BEST APPROXIMATIONS IN A CLASS OF LORENTZ IDEALS

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Dedicated to the memory of Jörg Eschmeier

Abstract. We consider the family of Lorentz ideals $\mathcal{C}_p^+$, $1 \le p < \infty$. Let $\mathcal{C}_p^{+(0)}$ be the $\|\cdot\|_p^+$-closure of the collection of finite-rank operators in $\mathcal{C}_p^+$. It is well known that $\mathcal{C}_p^{+(0)} \neq \mathcal{C}_p^+$. We show that $\mathcal{C}_p^{+(0)}$ is proximinal in $\mathcal{C}_p^+$. We further show that a classic approximation for Hankel operators [1, Theorem 3] does not generalize to this new context.

1. Introduction

Let X be a Banach space and let M be a closed linear subspace of X. An element $x \in X$ is said to have a best approximation in M if there is an $m \in M$ such that $\|x-m\| \le \|x-a\|$ for every $a \in M$. The subspace M is said to be *proximinal* in X if every $x \in X$ has a best approximation in M.

One of the most familiar and significant examples of such a pair is the case of $X = \mathcal{B}(\mathcal{H})$ and $M = \mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ is the collection of bounded operators on $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ is the collection of compact operators on $\mathcal{H}$. It is well known that $\mathcal{K}(\mathcal{H})$ is proximinal in $\mathcal{B}(\mathcal{H})$, which is a result in the influential book [11] by Gohberg and Krien. Given any $A \in \mathcal{B}(\mathcal{H})$, to find its best approximation in $\mathcal{K}(\mathcal{H})$, one takes the polar decomposition $A = U|A|$, where $|A| = (A^*A)^{1/2}$ and U is a partial isometry. Then from the spectral decomposition of $|A|$ one easily finds the best compact approximation to A. The moral of this example is that when looking for best approximations for operators on a Hilbert space, one should take advantage of spectral decomposition, which is not available on other Banach spaces.

The relation between $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ is that the latter is the closure of the collection of finite-rank operators in the former. On a Hilbert space $\mathcal{H}$, there are many pairs that fit this description, but with different norms. In particular, the *norm ideals* of Robert Schatten [16] are a good source for interesting examples of X and M.

Before getting to these examples, it is necessary to give a general introduction for norm ideals. For this we follow the approach in [11,19], because it offers the level of generality that is suitable for this paper.

As in [11], we write $\hat{c}$ for the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A *symmetric gauge function* is a map

$$\Phi : \hat{c} \to [0, \infty)$$

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that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \ldots, 0, \ldots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [11, page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \ldots, s_j(A), 0, \ldots, 0, \ldots\})$$

for bounded operators, where $s_1(A), \ldots, s_j(A), \ldots$ are the singular numbers of A. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{1.1}$$

is a norm ideal [11, page 68]. That is, $\mathcal{C}_\Phi$ has the following properties:

- For any B, $C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \leq \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \leq \|A\|_\Phi$, and the equality holds when $\mathrm{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|\cdot\|_\Phi$.

Given a symmetric gauge function Φ, we define $\mathcal{C}_\Phi^{(0)}$ to be the closure with respect to the norm $\|\cdot\|_\Phi$ of the collection of finite-rank operators in $\mathcal{C}_\Phi$.

Both ideals $\mathcal{C}_\Phi$ and $\mathcal{C}_\Phi^{(0)}$ are important in operator theory and operator algebras. For example, if one considers the problem of diagonalization under perturbation for single self-adjoint operators [13,14] or for commuting tuples of self-adjoint operators [2,18,19,20,21], then the natural perturbing operators come from ideals of the form $\mathcal{C}_\Phi^{(0)}$. If one studies Toeplitz operators or Hankel operators on various reproducing-kernel Hilbert spaces, then a natural question is the membership of these operators in ideals of the form $\mathcal{C}_\Phi$ [10,12,22].

For many symmetric gauge functions, we simply have $\mathcal{C}_\Phi^{(0)} = \mathcal{C}_\Phi$. For example, if we take any $1 \leq p < \infty$ and consider the symmetric gauge function

$$\Phi_p(\{a_j\}) = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{1/p}, \quad \{a_j\} \in \hat{c},$$

then the norm ideal $\mathcal{C}_{\Phi_p}$ defined according to (1.1) is simply the familiar Schatten p-class. It is well known and obvious that $\mathcal{C}_{\Phi_p}^{(0)} = \mathcal{C}_{\Phi_p}$.

From [11] we know that there also are many symmetric gauge functions for which $\mathcal{C}_\Phi^{(0)} \neq \mathcal{C}_\Phi$. The most noticeable of such examples is the symmetric gauge function

$$\Phi_\infty(\{a_j\}) = \sup_{j\in\mathbf{N}} |a_j|, \quad \{a_j\} \in \hat{c}.$$

Obviously, the norm $\|\cdot\|_{\Phi_\infty}$ is none other than the ordinary operator norm. Therefore (1.1) gives us $\mathcal{C}_{\Phi_\infty} = \mathcal{B}(\mathcal{H})$. It is also obvious that $\mathcal{C}_{\Phi_\infty}^{(0)} = \mathcal{K}(\mathcal{H})$. Thus the classic result that $\mathcal{K}(\mathcal{H})$ is proximinal in $\mathcal{B}(\mathcal{H})$ can be rephrased as the statement that $\mathcal{C}_{\Phi_\infty}^{(0)}$ is proximinal in $\mathcal{C}_{\Phi_\infty}$. Once one realizes that, it does not take too much imagination to propose

Problem 1.1. For a general symmetric gauge function Φ with the property $\mathcal{C}_\Phi^{(0)} \neq \mathcal{C}_\Phi$, is $\mathcal{C}_\Phi^{(0)}$ proximinal in $\mathcal{C}_\Phi$?

In such generality, Problem 1.1 does not appear to be easy, for it simply covers too many ideals of diverse properties. It is not too hard to convince oneself that to determine whether or not $\mathcal{C}_\Phi^{(0)}$ is proximinal in $\mathcal{C}_\Phi$, one needs to know the specifics of Φ. At this point, we do not see how to get a general answer using only properties (a)-(c) listed above plus the condition $\mathcal{C}_\Phi^{(0)} \neq \mathcal{C}_\Phi$.

But we are pleased to report that there is a family of symmetric gauge functions of common interest for which we are able to solve Problem 1.1 in the affirmative. Let us introduce these symmetric gauge functions and the corresponding ideals.

For each $1 \leq p < \infty$, let Φ_p^+ be the symmetric gauge function defined by the formula

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\geq 1} \frac{|a_{\pi(1)}| + |a_{\pi(2)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \cdots \geq |a_{\pi(j)}| \geq \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. The ideal $\mathcal{C}_{\Phi_p^+}$, which is defined by (1.1) using Φ_p^+, is often called a Lorentz ideal. It is well known that $\mathcal{C}_{\Phi_p^+} \neq \mathcal{C}_{\Phi_p^+}^{(0)}$ [11]. The ideal $\mathcal{C}_{\Phi_1^+}$ deserves special mentioning, because it is the domain of the Dixmier trace [4,6], which has wide-ranging connections [3,7,8,12,17].

The ideals $\mathcal{C}_{\Phi_p^+}$ and $\mathcal{C}_{\Phi_p^+}^{(0)}$, $1 \leq p < \infty$, are the main interest of this paper. Since they will appear so frequently in the sequel, let us introduce a simplified notation. From now on we will write

$$\text{(1.2)} \qquad \mathcal{C}_p^+ = \mathcal{C}_{\Phi_p^+}, \quad \mathcal{C}_p^{+(0)} = \mathcal{C}_{\Phi_p^+}^{(0)} \quad \text{and} \quad \|\cdot\|_p^+ = \|\cdot\|_{\Phi_p^+}$$

for $1 \leq p < \infty$. Here is our main result:

Theorem 1.2. *For every $1 \leq p < \infty$, $\mathcal{C}_p^{+(0)}$ is proximinal in $\mathcal{C}_p^+$.*

The result that $\mathcal{K}(\mathcal{H})$ is proximinal in $\mathcal{B}(\mathcal{H})$ has refinements within specific classes of operators [1]. One such class of operators are the Hankel operators $H_f : H^2 \to L^2$, where H^2 is the Hardy space on the unit circle $\mathbf{T} \subset \mathbf{C}$. We recall the following:

Theorem 1.3. [1, Theorem 3] *For each $f \in L^\infty$, the best compact approximation to the Hankel operator H_f can be realized in the form of a Hankel operator H_g.*

In other words, Theorem 1.3 says that H_f has a best compact approximation that is of the same kind, a Hankel operator. Using the method in [1], Theorem 1.3 can be easily generalized to Hankel operators on the Hardy space on the unit sphere in $\mathbf{C}^n$.

Since Theorem 1.2 tells us that each $\mathcal{C}_p^{+(0)}$ is proximinal in $\mathcal{C}_p^+$, we can obviously ask a more refined question along the line of Theorem 1.3: Suppose that we have an operator A in a natural class $\mathcal{N}$, and suppose we know that $A \in \mathcal{C}_p^+$, can we find a best $\mathcal{C}_p^{+(0)}$-approximation to A in the same class $\mathcal{N}$? In particular, what if $\mathcal{N}$ consists of Hankel operators? As we will see, the answer to this last question turns out to be negative.

The rest of the paper is organized as follows. We prove Theorems 1.2 in Section 2. Then in Section 3, we present the above-mentioned negative answer. Namely, we give an example of a Hankel operator on the unit sphere in $\mathbf{C}^2$ which is in the ideal $\mathcal{C}_4^+$ and which does not have any Hankel operator as its best $\mathcal{C}_4^{+(0)}$-approximation. This example requires some explicit calculation, which may be of independent interest.

2. Existence of best approximation

Recall that the starting domain for every symmetric gauge function Φ is the space $\hat{c}$, which consists of real sequences whose nonzero terms are finite in number. Our first order of business is to follow the standard practice to extend the domain of Φ to include every sequence. That is, for any sequence $\xi = \{\xi_j\}$ of complex numbers, we define

$$\Phi(\xi) = \sup_{k\geq 1} \Phi(\{|\xi_1|, |\xi_2|, \dots, |\xi_k|, 0, \dots, 0, \dots\}).$$

It is well known that the properties of Φ imply that if $|a_j| \leq |b_j|$ for every j, then

$$\Phi(\{a_j\}) \leq \Phi(\{b_j\}).$$

This fact will be used in many of our estimates below.

We will focus exclusively on the symmetric gauge functions Φ_p^+, $1 \leq p < \infty$. For the rest of the paper, p will always denote a positive number in $[1, \infty)$.

Definition 2.1. (1) Write c_p^+ for the collection of sequences ξ satisfying the condition $\Phi_p^+(\xi) < \infty$.
(2) Let $c_p^+(0)$ denote the Φ_p^+-closure of $\{a + ib : a, b \in \hat{c}\}$ in c_p^+.
(3) For each $\xi \in c_p^+$, denote $\Phi_{p,\text{ess}}^+(\xi) = \inf\{\Phi_p^+(\xi - \eta) : \eta \in c_p^+(0)\}$.
(4) Write d_p^+ for the collection of sequences $x = \{x_j\}$ in c_p^+ satisfying the conditions that $x_j \geq 0$ and that $x_j \geq x_{j+1}$ for every $j \in \mathbf{N}$.

In other words, d_p^+ consists of the non-negative, non-increasing sequences in c_p^+.

Proposition 2.2. *For every $\xi = \{\xi_j\} \in c_p^+$, we have*

$$\Phi_{p,\text{ess}}^+(\xi) = \lim_{m\to\infty} \Phi_p^+(\{\xi_{m+1}, \xi_{m+2}, \dots, \xi_{m+k}, \dots\}).$$

In particular, $\xi = \{\xi_j\} \in c_p^+(0)$ if and only if

$$\lim_{m\to\infty} \Phi_p^+(\{\xi_{m+1}, \xi_{m+2}, \dots, \xi_{m+k}, \dots\}) = 0.$$

Proof. From the above definitions it is obvious that

$$\Phi_{p,\mathrm{ess}}^{+}(\xi) \le \lim_{m\to\infty} \Phi_p^{+}(\{\xi_{m+1}, \xi_{m+2}, \dots, \xi_{m+k}, \dots\}).$$

On the other hand, for any $a, b \in \hat{c}$, there exist a $\nu \in \mathbf{N}$ and $\zeta_1, \dots, \zeta_\nu \in \mathbf{C}$ such that

$$\xi - a - ib = \{\zeta_1, \dots, \zeta_\nu, \xi_{\nu+1}, \xi_{\nu+2}, \dots, \xi_{\nu+k}, \dots\}.$$

Thus for every $m \ge \nu$ we have

$$\Phi_p^{+}(\xi - a - ib) \ge \Phi_p^{+}(\{\xi_{m+1}, \xi_{m+2}, \dots, \xi_{m+k}, \dots\}).$$

Since $c_p^{+}(0)$ is the Φ_p^{+}-closure of $\{a + ib : a, b \in \hat{c}\}$, it follows that

$$\Phi_{p,\mathrm{ess}}^{+}(\xi) \ge \lim_{m\to\infty} \Phi_p^{+}(\{\xi_{m+1}, \xi_{m+2}, \dots, \xi_{m+k}, \dots\}).$$

This completes the proof. □

Proposition 2.3. *For every $x = \{x_j\} \in d_p^{+}$ we have*

$$\Phi_{p,\mathrm{ess}}^{+}(x) = \limsup_{j\to\infty} \frac{x_1 + x_2 + \cdots + x_j}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}. \tag{2.1}$$

Proof. For $x = \{x_j\} \in d_p^{+}$, (2.1) trivially holds if $\sum_{j=1}^{\infty} x_j < \infty$. Suppose that $\sum_{j=1}^{\infty} x_j = \infty$. Then for every $m \in \mathbf{N}$ we have

$$\lim_{j\to\infty} \frac{x_{m+1} + \cdots + x_{m+j}}{x_1 + \cdots + x_j} = 1 - \lim_{j\to\infty} \frac{x_1 + \cdots + x_m}{x_1 + \cdots + x_j} + \lim_{j\to\infty} \frac{x_{1+j} + \cdots + x_{m+j}}{x_1 + \cdots + x_j} = 1.$$

Therefore

$$\limsup_{j\to\infty} \frac{x_1 + \cdots + x_j}{1^{-1/p} + \cdots + j^{-1/p}} \le \Phi_p^{+}(\{x_{m+1}, x_{m+2}, \dots, x_{m+k}, \dots\})$$

for every $m \in \mathbf{N}$. By Proposition 2.2, this means

$$\limsup_{j\to\infty} \frac{x_1 + \cdots + x_j}{1^{-1/p} + \cdots + j^{-1/p}} \le \Phi_{p,\mathrm{ess}}^{+}(x).$$

To prove the reverse inequality, note that for each $m \in \mathbf{N}$, there is a $k(m) \in \mathbf{N}$ such that

$$\Phi_p^{+}(\{x_{m+1}, x_{m+2}, \dots, x_{m+k}, \dots\}) \le \frac{x_{m+1} + \cdots + x_{m+k(m)}}{1^{-1/p} + \cdots + \{k(m)\}^{-1/p}} + \frac{1}{m}. \tag{2.2}$$

If there is a sequence $m_1 < m_2 < \cdots < m_i < \cdots$ in $\mathbf{N}$ such that $k(m_i) \to \infty$ as $i \to \infty$, then from Proposition 2.2 and (2.2) we obtain

$$\Phi_{p,\mathrm{ess}}^{+}(x) \le \limsup_{i\to\infty} \frac{x_1 + \cdots + x_{k(m_i)}}{1^{-1/p} + \cdots + \{k(m_i)\}^{-1/p}} \le \limsup_{j\to\infty} \frac{x_1 + \cdots + x_j}{1^{-1/p} + \cdots + j^{-1/p}}.$$

The only other possibility is that there is an $N \in \mathbf{N}$ such that $k(m) \le N$ for every $m \in \mathbf{N}$. Obviously, the membership $x \in d_p^+$ implies $\lim_{j\to\infty} x_j = 0$. Thus in the case $k(m) \le N$ for every $m \in \mathbf{N}$, from Proposition 2.2 and (2.2) we obtain

$$\Phi_{p,\text{ess}}^+(x) = \lim_{m\to\infty} \Phi_p^+(\{x_{m+1}, x_{m+2}, \dots, x_{m+k}, \dots\}) = 0 \le \limsup_{j\to\infty} \frac{x_1 + \cdots + x_j}{1^{-1/p} + \cdots + j^{-1/p}}.$$

This completes the proof. □

Proposition 2.4. *Let a $\xi = \{\xi_j\} \in c_p^+$ be given and denote*

$$N_i = \text{card}\{j \in \mathbf{N} : 2^{-i/p} < |\xi_j| \le 2^{-(i-1)/p}\}$$

for every $i \in \mathbf{N}$. If

$$\lim_{i\to\infty} 2^{-i} N_i = 0, \tag{2.3}$$

then $\xi \in c_p^+(0)$.

Proof. By (2.3), for every $k \in \mathbf{N}$, there is a natural number $i(k) > k + 3$ such that

$$N_i \le 2^{i-k} \quad \text{for every} \quad i \ge i(k). \tag{2.4}$$

For every $i \ge i(k)$, we also have

$$\text{card}\{j \in \mathbf{N} : 2^{i-k} \le j < 2^{i+1-k}\} = 2^{i+1-k} - 1 - 2^{i-k} + 1 = 2^{i-k}.$$

That is,

$$\text{card}\{j \in \mathbf{N} : 2^{-(i-1)/p} < 2^{-(k-2)/p} j^{-1/p} \le 2^{-(i-2)/p}\} = 2^{i-k}$$

when $i \ge i(k)$. Combining this with (2.4), we see that

$$\Phi_{p,\text{ess}}^+(\xi) \le \Phi_p^+(\{2^{-(k-2)/p} j^{-1/p}\}_{j\in\mathbf{N}}) = 2^{-(k-2)/p}.$$

Since this holds for every $k \in \mathbf{N}$, we conclude that $\Phi_{p,\text{ess}}^+(\xi) = 0$, i.e., $\xi \in c_p^+(0)$. □

Proposition 2.5. *For each $x \in d_p^+$, there is a decomposition $x = y + z$ such that $y \in d_p^+$ with*

$$\Phi_p^+(y) = \Phi_{p,\text{ess}}^+(x)$$

and $z = \{z_j\} \in c_p^+(0)$, where $z_j \ge 0$ for every $j \in \mathbf{N}$.

Proof. Obviously, it suffices to consider $x \in d_p^+$ with $\Phi_{p,\text{ess}}^+(x) = 1$. Write $x = \{x_j\}$. By the definition of d_p^+, we have $x_j \ge 0$ and $x_j \ge x_{j+1}$ for every $j \in \mathbf{N}$.

We define the desired sequences $y = \{y_j\}$ and $z = \{z_j\}$ inductively, starting with $j = 1$. If $x_1 \le 1$, we define $y_1 = x_1$ and $z_1 = x_1 - y_1 = 0$. If $x_1 > 1$, we define $y_1 = 1$ and $z_1 = x_1 - y_1 = x_1 - 1 > 0$.

Let $\nu \geq 1$ and suppose that we have defined $0 \leq y_j \leq x_j$ and $z_j = x_j - y_j$ for every $1 \leq j \leq \nu$ such that the following hold true: for every $1 \leq j \leq \nu$ we have

$$\frac{y_1 + \cdots + y_j}{1^{-1/p} + \cdots + j^{-1/p}} \leq 1,$$

and for each $j \in \{1, \dots, \nu\}$ with the property $y_j < x_j$ we have

$$\frac{y_1 + \cdots + y_j}{1^{-1/p} + \cdots + j^{-1/p}} = 1.$$

Then we define $y_{\nu+1}$ and $z_{\nu+1}$ as follows. Suppose that

$$\frac{y_1 + \cdots + y_\nu + x_{\nu+1}}{1^{-1/p} + \cdots + \nu^{-1/p} + (\nu+1)^{-1/p}} \leq 1.$$

In this case, we define $y_{\nu+1} = x_{\nu+1}$ and $z_{\nu+1} = x_{\nu+1} - y_{\nu+1} = 0$. Suppose that

$$\frac{y_1 + \cdots + y_\nu + x_{\nu+1}}{1^{-1/p} + \cdots + \nu^{-1/p} + (\nu+1)^{-1/p}} > 1.$$

Since we know that

$$\frac{y_1 + \cdots + y_\nu}{1^{-1/p} + \cdots + \nu^{-1/p}} \leq 1,$$

these two inequalities imply that there is a $y_{\nu+1} \in (0, x_{\nu+1})$ such that

$$\frac{y_1 + \cdots + y_\nu + y_{\nu+1}}{1^{-1/p} + \cdots + \nu^{-1/p} + (\nu+1)^{-1/p}} = 1.$$

This defines $y_{\nu+1}$. We then define $z_{\nu+1} = x_{\nu+1} - y_{\nu+1}$, which is greater than 0 in this case. Thus we have inductively defined the sequences $y = \{y_j\}$ and $z = \{z_j\}$ with the properties that $y_j \geq 0$, $z_j \geq 0$ and $y_j + z_j = x_j$ for every $j \in \mathbf{N}$. That is, $x = y + z$. The construction above ensures

$$\frac{y_1 + \cdots + y_j}{1^{-1/p} + \cdots + j^{-1/p}} \leq 1 \tag{2.5}$$

for every $j \in \mathbf{N}$. Moreover, the construction ensures that the equality

$$\frac{y_1 + \cdots + y_j}{1^{-1/p} + \cdots + j^{-1/p}} = 1 \tag{2.6}$$

holds for each $j \in \mathbf{N}$ with the property $y_j < x_j$.

Next we show that $y = \{y_j\}$ is a non-increasing sequence, i.e., $y_j \geq y_{j+1}$ for every $j \in \mathbf{N}$. First, consider the case $j = 1$. If $x_1 \leq 1$, then by definition $y_1 = x_1 \geq x_2 \geq y_2$, since $x_1 \geq x_2$. If $x_1 > 1$, then $y_1 = 1$ by definition, and (2.5) gives us $1 + y_2 \leq 1^{-1/p} + 2^{-1/p}$. Thus in the case $x_1 > 1$ we have $y_2 \leq 2^{-1/p} < 1 = y_1$.

Now consider any $j \ge 2$. If $y_j = x_j$, then we again have $y_j = x_j \ge x_{j+1} \ge y_{j+1}$, since x is a non-increasing sequence. If $y_j < x_j$, then (2.5) and (2.6) imply

$$\begin{aligned} y_1 + \cdots + y_{j-1} &\le 1^{-1/p} + \cdots + (j-1)^{-1/p}, \\ y_1 + \cdots + y_{j-1} + y_j &= 1^{-1/p} + \cdots + (j-1)^{-1/p} + j^{-1/p}, \\ y_1 + \cdots + y_{j-1} + y_j + y_{j+1} &\le 1^{-1/p} + \cdots + (j-1)^{-1/p} + j^{-1/p} + (j+1)^{-1/p}, \end{aligned}$$

from which we deduce

$$y_j \ge j^{-1/p} > (j+1)^{-1/p} \ge y_{j+1}.$$

Thus $y = \{y_j\}$ is indeed a non-increasing sequence. Combining this fact with (2.5), we conclude that $y \in d_p^+$ with $\Phi_p^+(y) \le 1$. What remains is to show that $z \in c_p^+(0)$.

To prove that $z \in c_p^+(0)$, we first consider the case where $1 < p < \infty$. Define

$$N_i = \mathrm{card}\{j \in \mathbf{N} : z_j \ge 2^{-i/p}\}$$

for each $i \in \mathbf{N}$. By Proposition 2.4, to prove that $z \in c_p^+(0)$, it suffices to show that

$$\lim_{i\to\infty} 2^{-i} N_i = 0. \tag{2.7}$$

Suppose that this failed. Then there would be a $\delta > 0$ and an increasing sequence

$$i_1 < i_2 < \cdots < i_\nu < \cdots$$

of natural numbers such that

$$2^{-i_\nu} N_{i_\nu} \ge \delta \quad \text{for every} \quad \nu \in \mathbf{N}. \tag{2.8}$$

We will show that this leads to a contradiction.

Write $a = \Phi_p^+(x)$. Let ν and j be any pair of natural numbers such that $z_j \ge 2^{-i_\nu/p}$. Since $z_j = x_j - y_j$ and $y_j \ge 0$, we have $x_j \ge 2^{-i_\nu/p}$. On the other hand, since x is a non-increasing sequence, we have

$$\frac{j x_j}{1^{-1/p} + \cdots + j^{-1/p}} \le \frac{x_1 + \cdots + x_j}{1^{-1/p} + \cdots + j^{-1/p}} \le a.$$

Writing $C_p = p/(p-1)$, the above facts lead to the inequality

$$j 2^{-i_\nu/p} \le j x_j \le C_p a j^{(p-1)/p}.$$

That is, if $z_j \ge 2^{-i_\nu/p}$, then $j \le (C_p a)^p 2^{i_\nu}$. For each $\nu \in \mathbf{N}$, let $j_\nu = \max\{j \in \mathbf{N} : z_j \ge 2^{-i_\nu/p}\}$. By (2.6), the fact $z_{j_\nu} > 0$ forces

$$\frac{y_1 + \cdots + y_{j_\nu}}{1^{-1/p} + \cdots + j_\nu^{-1/p}} = 1.$$

Thus

$$\begin{aligned}\frac{x_1+\cdots+x_{j_\nu}}{1^{-1/p}+\cdots+j_\nu^{-1/p}} &= \frac{y_1+\cdots+y_{j_\nu}}{1^{-1/p}+\cdots+j_\nu^{-1/p}} + \frac{z_1+\cdots+z_{j_\nu}}{1^{-1/p}+\cdots+j_\nu^{-1/p}} \\ &\geq 1+\frac{N_{i_\nu}2^{-i_\nu/p}}{C_p j_\nu^{(p-1)/p}} \geq 1+\frac{N_{i_\nu}2^{-i_\nu/p}}{C_p\{(C_pa)^p2^{i_\nu}\}^{(p-1)/p}} \\ &= 1+\frac{N_{i_\nu}}{C_p^pa^{p-1}2^{i_\nu}} \geq 1+\frac{\delta}{C_p^pa^{p-1}},\end{aligned}$$

where the last $\geq$ follows from (2.8). Since the above inequality supposedly holds for every $\nu \in \mathbf{N}$, by Proposition 2.3, it contradicts the condition $\Phi^+_{p,\mathrm{ess}}(x) = 1$. This proves (2.7). Applying Proposition 2.4, in the case $1 < p < \infty$ we have $z \in c_p^+(0)$.

Now consider the case $p = 1$, which is much more complicated. To prove $z \in c_1^+(0)$ in this case, pick an $\epsilon > 0$. Define the sequences $u = \{u_j\}$ and $v = \{v_j\}$ by the formulas

$$u_j = \begin{cases} z_j & \text{if} \quad z_j > \epsilon x_j \\ \\ 0 & \text{if} \quad z_j \leq \epsilon x_j \end{cases} \quad \text{and} \quad v_j = \begin{cases} 0 & \text{if} \quad z_j > \epsilon x_j \\ \\ z_j & \text{if} \quad z_j \leq \epsilon x_j \end{cases},$$

$j \in \mathbf{N}$. We have $z = u + v$ by design. Then note that $\Phi_1^+(v) \leq \epsilon\Phi_1^+(x)$. Since $\epsilon > 0$ is arbitrary, it suffices to show that $u \in c_1^+(0)$.

To prove that $u \in c_1^+(0)$, consider the set $N = \{j \in \mathbf{N} : u_j > 0\}$. If $\mathrm{card}(N) < \infty$, then we certainly have the membership $u \in c_1^+(0)$. Suppose that $\mathrm{card}(N) = \infty$. Then we enumerate the elements in N as a sequence

$$j(1) < j(2) < \cdots < j(k) < \cdots .$$

Keep in mind that $z_{j(k)} > \epsilon x_{j(k)}$ for every $k \in \mathbf{N}$.

Claim 1. If $k_1 < k_2 < \cdots < k_\nu < \cdots$ are natural numbers such that

$$\log k_\nu \geq \frac{1}{2}\log j(k_\nu) \tag{2.9}$$

for every $\nu \in \mathbf{N}$, then

$$\lim_{\nu\to\infty} \frac{x_{j(1)}+x_{j(2)}+\cdots+x_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+k_\nu^{-1}} = 0. \tag{2.10}$$

Indeed for each $\nu \in \mathbf{N}$, since $z_{j(k_\nu)} > 0$, i.e., $y_{j(k_\nu)} < x_{j(k_\nu)}$, we have

$$\begin{aligned}&\frac{x_1+x_2+\cdots+x_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}} \\ &= \frac{y_1+y_2+\cdots+y_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}} + \frac{z_1+z_2+\cdots+z_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}} \\ &\geq 1+\frac{z_{j(1)}+z_{j(2)}+\cdots+z_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}} \geq 1+\epsilon\frac{x_{j(1)}+x_{j(2)}+\cdots+x_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}} \\ &= 1+\epsilon\cdot\frac{1^{-1}+2^{-1}+\cdots+k_\nu^{-1}}{1^{-1}+2^{-1}+\cdots+\{j(k_\nu)\}^{-1}}\cdot\frac{x_{j(1)}+x_{j(2)}+\cdots+x_{j(k_\nu)}}{1^{-1}+2^{-1}+\cdots+k_\nu^{-1}}.\end{aligned}$$

Combining this with (2.9), we find that for $\nu \geq 3$,

$$\frac{x_1 + x_2 + \cdots + x_{j(k_\nu)}}{1^{-1} + 2^{-1} + \cdots + \{j(k_\nu)\}^{-1}} \geq 1 + \epsilon \cdot \frac{\log k_\nu}{2 \log j(k_\nu)} \cdot \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k_\nu)}}{1^{-1} + 2^{-1} + \cdots + k_\nu^{-1}}$$

$$\text{(2.11)} \qquad \geq 1 + \frac{\epsilon}{4} \cdot \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k_\nu)}}{1^{-1} + 2^{-1} + \cdots + k_\nu^{-1}}.$$

It follows from the condition $\Phi^+_{1,\mathrm{ess}}(x) = 1$ and Proposition 2.3 that

$$\text{(2.12)} \qquad \limsup_{\nu \to \infty} \frac{x_1 + x_2 + \cdots + x_{j(k_\nu)}}{1^{-1} + 2^{-1} + \cdots + \{j(k_\nu)\}^{-1}} \leq 1.$$

Obviously, (2.10) follows from (2.11) and (2.12). This proves Claim 1.

Claim 2. Let $E_1, \ldots, E_s, \ldots$ be finite subsets of $\mathbf{N}$ such that

$$\text{(2.13)} \qquad \lim_{s \to \infty} \mathrm{card}(E_s) = \infty.$$

Suppose that

$$\text{(2.14)} \qquad \log k < \frac{1}{2} \log j(k) \quad \text{for every} \quad k \in \bigcup_{s=1}^{\infty} E_s.$$

Then

$$\text{(2.15)} \qquad \lim_{s \to \infty} \sum_{k \in E_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(E_s)} \frac{1}{i} = 0.$$

To prove this, pick an $m \in \mathbf{N}$ and define $F_s = \{k \in E_s : mk \leq j(k)\}$ for each $s \in \mathbf{N}$. Note that (2.14) implies $j(k) > k^2$ for every $k \in \cup_{s=1}^{\infty} E_s$. Therefore $\mathrm{card}(E_s \backslash F_s) \leq m$ for every s. Thus it follows from (2.13) that

$$\lim_{s \to \infty} \left\{ \left(\sum_{k \in E_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(E_s)} \frac{1}{i} \right) - \left(\sum_{k \in F_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(F_s)} \frac{1}{i} \right) \right\} = 0.$$

Since $m \in \mathbf{N}$ is arbitrary, (2.15) will follow if we can show that

$$\text{(2.16)} \qquad \limsup_{s \to \infty} \sum_{k \in F_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(F_s)} \frac{1}{i} \leq \frac{\Phi^+_1(x)}{m}.$$

For each $s \in \mathbf{N}$, since $j(k) \geq mk$ for every $k \in F_s$ and since the sequence x is non-increasing, we have

$$\sum_{k \in F_s} x_{j(k)} \leq \sum_{k \in F_s} x_{mk} \leq \sum_{i=1}^{\mathrm{card}(F_s)} x_{mi} \leq \frac{1}{m} \sum_{i=1}^{m\mathrm{card}(F_s)} x_i \leq \frac{1}{m} \left(\sum_{i=1}^{m\mathrm{card}(F_s)} \frac{1}{i} \right) \Phi^+_1(x).$$

That is,

$$\sum_{k\in F_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(F_s)} \frac{1}{i} \le \frac{1}{m}\left\{ \sum_{i=1}^{m\mathrm{card}(F_s)} \frac{1}{i} \Big/ \sum_{i=1}^{\mathrm{card}(F_s)} \frac{1}{i} \right\} \Phi_1^+(x) \tag{2.17}$$

for every $s \in \mathbf{N}$. Since $\mathrm{card}(F_s) \to \infty$ as $s \to \infty$, (2.17) implies (2.16). This completes the proof of Claim 2.

Having proved Claims 1 and 2, we are now ready to prove the membership $u \in c_1^+(0)$. Recall that for every j for which $u_j \neq 0$, we have $u_j = z_j \le x_j$, and that the elements in $N = \{j \in \mathbf{N} : u_j > 0\}$ are listed as

$$j(1) < j(2) < \cdots < j(k) < \cdots .$$

Since x is non-increasing, by Proposition 2.3, the membership $u \in c_1^+(0)$ will follow if we can show that

$$\lim_{k\to\infty} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k)}}{1^{-1} + 2^{-1} + \cdots + k^{-1}} = 0.$$

Suppose that this limit did not hold. Then there would be a sequence

$$n_1 < n_2 < \cdots < n_s < \cdots$$

of natural numbers such that

$$\lim_{s\to\infty} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(n_s)}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}} = b \tag{2.18}$$

for some $b > 0$. Again, we will show that this leads to a contradiction.

For each $s \in \mathbf{N}$, define $A_s = \{k \in \{1, 2, \ldots, n_s\} : \log k \ge (1/2)\log j(k)\}$. If s is such that $A_s = \emptyset$, we define

$$G_s = \{1, 2, \ldots, n_s\}.$$

If s is such that $A_s \neq \emptyset$, we let k_s be the largest element in A_s and we define

$$G_s = \{1, 2, \ldots, n_s\} \backslash \{1, 2, \ldots, k_s\}.$$

Note that for each s, the definition of G_s guarantees that $\log k < (1/2)\log j(k)$ if $k \in G_s$. Denote $\Sigma = \{s \in \mathbf{N} : A_s \neq \emptyset\}$. For each $s \in \Sigma$, define

$$\alpha_s = \frac{1^{-1} + 2^{-1} + \cdots + k_s^{-1}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}} \quad \text{and} \quad \beta_s = \frac{\sum_{i=1}^{\mathrm{card}(G_s)} i^{-1}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}},$$

where β_s is understood to be 0 in the case $G_s = \emptyset$. For $s \in \Sigma$ with $G_s \neq \emptyset$, we have

$$\begin{aligned} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(n_s)}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}} &= \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k_s)} + \sum_{k\in G_s} x_{j(k)}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}} \\ &= \alpha_s \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k_s)}}{1^{-1} + 2^{-1} + \cdots + k_s^{-1}} + \beta_s \frac{\sum_{k\in G_s} x_{j(k)}}{\sum_{i=1}^{\mathrm{card}(G_s)} i^{-1}}. \end{aligned} \tag{2.19}$$

Suppose that $\Sigma \neq \emptyset$. Then $\Sigma = \{s \in \mathbf{N} : s \geq \ell\}$ for some $\ell \in \mathbf{N}$. Thus there is a sequence

$$s_1 < s_2 < \cdots < s_r < \cdots$$

contained in Σ such that both limits

$$\lim_{r\to\infty} \alpha_{s_r} \quad \text{and} \quad \lim_{r\to\infty} \beta_{s_r}$$

exist. By definition, $\log k_s \geq (1/2) \log j(k_s)$. By Claim 1, we have

$$\lim_{r\to\infty} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(k_{s_r})}}{1^{-1} + 2^{-1} + \cdots + k_{s_r}^{-1}} = 0 \quad \text{in the event} \quad \lim_{r\to\infty} \alpha_{s_r} \neq 0.$$

Recall that if $k \in G_s$, then $\log k < (1/2) \log j(k)$. By Claim 2, we have

$$\lim_{r\to\infty} \frac{\sum_{k\in G_{s_r}} x_{j(k)}}{\sum_{i=1}^{\mathrm{card}(G_{s_r})} i^{-1}} = 0 \quad \text{in the event} \quad \lim_{r\to\infty} \beta_{s_r} \neq 0.$$

Combining these facts with (2.19), we find that

$$\lim_{r\to\infty} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(n_{s_r})}}{1^{-1} + 2^{-1} + \cdots + n_{s_r}^{-1}} = 0,$$

which contradicts (2.18) in the case $\Sigma \neq \emptyset$. Suppose that $\Sigma = \emptyset$. Then by definition we have $G_s = \{1, 2, \ldots, n_s\}$ for every $s \in \mathbf{N}$. Thus we can apply Claim 2 to conclude that

$$\lim_{s\to\infty} \frac{x_{j(1)} + x_{j(2)} + \cdots + x_{j(n_s)}}{1^{-1} + 2^{-1} + \cdots + n_s^{-1}} = \lim_{s\to\infty} \sum_{k\in G_s} x_{j(k)} \Big/ \sum_{i=1}^{\mathrm{card}(G_s)} \frac{1}{i} = 0.$$

Thus (2.18) is also contradicted in the case $\Sigma = \emptyset$. This completes the proof of the proposition. □

Having only dealt with sequences so far, we now apply the above results to operators, which are the main interest of the paper. Let $\mathcal{H}$ be a Hilbert space. For any $u, v \in \mathcal{H}$, the notation $u \otimes v$ denotes the operator on $\mathcal{H}$ defined by the formula

$$u \otimes v f = \langle f, v \rangle u, \quad f \in \mathcal{H}.$$

It is well known that if A is a compact operator on an infinite-dimensional Hilbert space $\mathcal{H}$, then it admits the representation

$$A = \sum_{j=1}^{\infty} s_j(A) u_j \otimes v_j,$$

where $\{u_j : j \in \mathbf{N}\}$ and $\{v_j : j \in \mathbf{N}\}$ are orthonormal sets in $\mathcal{H}$. See, e.g., [5,11].

We remind the reader of our notation (1.2). For each each $A \in \mathcal{C}_p^+$, we define

$$\|A\|_{p,\mathrm{ess}}^+ = \inf\{\|A - K\|_p^+ : K \in \mathcal{C}_p^{+(0)}\}.$$

We think of $\|A\|_{p,\mathrm{ess}}^+$ as the *essential* $\|\cdot\|_p^+$-norm of A, hence the notation.

Proposition 2.6. *For every operator $A \in \mathcal{C}_p^+$, we have*

$$\|A\|_{p,\mathrm{ess}}^+ = \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\}) = \limsup_{j\to\infty} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}.$$

Proof. For an $A \in \mathcal{C}_p^+$, there are orthonormal sets $\{u_j : j \in \mathbf{N}\}$ and $\{v_j : j \in \mathbf{N}\}$ such that

$$A = \sum_{j=1}^{\infty} s_j(A) u_j \otimes v_j.$$

Therefore it is obvious that $\|A\|_{p,\mathrm{ess}}^+ \le \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\})$. To prove the reverse inequality, for every $k \in \mathbf{N}$ we define the orthogonal projection

$$E_k = \sum_{j=k}^{\infty} u_j \otimes u_j.$$

If F is a finite-rank operator, then $\|E_k F\|_p^+ \to 0$ as $k \to \infty$. Therefore

$$\begin{aligned}\|A - F\|_p^+ &\ge \limsup_{k\to\infty} \|E_k(A - F)\|_p^+ = \limsup_{k\to\infty} \|E_k A\|_p^+ \\ &= \lim_{k\to\infty} \Phi_p^+(\{s_k(A), s_{k+1}(A), \dots, s_{k+j}(A), \dots\}) = \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\}),\end{aligned}$$

where the last $=$ follows from Proposition 2.2. Since this inequality holds for every finite-rank operator F, we conclude that $\|A\|_{p,\mathrm{ess}}^+ \ge \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\})$. Recalling Proposition 2.3, the proof is complete. □

With the above preparation, we now prove Theorem 1.2 in a more explicit form:

Theorem 2.7. *For each $A \in \mathcal{C}_p^+$, there is a $K \in \mathcal{C}_p^{+(0)}$ such that*

$$\|A - K\|_p^+ = \|A\|_{p,\mathrm{ess}}^+ = \limsup_{j\to\infty} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}.$$

Proof. Given an $A \in \mathcal{C}_p^+$, we again represent it in the form

$$A = \sum_{j=1}^{\infty} s_j(A) u_j \otimes v_j,$$

where $\{u_j : j \in \mathbf{N}\}$ and $\{v_j : j \in \mathbf{N}\}$ are orthonormal sets. Applying Proposition 2.5 to the sequence $\{x_j\} = \{s_j(A)\}$, we obtain $y = \{y_j\} \in d_p^+$ and $z = \{z_j\} \in c_p^+(0)$ such that

$$s_j(A) = y_j + z_j \tag{2.20}$$

for every $j \in \mathbf{N}$ and $\Phi_p^+(y) = \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\})$. Define

$$K = \sum_{j=1}^{\infty} z_j u_j \otimes v_j.$$

The condition $z \in c_p^+(0)$ obviously implies $K \in \mathcal{C}_p^{+(0)}$. From (2.20) we obtain

$$A - K = \sum_{j=1}^{\infty} y_j u_j \otimes v_j.$$

Therefore

$$\|A - K\|_p^+ = \Phi_p^+(y) = \Phi_{p,\mathrm{ess}}^+(\{s_j(A)\}).$$

Now an application of Proposition 2.6 completes the proof. □

3. A contrast to the classic case

As we mentioned in the Introduction, the result that $\mathcal{K}(\mathcal{H})$ is proximinal in $\mathcal{B}(\mathcal{H})$ has refinements within specific classes of operators. One such class of operators are the Hankel operators $H_f : H^2 \to L^2$, where H^2 is the Hardy space on the unit circle $\mathbf{T} \subset \mathbf{C}$. Specifically, [1, Theorem 3] tells us that for $f \in L^\infty$, the best compact approximation to the Hankel operator $H_f : H^2 \to L^2$ can be realized in the form of a Hankel operator H_g.

In other words, [1, Theorem 3] says that H_f has a best compact approximation that is of the same kind, a Hankel operator. Using the method in [1], this result of best compact approximation can be easily generalized to Hankel operators on the Hardy space $H^2(S)$ on the unit sphere $S \subset \mathbf{C}^n$.

The fact that each $\mathcal{C}_p^{+(0)}$ is proximinal in $\mathcal{C}_p^+$ raises an obvious question: Suppose that we have an operator A in a natural class $\mathcal{N}$, and suppose we know that $A \in \mathcal{C}_p^+$, can we find a best $\mathcal{C}_p^{+(0)}$-approximation to A in the same class $\mathcal{N}$? In particular, what if $\mathcal{N}$ is the class of Hankel operators on $H^2(S)$?

In this section we show that the answer to the last question is negative. This negative answer provides a sharp contrast to the classic result [1, Theorem 3].

For the rest of the paper we assume $n \geq 2$. Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Write $d\sigma$ for the standard spherical measure on S with the normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the norm closure of the analytic polynomials $\mathbf{C}[z_1, \ldots, z_n]$ in $L^2(S, d\sigma)$ [15]. Let $P : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. Then the Hankel operator $H_f : H^2(S) \to L^2(S, d\sigma)$ is defined by the formula

$$H_f h = (1 - P)(fh), \quad h \in H^2(S).$$

For these Hankel operators, let us recall the following results:

Proposition 3.1. [9, Proposition 7.2] *If f is a Lipschitz function on S, then $H_f \in \mathcal{C}_{2n}^+$.*

Proposition 3.2. *When the complex dimension n is at least 2, for any $f \in L^2(S, d\sigma)$, if H_f is bounded and if $H_f \neq 0$, then $H_f \notin \mathcal{C}_{2n}^{+(0)}$.*

Proof. We apply [9, Theorem 1.6], which tells us that for $f \in L^2(S, d\sigma)$, if H_f is bounded and if $H_f \neq 0$, then there is an $\epsilon > 0$ such that

$$s_1(H_f) + \cdots + s_k(H_f) \geq \epsilon k^{(2n-1)/2n}$$

for every $k \in \mathbf{N}$. Thus it follows from Proposition 2.6 that $\|H_f\|_{2n,\mathrm{ess}}^+ > 0$, if $\|H_f\|_{2n}^+$ is finite to begin with. In any case, we have $H_f \notin \mathcal{C}_{2n}^{+(0)}$. □

As usual, we write $z_1, \ldots, z_n$ for the complex coordinate functions. Here is the main technical result of the section:

Theorem 3.3. *When the complex dimension n equals 2, we have $\|H_{\bar{z}_1}\|_4^+ > \|H_{\bar{z}_1}\|_{4,\mathrm{ess}}^+$.*

This leads to the negative answer promised above:

Example 3.4. Let the complex dimension n be equal to 2. By Theorem 2.7, $H_{\bar{z}_1}$ has a best approximation in $\mathcal{C}_4^{+(0)}$. On the other hand, it follows from the inequality $\|H_{\bar{z}_1}\|_4^+ > \|H_{\bar{z}_1}\|_{4,\mathrm{ess}}^+$ that if $K \in \mathcal{C}_4^{+(0)}$ is a best approximation of $H_{\bar{z}_1}$, then $K \neq 0$. The membership $K \in \mathcal{C}_4^{+(0)}$ implies that K is not a Hankel operator, for Proposition 3.2 tells us that $\mathcal{C}_4^{+(0)}$ does not contain any nonzero Hankel operators on $H^2(S)$ in the case $S \subset \mathbf{C}^2$. Thus for the class of Hankel operators on the Hardy space $H^2(S)$, $S \subset \mathbf{C}^2$, the analogue of Theorem 1.3 does not hold for the pair $\mathcal{C}_4^+$ and $\mathcal{C}_4^{+(0)}$, even though $\mathcal{C}_4^{+(0)}$ is proximinal in $\mathcal{C}_4^+$.

Having presented the principal conclusion of the section, we now turn to the proof of Theorem 3.3, which requires some calculation. We begin with the generality $n \geq 2$, and then specialize to the complex dimension $n = 2$.

We need to make one use of Toeplitz operators, whose definition we now recall. Given an $f \in L^\infty(S, d\sigma)$, the Toeplitz operator T_f is defined by the formula

$$T_f h = P(fh), \quad h \in H^2(S).$$

We need the following relation between Hankel operators and Toeplitz operators: We have

$$H_f^* H_f = T_{|f|^2} - T_{\bar{f}} T_f \tag{3.1}$$

for every $f \in L^\infty(S, d\sigma)$.

We follow the usual multi-index convention [15, page 3]. Then the standard orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ for $H^2(S)$ is given by the formula

$$e_\alpha(z) = \left\{ \frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \right\}^{1/2} z^\alpha, \quad \alpha \in \mathbf{Z}_+^n.$$

Consider the symbol function $\bar{z}_1$. Straightforward calculation using (3.1) shows that

$$\langle H^*_{\bar{z}_1} H_{\bar{z}_1} e_\alpha, e_\beta \rangle = 0 \quad \text{if} \quad \alpha \neq \beta$$

and that

$$\langle H^*_{\bar{z}_1} H_{\bar{z}_1} e_\alpha, e_\alpha \rangle = \frac{n-1+|\alpha|-\alpha_1}{(n-1+|\alpha|)(n+|\alpha|)} \quad \text{for every} \quad \alpha \in \mathbf{Z}^n_+, \tag{3.2}$$

where α_1 denotes the first component of α. Thus $H^*_{\bar{z}_1} H_{\bar{z}_1}$ is a diagonal operator with respect to the standard orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}^n_+\}$, and the above are the s-numbers of $H^*_{\bar{z}_1} H_{\bar{z}_1}$. Consequently, the s-numbers of $H_{\bar{z}_1}$ are a descending arrangement of

$$\left\{ \frac{n-1+|\alpha|-\alpha_1}{(n-1+|\alpha|)(n+|\alpha|)} \right\}^{1/2}, \quad \alpha \in \mathbf{Z}^n_+.$$

Lemma 3.5. *In the case where the complex dimension n equals 2, we have*

$$\|H_{\bar{z}_1}\|^+_{4,\text{ess}} = 6^{-1/4}.$$

Proof. For $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}^2_+$, note that $|\alpha| - \alpha_1 = \alpha_2$. Thus from (3.2) we obtain

$$(H^*_{\bar{z}_1} H_{\bar{z}_1})^{1/2} = \sum_{\alpha \in \mathbf{Z}^2_+} \left\{ \frac{1+\alpha_2}{(1+|\alpha|)(2+|\alpha|)} \right\}^{1/2} e_\alpha \otimes e_\alpha.$$

It is also easy to see that $(H^*_{\bar{z}_1} H_{\bar{z}_1})^{1/2} = Y + Z$, where $Z \in \mathcal{C}^{+(0)}_4$ and

$$Y = \sum_{\alpha \in \mathbf{Z}^2_+ \setminus \{0\}} \frac{\sqrt{\alpha_2}}{|\alpha|} e_\alpha \otimes e_\alpha.$$

Hence $\|H_{\bar{z}_1}\|^+_{4,\text{ess}} = \|(H^*_{\bar{z}_1} H_{\bar{z}_1})^{1/2}\|^+_{4,\text{ess}} = \|Y\|^+_{4,\text{ess}}$, and we need to figure out the latter.

To find $\|Y\|^+_{4,\text{ess}}$, consider $Q = \{(x, y) \in \mathbf{R}^2 : x \geq 0 \text{ and } y \geq 0\}$, the first quadrant in the xy-plane. For each $a > 0$, define

$$E_a = \{(x, y) \in Q : ay \geq (x + y)^2\}.$$

Solving the inequality $ay \geq (x + y)^2$ in Q, we find that

$$E_a = \{(x, y) \in Q : 0 \leq y \leq a \;\text{ and }\; 0 \leq x \leq \sqrt{ay} - y\}.$$

Let m_2 denote the natural 2-dimensional Lebesgue measure on Q. Then

$$m_2(E_a) = \int_0^a (\sqrt{ay} - y)dy = \frac{a^2}{6}.$$

For each $r > 1$ we define

$$N(r) = \mathrm{card}\{\alpha \in \mathbf{Z}_+^2 \backslash \{0\} : \sqrt{\alpha_2}/|\alpha| > 1/r\}.$$

To each $\alpha \in \mathbf{Z}_+^2$ we associate the square $\alpha + I^2$, where $I^2 = [0,1] \times [0,1]$. From this association we see that

$$N(r) = m_2(E_{r^2}) + o(r^4) = \frac{1}{6}r^4 + o(r^4). \tag{3.3}$$

We have

$$\begin{aligned}\iint_{E_{r^2}} \frac{\sqrt{y}}{x+y} dxdy &= \int_0^{r^2} \sqrt{y}\Big(\int_0^{r\sqrt{y}-y} \frac{dx}{x+y}\Big) dy = \int_0^{r^2} \sqrt{y} \log\left(\frac{r\sqrt{y}}{y}\right) dy \\ &= r^3 \int_0^1 \sqrt{u} \log \frac{1}{\sqrt{u}} du = 2r^3 \int_0^1 t^2 \log \frac{1}{t} dt = \frac{2}{9}r^3 = \frac{4}{3} \cdot \frac{1}{6} r^3.\end{aligned}$$

Denote $A_r = \{\alpha \in \mathbf{Z}_+^2 \backslash \{0\} : \sqrt{\alpha_2}/|\alpha| > 1/r\}$. Then

$$\begin{aligned}\sum_{j=1}^{N(r)} s_j(Y) &= \sum_{\alpha \in A_r} \frac{\sqrt{\alpha_2}}{|\alpha|} \\ &= \sum_{\alpha \in A_r} \iint_{\alpha + I^2} \frac{\sqrt{y}}{x+y} dxdy + \sum_{\alpha \in A_r} \iint_{\alpha+I^2} \left(\frac{\sqrt{\alpha_2}}{|\alpha|} - \frac{\sqrt{y}}{x+y}\right) dxdy \\ &= \iint_{E_{r^2}} \frac{\sqrt{y}}{x+y} dxdy + o(r^3) = \frac{4}{3} \cdot \frac{1}{6} r^3 + o(r^3).\end{aligned}$$

On the other hand, from (3.3) we obtain

$$\sum_{j=1}^{N(r)} j^{-1/4} = \frac{4}{3}\{N(r)\}^{3/4} + o(\{N(r)\}^{3/4}) = \frac{4}{3}\left(\frac{1}{6}\right)^{3/4} r^3 + o(r^3).$$

Combining these two identities, we find that

$$\lim_{r\to\infty} \frac{\sum_{j=1}^{N(r)} s_j(Y)}{\sum_{j=1}^{N(r)} j^{-1/4}} = \lim_{r\to\infty} \frac{\frac{4}{3} \cdot \frac{1}{6} r^3 + o(r^3)}{\frac{4}{3}\left(\frac{1}{6}\right)^{3/4} r^3 + o(r^3)} = \left(\frac{1}{6}\right)^{1/4}.$$

Thus the proof of the lemma will be complete if we can show that

$$\|Y\|_{4,\mathrm{ess}}^+ = \lim_{r\to\infty} \frac{\sum_{j=1}^{N(r)} s_j(Y)}{\sum_{j=1}^{N(r)} j^{-1/4}}. \tag{3.4}$$

To prove (3.4), first note that by Proposition 2.6, the left-hand side is greater than or equal to the right-hand side. Thus we only need to prove the reverse inequality. But for the reverse inequality, note that (3.3) gives us

$$N(\nu+1) - N(\nu) = o(\nu^4),$$

$\nu \in \mathbf{N}$. Hence

$$\sum_{j=N(\nu)+1}^{N(\nu+1)} s_j(Y) \leq \frac{1}{\nu}(N(\nu+1) - N(\nu)) = o(\nu^3).$$

For a large $k \in \mathbf{N}$, there is a $\nu(k) \in \mathbf{N}$ such that $N(\nu(k)) \leq k < N(\nu(k)+1)$. Thus

$$\begin{aligned}\frac{s_1(Y) + \cdots + s_k(Y)}{1^{-1/4} + \cdots + k^{-1/4}} &\leq \frac{\sum_{j=1}^{N(\nu(k)+1)} s_j(Y)}{\sum_{j=1}^{N(\nu(k))} j^{-1/4}} \\ &= \frac{\sum_{j=1}^{N(\nu(k))} s_j(Y)}{\sum_{j=1}^{N(\nu(k))} j^{-1/4}} + \frac{\sum_{j=N(\nu(k))+1}^{N(\nu(k)+1)} s_j(Y)}{\sum_{j=1}^{N(\nu(k))} j^{-1/4}} \\ &= \frac{\sum_{j=1}^{N(\nu(k))} s_j(Y)}{\sum_{j=1}^{N(\nu(k))} j^{-1/4}} + \frac{o(\{\nu(k)\}^3)}{(4/3)\{N(\nu(k))\}^{3/4} + o(\{N(\nu(k))\}^{3/4})}.\end{aligned}$$

Using (3.3) again, we find that

$$\limsup_{k\to\infty} \frac{s_1(Y) + \cdots + s_k(Y)}{1^{-1/4} + \cdots + k^{-1/4}} \leq \lim_{r\to\infty} \frac{\sum_{j=1}^{N(r)} s_j(Y)}{\sum_{j=1}^{N(r)} j^{-1/4}}.$$

Thus, by Proposition 2.6, the left-hand side of (3.4) is less than or equal to the right-hand side as promised. This completes the proof of the lemma. □

Proof of Theorem 3.3. Under the assumption $n = 2$, (3.2) gives us $\|H_{\bar{z}_1} 1\|^2 = 1/2$. Thus $\|H_{\bar{z}_1}\|_4^+ \geq \|H_{\bar{z}_1}\| \geq 2^{-1/2}$. On the other hand, Lemma 3.5 tells us that $\|H_{\bar{z}_1}\|_{4,\mathrm{ess}}^+ = 6^{-1/4}$. Since $2^{-1/2} > 6^{-1/4}$, it follows that $\|H_{\bar{z}_1}\|_4^+ > \|H_{\bar{z}_1}\|_{4,\mathrm{ess}}^+$. □

We choose to present Lemma 3.5 separately because its proof is more elementary than the general case. But the calculation in Lemma 3.5 can be generalized to all complex dimensions $n \geq 2$, which may be of independent interest:

Proposition 3.6. *In each complex dimension $n \geq 2$, we have*

$$\|H_{\bar{z}_1}\|_{2n,\mathrm{ess}}^+ = \left(\frac{1}{n!} \cdot \frac{n-1}{2n-1}\right)^{1/(2n)}.$$

Proof. We begin with some general volume calculation. For $j \geq 1$, let v_j denote the (real) j-dimensional volume measure. Let $k \geq 2$ and define

$$\Delta_k(t) = \{(x_1, \ldots, x_k) \in \mathbf{R}^k : x_1 \geq 0, \ldots, x_k \geq 0 \ \text{ and } \ x_1 + \cdots + x_k = t\}$$

for $t \geq 0$. Elementary calculation shows that $v_{k-1}(\Delta_k(1)) = \{(k-1)!\}^{-1}k^{1/2}$. Hence

$$(3.5) \qquad v_{k-1}(\Delta_k(t)) = \frac{\sqrt{k}}{(k-1)!}t^{k-1}$$

for all $t > 0$.

Consider the "first quadrant"

$$Q_n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 \geq 0, \ldots, x_n \geq 0\}$$

in $\mathbf{R}^n$. We write the elements in Q_n in the form (x, y), where $x \geq 0$ and $y = (y_1, \ldots, y_{n-1})$ with $y_j \geq 0$ for $1 \leq j \leq n-1$. For such a y, we denote

$$|y| = y_1 + \cdots + y_{n-1}$$

in this proof. Adapting the proof of Lemma 3.5 to general $n \geq 2$, we now define

$$E_a = \{(x, y) \in Q_n : a|y| \geq (x + |y|)^2\}$$

for $a > 0$. We claim that

$$(3.6) \qquad v_n(E_a) = \frac{a^n}{n!} \cdot \frac{n-1}{2n-1}.$$

To prove this, note that the condition $a|y| \geq (x+|y|)^2$ implies $a \geq x+|y|$ and, consequently, $a \geq x$ and $a \geq |y|$. For each $0 \leq t \leq a$, define

$$\Sigma_a(t) = \Delta_n(t) \cap E_a = \{(x, y) \in Q_n : x + |y| = t \ \text{ and } \ a|y| \geq t^2\}.$$

Obviously,

$$\Sigma_a(t) = \{(t - \rho, y) \in Q_n : t^2/a \leq \rho \leq t \ \text{ and } \ |y| = \rho\}.$$

For any $\lambda, \mu \in [t^2/a, t]$, the distance between the slices

$$\{(t - \lambda, y) \in Q_n : |y| = \lambda\} \quad \text{and} \quad \{(t - \mu, y) \in Q_n : |y| = \mu\}$$

is easily seen to be

$$\left(\{\mu - \lambda\}^2 + (n-1)\{(n-1)^{-1}\lambda - (n-1)^{-1}\mu\}^2\right)^{1/2} = \sqrt{\frac{n}{n-1}}|\lambda - \mu|.$$

Combining this fact with (3.5), when $n \geq 3$ we have

$$\begin{aligned} v_{n-1}(\Sigma_a(t)) &= \sqrt{\frac{n}{n-1}} \int_{t^2/a}^{t} v_{n-2}(\{(t-\rho, y) \in Q_n : |y| = \rho\})d\rho \\ &= \sqrt{\frac{n}{n-1}} \int_{t^2/a}^{t} \frac{\sqrt{n-1}}{(n-2)!}\rho^{n-2}d\rho = \frac{\sqrt{n}}{(n-1)!}\{t^{n-1} - (t^2/a)^{n-1}\}. \end{aligned}$$

When $n = 2$, we can omit the first two steps above and the last $=$ trivially holds. Let u be the unit vector $(n^{-1/2}, \dots, n^{-1/2})$ in $\mathbf{R}^n$. For $s, t \in [0, \infty)$, if $su \in \Delta_n(t)$, then $n^{1/2}s = t$. Since $x + |y| \le a$ for $(x, y) \in E_a$, integrating along the "u-axis" in $\mathbf{R}^n$, we have

$$
\begin{aligned}
v_n(E_a) &= \int_0^{n^{-1/2}a} v_{n-1}(\Sigma_a(n^{1/2}s))ds = \frac{1}{\sqrt{n}} \int_0^a v_{n-1}(\Sigma_a(t))dt \\
&= \frac{1}{(n-1)!} \int_0^a (t^{n-1} - a^{-(n-1)}t^{2n-2})dt = \frac{a^n}{(n-1)!} \left(\frac{1}{n} - \frac{1}{2n-1} \right).
\end{aligned}
$$

Then an obvious simplification of the right-hand side proves (3.6).

Let us again write each $\alpha \in \mathbf{Z}_+^n$ in the form $\alpha = (\alpha_1, \alpha_2)$, but keep in mind that this time we have $\alpha_2 \in \mathbf{Z}_+^{n-1}$. Accordingly, $|\alpha| - \alpha_1 = |\alpha_2|$. Thus from (3.2) we obtain

$$
(H_{\bar{z}_1}^* H_{\bar{z}_1})^{1/2} = \sum_{\alpha \in \mathbf{Z}_+^n} \left\{ \frac{n - 1 + |\alpha_2|}{(n - 1 + |\alpha|)(n + |\alpha|)} \right\}^{1/2} e_\alpha \otimes e_\alpha .
$$

Again, $(H_{\bar{z}_1}^* H_{\bar{z}_1})^{1/2} = Y + Z$, where $Z \in \mathcal{C}_{2n}^{+(0)}$ and

$$
Y = \sum_{\alpha \in \mathbf{Z}_+^n \backslash \{0\}} \frac{\sqrt{|\alpha_2|}}{|\alpha|} e_\alpha \otimes e_\alpha .
$$

Hence $\|H_{\bar{z}_1}\|_{2n,\text{ess}}^+ = \|(H_{\bar{z}_1}^* H_{\bar{z}_1})^{1/2}\|_{2n,\text{ess}}^+ = \|Y\|_{2n,\text{ess}}^+$, and we need to compute $\|Y\|_{2n,\text{ess}}^+$.

For each large $r > 1$ we define the set

$$
A_r = \{\alpha \in \mathbf{Z}_+^n \backslash \{0\} : \sqrt{|\alpha_2|}/|\alpha| > 1/r\}.
$$

To each $\alpha \in \mathbf{Z}_+^n$ we associate the cube $\alpha + I^n$, where $I^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n : 0 \le x_j \le 1$ for $j = 1, \dots, n\}$. Obviously, there is a constant $0 < C < \infty$ such that for any $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$ and any $(x, y) \in \alpha + I^n$, we have

$$
\left| \frac{\sqrt{|\alpha_2|}}{|\alpha|} - \frac{\sqrt{|y|}}{x + |y|} \right| \le \frac{C}{(1 + |\alpha_2|^{1/2})|\alpha|}. \tag{3.7}
$$

Write $N(r) = \text{card}(A_r)$. From (3.7) it is easy to deduce that $N(r) = v_n(E_{r^2}) + o(r^{2n})$. Combining this fact with (3.6), we obtain

$$
N(r) = \gamma_n r^{2n} + o(r^{2n}), \tag{3.8}
$$

where we denote

$$
\gamma_n = \frac{1}{n!} \cdot \frac{n-1}{2n-1}. \tag{3.9}
$$

For convenience let us write $dy = dy_1 \cdots dy_{n-1}$ on $\mathbf{R}^{n-1}$. We have

$$\begin{aligned}\iint_{E_{r^2}} \frac{\sqrt{|y|}}{x+|y|}dxdy &= \int_0^\infty v_n(\{(x,y) \in E_{r^2} : \sqrt{|y|}/(x+|y|) > t\})dt \\ &= \int_{1/r}^\infty v_n(E_{1/t^2})dt + \frac{1}{r}v_n(E_{r^2}) = \gamma_n \int_{1/r}^\infty \frac{1}{t^{2n}}dt + \frac{1}{r}\gamma_n r^{2n} \\ &= \frac{2n}{2n-1}\gamma_n r^{2n-1},\end{aligned}$$

where the third = follows from (3.6) and (3.9). Thus

$$\begin{aligned}\sum_{j=1}^{N(r)} s_j(Y) &= \sum_{\alpha \in A_r} \frac{\sqrt{|\alpha_2|}}{|\alpha|} \\ &= \sum_{\alpha \in A_r} \iint_{\alpha + I^n} \frac{\sqrt{|y|}}{x+|y|}dxdy + \sum_{\alpha \in A_r} \iint_{\alpha+I^n} \left(\frac{\sqrt{|\alpha_2|}}{|\alpha|} - \frac{\sqrt{|y|}}{x+|y|}\right)dxdy \\ &= \iint_{E_{r^2}} \frac{\sqrt{|y|}}{x+|y|}dxdy + o(r^{2n-1}) = \frac{2n}{2n-1}\gamma_n r^{2n-1} + o(r^{2n-1}).\end{aligned}$$

On the other hand, from (3.8) we obtain

$$\begin{aligned}\sum_{j=1}^{N(r)} j^{-1/(2n)} &= \frac{2n}{2n-1}\{N(r)\}^{(2n-1)/(2n)} + o(\{N(r)\}^{(2n-1)/(2n)}) \\ &= \frac{2n}{2n-1}\gamma_n^{(2n-1)/(2n)} r^{2n-1} + o(r^{2n-1}).\end{aligned}$$

Combining these two identities, we find that

$$\lim_{r\to\infty} \frac{\sum_{j=1}^{N(r)} s_j(Y)}{\sum_{j=1}^{N(r)} j^{-1/(2n)}} = \lim_{r\to\infty} \frac{\frac{2n}{2n-1}\gamma_n r^{2n-1} + o(r^{2n-1})}{\frac{2n}{2n-1}\gamma_n^{(2n-1)/(2n)} r^{2n-1} + o(r^{2n-1})} = \gamma_n^{1/(2n)}.$$

Recalling (3.9), the proof of the proposition will be complete if we can show that

$$\|Y\|_{2n,\mathrm{ess}}^+ = \lim_{r\to\infty} \frac{\sum_{j=1}^{N(r)} s_j(Y)}{\sum_{j=1}^{N(r)} j^{-1/(2n)}}. \tag{3.10}$$

As in the proof of Lemma 3.5, we first note that by Proposition 2.6, the left-hand side of (3.10) is greater than or equal to the right-hand side. Thus we only need to prove the reverse inequality. But for the reverse inequality, note that (3.8) gives us

$$N(\nu+1) - N(\nu) = o(\nu^{2n}),$$

$\nu \in \mathbf{N}$. Hence

$$\sum_{j=N(\nu)+1}^{N(\nu+1)} s_j(Y) \le \frac{1}{\nu}(N(\nu+1) - N(\nu)) = o(\nu^{2n-1}).$$

Once we have this, by the argument at the end of the proof of Lemma 3.5, the right-hand side of (3.10) is greater than or equal to the left-hand side. This completes the proof. □

The point that we try to make with Proposition 3.6 is that it is not easy to come up with functions f on $S \subset \mathbf{C}^n$ such that $\|H_f\|_{2n}^+ > \|H_f\|_{2n,\mathrm{ess}}^+$. Theorem 3.3 says that the function $\bar{z}_1$ on $S \subset \mathbf{C}^2$ has this property. So what about the function $\bar{z}_1$ on $S \subset \mathbf{C}^3$? In the case $n = 3$, Proposition 3.6 gives us

$$\|H_{\bar{z}_1}\|_{6,\mathrm{ess}}^+ = \left(\frac{1}{15}\right)^{1/6}.$$

On the other hand, the *obvious* lower bound that we obtain from (3.2) in the case $n = 3$ is $\|H_{\bar{z}_1}\|_6^+ \ge 3^{-1/2}$. Since $3^{-1/2} < 15^{-1/6}$, this is of no use to us. The difficulty here is to obtain an estimate of $\|H_{\bar{z}_1}\|_{2n}^+$ that is *close to its true value*. In view of this, it is somewhat surprising that we can actually calculate the essential norm $\|H_{\bar{z}_1}\|_{2n,\mathrm{ess}}^+$.

In the case $n = 2$, we do not know how close the lower bound $\|H_{\bar{z}_1}\|_4^+ \ge 2^{-1/2}$ is to the true value of $\|H_{\bar{z}_1}\|_4^+$. So it is really a matter of luck that the apparently crude lower bound $\|H_{\bar{z}_1}\|_4^+ \ge 2^{-1/2}$ in the case $n = 2$ is good enough to give us Example 3.4, which is the main purpose of the section. As of this writing, Example 3.4 is the only example of its kind that we are able to produce.

Data Availability. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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