If A is a trace class operator, then there are orthonormal sets $\{x_j : j \in \mathbf{N}\}$ and $\{y_j : j \in \mathbf{N}\}$ such that

$$A = \sum_{j=1}^{\infty} s_j(A) x_j \otimes y_j. \tag{13.5}$$

Therefore for each $\nu \in \mathbf{N}$,

$$\|A^{[\nu]}\|_\infty^- \leq \sum_{j=1}^{\infty} s_j(A)\|(x_j \otimes y_j)^{[\nu]}\|_\infty^- \leq \sum_{j=1}^{\infty} s_j(A)(1 + \log\nu) = (1 + \log\nu)\|A\|_1$$

as promised. □

Proposition 13.5. *Let $A \in \mathcal{C}_1$. Then there is a natural number $m(A) \in \mathbf{N}$ such that*

$$\|A^{[\nu]}\|_\infty^- \leq 3(1 + \log\nu)\|A\| \quad \textit{for every} \ \ \nu \geq m(A).$$

Proof. Given an $A \in \mathcal{C}_1$, we again write it in the form (13.5). There is an $m \in \mathbf{N}$ such that

$$\sum_{j=m+1}^{\infty} s_j(A) \le \|A\|.$$

Let us show that this natural number m suffices for the lemma. Indeed we decompose A in the form $A = A_1 + A_2$, where

$$A_1 = \sum_{j=1}^{m} s_j(A) x_j \otimes y_j \quad \text{and} \quad A_2 = \sum_{j=m+1}^{\infty} s_j(A) x_j \otimes y_j.$$

Since $\|A_2\|_1 \le \|A\|$, it follows from Lemma 13.4 that $\|A_2^{[\nu]}\|_\infty^- \le (1 + \log \nu)\|A\|$ for every $\nu \in \mathbf{N}$. On the other hand, since $\text{rank}(A_1) \le m$ and $\|A_1\| = \|A\|$, for each $\nu \ge m$ we have

$$\begin{aligned} \|A_1^{[\nu]}\|_\infty^- &\le \sum_{j=1}^{\nu m} \frac{\|A\|}{j} \le (1 + \log(\nu m))\|A\| \le (1 + \log(\nu^2))\|A\| \\ &= (1 + 2\log \nu)\|A\| \le 2(1 + \log \nu)\|A\|. \end{aligned}$$

Since $\|A^{[\nu]}\|_\infty^- \le \|A_1^{[\nu]}\|_\infty^- + \|A_2^{[\nu]}\|_\infty^-$, this completes the proof. □

Lemma 13.6. *There exists a sequence $\{f_k\}$ in $\mathcal{L}$ which has the properties that $\|f_k\|_\infty \le 1$ for every $k \in \mathbf{N}$ and that*

$$\bigoplus_{k=1}^{\infty} H_{f_k} \in \mathcal{C}_\infty^- \quad \textit{while} \quad \bigoplus_{k=1}^{\infty} H_{\bar{f}_k} \notin \mathcal{C}_\infty^-.$$

Proof. Applying Proposition 13.2, for each $j \in \mathbf{N}$ there is a $\varphi_j \in \mathcal{L}$ such that

$$\|H_{\varphi_j}\| \le 1 \quad \text{while} \quad \|H_{\bar{\varphi}_j}\| \ge j.$$

As we already mentioned, the membership $\varphi_j \in \mathcal{L}$ guarantees that $H_{\varphi_j} \in \mathcal{C}_1$. Thus for each $j \in \mathbf{N}$, there is a natural number $m(H_{\varphi_j})$ provided by Proposition 13.5 for H_{φ_j}. We pick a $\nu(j) \in \mathbf{N}$ satisfying the conditions

$$\nu(j) \ge m(H_{\varphi_j}) + 10 \quad \text{and} \quad \log \nu(j) \ge \|\varphi_j\|_\infty$$

for each $j \in \mathbf{N}$. It follows from Proposition 13.5 and the condition $\|H_{\varphi_j}\| \le 1$ that

$$\|H_{\varphi_j}^{[\nu(j)]}\|_\infty^- \le 3(1 + \log \nu(j)), \tag{13.6}$$

$j \in \mathbf{N}$. On the other hand, the condition $\|H_{\bar{\varphi}_j}\| \ge j$ implies that $s_\ell(H_{\bar{\varphi}_j}^{[\nu(j)]}) \ge j$ for $1 \le \ell \le \nu(j)$. Therefore

$$\|H_{\bar{\varphi}_j}^{[\nu(j)]}\|_\infty^- \ge j \sum_{\ell=1}^{\nu(j)} \frac{1}{\ell} \ge j \log \nu(j) \tag{13.7}$$

for every $j \in \mathbf{N}$. Now consider the operators

$$A = \bigoplus_{j=1}^{\infty} \frac{1}{j^2 \log \nu(j^3)} H_{\varphi_{j^3}}^{[\nu(j^3)]} \quad \text{and} \quad B = \bigoplus_{j=1}^{\infty} \frac{1}{j^2 \log \nu(j^3)} H_{\bar{\varphi}_{j^3}}^{[\nu(j^3)]}.$$

By (13.6), we have

$$\|A\|_{\infty}^{-} \leq \sum_{j=1}^{\infty} \frac{1}{j^2 \log \nu(j^3)} \|H_{\varphi_{j^3}}^{[\nu(j^3)]}\|_{\infty}^{-} \leq 3 \sum_{j=1}^{\infty} \frac{1 + \log \nu(j^3)}{j^2 \log \nu(j^3)} < \infty.$$

That is, $A \in \mathcal{C}_{\infty}^{-}$. On the other hand, for each $j \in \mathbf{N}$, it follows from (13.7) that

$$\|B\|_{\infty}^{-} \geq \frac{1}{j^2 \log \nu(j^3)} \|H_{\bar{\varphi}_{j^3}}^{[\nu(j^3)]}\|_{\infty}^{-} \geq \frac{j^3 \log \nu(j^3)}{j^2 \log \nu(j^3)} = j.$$

This means that $B \notin \mathcal{C}_{\infty}^{-}$. The choice of $\nu(j^3)$ ensures that $\|\varphi_{j^3}\|_{\infty}/\log \nu(j^3) \leq 1$. Thus if we let $\{f_k\}$ be a re-enumeration of the functions

$$\overbrace{\frac{1}{j^2 \log \nu(j^3)} \varphi_{j^3}, \cdots, \frac{1}{j^2 \log \nu(j^3)} \varphi_{j^3}}^{\nu(j^3) \text{ copies}}, \quad j \in \mathbf{N},$$

then the conclusion of the lemma holds. $\square$

For each $a \in \mathbf{C}$, we have the translation

$$\tau_a(z) = z - a, \quad z \in \mathbf{C},$$

of the complex plane. It is well known that for each $a \in \mathbf{C}$, the formula

$$V_a f = f \circ \tau_a \cdot k_a, \quad f \in L^2(\mathbf{C}, d\mu),$$

defines a unitary operator on $L^2(\mathbf{C}, d\mu)$, where $k_a(z) = e^{\bar{a}z} e^{-|a|^2/2}$. The restriction of V_a to $H^2(\mathbf{C}, d\mu)$ is also a unitary operator that maps the Fock space onto itself.

For any $f \in L^{\infty}(\mathbf{C})$, we will identify the Hankel operator H_f with the operator $(1 - P)M_f P$ on the space $L^2(\mathbf{C}, d\mu)$. Thus for $f, \varphi, \psi \in L^{\infty}(\mathbf{C})$, $M_{\varphi} H_f M_{\psi}$ means the operator $M_{\varphi}(1 - P)M_f P M_{\psi}$ on $L^2(\mathbf{C}, d\mu)$.

Proof of Theorem 1.7. Let $\{f_k\}$ be the sequence in $\mathcal{L}$ provided by Lemma 13.6. Then there is a sequence $\{\rho_k\}$ in $(0, \infty)$ such that $f_k = 0$ on $\mathbf{C} \backslash D(0, \rho_k)$ for every $k \in \mathbf{N}$. For each $k \in \mathbf{N}$, Lemma 13.3 allows us to pick a $\rho_k < r_k < \infty$ such that

$$\text{(13.8)} \qquad \begin{cases} \|H_{f_k} - M_{\chi_{D(0,r_k)}} H_{f_k} M_{\chi_{D(0,r_k)}}\|_1 \leq 2^{-k} & \text{and} \\ \|H_{\bar{f}_k} - M_{\chi_{D(0,r_k)}} H_{\bar{f}_k} M_{\chi_{D(0,r_k)}}\|_1 \leq 2^{-k} \end{cases}.$$

Thus the operators

$$\bigoplus_{k=1}^{\infty} H_{f_k} - \bigoplus_{k=1}^{\infty} M_{\chi_{D(0,r_k)}} H_{f_k} M_{\chi_{D(0,r_k)}} \quad \text{and} \quad \bigoplus_{k=1}^{\infty} H_{\bar{f}_k} - \bigoplus_{k=1}^{\infty} M_{\chi_{D(0,r_k)}} H_{\bar{f}_k} M_{\chi_{D(0,r_k)}}$$

are in the trace class. Applying Lemma 13.6, we have

$$\bigoplus_{k=1}^{\infty} M_{\chi_{D(0,r_k)}} H_{f_k} M_{\chi_{D(0,r_k)}} \in \mathcal{C}_\infty^- \quad \text{while} \quad \bigoplus_{k=1}^{\infty} M_{\chi_{D(0,r_k)}} H_{\bar{f}_k} M_{\chi_{D(0,r_k)}} \notin \mathcal{C}_\infty^-. \tag{13.9}$$

We can inductively select a sequence $\{a_k\}$ in $\mathbf{C}$ such that $D(a_k, r_k) \cap D(a_j, r_j) = \emptyset$ for all $j \neq k$. We have

$$V_{a_k} M_{\chi_{D(0,r_k)}} H_\varphi M_{\chi_{D(0,r_k)}} V_{a_k}^* = M_{\chi_{D(a_k,r_k)}} H_{\varphi \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}}$$

for every $\varphi \in L^\infty(\mathbf{C})$. Combining this unitary equivalence with (13.9), we see that

$$\bigoplus_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{f_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}} \in \mathcal{C}_\infty^- \quad \text{while} \quad \bigoplus_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{\bar{f}_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}} \notin \mathcal{C}_\infty^-.$$

Since $D(a_k, r_k) \cap D(a_j, r_j) = \emptyset$ for all $j \neq k$, the above implies that as operators on $L^2(\mathbf{C}, d\mu)$, we have

$$\sum_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{f_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}} \in \mathcal{C}_\infty^- \quad \text{while} \quad \sum_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{\bar{f}_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}} \notin \mathcal{C}_\infty^-. \tag{13.10}$$

Using the unitary operator V_{a_k} again, from (13.8) we obtain

$$\begin{cases} \|H_{f_k \circ \tau_{a_k}} - M_{\chi_{D(a_k,r_k)}} H_{f_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}}\|_1 \le 2^{-k} & \text{and} \\ \|H_{\bar{f}_k \circ \tau_{a_k}} - M_{\chi_{D(a_k,r_k)}} H_{\bar{f}_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}}\|_1 \le 2^{-k} & \end{cases},$$

$k \in \mathbf{N}$. Thus the operators

$$\sum_{k=1}^{\infty} H_{f_k \circ \tau_{a_k}} - \sum_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{f_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}} \quad \text{and}$$
$$\sum_{k=1}^{\infty} H_{\bar{f}_k \circ \tau_{a_k}} - \sum_{k=1}^{\infty} M_{\chi_{D(a_k,r_k)}} H_{\bar{f}_k \circ \tau_{a_k}} M_{\chi_{D(a_k,r_k)}}$$

are in the trace class. Combining this fact with (13.10), we see that

$$\sum_{k=1}^{\infty} H_{f_k \circ \tau_{a_k}} \in \mathcal{C}_\infty^- \quad \text{while} \quad \sum_{k=1}^{\infty} H_{\bar{f}_k \circ \tau_{a_k}} \notin \mathcal{C}_\infty^-. \tag{13.11}$$

The property that $f_k = 0$ on $\mathbf{C}\backslash D(0, r_k)$ implies that $f_k \circ \tau_{a_k} = 0$ on $\mathbf{C}\backslash D(a_k, r_k)$, $k \in \mathbf{N}$. Since $\|f_k\|_\infty \le 1$ for every k and since $D(a_k, r_k) \cap D(a_j, r_j) = \emptyset$ for $j \ne k$, the function

$$q = \sum_{k=1}^{\infty} f_k \circ \tau_{a_k}$$

is in $L^\infty(\mathbf{C})$. On $L^2(\mathbf{C}, d\mu)$, we have the obvious strong convergence $\sum_{k=1}^{\ell} M_{f_k \circ \tau_{a_k}} \to M_q$ as $\ell \to \infty$. Therefore

$$\sum_{k=1}^{\infty} H_{f_k \circ \tau_{a_k}} = H_q \quad \text{and} \quad \sum_{k=1}^{\infty} H_{\bar{f}_k \circ \tau_{a_k}} = H_{\bar{q}}.$$

Thus (13.11) tells us that $H_q \in \mathcal{C}_\infty^-$ while $H_{\bar{q}} \notin \mathcal{C}_\infty^-$. This completes the proof. □

14. Generalization

One may observe that Theorems 1.5, 1.6 and 1.7 all deal with "endpoint" cases of one kind or another, which may lead to the impression that it is rare for a norm ideal (in the sense of [12]) not to have the Berger-Coburn phenomenon. But the construction in Section 13 is so general that we can use it to produce, on a wholesale basis, norm ideals which do not have the Berger-Coburn phenomenon. In other words, with very little additional effort, the proof of Theorem 1.7 can be generalized to cover a class of ideals. To discuss this generalization, let us first introduce these ideals.

Let $\alpha = \{\alpha_j\}$ be a non-increasing sequence of positive numbers starting with $\alpha_1 = 1$. We assume that the sequence α is *binormalizing* [12, page 141], i.e.,

$$\sum_{j=1}^{\infty} \alpha_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \alpha_j = 0.$$

Such a sequence α gives rise to an operator ideal $\mathcal{C}_\alpha = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\alpha < \infty\}$, where the norm $\|\cdot\|_\alpha$ is defined by the formula

$$\|A\|_\alpha = \sum_{j=1}^{\infty} \alpha_j s_j(A).$$

See [12, Section III.15]. We assume that the sequence α satisfies the additional condition that there is a constant $0 < C = C(\alpha) < \infty$ such that

$$\sum_{j=1}^{\nu^2} \alpha_j \le C \sum_{j=1}^{\nu} \alpha_j \quad \text{for every} \ \ \nu \in \mathbf{N}. \tag{14.1}$$

Obviously, the sequence $\{j^{-1}\}$ is binormalizing and satisfies (14.1), and the corresponding ideal $\mathcal{C}_{\{j^{-1}\}}$ is just the Macaev ideal $\mathcal{C}_\infty^-$. For each $0 < t \le 1$, the sequence

$$\left\{\frac{1}{j(1+\log j)^t}\right\}$$

is also binormalizing, and it is easy to verify that it satisfies (14.1). Thus there are plenty of such α. We have the following generalization of Theorem 1.7:

Theorem 14.1. *Again, consider the case where $n = 1$. Let $\alpha = \{\alpha_j\}$ be any binormalizing sequence that satisfies condition* (14.1). *Then there exists an $f_\alpha \in L^\infty(\mathbf{C})$ such that $H_{f_\alpha} \in \mathcal{C}_\alpha$ while $H_{\bar{f}_\alpha} \notin \mathcal{C}_\alpha$.*

Proof. We will show that there is a sequence $\{f_k\}$ in $\mathcal{L}$ which has the properties that $\|f_k\|_\infty \leq 1$ for every $k \in \mathbf{N}$ and that

$$\bigoplus_{k=1}^{\infty} H_{f_k} \in \mathcal{C}_\alpha \quad \text{while} \quad \bigoplus_{k=1}^{\infty} H_{\bar{f}_k} \notin \mathcal{C}_\alpha. \tag{14.2}$$

Then, repeating the argument in the proof of Theorem 1.7, there is a sequence $\{a_k\}$ in $\mathbf{C}$ such that the function

$$f_\alpha = \sum_{k=1}^{\infty} f_k \circ \tau_{a_k}$$

is in $L^\infty(\mathbf{C})$ and has the property that $H_{f_\alpha} \in \mathcal{C}_\alpha$ while $H_{\bar{f}_\alpha} \notin \mathcal{C}_\alpha$.

To produce the sequence $\{f_k\}$ promised above, we first introduce the following notation. For each $\nu \in \mathbf{N}$, we write

$$\sigma(\nu) = \sum_{j=1}^{\nu} \alpha_j.$$

Thus (14.1) translates to

$$\sigma(\nu^2) \leq C\sigma(\nu) \quad \text{for every} \quad \nu \in \mathbf{N}.$$

Repeating the argument in the proof of Proposition 13.5 and using the above inequality, for each $A \in \mathcal{C}_1$ we obtain an $m(A) \in \mathbf{N}$ such that

$$\|A^{[\nu]}\|_\alpha \leq (1 + C)\sigma(\nu)\|A\| \quad \text{for every} \quad \nu \geq m(A). \tag{14.3}$$

The rest of the proof closely resembles that of Lemma 13.6.

For each $j \in \mathbf{N}$, we again apply Proposition 13.2 to obtain a $\varphi_j \in \mathcal{L}$ such that

$$\|H_{\varphi_j}\| \leq 1 \quad \text{while} \quad \|H_{\bar{\varphi}_j}\| \geq j.$$

As we know, the membership $\varphi_j \in \mathcal{L}$ guarantees that $H_{\varphi_j} \in \mathcal{C}_1$. Thus for each $j \in \mathbf{N}$, there is an $m(H_{\varphi_j}) \in \mathbf{N}$ for H_{φ_j} which has the property mentioned above.

Since α is binormalizing, we have $\sigma(\nu) \to \infty$ as $\nu \to \infty$. Thus for each $j \in \mathbf{N}$, we can pick a $\nu(j) \in \mathbf{N}$ satisfying the conditions

$$\nu(j) \geq m(H_{\varphi_j}) + 10 \quad \text{and} \quad \sigma(\nu(j)) \geq \|\varphi_j\|_\infty.$$

It follows from (14.3) and the condition $\|H_{\varphi_j}\| \leq 1$ that

$$\|H^{[\nu(j)]}_{\varphi_j}\|_\alpha \leq (1+C)\sigma(\nu(j)), \tag{14.4}$$

$j \in \mathbf{N}$. On the other hand, the condition $\|H_{\bar{\varphi}_j}\| \geq j$ implies that $s_\ell(H^{[\nu(j)]}_{\bar{\varphi}_j}) \geq j$ for $1 \leq \ell \leq \nu(j)$. Therefore

$$\|H^{[\nu(j)]}_{\bar{\varphi}_j}\|_\alpha \geq j\sigma(\nu(j)) \tag{14.5}$$

for every $j \in \mathbf{N}$. Now consider the operators

$$A = \bigoplus_{j=1}^{\infty} \frac{1}{j^2\sigma(\nu(j^3))} H^{[\nu(j^3)]}_{\varphi_{j^3}} \quad \text{and} \quad B = \bigoplus_{j=1}^{\infty} \frac{1}{j^2\sigma(\nu(j^3))} H^{[\nu(j^3)]}_{\bar{\varphi}_{j^3}}.$$

By (14.4), we have

$$\|A\|_\alpha \leq \sum_{j=1}^{\infty} \frac{1}{j^2\sigma(\nu(j^3))} \|H^{[\nu(j^3)]}_{\varphi_{j^3}}\|_\alpha \leq \sum_{j=1}^{\infty} \frac{(1+C)\sigma(\nu(j^3))}{j^2\sigma(\nu(j^3))} < \infty.$$

That is, $A \in \mathcal{C}_\alpha$. On the other hand, for each $j \in \mathbf{N}$, it follows from (14.5) that

$$\|B\|_\alpha \geq \frac{1}{j^2\sigma(\nu(j^3))} \|H^{[\nu(j^3)]}_{\bar{\varphi}_{j^3}}\|_\alpha \geq \frac{j^3\sigma(\nu(j^3))}{j^2\sigma(\nu(j^3))} = j.$$

This means that $B \notin \mathcal{C}_\alpha$. The choice of $\nu(j^3)$ ensures that $\|\varphi_{j^3}\|_\infty/\sigma(\nu(j^3)) \leq 1$. Thus if we let $\{f_k\}$ be a re-enumeration of the functions

$$\overbrace{\frac{1}{j^2\sigma(\nu(j^3))}\varphi_{j^3}, \cdots, \frac{1}{j^2\sigma(\nu(j^3))}\varphi_{j^3}}^{\nu(j^3) \text{ copies}}, \quad j \in \mathbf{N},$$

then (14.2) holds. This completes the proof. □

Appendix 1

The goal of this appendix is to give a proof of Proposition 9.2. This proof requires some preparation.

Definition A1.1. [19, Definition 4.3] For an analytic function h on $\mathbf{C}^n$, we write

$$\|h\|_* = \left(\int |h(\zeta)|^2 e^{-(1/2)|\zeta|^2} dV(\zeta) \right)^{1/2}.$$

Let $\mathcal{H}_*$ be the collection of analytic functions h on $\mathbf{C}^n$ satisfying the condition $\|h\|_* < \infty$.

For each $w \in \mathbf{C}^n$, define the unitary operator U_w on $L^2(\mathbf{C}^n, d\mu)$ by the formula

$$(U_w f)(\zeta) = f(w - \zeta)k_w(\zeta), \quad \zeta \in \mathbf{C}^n,$$

$f \in L^2(\mathbf{C}^n, d\mu)$. Obviously, U_w maps $H^2(\mathbf{C}^n, d\mu)$ to itself.

Lemma A1.2. [19, Lemma 4.4] *There is a constant* $0 < C_{A1.2} < \infty$ *such that the following estimate holds: Let* $\{e_u : u \in \mathbf{Z}^{2n}\}$ *be any orthonormal set and let* $h_u \in \mathcal{H}_*$, $u \in \mathbf{Z}^{2n}$, *be functions satisfying the condition* $\sup_{u \in \mathbf{Z}^{2n}} \|h_u\|_* < \infty$. *Then*

$$\left\| \sum_{u \in \mathbf{Z}^{2n}} (U_u h_u) \otimes e_u \right\| \le C_{A1.2} \sup_{u \in \mathbf{Z}^{2n}} \|h_u\|_*.$$

Lemma A1.3. [9, Lemma 3.3] *Let A and B be two bounded operators. Then the inequalities*

$$\||AB|^s\|_\Phi \le \|B\|^s \||A|^s\|_\Phi \quad \textit{and} \quad \||BA|^s\|_\Phi \le \|B\|^s \||A|^s\|_\Phi$$

hold for every symmetric gauge function Φ *and every* $0 < s \le 1$.

Lemma A1.4. [18, Lemma 3.1] *Suppose that* $A_1, \dots, A_m$ *are finite-rank operators on a Hilbert space* $\mathcal{H}$ *and let* $A = A_1 + \cdots + A_m$. *Then for every symmetric gauge function* Φ *and every* $0 < s \le 1$, *we have*

$$\||A|^s\|_\Phi \le 2^{1-s}(\||A_1|^s\|_\Phi + \cdots + \||A_m|^s\|_\Phi). \tag{A1.1}$$

Remark A1.5. As was explained in [9, Remark 3.5], (A1.1) actually holds for all bounded operators $A_1, \dots, A_m$ on any separable Hilbert space $\mathcal{H}$ and $A = A_1 + \cdots + A_m$.

Technically, it will be more convenient to prove a slightly stronger version of Proposition 9.2. For this reason we introduce the following.

For any Borel set E in $\mathbf{C}^n$, we define $L^2(E, d\mu)$ to be the collection of functions $f \in L^2(\mathbf{C}^n, d\mu)$ satisfying the condition $f = 0$ on $\mathbf{C}^n \backslash E$. We emphasize that we consider each element in $L^2(E, d\mu)$ as a function defined on the whole of the complex space $\mathbf{C}^n$.

For each $z \in \mathbf{C}^n$, let $\mathcal{B}_z$ be the collection of functions h in $L^2(W + z, d\mu)$ that are analytic on $W + z$ (see (8.1)). In other words, $\mathcal{B}_z$ consists of functions in $L^2(\mathbf{C}^n, d\mu)$ that are analytic on $W + z$ and identically zero on $\mathbf{C}^n \backslash \{W + z\}$. Obviously, $\mathcal{B}_z$ is a closed linear subspace of $L^2(\mathbf{C}^n, d\mu)$. One may think of $\mathcal{B}_z$ as a kind of "Bergman space", but keep in mind that the measure in question is the restriction of the Gaussian measure $d\mu$ to $W + z$, and this choice of measure is crucial for the estimates below.

For each $z \in \mathbf{C}^n$, let

$$P_z : L^2(\mathbf{C}^n, d\mu) \to \mathcal{B}_z$$

be the orthogonal projection. For all $f \in \mathcal{T}(\mathbf{C}^n)$ and $z \in \mathbf{C}^n$, we define

$$\tilde{M}(f; z) = \left(\int_{W+z} |f(\zeta) - k_z^{-1}(\zeta)(P_z f k_z)(\zeta)|^2 dV(\zeta) \right)^{1/2}. \tag{A1.2}$$

A comparison of (A1.2) and (9.2) gives us the inequality

$$M(f;z) \le \tilde{M}(f;z)$$

for $z \in \mathbf{Z}^{2n}$. Thus Proposition 9.2 is an immediate consequence of the following:

Proposition A1.6. *Let $0 < s \le 1$. Then there is a constant $0 < C_{\mathrm{A1.6}} < \infty$ that depends only on s and the complex dimension n such that*

$$\Phi(\{\tilde{M}^s(f;z)\}_{z\in\mathbf{Z}^{2n}}) \le C_{\mathrm{A1.6}}\||H_f|^s\|_\Phi$$

for every $f \in \mathcal{T}(\mathbf{C}^n)$ and every symmetric gauge function Φ.

Proof. As we have already explained in Section 9, this is essentially an easier version of the proof of [9, Proposition 6.8]. For any natural number $R \ge 10$, define

$$(R\mathbf{Z})^{2n} = \{(j_1R + ik_1R, \dots, j_nR + ik_nR) : j_1, \dots, j_n, k_1, \dots, k_n \in \mathbf{Z}\} \quad \text{and}$$
$$\Lambda_R = \{(j_1 + ik_1, \dots, j_n + ik_n) : j_1, \dots, j_n, k_1, \dots, k_n \in \{0, 1, 2, \dots, R-1\}\}.$$

Let $0 < s \le 1$ be given. Then define

$$\delta(R) = \sum_{x\in(R\mathbf{Z})^{2n}\setminus\{0\}} e^{-(s/4)|x|^2}. \tag{A1.3}$$

Note that for $x \in (R\mathbf{Z})^{2n}$, if $x \ne 0$, then $|x|^2 \ge R^2$. Hence

$$\delta(R) \le e^{-(s/8)R^2} \sum_{x\in(R\mathbf{Z})^{2n}\setminus\{0\}} e^{-(s/8)|x|^2} \le e^{-(s/8)R^2} \sum_{x\in\mathbf{Z}^{2n}} e^{-(s/8)|x|^2}.$$

This shows that $\delta(R) \to 0$ as $R \to \infty$. This allows us to pick an R satisfying the condition

$$4e^{18sn}\delta(R) \le 1/2 \tag{A1.4}$$

as well as the condition $R \ge 10$. With R so fixed, we have the partition

$$\mathbf{Z}^{2n} = \bigcup_{\lambda\in\Lambda_R} \{(R\mathbf{Z})^{2n} + \lambda\}.$$

Fix a $\lambda \in \Lambda_R$ for the moment, and let Γ be any *finite* subset of $(R\mathbf{Z})^{2n} + \lambda$.

Let $\{e_u : u \in \mathbf{Z}^{2n}\}$ be an orthonormal set in $L^2(\mathbf{C}^n, d\mu)$. Define the operators

$$G\varphi = \sum_{z\in\Gamma}(\varphi - P_z\varphi)\chi_{W+z} \quad \text{and} \quad F\varphi = \sum_{z\in\Gamma}\langle\varphi, e_z\rangle k_z, \quad \varphi \in L^2(\mathbf{C}^n, d\mu).$$

Since $R \ge 10$, we have $\{W + z\} \cap \{W + w\} = \emptyset$ for $z \ne w$ in Γ. Hence $\|G\| \le 1$. Since $k_z = U_z1$ for $z \in \Gamma$, it follows from Lemma A1.2 that $\|F\| \le C_{\mathrm{A1.2}}\|1\|_* = C$.

Let $f \in \mathcal{T}(\mathbf{C}^n)$ and symmetric gauge function Φ be given. By Lemma A1.3 we have

$$\||GH_fF|^s\|_\Phi \le C^s\||H_f|^s\|_\Phi. \tag{A1.5}$$

On the other hand, we have

$$GH_fF = A + B,$$

where

$$A = \sum_{z\in\Gamma}\{(H_fk_z - P_zH_fk_z)\chi_{W+z}\} \otimes e_z \quad \text{and}$$

$$B = \sum_{\substack{(z,u)\in\Gamma\times\Gamma \\ z\ne u}}\{(H_fk_u - P_zH_fk_u)\chi_{W+z}\} \otimes e_u.$$

For $z \ne w$ in Γ we have $\{W+z\}\cap\{W+w\} = \emptyset$. Thus, applying Lemma A1.4 and (A1.5),

$$\begin{aligned}\Phi(\{\|(H_fk_z - P_zH_fk_z)\chi_{W+z}\|^s\}_{z\in\Gamma}) = \||A|^s\|_\Phi &\le 2\||GH_fF|^s\|_\Phi + 2\||B|^s\|_\Phi \\ &\le 2C^s\||H_f|^s\|_\Phi + 2\||B|^s\|_\Phi.\end{aligned} \tag{A1.6}$$

On the other hand, if $h \in H^2(\mathbf{C}^n, d\mu)$, then obviously $P_wh = h\chi_{W+w}$ for every $w \in \mathbf{C}^n$. It follows that $(H_fk_z - P_zH_fk_z)\chi_{W+z} = (fk_z - P_z(fk_z))\chi_{W+z}$ for every $z \in \Gamma$. Thus

$$\begin{aligned}\|(H_fk_z - P_zH_fk_z)\chi_{W+z}\|^2 &= \int_{W+z}|fk_z - P_z(fk_z)|^2d\mu \\ &= \frac{1}{\pi^n}\int_{W+z}|f(\zeta) - k_z^{-1}(\zeta)(P_zfk_z)(\zeta)|^2e^{-|\zeta-z|^2}dV(\zeta).\end{aligned}$$

If $\zeta \in W + z$, then $\zeta - z \in W$, which implies $|\zeta - z|^2 \le 18n$. Hence the above implies

$$\begin{aligned}\|(H_fk_z - P_zH_fk_z)\chi_{W+z}\|^2 &\ge \frac{1}{\pi^ne^{18n}}\int_{W+z}|f(\zeta) - k_z^{-1}(\zeta)(P_zfk_z)(\zeta)|^2dV(\zeta) \\ &= \frac{1}{\pi^ne^{18n}}\tilde{M}^2(f;z).\end{aligned}$$

Combining this with (A1.6), we obtain

$$\Phi(\{\tilde{M}^s(f;z)\}_{z\in\Gamma}) \le 2\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi + 2\pi^{sn/2}e^{9sn}\||B|^s\|_\Phi. \tag{A1.7}$$

Next we bound $\||B|^s\|_\Phi$ from above.

Consider any pair of $z, u \in \Gamma$. Because $P_zh = h\chi_{W+z}$ for every $h \in H^2(\mathbf{C}^n, d\mu)$, we have $(H_fk_u - P_zH_fk_u)\chi_{W+z} = (fk_u - P_z(fk_u))\chi_{W+z}$. Since $\chi_{W+z}P_z(fk_u) = P_z(fk_u)$ is the orthogonal projection of $fk_u\chi_{W+z}$ on $\mathcal{B}_z$, we have

$$\begin{aligned}\|(H_fk_u - P_zH_fk_u)\chi_{W+z}\|^2 &= \|(fk_u - P_z(fk_u))\chi_{W+z}\|^2 \\ &\le \|(fk_u - k_uk_z^{-1}P_z(fk_z))\chi_{W+z}\|^2 \\ &= \frac{1}{\pi^n}\int_{W+z}|f(\zeta) - k_z^{-1}(\zeta)(P_zfk_z)(\zeta)|^2e^{-|\zeta-u|^2}dV(\zeta).\end{aligned}$$

If $\zeta \in W + z$, then $|\zeta - u|^2 \geq (1/2)|z - u|^2 - 18n$. Thus the above implies

$$\text{(A1.8)} \qquad \|(H_f k_u - P_z H_f k_u)\chi_{W+z}\|^2 \leq e^{18n}\pi^{-n}e^{-(1/2)|z-u|^2}\tilde{M}^2(f;z)$$

for every pair of $z, u \in \Gamma$. Note that for $z, u \in \Gamma$, we have $z - u \in (R\mathbf{Z})^{2n}$. Thus we can rewrite the operator B in the form

$$B = \sum_{x \in (R\mathbf{Z})^{2n}\backslash\{0\}} B_x,$$

where

$$B_x = \sum_{z \in \Gamma \cap \{\Gamma + x\}} \{(H_f k_{z-x} - P_z H_f k_{z-x})\chi_{W+z}\} \otimes e_{z-x}.$$

Again, for $z \neq w$ in Γ we have $\{W + z\} \cap \{W + w\} = \emptyset$. Thus for every $x \in (R\mathbf{Z})^{2n}\backslash\{0\}$,

$$\||B_x|^s\|_\Phi = \Phi(\{\|(H_f k_{z-x} - P_z H_f k_{z-x})\chi_{W+z}\|^s\}_{z \in \Gamma \cap \{\Gamma + x\}}).$$

Applying (A1.8), for every $x \in (R\mathbf{Z})^{2n}\backslash\{0\}$ we have

$$\||B_x|^s\|_\Phi \leq e^{9sn}\pi^{-sn/2}e^{-(s/4)|x|^2}\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}).$$

Recalling Lemma A1.4, we obtain

$$\text{(A1.9)} \qquad \||B|^s\|_\Phi \leq 2\sum_{x \in (R\mathbf{Z})^{2n}\backslash\{0\}} \||B_x|^s\|_\Phi \leq 2e^{9sn}\pi^{-sn/2}\delta(R)\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}),$$

where $\delta(R)$ is defined by (A1.3). Substituting (A1.9) in (A1.7), we find that

$$\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}) \leq 2\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi + 4e^{18sn}\delta(R)\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}).$$

Applying (A1.4), the above becomes

$$\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}) \leq 2\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi + (1/2)\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}).$$

Since Γ is a finite set, the quantity $\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma})$ is finite. Thus we can cancel out $(1/2)\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma})$ from both sides to obtain

$$\Phi(\{\tilde{M}^s(f;z)\}_{z \in \Gamma}) \leq 4\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi.$$

Since this holds for every finite subset Γ of $(R\mathbf{Z})^{2n} + \lambda$, by (6.3) and (6.4) this means

$$\Phi(\{\tilde{M}^s(f;z)\}_{z \in (R\mathbf{Z})^{2n}+\lambda}) \leq 4\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi.$$

Finally, since this holds for every $\lambda \in \Lambda_R$ and $\text{card}(\Lambda_R) = R^{2n}$, the property of Φ leads to

$$\Phi(\{\tilde{M}^s(f;z)\}_{z \in \mathbf{Z}^{2n}}) \leq \sum_{\lambda \in \Lambda_R} \Phi(\{\tilde{M}^s(f;z)\}_{z \in (R\mathbf{Z})^{2n}+\lambda}) \leq 4R^{2n}\pi^{sn/2}e^{9sn}C^s\||H_f|^s\|_\Phi.$$

This completes the proof of the proposition. □

Appendix 2

We will now prove Proposition 12.1.

Write $A = \bar{\partial}$ and $C = -\partial + \bar{z}$, both of which are considered as operators on the polynomial ring $\mathbf{C}[z, \bar{z}]$. Taking inner product in $L^2(\mathbf{C}, d\mu)$, it is easy to verify that

$$\langle Au, v \rangle = \langle u, Cv \rangle \quad \text{for all} \quad u, v \in \mathbf{C}[z, \bar{z}].$$

That is, $C = A^*$ on $\mathbf{C}[z, \bar{z}]$. Moreover, it is easy to see that $[A, C] = 1$. Now let $H_0 = \{u \in \mathbf{C}[z, \bar{z}] : Au = 0\}$ and $H_k = C^k H_0$ for every $k \in \mathbf{N}$. From the relation $[A, C] = 1$ we deduce $[A, C^k] = kC^{k-1}$ for every $k \in \mathbf{N}$. Hence

$$ACv = (k+1)v \quad \text{if} \quad v \in H_k,$$

$k = 0, 1, 2, \ldots$. It follows from this and the "self-adjointness" of AC on $\mathbf{C}[z, \bar{z}]$ that we have the orthogonality $H_j \perp H_k$ for every pair of $j \neq k$ in $\{0, 1, 2, \ldots, m, \ldots\}$.

Note that $H_0 = \mathbf{C}[z]$. We pick an orthonormal set $\mathcal{B}_0$ in H_0 such that $\text{span}(\mathcal{B}_0) = H_0$. Suppose that $k \geq 0$ and that we have inductively defined orthonormal sets $\mathcal{B}_0, \ldots, \mathcal{B}_k$ such that $\text{span}(\mathcal{B}_j) = H_j$ for every $0 \leq j \leq k$. Then we define

$$\mathcal{B}_{k+1} = \{(k+1)^{-1/2} Cb : b \in \mathcal{B}_k\}.$$

Since $H_{k+1} = CH_k$, we have $\mathcal{B}_{k+1} \subset H_{k+1}$ and that $\text{span}(\mathcal{B}_{k+1}) = H_{k+1}$. Let us verify that $\mathcal{B}_{k+1}$ is also an orthonormal set. Indeed for any $b, b' \in \mathcal{B}_k$, we have

$$\langle (k+1)^{-1/2} Cb, (k+1)^{-1/2} Cb' \rangle = (k+1)^{-1} \langle ACb, b' \rangle = (k+1)^{-1}(k+1)\langle b, b' \rangle,$$

which equals 1 or 0 depending on whether b equals b' or not. Hence $\mathcal{B}_{k+1}$ is also an orthonormal set. Thus we have inductively constructed orthonormal sets $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \ldots$.

Since $H_j \perp H_k$ for every pair of $j \neq k$ in $\{0, 1, 2, \ldots, m, \ldots\}$, we have $\mathcal{B}_j \perp \mathcal{B}_k$ whenever $j \neq k$. Therefore

$$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k \cup \cdots$$

is an orthonormal set in $L^2(\mathbf{C}, d\mu)$. For each $k \geq 1$, we have $C^k = \sum_{j=0}^{k} \frac{(-1)^{k-j} k!}{j!(k-j)!} \bar{z}^j \partial^{k-j}$. Also, $H_0 = \mathbf{C}[z]$. Thus a simple induction on the power of $\bar{z}$ proves that

$$\text{span}\{H_0, H_1, H_2, \ldots, H_k, \ldots\} = \mathbf{C}[z, \bar{z}].$$

Combining this with the fact that $\text{span}(\mathcal{B}_k) = H_k$ for every $k \geq 0$, we conclude that $\mathcal{B}$ is an orthonormal basis for $L^2(\mathbf{C}, d\mu)$. Obviously, $\mathcal{B}_0$ is an orthonormal basis for the one-variable Fock space $H^2(\mathbf{C}, d\mu)$.

(1) First, suppose that $f \in C_c^\infty(\mathbf{C})$. We have $\langle f - Pf, b\rangle = 0$ for $b \in \mathcal{B}_0$ and $\langle Pf, b\rangle = 0$ if $b \in \cup_{k=1}^\infty \mathcal{B}_k$. Therefore

$$\|f - Pf\|^2 = \sum_{b\in\mathcal{B}} |\langle f - Pf, b\rangle|^2 = \sum_{k=1}^\infty \sum_{b_k\in\mathcal{B}_k} |\langle f, b_k\rangle|^2. \tag{A2.1}$$

On the other hand, for every $u \in \mathbf{C}[z, \bar{z}]$, simple integration by parts gives us

$$\langle \bar{\partial} f, u\rangle = \langle f, Cu\rangle.$$

Combining this with the fact that $\mathcal{B}_{k+1} = \{(k+1)^{-1/2}Cb : b \in \mathcal{B}_k\}$, we have

$$\begin{aligned}\|\bar{\partial} f\|^2 &= \sum_{b\in\mathcal{B}} |\langle \bar{\partial} f, b\rangle|^2 = \sum_{b\in\mathcal{B}} |\langle f, Cb\rangle|^2 = \sum_{k=0}^\infty \sum_{b_k\in\mathcal{B}_k} |\langle f, Cb_k\rangle|^2 \\ &= \sum_{k=0}^\infty (k+1) \sum_{b_{k+1}\in\mathcal{B}_{k+1}} |\langle f, b_{k+1}\rangle|^2 \geq \sum_{k=1}^\infty \sum_{b_k\in\mathcal{B}_k} |\langle f, b_k\rangle|^2.\end{aligned}$$

Comparing this with (A2.1), we obtain $\|f - Pf\| \leq \|\bar{\partial} f\|$ in the case $f \in C_c^\infty(\mathbf{C})$.

(2) Now we consider a general $f \in C^\infty(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, d\mu)$. If $\|\bar{\partial} f\| = \infty$, then, of course, we have $\|f - Pf\| \leq \|\bar{\partial} f\|$. Thus we may assume $\|\bar{\partial} f\| < \infty$, i.e., $\bar{\partial} f \in L^2(\mathbf{C}, d\mu)$. Let φ be a function in $C_c^\infty(\mathbf{C})$ such that $\varphi(\zeta) = 1$ when $|\zeta| \leq 1$ and $\varphi(\zeta) = 0$ when $|\zeta| \geq 2$. For each $\epsilon > 0$, define the function φ_ϵ by the formula

$$\varphi_\epsilon(\zeta) = \varphi(\epsilon\zeta), \quad \zeta \in \mathbf{C}.$$

Since $\varphi_\epsilon f \in C_c^\infty(\mathbf{C}^n)$, by the conclusion in (1), we have

$$\|\varphi_\epsilon f - P(\varphi_\epsilon f)\| \leq \|\bar{\partial}(\varphi_\epsilon f)\| \tag{A2.2}$$

for every $\epsilon > 0$. From the condition $\varphi(\zeta) = 1$ for $|\zeta| \leq 1$ we deduce that $\|\varphi_\epsilon f - f\| \to 0$ as $\epsilon \downarrow 0$. Similarly, since $\bar{\partial} f \in L^2(\mathbf{C}^n, d\mu)$, we have $\|\varphi_\epsilon \bar{\partial} f - \bar{\partial} f\| \to 0$ as $\epsilon \downarrow 0$. Also, since $\|\bar{\partial}\varphi_\epsilon\|_\infty = \epsilon\|\bar{\partial}\varphi\|_\infty$, we have $\|\bar{\partial}\varphi_\epsilon\|_\infty \to$ as $\epsilon \downarrow 0$. Since $\bar{\partial}(\varphi_\epsilon f) = f\bar{\partial}\varphi_\epsilon + \varphi_\epsilon\bar{\partial} f$, we conclude that $\|\bar{\partial}(\varphi_\epsilon f) - \bar{\partial} f\| \to 0$ as $\epsilon \downarrow 0$. Thus, letting ϵ descend to 0 in (A2.2), we obtain the inequality $\|f - Pf\| \leq \|\bar{\partial} f\|$ for general $f \in C^\infty(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, d\mu)$. This completes the proof of Proposition 12.1.

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Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260

E-mail: jxia@acsu.buffalo.edu