TRACE-CLASS MEMBERSHIP FOR ANTISYMMETRIC SUMS ON QUOTIENT MODULES OF THE HARDY MODULE

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Abstract. We consider the quotient module $\mathcal{Q}$ of the Hardy module $H^2(S)$ defined by an analytic set $\tilde{M}$ satisfying certain conditions. Denote $d = \dim_{\mathbf{C}} \tilde{M}$. When $d = 1$, $\mathcal{Q}$ was shown to be 1-essentially normal in [24]. An analogous problem for the case $d \geq 2$ was proposed in [24], which asks whether $2d$-antisymmetric sums of certain module operators are in the trace class. In this paper we solve this problem in the affirmative. In the case $d = 1$, we derive a trace formula on $\mathcal{Q}$, which answers another question raised in [24].

1. Introduction

This paper is a sequel to [23,24]. Naturally, we will try to keep our notations consistent with [23,24]. Let us first review what led us here.

Denote $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$ and $S = \{z \in \mathbf{C}^n : |z| = 1\}$ as usual. Let $d\sigma$ be the spherical measure on S with the normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is just the closure of the analytic polynomials $\mathbf{C}[\zeta_1, \ldots, \zeta_n]$ in $L^2(S, d\sigma)$ [20].

Consider an analytic subset $\tilde{M}$ of an open neighborhood of $\overline{\mathbf{B}}$ with $1 \leq d \leq n-1$, where $d = \dim_{\mathbf{C}} \tilde{M}$. We assume that $\tilde{M}$ has no singular points on S and that $\tilde{M}$ intersects S transversely. Denote $M = \mathbf{B} \cap \tilde{M}$. Then we have a submodule

$$\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$$

of $H^2(S)$. The corresponding quotient module is

$$\mathcal{Q} = H^2(S) \ominus \mathcal{R}.$$

Both $\mathcal{R}$ and $\mathcal{Q}$ are the focus of the Arveson-Douglas conjecture [1,2,9], which commands intense current research interest [3,10-15,18,22]. As it was the case for [23,24], the quotient module $\mathcal{Q}$ will be the focus of this paper.

Let $Q : L^2(S, d\sigma) \to \mathcal{Q}$ be the orthogonal projection. For $f \in L^\infty(S, d\sigma)$, we define

$$Q_f = QM_f|\mathcal{Q}.$$

Obviously, Q_f is the analogue of Toeplitz operator for the quotient module $\mathcal{Q}$. Let us write $\zeta_1, \ldots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$. Then $Q_{\zeta_1}, \ldots, Q_{\zeta_n}$ are the module operators for the quotient module $\mathcal{Q}$.

Keywords: Arveson-Douglas conjecture, quotient module, antisymmetric sum.

In [23], the geometric Arveson-Douglas conjecture was proved for $\mathcal{Q}$. More precisely, it was shown that that for all $i, j \in \{1, \dots, n\}$, the commutator $[Q_{\zeta_i}, Q_{\zeta_j}^*]$ is in the Schatten class $\mathcal{C}_p$ for all $p > d$. For the special case $d = 1$, there is a stronger result:

Theorem 1.1. [24, Theorem 1.5] *In the case $d = 1$, the quotient module $\mathcal{Q}$ is 1-essentially normal. That is, if $d = 1$, then for every pair of $i, j \in \{1, \dots, n\}$, the commutator $[Q_{\zeta_i}, Q_{\zeta_j}^*]$ belongs to the trace class $\mathcal{C}_1$.*

Since Theorem 1.1 is stronger than the prediction of the geometric Arveson-Douglas conjecture for the case $d = 1$, it naturally leads to the question, what about the case $d \geq 2$? First of all, easy examples show that if $d \geq 2$, then in general $[Q_{\zeta_i}, Q_{\zeta_j}^*] \notin \mathcal{C}_d$. In other words, in the case $d \geq 2$, the geometric Arveson-Douglas conjecture is sharp in terms of the Schatten-class membership for $[Q_{\zeta_i}, Q_{\zeta_j}^*]$.

The proper analogue of Theorem 1.1 for the case $d \geq 2$ must be stated not in terms of commutators, but in terms of *antisymmetric sums*. Given operators $A_1, \dots, A_k$ on a Hilbert space $\mathcal{H}$, one has the antisymmetric sum

$$[A_1, \dots, A_k] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)}.$$

This was first introduced by Helton and Howe in [17], and has since become an important part of operator theory and non-commutative geometry [7]. See [21] for the latest development in the study of antisymmetric sums.

Inspired by Theorem 1.1 and by what is known about antisymmetric sums of Toeplitz operators on $H^2(S)$ and on the Bergman space $L^2_a(\mathbf{B})$, the following was proposed in [24]:

Problem 1.2. [24, Problem 12.3] Suppose that $2 \leq d \leq n - 1$. For analytic polynomials $p_1, \dots, p_d, q_1, \dots, q_d \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, is the antisymmetric sum

$$[Q_{p_1}, Q_{q_1}^*, \dots, Q_{p_d}, Q_{q_d}^*] \tag{1.1}$$

in the trace class? If it is in the trace class, is there a formula for its trace, say in terms of some integral on M? In other words, is there an analogue of the Helton-Howe trace formula [17] for the above antisymmetric sum?

We are pleased to report that we are now able to solve the first half of Problem 1.2 in the affirmative. That is, we can now show that (1.1) is indeed in the trace class, and more. Even though we still do not have an integral formula for the trace of (1.1), proving its membership in $\mathcal{C}_1$ is a significant step forward:

Theorem 1.3. *Suppose that $2 \leq d \leq n - 1$. For analytic polynomials $p_1, \dots, p_d, q_1, \dots, q_d \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, the antisymmetric sum*

$$[Q_{p_1}, Q_{q_1}^*, \dots, Q_{p_d}, Q_{q_d}^*]$$

is in the trace class. Moreover, for all polynomials $f_1, f_2, \dots, f_{2d} \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \dots, \zeta_n, \bar{\zeta}_n]$, the antisymmetric sum

$$[Q_{f_1}, Q_{f_2}, \dots, Q_{f_{2d}}]$$

is in the trace class.

In the case $d = 1$, a question about trace formula was also raised in [24]. Recall the following:

Theorem 1.4. [24, Theorem 11.12] *When $d = 1$, we have*

$$\operatorname{tr}[Q_p^* Q_q, Q_r^* Q_s] = \operatorname{tr}[M_p^* M_q, M_r^* M_s] \tag{1.2}$$

for all $p, q, r, s \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$.

In (1.2), M_p, M_q, M_r, M_s are multiplication operators on the range space $\mathcal{P}$ associated with $\mathcal{Q}$ [24, Section 10]. Thus Theorem 1.4 identifies a trace on the quotient module $\mathcal{Q}$ with a trace in terms of multiplication operators. As was mentioned on page 46 in [24], the hope was that the right-hand side of (1.2) was more computable, and consequently there might be an explicit formula for (1.2) in terms of some integral involving p, q, r, s. As the second result of the paper, we are pleased to report that the right-hand side of (1.2) can indeed be computed. Consequently, we obtain an integral formula for (1.2) in terms of p, q, r, s.

To present this integral formula, we begin with $X = \tilde{M} \cap S$. Under our assumption on $\tilde{M}$, X is a compact, smooth manifold of real dimension $2d - 1$, which only has a finite number of connected components. If $d = 1$, then the real dimension of X is 1. Thus when $d = 1$, X admits a decomposition

$$X = \Gamma_1 \cup \cdots \cup \Gamma_\ell$$

in terms of connected components, where each component Γ_j is diffeomorphic to the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, $1 \le j \le \ell$.

Theorem 1.5. *Suppose that $d = 1$. Then there exist integers $c_1, \ldots, c_\ell \in \mathbf{Z}$ such that*

$$\operatorname{tr}[Q_p^* Q_q, Q_r^* Q_s] = \frac{1}{2\pi i} \sum_{j=1}^{\ell} c_j \int_{\Gamma_j} \bar{p} q d\bar{r} s \tag{1.3}$$

for all analytic polynomials $p, q, r, s \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$.

The integrals in (1.3) are defined in the following way. For each $1 \le j \le \ell$, we choose an orientation for Γ_j as the positive direction, which determines the sign of the corresponding integer c_j. Then $\int_{\Gamma_j} \bar{p} q d\bar{r} s$ is just the Riemann-Stieltjes integral of $\bar{p} q$ against $\bar{r} s$ on Γ_j in the chosen positive direction.

We also have the following variant of Theorem 1.5:

Theorem 1.6. *Suppose that $d = 1$. Then for all polynomials $f, g \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, the commutator $[Q_f, Q_g]$ is in the trace class, and we have*

$$\operatorname{tr}[Q_f, Q_g] = \frac{1}{2\pi i} \sum_{j=1}^{\ell} c_j \int_{\Gamma_j} f dg,$$

where the integers $c_1, \dots, c_\ell \in \mathbf{Z}$ *are the same as in Theorem* 1.5.

As we will see, the integers $c_1, \dots, c_\ell \in \mathbf{Z}$ in Theorems 1.5 and 1.6 are -1 times the indices of certain explicit "Toeplitz operators" on the range space $\mathcal{P}$.

We will present a family of analytic sets $\tilde{M}$ for which the integers $c_1, \dots, c_\ell$ in Theorems 1.5 and 1.6 can be explicitly determined.

Consider the case $n = 2$, i.e., the case where $S = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 = 1\}$. Let $p \geq 2$ and $q \geq 2$ be natural numbers that are *relatively prime*. For convenience, we assume $p < q$. Define

$$\tilde{M}_{p,q} = \{(z, w) \in \mathbf{C}^2 : z^p - w^q = 0\}.$$

Obviously, $\tilde{M}_{p,q}$ is an analytic subset of $\mathbf{C}^2$ with the point $(0, 0)$ as its only singularity. It is easy to verify that $\tilde{M}_{p,q}$ intersects S transversely. Define

$$\begin{aligned} M_{p,q} &= \{(z, w) \in \tilde{M}_{p,q} : |z|^2 + |w|^2 < 1\} \quad \text{and} \\ \mathcal{Q}_{p,q} &= H^2(S) \ominus \{f \in H^2(S) : f = 0 \text{ on } M_{p,q}\}. \end{aligned}$$

Then $\mathcal{Q}_{p,q}$ is a quotient module of $H^2(S)$ to which Theorems 1.5 and 1.6 can be applied. Thus for all $f, g \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2]$, the operators Q_f, Q_g on $\mathcal{Q}_{p,q}$ have the property $[Q_f, Q_g] \in \mathcal{C}_1$. For these Q_f, Q_g on $\mathcal{Q}_{p,q}$ we have the explicit formula

$$\text{tr}[Q_f, Q_g] = \frac{1}{2\pi i} \int_0^{2\pi} f((1 - b^2)^{1/2} e^{iqt}, be^{ipt}) dg((1 - b^2)^{1/2} e^{iqt}, be^{ipt}), \tag{1.4}$$

where b is the unique number in $(0, 1)$ that satisfies the equation

$$1 - b^2 - b^{2q/p} = 0.$$

To conclude the introduction, let us discuss the organization of the paper and the main ideas involved in the proofs. First of all, Section 2 contains the necessary preliminaries required by the proofs in the paper. We call the reader's attention to the range space $\mathcal{P}$ reviewed in Section 2, which is needed in the proofs of all the theorems in this paper.

The proof of Theorem 1.3 is based on a very simple idea. Suppose that f is analytic on $\{z \in \mathbf{C}^n : |z| < 1 + s\}$ for some $s > 0$. By the first order Taylor expansion,

$$f(\zeta) - f(w) = \langle (\partial f)(\zeta), \overline{\zeta - w} \rangle + O(|\zeta - w|^2).$$

Now if we consider a commutator of the form $[M_f, T]$, where T has an integral kernel in some negative power of $1 - \langle \zeta, w \rangle$, then the $O(|\zeta - w|^2)$ above results in a "small perturbation" to the main term. But the main term has the form $\sum_j M_{\partial_j f}[M_{\zeta_j}, T]$, where f is not involved in commutation. In Section 3, we show that there is a local version of this on $\tilde{M}$ near S. By choosing the right local frame, we further show that one of the summands in the main term is itself a "small perturbation". In other words, locally on $\tilde{M}$, there are only $d - 1$ summands in the "main term" of such a commutator. Using this fact, in Section 4 we show that the right antisymmetrization leads to trace-class membership. With these preparations, the proof of Theorem 1.3 is completed in Section 5.

After that, we turn to the proofs of Theorems 1.5 and 1.6. First, in Section 6, we revisit the Helton-Howe trace formula in [16]. Our main point is to re-examine this classic formula from the perspective of the Carey-Pincus theory of principal functions. The result of this re-examination is Proposition 6.3, which can be viewed as a special version of Green's theorem for Lipschitz curves, which allow self-intersections. In Section 7, we first present the Fredholm theory for Toeplitz operators on the range space $\mathcal{P}$. Then, using the Carey-Pincus theory, we derive a trace formula for the commutators of these operators. With the preparations in Sections 6 and 7, we prove Theorems 1.5 and 1.6 in Section 8.

Finally, in Section 9, we present the details of the family of examples $\mathcal{Q}_{p,q}$ introduced above, and we prove (1.4).

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2. Preliminaries

The analytic sets, submodules and quotient modules, etc, involved in this paper are exactly the same as those in [23,24]. But in the interest of being precise, we repeat the necessary definitions and notations below.

Definition 2.1. [6] Let Ω be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of Ω if for each point $a \in \Omega$ there are a neighborhood U of a and functions $f_1, \cdots, f_N$ analytic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a *singular point* of A if it is not regular.

Assumption 2.2. Let $\tilde{M}$ be an analytic subset in an open neighborhood of the closed ball $\overline{\mathbf{B}}$. Furthermore, $\tilde{M}$ satisfies the following conditions:
(1) $\tilde{M}$ intersects $\partial\mathbf{B}$ transversely.
(2) $\tilde{M}$ has no singular points on $\partial\mathbf{B}$.
(3) $\tilde{M}$ is of pure dimension d, where $1 \le d \le n-1$.

Given such an $\tilde{M}$, we fix M, $\mathcal{R}$, $\mathcal{Q}$ and Q as follows.

Notation 2.3. (a) Let $M = \tilde{M} \cap \mathbf{B}$.
(b) Denote $\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$.
(c) Denote $\mathcal{Q} = H^2(S) \ominus \mathcal{R}$.
(d) Let Q be the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{Q}$.

By Assumption 2.2, there is an $s \in (0,1)$ such that

$$\mathcal{M} = \{z \in \tilde{M} : 1-s < |z| < 1+s\} \tag{2.1}$$

is a complex manifold of complex dimension d and of finite volume. We take the value $s \in (0,1)$ so small that the closure of $\mathcal{M}$ is contained in the regular part of $\tilde{M}$.

Definition 2.4. (a) We define the measure v_M on $M = \tilde{M} \cap \mathbf{B}$ by the formula $v_M(E) = v_{\mathcal{M}}(E \cap \mathcal{M})$ for Borel sets $E \subset M$, where $v_{\mathcal{M}}$ is the natural volume measure on $\mathcal{M}$.
(b) We define the measure μ on M by the formula

$$d\mu(w) = (1 - |w|^2)^{n-1-d} dv_M(w).$$

We further extend μ to a measure on $\mathbf{B}$ by setting $\mu(\mathbf{B} \backslash M) = 0$.

Next we recall certain results from previous investigations. First of all, for the measure μ in Definition 2.4(b), we have the following Forelli-Rudin estimate:

Lemma 2.5. [23, Lemma 2.10] *Given any $a > 0$ and $\kappa > -1$, there is a $0 < C_{2.5} < \infty$ such that*

$$\int_M \frac{(1 - |z|^2)^a (1 - |w|^2)^\kappa}{|1 - \langle w, z \rangle|^{d+1+a+\kappa}} dv_M(w) \le C_{2.5}$$

for every $z \in M$.

Moreover, it is known that if $\kappa > -1$, then

$$\int_M (1 - |w|^2)^\kappa dv_M(w) < \infty$$

[23, page 15]. This finiteness is due to the fact that we can use the function $1 - |w|^2$ as one of the $2d$ real coordinates on M for $w \in M$ near S.

We have a Toeplitz operator with the measure μ in Definition 2.4(b) as its symbol. In other words, we define

$$(T_\mu f)(z) = \int \frac{f(w)}{(1 - \langle z, w \rangle)^n} d\mu(w),$$

$f \in H^2(S)$. Alternatively, we can rewrite this operator in the form

$$T_\mu = \int K_w \otimes K_w d\mu(w), \tag{2.2}$$

where $K_w(z) = (1 - \langle z, w \rangle)^{-n}$, the reproducing kernel for $H^2(S)$. It is easy to see that

$$\langle T_\mu f, f \rangle = \int |f(w)|^2 d\mu(w) \tag{2.3}$$

for every $f \in H^2(S)$.

Theorem 2.6. [23, Theorem 3.5] *There are scalars $0 < c \le C < \infty$ such that the operator inequality*

$$cQ \le T_\mu \le CQ$$

holds on $L^2(S, d\sigma)$.

Proposition 2.7. [23, Proposition 8.4] *For any Lipschitz function f on S, the commutator $[M_f, Q]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

Theorem 2.8. [24, Theorem 1.1] *For any Lipschitz functions f, g on S, the double commutator $[M_f, [M_g, Q]]$ is in the Schatten class $\mathcal{C}_p$ for every $p > d$.*

The measure in Definition 2.4(b) also gives rise to $L^2(\mu) = L^2(M, d\mu)$, the Hilbert space of measurable functions on M that are square-integrable with respect to $d\mu$. This allows us to introduce the *range space* as in [24].

Let $f \in \mathcal{Q}$. Since f is an analytic function on $\mathbf{B}$, we define Jf to be the restriction of this analytic function to the subset M of $\mathbf{B}$. By (2.3) we have

$$\int_M |(Jf)(w)|^2 d\mu(w) = \int_M |f(w)|^2 d\mu(w) = \langle T_\mu f, f\rangle \tag{2.4}$$

for every $f \in \mathcal{Q}$. Thus, by the upper bound in Theorem 2.6, J is a bounded operator that maps $\mathcal{Q}$ into $L^2(\mu)$. By the lower bound in Theorem 2.6 and (2.4), we have

$$\|Jf\|^2 \geq c\|f\|^2 \quad \text{for every} \quad f \in \mathcal{Q}. \tag{2.5}$$

Therefore the range of J is a closed linear subspace of $L^2(\mu)$.

Definition 2.9. (a) Write $\mathcal{P}$ for the range of the restriction operator J introduced above. (b) Let E denote the orthogonal projection from $L^2(\mu)$ onto $\mathcal{P}$.

Obviously, (2.4) is equivalent to the statement that

$$J^*Jf = T_\mu f \quad \text{for every} \quad f \in \mathcal{Q}. \tag{2.6}$$

Moreover, (2.5) says that J is an invertible operator from $\mathcal{Q}$ to $\mathcal{P}$.

If $f \in \mathcal{R}$, then its restriction to M is the zero function. Since $H^2(S) = \mathcal{R} \oplus \mathcal{Q}$, we see that the range space $\mathcal{P}$ is actually the collection of the restrictions of all $f \in H^2(S)$ to M.

Definition 2.10. For $f \in L^\infty(\mu)$, $\hat{M}_f$ denotes the operator of multiplication by the function f on $L^2(\mu)$.

Recall that we write $\zeta_1, \ldots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$.

Proposition 2.11. [24, Proposition 10.5] *For each $j \in \{1, \ldots, n\}$, $\mathcal{P}$ is an invariant subspace for $\hat{M}_{\zeta_j}$.*

Proposition 2.11 makes it possible for us to introduce

Definition 2.12. For each $j \in \{1, \ldots, n\}$, let M_{ζ_j} denote the restriction of the operator $\hat{M}_{\zeta_j}$ to the invariant subspace $\mathcal{P}$. More generally, for $r \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, M_r denotes the restriction of the operator $\hat{M}_r$ to the invariant subspace $\mathcal{P}$.

Proposition 2.13. [24, Corollary 10.7] *We have $JQ_{\zeta_j} = M_{\zeta_j}J$ for every $j \in \{1, \ldots, n\}$.*

We define the operator $\hat{T}_\mu$ on $L^2(\mu)$ by the formula

$$(\hat{T}_\mu\varphi)(\zeta) = \int_M \varphi(w)K_w(\zeta)d\mu(w), \quad \zeta \in M, \tag{2.7}$$

$\varphi \in L^2(\mu)$.

Proposition 2.14. [24, Proposition 10.3] (a) *$\hat{T}_\mu$ is a bounded operator on $L^2(\mu)$.*
(b) *$\hat{T}_\mu$ maps $L^2(\mu)$ into $\mathcal{P}$.*
(c) *Let $\tilde{T}_\mu$ denote the restriction of $\hat{T}_\mu$ to the subspace $\mathcal{P}$. Then $\tilde{T}_\mu = JJ^*$. In particular, $\tilde{T}_\mu$ is invertible on $\mathcal{P}$.*
(d) *With respect to the orthogonal decomposition $L^2(\mu) = \mathcal{P} \oplus \mathcal{P}^\perp$, we have $\hat{T}_\mu = \tilde{T}_\mu \oplus 0$.*

Lemma 2.15. [24, Lemma 10.8] *Let $G(\zeta, w)$ be a Borel function on $M \times M$. Consider the operator on $L^2(\mu)$ given by the formula*

$$(A_G\varphi)(\zeta) = \int_M \varphi(w)G(\zeta, w)K_w(\zeta)d\mu(w), \tag{2.8}$$

$\varphi \in L^2(\mu)$. If G satisfies the condition

$$\iint |G(\zeta, w)|^p|K_w(\zeta)|^2 d\mu(w)d\mu(\zeta) < \infty$$

for some $2 \le p < \infty$, then A_G belongs to the Schatten class $\mathcal{C}_p$.

Proposition 2.16. *If f is a Lipschitz function on M, then for every $p > 2d$, we have $[\hat{M}_f, \hat{T}_\mu] \in \mathcal{C}_p$, $[\hat{M}_f, \hat{T}_\mu^{1/2}] \in \mathcal{C}_p$ and $[\hat{M}_f, E] \in \mathcal{C}_p$.*

Proof. Obviously, $[\hat{M}_f, \hat{T}_\mu] = A_G$, where $G(\zeta, w) = f(\zeta) - f(w)$. Since $f \in \mathrm{Lip}(M)$, we have

$$\begin{aligned}\iint |G(\zeta, w)|^p|K_w(\zeta)|^2 d\mu(w)d\mu(\zeta) &\le C_1 \iint \frac{|\zeta - w|^p}{|1 - \langle \zeta, w\rangle|^{2+2d}} dv_M(w)dv_M(\zeta) \\ &\le C_2 \iint \frac{|1 - \langle \zeta, w\rangle|^{p/2}}{|1 - \langle \zeta, w\rangle|^{d+1+(d+1)}} dv_M(w)dv_M(\zeta).\end{aligned}$$

The condition $p > 2d$ leads to $d + 1 - (p/2) < 1$. Thus by Lemma 2.5, the above is finite. By Lemma 2.15, this means that $[\hat{M}_f, \hat{T}_\mu] \in \mathcal{C}_p$.

It follows from Proposition 2.14, (2.6), and Theorem 2.6 that the spectrum of $\hat{T}_\mu$ is contained in $\{0\} \cup [c, C]$, and that the spectral projection of $\hat{T}_\mu$ corresponding to the interval $[c, C]$ equals E. Therefore there are $h, \eta \in C_c^\infty(\mathbf{R})$ such that $E = h(\hat{T}_\mu)$ and $\hat{T}_\mu^{1/2} = \eta(\hat{T}_\mu)$. By the membership $[\hat{M}_f, \hat{T}_\mu] \in \mathcal{C}_p$ and the standard smooth functional calculus, we have $[\hat{M}_f, \hat{T}_\mu^{1/2}] \in \mathcal{C}_p$ and $[\hat{M}_f, E] \in \mathcal{C}_p$. $\square$

Proposition 2.17. *Suppose that $d \ge 2$.*

(a) *Let G be a measurable function on $M \times M$ for which there is a $0 < C < \infty$ such that $|G(\zeta, w)| \le C|1 - \langle \zeta, w\rangle|$ for all $\zeta, w \in M$. Then $A_G \in \mathcal{C}_p$ for every $p > d$.*
(b) *Let Y be a measurable function on $M \times M$ for which there is a $0 < C' < \infty$ such that $|Y(\zeta, w)| \le C'|1 - \langle \zeta, w\rangle|^{3/2}$ for all $\zeta, w \in M$. Then $EA_Y \in \mathcal{C}_p$ for every $p > 2d/3$.*

Proof. (a) For $p > d$, we have

$$\iint |G(\zeta, w)|^p |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) \le C_1 \iint \frac{|1 - \langle \zeta, w\rangle|^p}{|1 - \langle \zeta, w\rangle|^{2+2d}} dv_M(w) dv_M(\zeta)$$
$$= C_1 \iint \frac{1}{|1 - \langle \zeta, w\rangle|^{d+1+a}} dv_M(w) dv_M(\zeta),$$

where $a = d + 1 - p$. The condition $p > d$ means that $a < 1$. Therefore, by Lemma 2.5, the above is finite. By Lemma 2.15, this means that $A_G \in \mathcal{C}_p$.

(b) If $d \ge 3$, then $2d/3 \ge 2$, and Lemma 2.15 applies to every $p > 2d/3$. Thus in the case $d \ge 3$, (b) is proved in the same way (a) was proved above.

Let us consider the case $d = 2$. Denote $\rho(\zeta) = 1 - |\zeta|^2$. Pick any $p > 4$. Since $p/2 > 2$, we can pick an $r \in (0, 1)$ such that $rp/2 > 2$. We first show that $E\hat{M}_{\rho^{r/2}} \in \mathcal{C}_p$. By Proposition 2.14 and the fact that $\hat{T}_\mu \hat{M}_{\rho^{r/2}} = \{\hat{M}_{\rho^{r/2}} \hat{T}_\mu\}^*$, it suffices to show that $\hat{M}_{\rho^{r/2}} \hat{T}_\mu \in \mathcal{C}_p$. In terms of (2.8), we have $\hat{M}_{\rho^{r/2}} \hat{T}_\mu = A_G$ with $G(\zeta, w) = (1 - |\zeta|^2)^{r/2}$. Applying Definition 2.4(b) to the case $d = 2$, we have

$$\iint |G(\zeta, w)|^p |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) \le C_1 \iint \frac{(1 - |\zeta|^2)^{rp/2}}{|1 - \langle \zeta, w\rangle|^6} dv_M(w) dv_M(\zeta)$$
$$\le C_2 \iint \frac{1}{|1 - \langle \zeta, w\rangle|^{2+1+\{3-(rp/2)\}}} dv_M(w) dv_M(\zeta).$$

Since $rp/2 > 2$, we have $3 - (rp/2) < 1$. Thus, by Lemma 2.5, the above is finite. By Lemma 2.15, we have $\hat{M}_{\rho^{r/2}} \hat{T}_\mu \in \mathcal{C}_p$.

Next we show that $\hat{M}_{\rho^{-r/2}} A_Y \in \mathcal{C}_2$. Indeed $\hat{M}_{\rho^{-r/2}} A_Y = A_H$, where

$$H(\zeta, w) = (1 - |\zeta|^2)^{-r/2} Y(\zeta, w).$$

By the assumption on Y, we have

$$\|A_H\|_2^2 \le C_3 \iint \frac{|1 - \langle \zeta, w\rangle|^3}{(1 - |\zeta|^2)^r |1 - \langle \zeta, w\rangle|^6} dv_M(w) dv_M(\zeta).$$

Since $r < 1$, by Lemma 2.5, this is finite. Hence $\hat{M}_{\rho^{-r/2}} A_Y \in \mathcal{C}_2$.

By the factorization
$$EA_Y = E\hat{M}_{\rho^{r/2}} \cdot \hat{M}_{\rho^{-r/2}} A_Y$$
and the conclusions of the two paragraphs above, we have $EA_Y \in \mathcal{C}_t$, where $1/t = (1/p) + (1/2)$. Since $p > 4$ is arbitrary, this means that $EA_Y \in \mathcal{C}_t$ for every $t > 4/3 = 2 \cdot 2/3$. $\square$

Proposition 2.18. *Suppose that $d \geq 2$. Then for Lipschitz functions f, g on M, the double commutators $[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]]$, $[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu^{1/2}]]$ and $[\hat{M}_g, [\hat{M}_f, E]]$ all belong to the Schatten class $\mathcal{C}_p$ for every $p > d$.*

Proof. Obviously, $[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]] = A_G$, where $G(\zeta, w) = (g(\zeta) - g(w))(f(\zeta) - f(w))$. By the Lipschitz conditions for f, g, we have

$$|G(\zeta, w)| \leq C_1|\zeta - w|^2 \leq 2C_1|1 - \langle\zeta, w\rangle|.$$

Thus it follows from Proposition 2.17(a) that $[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]] \in \mathcal{C}_p$ for every $p > d$. To prove the other two Schatten-class memberships, this time we use Riesz functional calculus.

As we already mentioned, the spectrum of $\hat{T}_\mu$ is contained in $\{0\} \cup [c, C]$, and the spectral projection of $\hat{T}_\mu$ corresponding to the interval $[c, C]$ equals E. Consider $H_+ = \{\lambda \in \mathbf{C} : \mathrm{Re}(\lambda) > 0\}$, the right half-plane. Let γ be a simple Jordan curve in $H_+ \backslash [c, C]$ whose winding number about every $x \in [c, C]$ is 1. Taking advantage of the fact that the square-root function $\lambda^{1/2}$ is analytic on H_+, we have

$$\hat{T}_\mu^{1/2} = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}(\lambda - \hat{T}_\mu)^{-1} d\lambda.$$

It is easy to see that

$$[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu^{1/2}]] = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}\{A(\lambda) + B(\lambda) + C(\lambda)\} d\lambda,$$

where

$$\begin{aligned}
A(\lambda) &= (\lambda - \hat{T}_\mu)^{-1}[\hat{M}_g, \hat{T}_\mu](\lambda - \hat{T}_\mu)^{-1}[\hat{M}_f, \hat{T}_\mu](\lambda - \hat{T}_\mu)^{-1}, \\
B(\lambda) &= (\lambda - \hat{T}_\mu)^{-1}[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]](\lambda - \hat{T}_\mu)^{-1} \quad \text{and} \\
C(\lambda) &= (\lambda - \hat{T}_\mu)^{-1}[\hat{M}_f, \hat{T}_\mu](\lambda - \hat{T}_\mu)^{-1}[\hat{M}_g, \hat{T}_\mu](\lambda - \hat{T}_\mu)^{-1}.
\end{aligned}$$

Applying the preceding paragraph to $B(\lambda)$ and Proposition 2.16 to $A(\lambda)$ and $C(\lambda)$, we obtain the membership $[\hat{M}_g, [\hat{M}_f, \hat{T}_\mu^{1/2}]] \in \mathcal{C}_p$ for $p > d$.

For the double commutator $[\hat{M}_g, [\hat{M}_f, E]]$, we use the representation

$$E = \frac{1}{2\pi i}\int_\gamma (\lambda - \hat{T}_\mu)^{-1} d\lambda,$$

and the rest of the argument is similar to the above paragraph. $\square$

Proposition 2.19. *Suppose that $d \geq 2$. Then for Lipschitz functions f, g, h on M, the operators $E[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]]]$ and $E[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu^{1/2}]]]$ belong to the Schatten class $\mathcal{C}_p$ for every $p > 2d/3$.*

Proof. Recalling (2.8), we have $[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]]] = A_Y$, where

$$Y(\zeta, w) = (h(\zeta) - h(w))(g(\zeta) - g(w))(f(\zeta) - f(w)).$$

The Lipschitz conditions for f, g, h imply

$$|Y(\zeta, w)| \le C|\zeta - w|^3 \le 2^{3/2}C|1 - \langle \zeta, w\rangle|^{3/2}.$$

Thus Proposition 2.17(b) tells us that $E[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]]] = EA_Y \in \mathcal{C}_p$ for $p > 2d/3$.

But once we have $E[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu]]] \in \mathcal{C}_p$, the membership $E[\hat{M}_h, [\hat{M}_g, [\hat{M}_f, \hat{T}_\mu^{1/2}]]] \in \mathcal{C}_p$ is obtained by using the Riesz functional calculus in the proof of Proposition 2.18. □

Let

$$J^* = U|J^*| \tag{2.9}$$

be the polar decomposition of the operator J^*. We know that $J^* : \mathcal{P} \to \mathcal{Q}$ is invertible. Therefore the U above is a unitary operator. Also, by Proposition 2.14(c), we have $|J^*| = (JJ^*)^{1/2} = \tilde{T}_\mu^{1/2}$. Combining this with Proposition 2.13, we find that

$$Q_{\zeta_j} = J^{-1}M_{\zeta_j}J = U\tilde{T}_\mu^{-1/2}M_{\zeta_j}\tilde{T}_\mu^{1/2}U^* = U(M_{\zeta_j} + Z_j)U^* \tag{2.10}$$

for each $j \in \{1, \dots, n\}$, where

$$Z_j = \tilde{T}_\mu^{-1/2}[M_{\zeta_j}, \tilde{T}_\mu^{1/2}]. \tag{2.11}$$

We alert the reader that (2.10) is a crucial identity.

The above identities suggest that we also need the operator

$$\mathcal{T} = \tilde{T}_\mu^{-1/2} \oplus 0, \tag{2.12}$$

which corresponds to the space decomposition $L^2(\mu) = \mathcal{P} \oplus \mathcal{P}^\perp$.

Proposition 2.20. *If f is a Lipschitz function on M, then $[\hat{M}_f, \mathcal{T}] \in \mathcal{C}_p$ for every $p > 2d$. If f, g are Lipschitz functions on M, then $[\hat{M}_g, [\hat{M}_f, \mathcal{T}]] \in \mathcal{C}_p$ for every $p > d$.*

Proof. This is obtained from the Schatten-class memberships provided in Propositions 2.16 and 2.18 by using the representation

$$\mathcal{T} = \frac{1}{2\pi i}\int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1}d\lambda,$$

where the contour γ is the same as in the proof of Proposition 2.18. □

Lemma 2.21. *Let $\varphi \in L^\infty(\mu)$. Suppose that there is a $0 < t < 1$ such that $\varphi = 0$ on $N_t = \{\zeta \in M : t < |\zeta| < 1\}$. Then $\hat{M}_\varphi E \in \mathcal{C}_1$.*

Proof. By Proposition 2.14, $E = \hat{T}_\mu(\tilde{T}_\mu^{-1} \oplus 0)$. Therefore it suffices to show that $\hat{M}_\varphi \hat{T}_\mu \in \mathcal{C}_1$. For each $k \in \mathbf{Z}_+$, define the operator A_k on $L^2(\mu)$ by the formula

$$(A_k f)(\zeta) = \varphi(\zeta) \int \langle \zeta, w \rangle^k f(w) d\mu(w), \quad f \in L^2(\mu).$$

From the conditions $\varphi \in L^\infty(\mu)$ and $\varphi = 0$ on N_t it is easy to deduce that $\|A_k\| \leq Ct^k$. By elementary combinatorics, $\text{rank}(A_k) \leq \{(n-1)!k!\}^{-1}(k+n-1)!$. Hence

$$\|A_k\|_1 \leq Ct^k \frac{(k+n-1)!}{(n-1)!k!} \quad \text{for every} \quad k \in \mathbf{Z}_+. \tag{2.13}$$

On the other hand, by the expansion for $(1-u)^{-n}$ on the unit disc, we have

$$\hat{M}_\varphi \hat{T}_\mu = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{(n-1)!k!} A_k. \tag{2.14}$$

Obviously, (2.13) and (2.14) together imply the membership $\hat{M}_\varphi \hat{T}_\mu \in \mathcal{C}_1$. $\square$

3. Local analysis

For $z \in \mathbf{C}^n$ and $r > 0$, we write $B(z, r) = \{u \in \mathbf{C}^n : |z - u| < r\}$, the standard Euclidean ball in $\mathbf{C}^n$. Similarly, $B_d(z, r)$ denotes the standard Euclidean ball in $\mathbf{C}^d$.

For each $z \in \mathcal{M}$, let T_z be the tangent space to $\mathcal{M}$ at the point z, viewed as a natural subspace of $\mathbf{C}^n$. For each $z \in \mathcal{M}$, let p_z be the orthogonal projection of z on T_z. Condition (1) in Assumption 2.2 implies that if $z \in \tilde{M} \cap S$, then $p_z \neq 0$. Thus there exist an $s_0 \in (0, s)$ and a $\gamma > 0$ such that if we define

$$\mathcal{M}_0 = \{z \in \tilde{M} : 1 - s_0 < |z| < 1 + s_0\}, \tag{3.1}$$

then

$$|p_z| \geq \gamma \quad \text{for every} \quad z \in \mathcal{M}_0. \tag{3.2}$$

Since $\mathcal{M}_0$ is a complex manifold, for each $z \in \mathcal{M}_0$, there exist an open neighborhood U_z of z in $\mathcal{M}_0$, an $a > 0$ and a biholomorphic map

$$G : B_d(0, a) \to U_z \quad \text{with} \quad G(0) = z.$$

Reducing the value of a if necessary, we may assume that there are $0 < c \leq C < \infty$ such that DG, the complex derivative of G, satisfies the matrix bound

$$c \leq (DG)^*(u)(DG)(u) \leq C \quad \text{for every} \quad u \in B_d(0, a). \tag{3.3}$$

For each $u \in B_d(0,a)$, we have the polarization

$$(DG)(u) = V(u)\{(DG)^*(u)(DG)(u)\}^{1/2}.$$

By (3.3), V smoothly maps $B_d(0,a)$ into $M_{n\times d}$, the collection of $n \times d$ matrices. For each $u \in B_d(0,a)$, the range of $V(u)$ is obviously the tangent space $T_{G(u)}$. Define

$$F_\zeta = V(G^{-1}(\zeta))V^*(G^{-1}(\zeta))$$

for each $\zeta \in U_z$. Then F_ζ is the orthogonal projection from $\mathbf{C}^n$ onto the tangent space T_ζ. Moreover, the map $\zeta \mapsto F_\zeta$ from U_z into $M_{n\times n}$ is smooth with respect to the smooth structure on $\mathcal{M}_0$.

By (3.2), the formula

$$e_1(\zeta) = p_\zeta/|p_\zeta|, \quad \zeta \in \mathcal{M}_0,$$

defines a global cross section of the complex tangent bundle of $\mathcal{M}_0$. This allows us to further define

$$F_\zeta^{(1)} = F_\zeta - e_1(\zeta) \otimes e_1(\zeta), \quad \zeta \in U_z.$$

Then the map $\zeta \mapsto F_\zeta^{(1)}$ is again smooth on U_z, and, for each $\zeta \in U_z$, $F_\zeta^{(1)}$ is the orthogonal projection from $\mathbf{C}^n$ onto the orthogonal complement of $\{\lambda e_1(\zeta) : \lambda \in \mathbf{C}\}$ in T_ζ.

For $d \geq 2$, $T_z \ominus \{\lambda e_1(z) : \lambda \in \mathbf{C}\} \neq \{0\}$. Thus there is a $v_2 \in T_z$ with $|v_2| = 1$ such that $v_2 \perp e_1(z)$. The map $\zeta \mapsto F_\zeta^{(1)} v_2$ is smooth on U_z, and we have $F_z^{(1)} v_2 = v_2$. Thus there is an open neighborhood $U_z^{(2)}$ of z in $\mathcal{M}_0$, $U_z^{(2)} \subset U_z$, such that $F_\zeta^{(1)} v_2 \neq 0$ for every $\zeta \in U_z^{(2)}$. Define

$$e_2(\zeta) = F_\zeta^{(1)} v_2/|F_\zeta^{(1)} v_2|, \quad \zeta \in U_z^{(2)}.$$

Then e_2 is a smooth cross section of the tangent bundle over $U_z^{(2)}$. Moreover, $e_2(\zeta) \perp e_1(\zeta)$ for every $\zeta \in U_z^{(2)}$. Accordingly, we define

$$F_\zeta^{(2)} = F_\zeta - e_1(\zeta) \otimes e_1(\zeta) - e_2(\zeta) \otimes e_2(\zeta), \quad \zeta \in U_z^{(2)}.$$

In the case $d \geq 3$, we have $T_z \ominus \mathrm{span}\{e_1(z), e_2(z)\} \neq \{0\}$, and we can pick a unit vector $v_3 \in T_z \ominus \mathrm{span}\{e_1(z), e_2(z)\}$ to repeat this process. Thus, inductively, we obtain a smooth local frame for the tangent bundle near z. We summarize the result as follows:

Proposition 3.1. *For each $z \in \mathcal{M}_0$, there exist an open neighborhood V_z of z in $\mathcal{M}_0$ and vectors $\{e_1(\zeta), \dots, e_d(\zeta)\} \subset \mathbf{C}^n$, $\zeta \in V_z$, which have the following properties*:

(1) *For each $1 \leq i \leq d$, the map $\zeta \mapsto e_i(\zeta)$ is smooth on V_z.*
(2) *For every $\zeta \in V_z$, $\{e_1(\zeta), \dots, e_d(\zeta)\}$ is an orthonormal basis for T_ζ.*
(3) *We have $e_1(\zeta) = p_\zeta/|p_\zeta|$ for every $\zeta \in V_z$.*

Obviously, the above construction of local frame is just a smoothly parametrized version of the Gram-Schmidt process with a privileged $e_1(\zeta)$.

Since $G(0) = z$, once we have the open neighborhood V_z of z in Proposition 3.1, there is an $a_1 \in (0, a)$ such that $GB_d(0, a_1) \subset V_z$. By the open mapping theorem, $GB_d(0, a_1) \supset \mathcal{M}_0 \cap B(z, b)$ for some $b > 0$. That is, there are $0 < a_1 < a$ and $b > 0$ such that

$$\mathcal{M}_0 \cap B(z, b) \subset GB_d(0, a_1) \subset V_z \subset \mathcal{M}_0. \tag{3.4}$$

Proposition 3.2. *We have* $\zeta - w - F_\zeta(\zeta - w) = O(|\zeta - w|^2)$ *for* $\zeta, w \in \mathcal{M}_0 \cap B(z, b)$.

Proof. By (3.4), there are $x, y \in B_d(0, a_1)$ such that $\zeta = G(x)$ and $w = G(y)$. By the first order Taylor expansion,

$$w-\zeta = G(y)-G(x) = (DG)(x)(y-x)+\int_0^1 \{(DG)(x+t(y-x))-(DG)(x)\}(y-x)dt. \tag{3.5}$$

Since $\zeta = G(x)$, we have $(DG)(x)(y - x) \in T_\zeta$. Consequently,

$$F_\zeta(w - \zeta) = (DG)(x)(y - x) + \int_0^1 F_\zeta\{(DG)(x + t(y - x)) - (DG)(x)\}(y - x)dt. \tag{3.6}$$

From (3.3) it is easy to deduce that $y - x = O(|\zeta - w|)$. Thus, subtracting (3.5) from (3.6), the desired conclusion follows. □

Proposition 3.3. *Let f be an analytic function on* $B(0, 1 + s)$ (*see* (2.1)). *Then*

$$f(\zeta) - f(w) - \langle(\partial f)(\zeta), \overline{F_\zeta(\zeta - w)}\rangle = O(|\zeta - w|^2)$$

for $\zeta, w \in \mathcal{M}_0 \cap B(z, b)$.

Proof. By Proposition 3.2, it suffices to show that

$$f(\zeta) - f(w) - \langle(\partial f)(\zeta), \overline{\zeta - w}\rangle = O(|\zeta - w|^2) \quad \text{for} \quad \zeta, w \in \mathcal{M}_0 \cap B(z, b). \tag{3.7}$$

Again, let $x, y \in B_d(0, a_1)$ be such that $\zeta = G(x)$ and $w = G(y)$. Denote $g = f \circ G$. Then

$$\begin{aligned} f(w) - f(\zeta) &= g(y) - g(x) \\ &= \langle(\partial g)(x), \overline{y - x}\rangle + \int_0^1 \langle(\partial g)(x + t(y - x)) - (\partial g)(x), \overline{y - x}\rangle dt. \end{aligned} \tag{3.8}$$

By the chain rule of differentiation,

$$\langle(\partial g)(x), \overline{y - x}\rangle = \langle(\partial f)(G(x)), \overline{(DG)(x)(y - x)}\rangle = \langle(\partial f)(\zeta), \overline{w - \zeta}\rangle + O(|\zeta - w|^2),$$

where the second = follows from (3.5). Substituting this in (3.8), we obtain (3.7). □

Proposition 3.4. *Let f be an analytic function on* $B(0, 1 + s)$. *Then*

$$\langle(\partial f)(\zeta), \overline{F_\zeta(\zeta - w)}\rangle = \langle F_\zeta(\zeta - w), \overline{(\partial f)(\zeta)}\rangle = \sum_{i=1}^d \langle \zeta - w, e_i(\zeta)\rangle\langle(\partial f)(\zeta), \overline{e_i(\zeta)}\rangle$$

for $\zeta, w \in \mathcal{M}_0 \cap B(z,b)$.

Proof. This follows immediately from the fact that $F_\zeta = \sum_{i=1}^d e_i(\zeta) \otimes e_i(\zeta)$, $\zeta \in V_z$. □

Proposition 3.5. *For* $\zeta, w \in \mathcal{M}_0 \cap B(z,b) \cap \mathbf{B}$, *we have* $\langle \zeta - w, e_1(\zeta)\rangle = O(|1 - \langle \zeta, w\rangle|)$.

Proof. By (3.2), it suffices to show that $\langle \zeta - w, p_\zeta\rangle = O(|1 - \langle \zeta, w\rangle|)$. By definition, $p_\zeta = F_\zeta \zeta$. Applying Proposition 3.2, we have

$$\begin{aligned}\langle \zeta - w, p_\zeta\rangle &= \langle \zeta - w, F_\zeta \zeta\rangle = \langle F_\zeta(\zeta - w), \zeta\rangle = \langle \zeta - w, \zeta\rangle + O(|\zeta - w|^2)\\ &= (1 - \langle w, \zeta\rangle) - (1 - |\zeta|^2) + O(|\zeta - w|^2).\end{aligned}$$

This completes the proof. □

We now pick a pair of $0 < \delta < \delta_1 < b$ and fix the following:

Definition 3.6. (1) Let φ_z be a C^∞-function on $\mathbf{C}^n$ satisfying the conditions $0 \le \varphi_z \le 1$ on $\mathbf{C}^n$, $\varphi_z = 1$ on $B(z,\delta)$, and $\varphi_z = 0$ on $\mathbf{C}^n \backslash B(z,\delta_1)$.
(2) Denote $W_z = \mathcal{M}_0 \cap B(z,\delta)$.

We further introduce

Definition 3.7. (1) We extend the $e_1, \dots, e_d$ in Proposition 3.1 to vector-valued functions on the entire $\tilde{M}$ by setting $e_i = 0$ on $\tilde{M} \backslash V_z$, $1 \le i \le d$.
(2) With the definition of $e_1, \dots, e_d$ extended as in (1), we define the functions $\epsilon_1, \dots, \epsilon_d$ on $\tilde{M}$ by the formula $\epsilon_i = \varphi_z e_i$ for $1 \le i \le d$.

Under Definition 3.7, $\epsilon_1, \dots, \epsilon_d$ are vector-valued Lipschitz functions on M.

Definition 3.8. For any analytic function f on $B(0, 1+s)$, we define the functions $D_1 f, \dots, D_d f$ on M by the formula

$$(D_i f)(\zeta) = \langle (\partial f)(\zeta), \overline{\epsilon_i(\zeta)}\rangle$$

for $\zeta \in M$ and $1 \le i \le d$.

Definition 3.9. Let A be a bounded operator on $L^2(\mu)$. For each $1 \le i \le d$, we write

$$C_i(A) = \sum_{j=1}^n \hat{M}_{\bar{\epsilon}_{i,j}} [\hat{M}_{\zeta_j}, A],$$

where $\epsilon_{i,1}, \dots, \epsilon_{i,n}$ are the components of the vector-valued function ϵ_i.

Proposition 3.10. *Let f be an analytic function on $B(0,1+s)$. Then*

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, \hat{T}_\mu]\hat{M}_{\varphi_z} = \sum_{i=2}^d \hat{M}_{D_i f} C_i(\hat{T}_\mu)\hat{M}_{\varphi_z} + L, \tag{3.9}$$

where $L \in \mathcal{C}_p$ for every $p > d$. Moreover, the operator L has the property that for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$, $E[\hat{M}_h, L] \in \mathcal{C}_t$.

Proof. Obviously, $\hat{M}_{\varphi_z^2}[\hat{M}_f, \hat{T}_\mu]\hat{M}_{\varphi_z}$ is the operator on $L^2(\mu)$ with the function

$$\frac{\varphi_z^2(\zeta)(f(\zeta) - f(w))\varphi_z(w)}{(1 - \langle \zeta, w\rangle)^n} \tag{3.10}$$

as its integral kernel. Similarly, $\sum_{i=1}^d \hat{M}_{D_i f} C_i(\hat{T}_\mu)\hat{M}_{\varphi_z}$ is the operator on $L^2(\mu)$ with the function

$$\sum_{i=1}^d \frac{(D_i f)(\zeta)\langle \zeta - w, \epsilon_i(\zeta)\rangle\varphi_z(w)}{(1 - \langle \zeta, w\rangle)^n} \tag{3.11}$$

as its integral kernel. If we write the difference of (3.10) and (3.11) as

$$\frac{u(\zeta, w)}{(1 - \langle \zeta, w\rangle)^n}, \tag{3.12}$$

then it follows from Propositions 3.4 and 3.3 that $u(\zeta, w) = O(|1 - \langle \zeta, w\rangle|)$. Denote

$$L_1 = \hat{M}_{\varphi_z}[\hat{M}_f, \hat{T}_\mu]\hat{M}_{\varphi_z} - \sum_{i=1}^d \hat{M}_{D_i f} C_i(\hat{T}_\mu)\hat{M}_{\varphi_z}.$$

Then (3.12) is the integral kernel for L_1. By Proposition 2.17, the fact $u(\zeta, w) = O(|1 - \langle \zeta, w\rangle|)$ implies

(a) $L_1 \in \mathcal{C}_p$ for every $p > d$;

(b) $E[\hat{M}_h, L_1] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$.

Denote $L_2 = \hat{M}_{D_1 f} C_1(\hat{T}_\mu)\hat{M}_{\varphi_z}$. Then L_2 has the function

$$\frac{(D_1 f)(\zeta)\langle \zeta - w, \epsilon_1(\zeta)\rangle\varphi_z(w)}{(1 - \langle \zeta, w\rangle)^n}$$

as its integral kernel. By Proposition 3.5, $(D_1 f)(\zeta)\langle \zeta - w, \epsilon_1(\zeta)\rangle\varphi_z(w) = O(|1 - \langle \zeta, w\rangle|)$. Thus it follows from Proposition 2.17 that

(a) $L_2 \in \mathcal{C}_p$ for every $p > d$;

(b) $E[\hat{M}_h, L_2] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$.

Since (3.9) holds for $L = L_1 + L_2$, the proposition is proved. □

Proposition 3.11. *Let f be an analytic function on $B(0, 1 + s)$. Then for every $A \in \{\hat{T}_\mu^{1/2}, \mathcal{T}, E\}$, we have*

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, A]\hat{M}_{\varphi_z} = \sum_{i=2}^d \hat{M}_{D_i f} C_i(A)\hat{M}_{\varphi_z} + L_A,$$

where $L_A \in \mathcal{C}_p$ for every $p > d$. Moreover, the operator L_A has the property that for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$, $E[\hat{M}_h, L_A] \in \mathcal{C}_t$.

Proof. We will work out the details for the case $A = \mathcal{T}$; the other two cases are similar. As before, we write $\mathcal{T}$ in the form

$$\mathcal{T} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1} d\lambda,$$

where the contour γ is the same as in the proof of Proposition 2.18. Then

$$[\hat{M}_f, \mathcal{T}] = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1}[\hat{M}_f, \hat{T}_\mu](\lambda - \hat{T}_\mu)^{-1} d\lambda.$$

Since $\varphi_z \in \mathrm{Lip}(M)$, it follows from Proposition 2.16 that

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, \mathcal{T}]\hat{M}_{\varphi_z} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1}\hat{M}_{\varphi_z^2}[\hat{M}_f, \hat{T}_\mu]\hat{M}_{\varphi_z}(\lambda - \hat{T}_\mu)^{-1} d\lambda + L_1,$$

where $L_1 \in \mathcal{C}_p$ for every $p > d$. Moreover, by Propositions 2.16 and 2.18, L_1 has the property that $[\hat{M}_h, L_1] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$. Applying Proposition 3.10,

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, \mathcal{T}]\hat{M}_{\varphi_z} = \sum_{i=2}^{d} \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1}\hat{M}_{D_i f} C_i(\hat{T}_\mu)\hat{M}_{\varphi_z}(\lambda - \hat{T}_\mu)^{-1} d\lambda + L_1 + L_2,$$

where

$$L_2 = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{-1/2}(\lambda - \hat{T}_\mu)^{-1} L (\lambda - \hat{T}_\mu)^{-1} d\lambda.$$

By Propositions 3.10 and 2.16, we have $L_2 \in \mathcal{C}_p$ for every $p > d$ and $E[\hat{M}_h, L_2] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$. Since the functions $D_i f$, $\bar{\epsilon}_{i,j}$ and φ_z are Lipschitz on M, we can "move the corresponding multiplication operators to the other side of $(\lambda - \hat{T}_\mu)^{-1}$". That is, by Propositions 2.16 and 2.18, we have

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, \mathcal{T}]\hat{M}_{\varphi_z} = \sum_{i=2}^{d} \hat{M}_{D_i f} C_i(\mathcal{T})\hat{M}_{\varphi_z} + L_1 + L_2 + L_3,$$

where L_3 has the properties that $L_3 \in \mathcal{C}_p$ for every $p > d$ and that $[\hat{M}_h, L_3] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$. This completes the proof. $\square$

Proposition 3.12. *Let f be an analytic function on $B(0, 1+s)$. Then for every Lipschitz function g on M, we have*

$$E\hat{M}_{\varphi_z^2}[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]]\hat{M}_{\varphi_z} = E\sum_{i=2}^{d} \hat{M}_{D_i f} C_i([\hat{M}_g, \hat{T}_\mu^{1/2}])\hat{M}_{\varphi_z} + \Lambda,$$

where $\Lambda \in \mathcal{C}_p$ *for every* $p > 2d/3$.

Proof. Note that $\hat{M}_{\varphi_z^2}[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]]\hat{M}_{\varphi_z} = [\hat{M}_g, \hat{M}_{\varphi_z^2}[\hat{M}_f, \hat{T}_\mu^{1/2}]\hat{M}_{\varphi_z}]$. Thus it follows from Proposition 3.11 that

$$\hat{M}_{\varphi_z^2}[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]]\hat{M}_{\varphi_z} = \sum_{i=2}^{d}[\hat{M}_g, \hat{M}_{D_i f}C_i(\hat{T}_\mu^{1/2})\hat{M}_{\varphi_z}] + [\hat{M}_g, L_{\hat{T}_\mu^{1/2}}], \tag{3.13}$$

where $L_{\hat{T}_\mu^{1/2}}$ has the property that $E[\hat{M}_h, L_{\hat{T}_\mu^{1/2}}] \in \mathcal{C}_t$ for all $h \in \mathrm{Lip}(M)$ and $t > 2d/3$. In particular, if we let $\Lambda = E[\hat{M}_g, L_{\hat{T}_\mu^{1/2}}]$, then $\Lambda \in \mathcal{C}_t$ for every $t > 2d/3$. Then note that

$$[\hat{M}_g, \hat{M}_{D_i f}C_i(\hat{T}_\mu^{1/2})\hat{M}_{\varphi_z}] = \hat{M}_{D_i f}C_i([\hat{M}_g, \hat{T}_\mu^{1/2}])\hat{M}_{\varphi_z}$$

for every $2 \le i \le d$. Substituting this in (3.13), the proof is complete. $\square$

4. Products of commutators

For each $f \in L^\infty(\mu)$, we define

$$\tilde{T}_f = E\hat{M}_f|\mathcal{P},$$

which can be thought of as a Toeplitz operator on $\mathcal{P}$. Let us also introduce

$$A_f = \tilde{T}_f + \tilde{T}_\mu^{-1/2}[\tilde{T}_f, \tilde{T}_\mu^{1/2}] \quad \text{and} \quad B_f = \tilde{T}_f - [\tilde{T}_f, \tilde{T}_\mu^{1/2}]\tilde{T}_\mu^{-1/2}. \tag{4.1}$$

By (2.10) and (2.11), we have

$$[Q_{\zeta_i}, Q^*_{\zeta_j}] = U[A_{\zeta_i}, B_{\bar{\zeta}_j}]U^*$$

for $i, j \in \{1, \dots, n\}$. This suggests that it will be beneficial to expand the general commutator $[A_f, B_g]$ in terms of operators defined on $L^2(\mu)$. For $f, g \in L^\infty(\mu)$, we have

$$\begin{aligned}
[A_f, B_g] &= [\tilde{T}_f, \tilde{T}_g] - [\tilde{T}_f, [\tilde{T}_g, \tilde{T}_\mu^{1/2}]\tilde{T}_\mu^{-1/2}] + [\tilde{T}_\mu^{-1/2}[\tilde{T}_f, \tilde{T}_\mu^{1/2}], \tilde{T}_g] \\
&\quad - [\tilde{T}_\mu^{-1/2}[\tilde{T}_f, \tilde{T}_\mu^{1/2}], [\tilde{T}_g, \tilde{T}_\mu^{1/2}]\tilde{T}_\mu^{-1/2}] \\
&= [\hat{M}_f, E](1-E)[\hat{M}_g, E] - [\hat{M}_g, E](1-E)[\hat{M}_f, E] \\
&\quad - E[\hat{M}_f, E][\hat{M}_g, \hat{T}_\mu^{1/2}]\mathcal{T} - E[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]]\mathcal{T} - E[\hat{M}_g, \hat{T}_\mu^{1/2}][\hat{M}_f, \mathcal{T}]E \\
&\quad + E[\mathcal{T}, \hat{M}_g][\hat{M}_f, \hat{T}_\mu^{1/2}]E + \mathcal{T}[[\hat{M}_f, \hat{T}_\mu^{1/2}], \hat{M}_g]E + \mathcal{T}[\hat{M}_f, \hat{T}_\mu^{1/2}][E, \hat{M}_g]E \\
&\quad - \mathcal{T}[\hat{M}_f, \hat{T}_\mu^{1/2}]E[\hat{M}_g, \hat{T}_\mu^{1/2}]\mathcal{T} + E[\hat{M}_g, \hat{T}_\mu^{1/2}]\mathcal{T}^2[\hat{M}_f, \hat{T}_\mu^{1/2}]E.
\end{aligned}$$

We enumerate the ten terms after the second $=$ as $H_1(f, g), \dots, H_{10}(f, g)$. Thus

$$[A_f, B_g] = H_1(f, g) + \cdots + H_{10}(f, g). \tag{4.2}$$

It follows from Propositions 2.16 and 2.18 that for all $f, g \in \mathrm{Lip}(M)$ and $1 \leq \nu \leq 10$,

$$H_\nu(f,g) \in \mathcal{C}_p \quad \text{for every} \quad p > d. \tag{4.3}$$

It follows from Propositions 2.16 and 2.18-2.20 that for all $f, g, h \in \mathrm{Lip}(M)$ and $1 \leq \nu \leq 10$,

$$[\hat{M}_h, H_\nu(f,g)] \in \mathcal{C}_t \quad \text{for every} \quad t > 2d/3. \tag{4.4}$$

Next we apply the results in Section 3 to the operators $H_1(f,g), \dots, H_{10}(f,g)$.

Lemma 4.1. *Consider any $\nu \in \{1, \dots, 10\}$. If $g \in \mathrm{Lip}(M)$ and f is an analytic function on $B(0, 1+s)$, then*

$$\hat{M}_{\varphi_z^2} H_\nu(f,g) \hat{M}_{\varphi_z} = \sum_{i=2}^{d} \hat{M}_{D_i f} H_{\nu,i}(g) + L,$$

where $L \in \mathcal{C}_t$ for every $t > 2d/3$, and the operators $H_{\nu,2}(g), \dots, H_{\nu,d}(g)$ have the following properties:

(1) *$H_{\nu,2}(g), \dots, H_{\nu,d}(g)$ are independent of f.*
(2) *$H_{\nu,i}(g) \in \mathcal{C}_p$ for all $p > d$ and $2 \leq i \leq d$.*
(3) *If $h \in \mathrm{Lip}(M)$, then $[\hat{M}_h, H_{\nu,i}(g)] \in \mathcal{C}_t$ for all $t > 2d/3$ and $2 \leq i \leq d$.*

Proof. The argument is similar for all $\nu \in \{1, \dots, 10\}$. Therefore we will present the details only for two of the ν's. Below, the symbols $L_1, L_2, \dots$ represent operators that belong to $\mathcal{C}_t$ for every $t > 2d/3$.

Consider, for example, the case $\nu = 4$. Then

$$\begin{aligned} \hat{M}_{\varphi_z^2} H_4(f,g) \hat{M}_{\varphi_z} &= -\hat{M}_{\varphi_z^2} E[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]] \mathcal{T} \hat{M}_{\varphi_z} \\ &= -E \hat{M}_{\varphi_z^2} [\hat{M}_f, [\hat{M}_g, \hat{T}_\mu^{1/2}]] \hat{M}_{\varphi_z} \mathcal{T} + L_1 \\ &= -E \sum_{i=2}^{d} \hat{M}_{D_i f} C_i([\hat{M}_g, \hat{T}_\mu^{1/2}]) \hat{M}_{\varphi_z} \mathcal{T} + L_2 \\ &= -\sum_{i=2}^{d} \hat{M}_{D_i f} E C_i([\hat{M}_g, \hat{T}_\mu^{1/2}]) \hat{M}_{\varphi_z} \mathcal{T} + L_3. \end{aligned}$$

where the second and the fourth $=$ follow from Propositions 2.16, 2.18 and 2.20, and the third $=$ follows from Proposition 3.12. Define

$$H_{4,i}(g) = -E C_i([\hat{M}_g, \hat{T}_\mu^{1/2}]) \hat{M}_{\varphi_z} \mathcal{T}, \quad 2 \leq i \leq d.$$

The fact that the operators $H_{4,2}(g), \dots, H_{4,d}(g)$ have properties (2) and (3) also follows from Propositions 2.16, 2.18, 2.19 and 2.20.

As a second case, consider $\nu = 5$. Then

$$\begin{aligned}\hat{M}_{\varphi_z^2} H_5(f,g)\hat{M}_{\varphi_z} &= -\hat{M}_{\varphi_z^2} E[\hat{M}_g, \hat{T}_\mu^{1/2}][\hat{M}_f, \mathcal{T}]E\hat{M}_{\varphi_z} \\ &= -E[\hat{M}_g, \hat{T}_\mu^{1/2}]\hat{M}_{\varphi_z^2}[\hat{M}_f, \mathcal{T}]\hat{M}_{\varphi_z}E + L_4 \\ &= -E[\hat{M}_g, \hat{T}_\mu^{1/2}]\sum_{i=2}^{d} \hat{M}_{D_i f} C_i(\mathcal{T})\hat{M}_{\varphi_z}E + L_5 \\ &= -\sum_{i=2}^{d} \hat{M}_{D_i f} E[\hat{M}_g, \hat{T}_\mu^{1/2}]C_i(\mathcal{T})\hat{M}_{\varphi_z}E + L_6,\end{aligned}$$

where the second and the fourth $=$ follow from Propositions 2.16, 2.18 and 2.20, and the third $=$ follows from Propositions 3.11 and 2.16. Define

$$H_{5,i}(g) = -E[\hat{M}_g, \hat{T}_\mu^{1/2}]C_i(\mathcal{T})\hat{M}_{\varphi_z}E, \quad 2 \le i \le d.$$

The fact that the operators $H_{5,2}(g), \dots, H_{5,d}(g)$ have properties (2) and (3) also follows from Propositions 2.16, 2.18, and 2.20. $\square$

For operators A and B, the notation $A \sim_1 B$ means that $A - B$ is in the trace class.

Lemma 4.2. *Let $f_1, \dots, f_d$ be analytic functions on $B(0, 1+s)$. Let z, φ_z, etc, be the same as in Section* 3. *Then for all $g_1, \dots, g_d \in \mathrm{Lip}(M)$, the operator*

$$\hat{M}_{\varphi_z^{3d}} \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}]$$

is in the trace class.

Proof. Let $\nu_1, \dots, \nu_d \in \{1, \dots, 10\}$. Applying (4.3), (4.4) and Lemma 4.1, we have

$$\begin{aligned}\hat{M}_{\varphi_z^{3d}} H_{\nu_1}(f_1, g_1) \cdots H_{\nu_d}(f_d, g_d) &\sim_1 \{\hat{M}_{\varphi_z^2} H_{\nu_1}(f_1, g_1)\hat{M}_{\varphi_z}\} \cdots \{\hat{M}_{\varphi_z^2} H_{\nu_d}(f_d, g_d)\hat{M}_{\varphi_z}\} \\ &\sim_1 \sum_{i_1=2}^{d} \cdots \sum_{i_d=2}^{d} \hat{M}_{D_{i_1} f_1} H_{\nu_1, i_1}(g_1) \cdots \hat{M}_{D_{i_d} f_d} H_{\nu_d, i_d}(g_d) \\ &\sim_1 \sum_{i_1=2}^{d} \cdots \sum_{i_d=2}^{d} \hat{M}_{D_{i_1} f_1} \cdots \hat{M}_{D_{i_d} f_d} H_{\nu_1, i_1}(g_1) \cdots H_{\nu_d, i_d}(g_d). \qquad (4.5)\end{aligned}$$

For each choice of $i_1, \dots, i_d \in \{2, \dots, d\}$, there are $j \ne k$ in $\{1, \dots, d\}$ such that $i_j = i_k$, i.e., $D_{i_j} = D_{i_k}$. Therefore

$$\sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) D_{i_1} f_{\sigma(1)} \cdots D_{i_d} f_{\sigma(d)} = 0.$$

Taking the antisymmetrization of $f_1, \dots, f_d$ in (4.5), we find that

$$\hat{M}_{\varphi_z^{3d}} \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) H_{\nu_1}(f_{\sigma(1)}, g_1) \cdots H_{\nu_d}(f_{\sigma(d)}, g_d) \in \mathcal{C}_1 \qquad (4.6)$$

for any choice of $\nu_1, \ldots, \nu_d \in \{1, \ldots, 10\}$. By (4.2), we have

$$\hat{M}_{\varphi_z^{3d}} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}]$$

$$= \sum_{\nu_1=1}^{10} \cdots \sum_{\nu_d=1}^{10} \hat{M}_{\varphi_z^{3d}} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) H_{\nu_1}(f_{\sigma(1)}, g_1) \cdots H_{\nu_d}(f_{\sigma(d)}, g_d). \tag{4.7}$$

Combining (4.7) and (4.6), the lemma is proved. □

Once Lemma 4.2 is proved, the operator $\hat{M}_{\varphi_z^{3d}}$ can be removed by a standard argument:

Proposition 4.3. *Let $f_1, \ldots, f_d$ be analytic functions on $B(0, 1+s)$. Then for all $g_1, \ldots, g_d \in \mathrm{Lip}(M)$, the operator*

$$\sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}]$$

is in the trace class.

Proof. Pick an $s_1 \in (0, s_0)$ and define $N = \{z \in \tilde{M} : 1 - s_1 \le |z| < 1\}$. Then $\overline{N} = \{z \in \tilde{M} : 1 - s_1 \le |z| \le 1\}$, which is a compact subset of $\mathcal{M}_0$. For each $z \in \overline{N}$, we have the function φ_z and the open set W_z given in Definition 3.6. Since $z \in W_z$ and since $\overline{N}$ is compact, there is a finite subset F of $\overline{N}$ such that $\cup_{z \in F} W_z \supset \overline{N}$.

By Lemma 4.2, we have

$$\sum_{z \in F} \hat{M}_{\varphi_z^{3d}} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}] \in \mathcal{C}_1. \tag{4.8}$$

Recall from Definition 3.6 that $\varphi_z = 1$ on W_z and $0 \le \varphi_z \le 1$ on $\mathbf{C}^n$. Since $\cup_{z \in F} W_z \supset \overline{N}$, we have $\sum_{z \in F} \varphi_z^{3d} \ge \chi_{\overline{N}}$. Therefore (4.8) implies

$$\hat{M}_{\chi_N} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}] \in \mathcal{C}_1.$$

On the other hand, since $s_1 > 0$, it follows from Lemma 2.21 that

$$\hat{M}_{\chi_{M \setminus N}} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_1}] \cdots [A_{f_{\sigma(d)}}, B_{g_d}] \in \mathcal{C}_1.$$

Obviously, the proposition follows from these two memberships. □

Lemma 4.4. *Let $X_1, \ldots, X_k$ be operators such that $[X_i, X_j] = 0$ for all $i, j \in \{1, \ldots, k\}$. Then for any operators $Y_1, \ldots, Y_k$,*

$$[X_1, Y_1, \ldots, X_k, Y_k] = \sum_{\sigma \in S_k} \sum_{\lambda \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\lambda)[X_{\sigma(1)}, Y_{\lambda(1)}] \cdots [X_{\sigma(k)}, Y_{\lambda(k)}].$$

Proof. For each $1 \le j \le k$, let τ_j be the transposition that flips the pair $2j-1$, $2j$. That is, $\tau_j(2j-1) = 2j$, $\tau_j(2j) = 2j-1$, and $\tau_j(i) = i$ for every $i \in \{1, \dots, 2k\} \backslash \{2j-1, 2j\}$. Let T_{2k} be the subgroup of S_{2k} generated by $\tau_1, \dots, \tau_k$. Then

$$[B_1, B_2] \cdots [B_{2k-1}, B_{2k}] = \sum_{\tau \in T_{2k}} \mathrm{sgn}(\tau) B_{\tau(1)} B_{\tau(2)} \cdots B_{\tau(2k-1)} B_{\tau(2k)}$$

for all operators $B_1, B_2, \dots, B_{2k}$. Consequently,

$$\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma) [B_{\sigma(1)}, B_{\sigma(2)}] \cdots [B_{\sigma(2k-1)}, B_{\sigma(2k)}] = 2^k [B_1, B_2, \dots, B_{2k}]. \tag{4.9}$$

Define $A_1, \dots, A_{2k}$ such that $A_{2j-1} = X_j$ and $A_{2j} = Y_j$ for every $1 \le j \le k$. By the commutation property of $X_1, \dots, X_k$, for any $\sigma \in S_{2k}$, we have

$$[A_{\sigma(1)}, A_{\sigma(2)}] \cdots [A_{\sigma(2k-1)}, A_{\sigma(2k)}] = 0$$

unless σ has the property that for every $1 \le j \le k$, the set $\{\sigma(2j-1), \sigma(2j)\}$ contains both an odd number and an even number. For every σ that has this property, define

$$S_{2k}(\sigma) = \{\sigma' \in S_{2k} : \{\sigma'(2j-1), \sigma'(2j)\} = \{\sigma(2j-1), \sigma(2j)\} \text{ for every } 1 \le j \le k\}.$$

Then $\mathrm{card}(S_{2k}(\sigma)) = 2^k$, and there is a unique $\sigma^* \in S_{2k}(\sigma)$ such that $\sigma^*(2j-1)$ is odd and $\sigma^*(2j)$ is even for every $1 \le j \le k$. Obviously, for every $\sigma' \in S_{2k}(\sigma)$,

$$\begin{aligned} \mathrm{sgn}(\sigma')[A_{\sigma'(1)}, &A_{\sigma'(2)}] \cdots [A_{\sigma'(2k-1)}, A_{\sigma'(2k)}] \\ &= \mathrm{sgn}(\sigma^*)[A_{\sigma^*(1)}, A_{\sigma^*(2)}] \cdots [A_{\sigma^*(2k-1)}, A_{\sigma^*(2k)}]. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)[A_{\sigma(1)}, &A_{\sigma(2)}] \cdots [A_{\sigma(2k-1)}, A_{\sigma(2k)}] \\ &= 2^k \sum_{\text{such } \sigma^*} \mathrm{sgn}(\sigma^*)[A_{\sigma^*(1)}, A_{\sigma^*(2)}] \cdots [A_{\sigma^*(2k-1)}, A_{\sigma^*(2k)}] \\ &= 2^k \sum_{\epsilon \in S_k} \sum_{\lambda \in S_k} \mathrm{sgn}(\epsilon)\mathrm{sgn}(\lambda)[X_{\epsilon(1)}, Y_{\lambda(1)}] \cdots [X_{\epsilon(k)}, Y_{\lambda(k)}]. \end{aligned}$$

Combining this with (4.9), the lemma is proved. $\square$

5. Antisymmetric sums on $\mathcal{Q}$

With the preparations in the previous sections, we can now deal with antisymmetric sums on the quotient module $\mathcal{Q}$.

Lemma 5.1. *Let $k, m \in \mathbf{Z}_+$ be such that $k + m = 2d$. Then for any $i_1, \dots, i_k, r_1, \dots, r_m \in \{1, \dots, n\}$, the antisymmetric sum*

$$[Q_{\zeta_{i_1}}, \dots, Q_{\zeta_{i_k}}, Q^*_{\zeta_{r_1}}, \dots, Q^*_{\zeta_{r_m}}]$$

is in the trace class.

Proof. By identity (4.9), this antisymmetric sum is 0 unless $k = m = d$. Thus it suffices to show that for any $i_1, \ldots, i_d$ and $r_1, \ldots, r_d$ in $\{1, \ldots, n\}$, we have

$$[Q_{\zeta_{i_1}}, Q^*_{\zeta_{r_1}}, \ldots, Q_{\zeta_{i_d}}, Q^*_{\zeta_{r_d}}] \in \mathcal{C}_1.$$

To prove this membership, we define

$$f_j(\zeta) = \zeta_{i_j} \quad \text{and} \quad g_j(\zeta) = \bar{\zeta}_{r_j}$$

for $1 \le j \le d$. It follows from (2.10), (2.11) and (4.1) that

$$U^*[Q_{\zeta_{i_1}}, Q^*_{\zeta_{r_1}}, \ldots, Q_{\zeta_{i_d}}, Q^*_{\zeta_{r_d}}]U = [A_{f_1}, B_{g_1}, \ldots, A_{f_d}, B_{g_d}].$$

Since $Q_{\zeta_{i_j}} = UA_{f_j}U^*$ and $Q^*_{\zeta_{r_j}} = UB_{g_j}U^*$ for every $1 \le j \le d$, we have $[A_{f_j}, A_{f_k}] = 0$ and $[B_{g_j}, B_{g_k}] = 0$ for all $j, k \in \{1, \ldots, d\}$. Applying Lemma 4.4, we have

$$U^*[Q_{\zeta_{i_1}}, Q^*_{\zeta_{r_1}}, \ldots, Q_{\zeta_{i_d}}, Q^*_{\zeta_{r_d}}]U = \sum_{\lambda \in S_d} \operatorname{sgn}(\lambda) \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)[A_{f_{\sigma(1)}}, B_{g_{\lambda(1)}}] \cdots [A_{f_{\sigma(d)}}, B_{g_{\lambda(d)}}].$$

Proposition 4.3 tells us that this operator is in the trace class. □

Proposition 5.2. [24, Proposition 7.2] *For $f, g \in \operatorname{Lip}(S)$, we have $[Q_f, Q_g] \in \mathcal{C}_p$ for every $p > d$.*

Proposition 5.3. [24, Proposition 7.3] *Suppose that $d \ge 2$. Then for $f, g, h \in \operatorname{Lip}(S)$ we have $[Q_h, [Q_f, Q_g]] \in \mathcal{C}_p$ for every $p > 2d/3$.*

Proposition 5.4. *For $f, g \in \operatorname{Lip}(S)$, we have $Q_{fg} - Q_f Q_g \in \mathcal{C}_p$ for every $p > d$.*

Proof. This follows from the identity

$$Q_{fg} - Q_f Q_g = QM_f(1 - Q)M_g Q = [Q, M_f](1 - Q)[M_g, Q] \tag{5.1}$$

and Proposition 2.7. □

Proposition 5.5. *Let $f, g, h \in \operatorname{Lip}(S)$. If $d \ge 2$, then $[Q_h, Q_{fg} - Q_f Q_g] \in \mathcal{C}_p$ for every $p > 2d/3$. If $d = 1$, then $[Q_h, Q_{fg} - Q_f Q_g] \in \mathcal{C}_1$.*

Proof. Continuing with (5.1), we have

$$\begin{aligned}
[Q_h, Q_{fg} - Q_f Q_g] &= Q[M_h, [Q, M_f](1 - Q)[M_g, Q]]Q \\
&= Q[M_h, [Q, M_f]](1 - Q)[M_g, Q]Q - Q[Q, M_f][M_h, Q][M_g, Q]Q \\
&\quad + Q[Q, M_f](1 - Q)[M_h, [M_g, Q]]Q.
\end{aligned}$$

Thus the desired conclusion follows from Proposition 2.7 and Theorem 2.8. □

Lemma 5.6. *Let $k \in \mathbf{N}$. For every pair of $1 \le i \le 2d$ and $1 \le j \le k$, let $A_{i,j}$ be an operator in the collection $\{1, Q_{\zeta_1}, \dots, Q_{\zeta_n}, Q^*_{\zeta_1}, \dots, Q^*_{\zeta_n}\}$. Define*

$$A_i = A_{i,1} \cdots A_{i,k}$$

for $i = 1, 2, \dots, 2d$. Then the antisymmetric sum $[A_1, A_2, \dots, A_{2d}]$ is in the trace class.

Proof. For every pair of $1 \le i \le 2d$ and $1 \le j \le k$, define

$$B_{i,j} = A_{i,1} \cdots A_{i,j-1} A_{i,j+1} \cdots A_{i,k}.$$

In other words, $B_{i,j}$ is obtained from A_i by replacing the factor $A_{i,j}$ by 1. It follows from the "product rule" for commutators and Propositions 5.2 and 5.3 that

$$\begin{aligned}
&[A_1, A_2] \cdots [A_{2d-1}, A_{2d}] \\
&\qquad = [A_{1,1} \cdots A_{1,k}, A_{2,1} \cdots A_{2,k}] \cdots [A_{2d-1,1} \cdots A_{2d-1,k}, A_{2d,1} \cdots A_{2d,k}] \\
&\qquad \sim_1 \sum_{j_1,\dots,j_{2d}=1}^{k} [A_{1,j_1}, A_{2,j_2}] \cdots [A_{2d-1,j_{2d-1}}, A_{2d,j_{2d}}] B_{1,j_1} B_{2,j_2} \cdots B_{2d,j_{2d}}.
\end{aligned}$$

Let $\sigma \in S_{2d}$. Then the map $(j_1, \dots, j_{2d}) \mapsto (j_{\sigma(1)}, \dots, j_{\sigma(2d)})$ is injective on the product set $\{1, \dots, k\}^{2d}$, hence surjective also. Therefore

$$\begin{aligned}
&[A_{\sigma(1)}, A_{\sigma(2)}] \cdots [A_{\sigma(2d-1)}, A_{\sigma(2d)}] \\
&\qquad \sim_1 \sum_{j_1,\dots,j_{2d}=1}^{k} [A_{\sigma(1),j_1}, A_{\sigma(2),j_2}] \cdots [A_{\sigma(2d-1),j_{2d-1}}, A_{\sigma(2d),j_{2d}}] \\
&\qquad \times B_{\sigma(1),j_1} B_{\sigma(2),j_2} \cdots B_{\sigma(2d),j_{2d}} \\
&\qquad = \sum_{j_1,\dots,j_{2d}=1}^{k} [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2d-1),j_{\sigma(2d-1)}}, A_{\sigma(2d),j_{\sigma(2d)}}] \\
&\qquad \times B_{\sigma(1),j_{\sigma(1)}} B_{\sigma(2),j_{\sigma(2)}} \cdots B_{\sigma(2d),j_{\sigma(2d)}} \\
&\qquad \sim_1 \sum_{j_1,\dots,j_{2d}=1}^{k} [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2d-1),j_{\sigma(2d-1)}}, A_{\sigma(2d),j_{\sigma(2d)}}] \\
&\qquad \times B_{1,j_1} B_{2,j_2} \cdots B_{2d,j_{2d}},
\end{aligned}$$

where the second $\sim_1$ follows from Proposition 5.2. By (4.9), we have

$$\begin{aligned}
&[A_1, A_2, \dots, A_{2d}] = 2^{-d} \sum_{\sigma \in S_{2d}} \operatorname{sgn}(\sigma) [A_{\sigma(1)}, A_{\sigma(2)}] \cdots [A_{\sigma(2d-1)}, A_{\sigma(2d)}] \\
&\quad \sim_1 2^{-d} \sum_{j_1,\dots,j_{2d}=1}^{k} \sum_{\sigma \in S_{2d}} \operatorname{sgn}(\sigma) [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2d-1),j_{\sigma(2d-1)}}, A_{\sigma(2d),j_{\sigma(2d)}}] \\
&\quad \times B_{1,j_1} B_{2,j_2} \cdots B_{2d,j_{2d}} \\
&\quad = \sum_{j_1,\dots,j_{2d}=1}^{k} [A_{1,j_1}, A_{2,j_2}, \dots, A_{2d,j_{2d}}] B_{1,j_1} B_{2,j_2} \cdots B_{2d,j_{2d}}.
\end{aligned}$$

Applying Lemma 5.1 to each $[A_{1,j_1}, A_{2,j_2}, \ldots, A_{2d,j_{2d}}]$, we obtain the membership $[A_1, A_2, \ldots, A_{2d}] \in \mathcal{C}_1$. $\square$

Lemma 5.7. *Suppose that A_1, A_2 and $B_2, \ldots, B_{2d}$ are finite products of operators in the collection $\{Q_h : h \in \mathrm{Lip}(S)\}$. Then for every pair of $f, g \in \mathrm{Lip}(S)$, we have*

$$[A_1(Q_{fg} - Q_f Q_g)A_2, B_2, \ldots, B_{2d}] \in \mathcal{C}_1.$$

Proof. Denote $B_1 = A_1(Q_{fg} - Q_f Q_g)A_2$. By (4.9),

$$[B_1, B_2, \ldots, B_{2d}] = 2^{-d} \sum_{\sigma \in S_{2d}} \mathrm{sgn}(\sigma)[B_{\sigma(1)}, B_{\sigma(2)}] \cdots [B_{\sigma(2d-1)}, B_{\sigma(2d)}]. \tag{5.2}$$

By Propositions 5.2, 5.4 and 5.5, for each $\sigma \in S_{2d}$, one of the commutators among

$$[B_{\sigma(1)}, B_{\sigma(2)}], \ldots, [B_{\sigma(2d-1)}, B_{\sigma(2d)}]$$

is in $\mathcal{C}_t$ for every $t > 2d/3$, while every other commutator is in $\mathcal{C}_p$ for every $p > d$. Therefore every term on the right-hand side of (5.2) is in the trace class. Consequently, so is $[B_1, B_2, \ldots, B_{2d}]$. $\square$

Proof of Theorem 1.3. For $p_1, \ldots, p_d, q_1, \ldots, q_d \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, the membership

$$[Q_{p_1}, Q^*_{q_1}, \ldots, Q_{p_d}, Q^*_{q_d}] \in \mathcal{C}_1$$

is an immediate consequence of Lemma 5.6. For $u_1, \ldots, u_{2d}, v_1, \ldots, v_{2d} \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, Lemma 5.6 also implies that

$$[Q_{u_1}Q^*_{v_1}, Q_{u_2}Q^*_{v_2}, \ldots, Q_{u_{2d}}Q^*_{v_{2d}}] \in \mathcal{C}_1.$$

Combining this with Lemma 5.7, we find that

$$[Q_{u_1\bar{v}_1}, Q_{u_2\bar{v}_2}, \ldots, Q_{u_{2d}\bar{v}_{2d}}] \in \mathcal{C}_1.$$

By linearity, this implies that for $f_1, f_2, \ldots, f_{2d} \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, we have

$$[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2d}}] \in \mathcal{C}_1.$$

This completes the proof. $\square$

6. Green's theorem for non-simple curves

First of all, the material in this section should be considered as expository.

Denote $D = \{z \in \mathbf{C} : |z| < 1\}$, the unit disc in $\mathbf{C}$. Suppose that $f, g \in C^1(\tilde{D})$, where $\tilde{D} = \{z \in \mathbf{C} : |z| < \rho\}$ for some $\rho > 1$. In [16], Helton and Howe proved the trace formula

$$\mathrm{tr}[T_f, T_g] = \frac{1}{2\pi i} \int_D df \wedge dg \tag{6.1}$$

for Toeplitz operators. This was what motivated Theorem 1.4, and the hope in [24] was that there might be an analogue of (6.1) for (1.2). But for (1.2), this is certainly a tricky proposition, because, unlike the unit disc D, M may have singularities, which would be a problem for integration.

On the other hand, by Green's theorem, one can rewrite (6.1) in the form

$$\operatorname{tr}[T_f, T_g] = \frac{1}{2\pi i}\int_{\mathbf{T}} f dg. \tag{6.2}$$

For the quotient module $\mathcal{Q}$ and the range space $\mathcal{P}$, the analogue of $\mathbf{T}$ is $X = \tilde{M} \cap S$, which is a smooth manifold by our assumption on $\tilde{M}$, and which does not present a problem for integration. Thus it makes more sense to look for an analogue of (6.2) for (1.2), and this was what eventually led to Theorems 1.5 and 1.6.

Let H^2 be the classic Hardy space on the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. As usual, we respectively write T_f and H_f for the Toeplitz operator and the Hankel operator with symbol f on H^2. It is well known that

$$[T_f, T_g] = H^*_{\bar{g}} H_f - H^*_{\bar{f}} H_g = P[M_f, P][M_g, P]P - P[M_g, P][M_f, P]P \tag{6.3}$$

for all $f, g \in L^\infty(\mathbf{T})$, where $P : L^2 \to H^2$ is the orthogonal projection.

Let us retrace the steps on [16, pages 150-151] in detail. Write z for the coordinate function on $\mathbf{T}$. From (6.3) it is easy to deduce that for all $j, k \in \mathbf{Z}_+$, the commutator $[T_{\bar{z}^j}, T_{z^k}]$ is a finite-rank operator. Moreover, if $j \neq k$, then $\operatorname{tr}[T_{\bar{z}^j}, T_{z^k}] = 0$, and $\operatorname{tr}[T_{\bar{z}^j}, T_{z^j}] = j$ for every $j \in \mathbf{Z}_+$. Hence

$$\operatorname{tr}[T_{\bar{p}}, T_p] = \sum_{j=0}^{m} j|c_j|^2 \quad \text{for polynomial} \quad p(z) = \sum_{j=0}^{m} c_j z^j. \tag{6.4}$$

Alternatively, we can write

$$\operatorname{tr}[T_{\bar{p}}, T_p] = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{p} dp \tag{6.5}$$

for every $p \in \mathbf{C}[z]$, where the integral on the right-hand side is taken in the sense of Riemann-Stieltjes. From (6.4) and (6.3) we deduce

$$\|H_{\bar{p}}\|_2^2 = \sum_{j=0}^{m} j|c_j|^2 \quad \text{if} \quad p(z) = \sum_{j=0}^{m} c_j z^j, \tag{6.6}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

Now consider any C^1-function f on $\mathbf{T}$. We have the representation

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j.$$

For each $k \in \mathbf{N}$, define

$$f_k(z) = \sum_{j=-k}^{k} a_j z^j.$$

Then it follows from (6.6) that

$$\lim_{k\to\infty} \|H_{\bar{f}} - H_{\bar{f}_k}\|_2 = 0 \quad \text{and} \quad \lim_{k\to\infty} \|H_f - H_{f_k}\|_2 = 0. \tag{6.7}$$

In particular, H_f and $H_{\bar{f}}$ are Hilbert-Schmidt operators. By (6.3), the commutator $[T_{\bar{f}}, T_f]$ is in the trace class. Moreover, it follows from (6.7) and (6.3) that

$$\lim_{k\to\infty} \|[T_{\bar{f}}, T_f] - [T_{\bar{f}_k}, T_{f_k}]\|_1 = 0.$$

From (6.5) it is easy to deduce that

$$\mathrm{tr}[T_{\bar{f}_k}, T_{f_k}] = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f}_k df_k$$

for every $k \in \mathbf{N}$. It is obvious that

$$\lim_{k\to\infty} \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f}_k df_k = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f} df.$$

Summarizing the above, we have

Lemma 6.1. *If $f \in C^1(\mathbf{T})$, then the commutator $[T_{\bar{f}}, T_f]$ is in the trace class with*

$$\mathrm{tr}[T_{\bar{f}}, T_f] = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f} df.$$

By approximation, the condition $f \in C^1(\mathbf{T})$ in Lemma 6.1 can be weakened:

Proposition 6.2. *Let f be a Lipschitz function on $\mathbf{T}$. Then the commutator $[T_{\bar{f}}, T_f]$ is in the trace class, and we have*

$$\mathrm{tr}[T_{\bar{f}}, T_f] = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f} df. \tag{6.8}$$

Proof. For $g \in L^1(\mathbf{T})$ and $0 \le r < 1$, we denote

$$g_r(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i(t+s)})\frac{1-r^2}{|1-re^{is}|^2} ds.$$

Let $f \in \mathrm{Lip}(\mathbf{T})$. Then $f_r \in C^1(\mathbf{T})$ for $0 < r < 1$. By Lemma 6.1, we have $[T_{\bar{f}_r}, T_{f_r}] \in \mathcal{C}_1$ and

$$\mathrm{tr}[T_{\bar{f}_r}, T_{f_r}] = \frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f}_r df_r. \tag{6.9}$$

If we write $L(f)$ and $L(f_r)$ for the Lipschitz constants of f and f_r, then it is easy to see that $L(f_r) \leq L(f)$. Let $K(e^{it}, e^{iu})$ and $K_r(e^{it}, e^{iu})$ be the integral kernels for the commutators $[M_f, P]$ and $[M_{f_r}, P]$ respectively. It is easy to see that

$$|K(e^{it}, e^{iu})| \leq L(f) \quad \text{and} \quad |K_r(e^{it}, e^{iu})| \leq L(f_r) \leq L(f)$$

when $e^{it} \neq e^{iu}$. Also, when $e^{it} \neq e^{iu}$, we have $K_r(e^{it}, e^{iu}) \to K(e^{it}, e^{iu})$ as $r \uparrow 1$. Thus it follows from the dominated convergence theorem that

$$\lim_{r\uparrow 1} \|[M_f, P] - [M_{f_r}, P]\|_2 = 0 \ \text{ and, consequently, } \ \lim_{r\uparrow 1} \|[M_{\bar{f}}, P] - [M_{\bar{f}_r}, P]\|_2 = 0.$$

Combining this with (6.3), we find that

$$\lim_{r\uparrow 1} \|[T_{\bar{f}}, T_f] - [T_{\bar{f}_r}, T_{f_r}]\|_1 = 0. \tag{6.10}$$

In particular, $[T_{\bar{f}}, T_f] \in \mathcal{C}_1$.

For any $\varphi \in \mathrm{Lip}(\mathbf{T})$, define

$$(D\varphi)(e^{it}) = \lim_{\delta \to 0} \frac{1}{\delta}(\varphi(e^{i(t+\delta)}) - \varphi(e^{it}))$$

at each $e^{it} \in \mathbf{T}$ where the above limit exists. The condition $\varphi \in \mathrm{Lip}(\mathbf{T})$ implies $D\varphi \in L^\infty(\mathbf{T})$. By the Poisson integral formula, the Lipschitz condition for φ and the dominated convergence theorem, we have $D(\varphi_r) = (D\varphi)_r$ for every $0 < r < 1$. Consequently, $\|D(\varphi_r)\|_\infty \leq \|D\varphi\|_\infty$ for every $0 < r < 1$. By the properties of the Poisson integral,

$$\lim_{r\uparrow 1} (D\varphi)_r = D\varphi \quad \text{a.e. on} \quad \mathbf{T}.$$

Applying these facts to the f under consideration and using the properties of Riemann-Stieltjes integral, we have

$$\begin{aligned} \lim_{r\uparrow 1} \frac{1}{2\pi i} \int_{\mathbf{T}} \bar{f}_r df_r &= \lim_{r\uparrow 1} \frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{f_r(e^{it})}(Df_r)(e^{it})dt = \lim_{r\uparrow 1} \frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{f_r(e^{it})}(Df)_r(e^{it})dt \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{f(e^{it})}(Df)(e^{it})dt = \frac{1}{2\pi i} \int_{\mathbf{T}} \bar{f} df, \end{aligned} \tag{6.11}$$

where the third $=$ follows from the dominated convergence theorem. Combining (6.9), (6.10) and (6.11), we obtain (6.8). $\square$

Next we review the Carey-Pincus theory of principal functions [4,5,19]. Suppose that A, B are bounded self-adjoint operators such that the commutator $[A, B]$ is in the trace class. Then there exists a real-valued $g \in L^1(\mathbf{R}^2)$, which is called the *principal function* for the pair A, B, such that

$$\mathrm{tr}[p(A, B), q(A, B)] = \frac{-1}{2\pi i} \iint \{p, q\}(x, y) g(y, x) dx dy \tag{6.12}$$

for all polynomials $p, q \in \mathbf{C}[x, y]$, where

$$\{p, q\}(x, y) = \frac{\partial p}{\partial x}(x, y)\frac{\partial q}{\partial y}(x, y) - \frac{\partial p}{\partial y}(x, y)\frac{\partial q}{\partial x}(x, y).$$

It is known that g is supported on the spectrum of the operator $T = A + iB$. More important, for each point (x, y) such that $x + iy$ is not in the essential spectrum of T,

$$g(y, x) = \mathrm{index}(T - (x + iy)). \tag{6.13}$$

See [4, Theorem 4], or [5, Theorem 8.1].

We now apply the Carey-Pincus theory to Toeplitz operators on H^2. We begin with any Lipschitz function f on $\mathbf{T}$. Consider the operators

$$A = T_{\mathrm{Re}(f)}, \quad B = T_{\mathrm{Im}(f)} \quad \text{and} \quad T = A + iB.$$

Then, of course, $T = T_f$. By Proposition 6.2, we have $[A, B] = (2i)^{-1}[T_{\bar{f}}, T_f] \in \mathcal{C}_1$.

Since $f \in \mathrm{Lip}(\mathbf{T})$, the two-dimensional Lebesgue measure of the set $f(\mathbf{T}) = \{f(z) : z \in \mathbf{T}\}$ is zero. It is well known that the essential spectrum of T_f equals $f(\mathbf{T})$ [8]. Thus the two-dimensional Lebesgue measure of the essential spectrum of T_f is zero. Let g be the principal function for this pair A, B. By (6.13), for each $(x, y) \in \mathbf{R}^2$ such that $x + iy \notin f(\mathbf{T})$, we have

$$g(y, x) = \mathrm{index}(T_f - (x + iy)) = -\{\text{the winding number of } f \text{ about } x + iy\},$$

where the second = is a well-known fact about Toeplitz operators on H^2 [8]. This motivates us to define the function

$$w(f; x + iy) = \begin{cases} \text{the winding number of } f \text{ about } x + iy & \text{if} \quad x + iy \notin f(\mathbf{T}) \\ 0 & \text{if} \quad x + iy \in f(\mathbf{T}) \end{cases}. \tag{6.14}$$

With this function, we can rewrite the above identity in the form

$$g(y, x) = -w(f; x + iy) \quad \text{for a.e.} \quad (x, y) \in \mathbf{R}^2.$$

Applying (6.12) to the case $p(x, y) = x$ and $q(x, y) = y$, we now have

$$\mathrm{tr}[T_{\bar{f}}, T_f] = 2i\mathrm{tr}[A, B] = \frac{-1}{\pi}\iint g(y, x)dxdy = \frac{1}{\pi}\iint w(f; x + iy)dxdy.$$

Combining this with Proposition 6.2, we obtain

Proposition 6.3. *Let $f \in \mathrm{Lip}(\mathbf{T})$. Then the function $w(f; x + iy)$ defined by* (6.14) *is in $L^1(\mathbf{R}^2)$, and we have the identify*

$$\frac{1}{2\pi i}\int_{\mathbf{T}} \bar{f}df = \frac{1}{\pi}\iint w(f; x + iy)dxdy. \tag{6.15}$$

If the function $f : \mathbf{T} \to \mathbf{C}$ is C^1 and one-to-one, i.e, a smooth Jordan curve, then (6.15) obviously follows from Green's theorem. Thus one can regard Proposition 6.3 as a specialized version of Green's theorem for general Lipschitz curves, which can have plenty of self-intersections.

7. A weighted Bergman space on M

For this section we assume $d = 1$, i.e., $\dim_{\mathbf{C}} \tilde{M} = 1$. Recall that we denote $X = \tilde{M} \cap S$. Write $\overline{M}$ for the closure of M in $\mathbf{C}^n$.

Recall from Definition 2.10 that for $f \in L^\infty(M)$, we write $\hat{M}_f$ for the operator on $L^2(\mu)$ of multiplication by f. For such an f we also have the operator $\tilde{T}_f h = E\hat{M}_f|\mathcal{P}$ introduced in Section 4. We think of $\tilde{T}_f$ as a Toeplitz operator on $\mathcal{P}$.

Lemma 7.1. *For any $f \in C(\overline{M})$, the commutator $[\hat{M}_f, E]$ is compact.*

Proof. This is an immediate consequence of Proposition 2.16. □

The following is an immediate consequence of Lemma 2.21:

Corollary 7.2. *If $f \in C(\overline{M})$ and if $f = 0$ on X, then $\tilde{T}_f$ is a compact operator.*

Lemma 7.3. *For each $f \in C(\overline{M})$, the essential spectrum of $\tilde{T}_f$ is contained in $f(X)$.*

Proof. Let $f \in C(\overline{M})$. If $\lambda \in \mathbf{C}\backslash f(X)$, then there is a $g \in C(\overline{M})$ such that $(f - \lambda)g = 1$ on X. Thus it follows from Lemma 7.1 and Corollary 7.2 that

$$(\tilde{T}_f - \lambda)\tilde{T}_g = 1 + K_1 \quad \text{and} \quad \tilde{T}_g(\tilde{T}_f - \lambda) = 1 + K_2,$$

where K_1 and K_2 are compact operators. This means that λ is not in the essential spectrum of $\tilde{T}_f$. □

We write $X = \tilde{M} \cap S$ as the union of its connected components:

$$X = \Gamma_1 \cup \cdots \cup \Gamma_\ell. \tag{7.1}$$

Under the assumption $d = 1$, X is a compact manifold of real dimension 1. Thus each Γ_j is diffeomorphic to $\mathbf{T}$, $1 \le j \le \ell$. Obviously, each Γ_j has two opposite orientations. We fix an orientation for each Γ_j. Thus if $\gamma : \Gamma_j \to \mathbf{C}$ is any continuous function, then it has a winding number, whose sign depends on our choice of orientation for Γ_j, about every $\lambda \in \mathbf{C}\backslash\gamma(\Gamma_j)$. Let $f \in C(\overline{M})$. For every $\lambda \in \mathbf{C}\backslash f(X)$ and every $1 \le j \le \ell$, we denote

$$w_j(f; \lambda) = \text{the winding number of } f : \Gamma_j \to \mathbf{C} \text{ about } \lambda. \tag{7.2}$$

For each $j \in \{1, \ldots, \ell\}$, there is a $\varphi_j \in C(X)$ such that $\varphi_j = 1$ on Γ_k for every $k \ne j$, $|\varphi_j| = 1$ on Γ_j, and $w_j(\varphi_j, 0) = 1$. By the Tietze extension theorem, there is a $\psi_j \in C(\overline{M})$ such that $\varphi_j = \psi_j|X$. Denote

$$c_j = -\text{index}(\tilde{T}_{\psi_j}). \tag{7.3}$$

This defines the integers $c_1, \ldots, c_\ell$.

Proposition 7.4. *Let $f \in C(\overline{M})$. If $0 \notin f(X)$, then*

$$\mathrm{index}(\tilde{T}_f) = -\sum_{j=1}^{\ell} c_j w_j(f;0),$$

where $c_1, \ldots, c_\ell$ are the integers defined by (7.3).

Proof. Let $f \in C(\overline{M})$ be given, and suppose that $0 \notin f(X)$. For each $1 \le j \le \ell$, denote $\nu_j = w_j(f;0)$. For each $1 \le j \le \ell$, we define

$$\eta_j = \begin{cases} \psi_j & \text{if} \quad \nu_j \ge 1 \\ \bar{\psi}_j & \text{if} \quad \nu_j \le -1 \\ 1 & \text{if} \quad \nu_j = 0 \end{cases}.$$

By (7.3), we have

$$\mathrm{index}(\tilde{T}_{\eta_j}) = \begin{cases} -c_j & \text{if} \quad \nu_j \ge 1 \\ c_j & \text{if} \quad \nu_j \le -1 \\ 0 & \text{if} \quad \nu_j = 0 \end{cases}.$$

By what we know about winding numbers, there is a continuous function $F : [0,1] \times X \to \mathbf{C}\backslash\{0\}$ such that

$$F(0,x) = f(x) \quad \text{and} \quad F(1,x) = \prod_{j=1}^{\ell} \{\eta_j(x)\}^{|\nu_j|}$$

for every $x \in X$. Applying the Tietze extension theorem again, there is a continuous $G : [0,1] \times \overline{M} \to \mathbf{C}$ such that F is the restriction of G to the subset $[0,1] \times X$. We now define

$$g_t(z) = G(t,z), \quad z \in \overline{M},$$

for each $t \in [0,1]$. By Lemma 7.3, each $\tilde{T}_{g_t}$ is a Fredholm operator. Since the map $t \mapsto \tilde{T}_{g_t}$ is continuous with respect to the operator norm, $\mathrm{index}(\tilde{T}_{g_t})$ remains constant as t varies in $[0,1]$. Note that the functions

$$f - g_0 \quad \text{and} \quad g_1 - \prod_{j=1}^{\ell} \eta_j^{|\nu_j|}$$

vanish on X. Applying Corollary 7.2 and Lemma 7.1, we have

$$\mathrm{index}(\tilde{T}_f) = \mathrm{index}(\tilde{T}_{g_0}) = \mathrm{index}(\tilde{T}_{g_1}) = \mathrm{index}\left(\prod_{j=1}^{\ell} \tilde{T}_{\eta_j}^{|\nu_j|}\right)$$

$$= \sum_{j=1}^{\ell} |\nu_j| \mathrm{index}(\tilde{T}_{\eta_j}) = -\sum_{j=1}^{\ell} c_j w_j(f;0)$$

as promised. □

The condition $d = 1$ entails the following Schatten-class memberships:

Proposition 7.5. [24, Proposition 11.2] *For every $f \in \mathrm{Lip}(M)$, the commutators $[\hat{M}_f, \hat{T}_\mu]$ and $[\hat{M}_f, E]$ are in the Hilbert-Schmidt class $\mathcal{C}_2$.*

Corollary 7.6. *For all $f, g \in \mathrm{Lip}(M)$, the commutator $[\tilde{T}_f, \tilde{T}_g]$ is in the trace class.*

Proof. By the identity

$$[\tilde{T}_f, \tilde{T}_g] = E[\hat{M}_f, E][\hat{M}_g, E]E - E[\hat{M}_g, E][\hat{M}_f, E]E,$$

the membership $[\tilde{T}_f, \tilde{T}_g] \in \mathcal{C}_1$ follows from Proposition 7.5. □

Proposition 7.7. *For every $f \in \mathrm{Lip}(\overline{M})$, we have $[\tilde{T}_{\bar{f}}, \tilde{T}_f] \in \mathcal{C}_1$* and

$$\mathrm{tr}[\tilde{T}_{\bar{f}}, \tilde{T}_f] = \frac{1}{2\pi i} \sum_{j=1}^{\ell} c_j \int_{\Gamma_j} \bar{f} df,$$

where $\Gamma_1, \dots, \Gamma_\ell$ and $c_1, \dots, c_\ell$ are given by (7.1) *and* (7.3) *respectively.*

Proof. For $f \in \mathrm{Lip}(\overline{M})$, Corollary 7.6 tells us that the commutator $[\tilde{T}_{\bar{f}}, \tilde{T}_f]$ is in the trace class. To compute its trace, we again resort to the Carey-Pincus theory, which was reviewed in Section 6. Define

$$A = \tilde{T}_{\mathrm{Re}(f)}, \quad B = \tilde{T}_{\mathrm{Im}(f)} \quad \text{and} \quad T = A + iB = \tilde{T}_f.$$

We have $[A, B] = (2i)^{-1}[\tilde{T}_{\bar{f}}, \tilde{T}_f] \in \mathcal{C}_1$.

Let g be the principal function for this pair A, B. By (6.12),

$$\mathrm{tr}[\tilde{T}_{\bar{f}}, \tilde{T}_f] = 2i\mathrm{tr}[A, B] = \frac{-1}{\pi} \iint g(y, x) dx dy.$$

Recall from Lemma 7.3 that the essential spectrum of $\tilde{T}_f$ is contained in $f(X)$. Since $f \in \mathrm{Lip}(\overline{M})$ and since $\dim_{\mathbf{R}} X = 1$, the two-dimensional Lebesgue measure of the set $f(X)$ is 0. It follows from (6.13) and Proposition 7.4 that if $x + iy \in \mathbf{C} \backslash f(X)$, then

$$g(y, x) = \mathrm{index}(\tilde{T}_f - (x + iy)) = -\sum_{j=1}^{\ell} c_j w_j(f - (x + iy); 0) = -\sum_{j=1}^{\ell} c_j w_j(f; x + iy).$$

Therefore

$$\mathrm{tr}[\tilde{T}_{\bar{f}}, \tilde{T}_f] = \sum_{j=1}^{\ell} \frac{c_j}{\pi} \iint w_j(f; x + iy) dx dy. \tag{7.4}$$

For each $1 \le j \le \ell$, let $h_j : \mathbf{T} \to \Gamma_j$ be an orientation-preserving diffeomorphism. Then

$$w_j(f; x+iy) = w(f \circ h_j; x+iy) \quad \text{for} \quad x+iy \in \mathbf{C}\backslash f(X).$$

Thus, applying Proposition 6.3, for each $1 \le j \le \ell$ we have

$$\begin{aligned} &\frac{1}{\pi}\iint w_j(f; x+iy)dxdy \\ &\quad = \frac{1}{\pi}\iint w(f\circ h_j; x+iy)dxdy = \frac{1}{2\pi i}\int_{\mathbf{T}} \overline{f\circ h_j} df \circ h_j = \frac{1}{2\pi i}\int_{\Gamma_j} \bar{f} df. \end{aligned}$$

Substituting this in (7.4), the proof is complete. □

Corollary 7.8. *For every pair of $f, g \in \mathrm{Lip}(\overline{M})$, we have*

$$\operatorname{tr}[\tilde{T}_f, \tilde{T}_g] = \frac{1}{2\pi i}\sum_{j=1}^{\ell} c_j \int_{\Gamma_j} f dg, \tag{7.5}$$

where $\Gamma_1, \dots, \Gamma_\ell$ and $c_1, \dots, c_\ell$ are given by (7.1) *and* (7.3) *respectively.*

Proof. If $f, g \in \mathrm{Lip}(\overline{M})$ are real valued, then (7.5) follows Proposition 7.7 by considering the function $f + ig$. But once (7.5) is established for real-valued $f, g \in \mathrm{Lip}(\overline{M})$, the general case follows from the linearity of both sides. □

Proposition 7.9. *For the integers $c_1, \dots, c_\ell$ defined by* (7.3), *we have*

$$|c_1| + \cdots + |c_\ell| > 0.$$

Proof. For each $1 \le j \le n$, the M_{ζ_j} on $\mathcal{P}$ is obviously a subnormal operator. Therefore $[M^*_{\zeta_j}, M_{\zeta_j}] \ge 0$. If it were true that $|c_1| + \cdots + |c_\ell| = 0$, then by Proposition 7.7 we would have

$$\operatorname{tr}[M^*_{\zeta_j}, M_{\zeta_j}] = \operatorname{tr}[\tilde{T}_{\bar{\zeta}_j}, \tilde{T}_{\zeta_j}] = 0$$

for every $1 \le j \le n$. Since $[M^*_{\zeta_j}, M_{\zeta_j}] \ge 0$, this means $[M^*_{\zeta_j}, M_{\zeta_j}] = 0$. Thus for all $1 \le j \le n$ and $h \in \mathcal{P}$, we have

$$\begin{aligned} \|E\hat{M}_{\bar{\zeta}_j}h\|^2 &= \|M^*_{\zeta_j}h\|^2 = \langle M_{\zeta_j}M^*_{\zeta_j}h, h\rangle = \langle M^*_{\zeta_j}M_{\zeta_j}h, h\rangle \\ &= \|M_{\zeta_j}h\|^2 = \|\hat{M}_{\zeta_j}h\|^2 = \|\hat{M}_{\bar{\zeta}_j}h\|^2 = \|E\hat{M}_{\bar{\zeta}_j}h\|^2 + \|(1-E)\hat{M}_{\bar{\zeta}_j}h\|^2. \end{aligned}$$

We conclude that $(1-E)\hat{M}_{\bar{\zeta}_j}h = 0$ for all $1 \le j \le n$ and $h \in \mathcal{P}$. In other words, $\hat{M}_{\bar{\zeta}_j}h \in \mathcal{P}$ for all $1 \le j \le n$ and $h \in \mathcal{P}$. This means that $\mathcal{P}$ contains the closure of $\mathbf{C}[\zeta_1, \bar{\zeta}_1, \dots, \zeta_n, \bar{\zeta}_n]$ in $L^2(\mu)$. That is, $\mathcal{P} = L^2(\mu)$. But this is obviously a contradiction, because $\mathcal{P}$ consists of the restrictions to M of the functions in $H^2(S)$, which, for example, rules out any real-valued functions that are not locally constant on the regular part of M. □

8. Proofs of Theorems 1.5 and 1.6

In this section we continue to assume $d = 1$.

Lemma 8.1. [17, Lemma 1.3] *Suppose that X is a self-adjoint operator and C is a compact operator. If $[X, C]$ is in the trace class, then* $\mathrm{tr}[X, C] = 0$.

Lemma 8.2. *For $f, g, h_1, \ldots, h_k \in \mathrm{Lip}(S)$, the commutator $[Q_{fg} - Q_f Q_g, Q_{h_1} \cdots Q_{h_k}]$ is in the trace class with zero trace.*

Proof. Obviously, the membership

$$[Q_{fg} - Q_f Q_g, Q_{h_1} \cdots Q_{h_k}] \in \mathcal{C}_1 \tag{8.1}$$

follows from Proposition 5.5 and the "product rule" for commutators. Similarly, we have

$$[Q_{fg} - Q_f Q_g, \{Q_{h_1} \cdots Q_{h_k}\}^*] = [Q_{fg} - Q_f Q_g, Q_{\bar{h}_k} \cdots Q_{\bar{h}_1}] \in \mathcal{C}_1.$$

Since $Q_{fg} - Q_f Q_g$ is compact, the fact $\mathrm{tr}[Q_{fg} - Q_f Q_g, Q_{h_1} \cdots Q_{h_k}] = 0$ now follows from Lemma 8.1. □

We know that for all $p, q, r, s \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, the commutator $[Q_p^* Q_q, Q_r^* Q_s]$ is in the trace class. See [24, page 45].

Lemma 8.3. *Let $p, q, r, s \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$. Then the commutator $[Q_{\bar{p}q}, Q_{\bar{r}s}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[Q_{\bar{p}q}, Q_{\bar{r}s}] = \mathrm{tr}[Q_p^* Q_q, Q_r^* Q_s] = \mathrm{tr}[M_p^* M_q, M_r^* M_s] = \mathrm{tr}[\tilde{T}_{\bar{p}q}, \tilde{T}_{\bar{r}s}]. \tag{8.2}$$

Proof. Since $Q_p^* = Q_{\bar{p}}$ and $Q_r^* = Q_{\bar{r}}$, we have

$$\begin{aligned}[Q_{\bar{p}q}, Q_{\bar{r}s}] &= [Q_p^* Q_q, Q_{\bar{r}s}] + [Q_{\bar{p}q} - Q_{\bar{p}} Q_q, Q_{\bar{r}s}] \\ &= [Q_p^* Q_q, Q_r^* Q_s] + [Q_{\bar{p}} Q_q, Q_{\bar{r}s} - Q_{\bar{r}} Q_s] + [Q_{\bar{p}q} - Q_{\bar{p}} Q_q, Q_{\bar{r}s}].\end{aligned}$$

By the fact $[Q_p^* Q_q, Q_r^* Q_s] \in \mathcal{C}_1$ and (8.1), we have the membership $[Q_{\bar{p}q}, Q_{\bar{r}s}] \in \mathcal{C}_1$. The first = in (8.2) is obtained by applying Lemma 8.2 in the above identity. The second = in (8.2) is provided by Theorem 1.4. Finally, the third = in (8.2) holds because $M_p^* M_q = \tilde{T}_{\bar{p}q}$ and $M_r^* M_s = \tilde{T}_{\bar{r}s}$ (see Definition 2.12). This completes the proof. □

Proof of Theorem 1.5. This follows immediately from Lemma 8.3 and Corollary 7.8. □

Proof of Theorem 1.6. For $f, g \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \ldots, \zeta_n, \bar{\zeta}_n]$, there are analytic polynomials p_ν, q_ν, r_ν, $s_\nu \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, $1 \le \nu \le k$, such that $f = \sum_{\nu=1}^k \bar{p}_\nu q_\nu$ and $g = \sum_{\nu=1}^k \bar{r}_\nu s_\nu$. Thus

$$[Q_f, Q_g] = \sum_{m=1}^k \sum_{\nu=1}^k [Q_{\bar{p}_m q_m}, Q_{\bar{r}_\nu s_\nu}],$$

which, according to Lemma 8.3, is in the trace class. Furthermore, it follows from Lemma 8.3 and Corollary 7.8 that

$$\begin{aligned}\mathrm{tr}[Q_f, Q_g] = \sum_{m=1}^{k}\sum_{\nu=1}^{k}\mathrm{tr}[\tilde{T}_{\bar{p}_m q_m}, \tilde{T}_{\bar{r}_\nu s_\nu}] &= \sum_{m=1}^{k}\sum_{\nu=1}^{k}\frac{1}{2\pi i}\sum_{j=1}^{\ell} c_j \int_{\Gamma_j} \bar{p}_m q_m d\bar{r}_\nu s_\nu \\ &= \frac{1}{2\pi i}\sum_{j=1}^{\ell} c_j \int_{\Gamma_j} f dg.\end{aligned}$$

This completes the proof. □

9. A family of examples

We now apply Theorems 1.5 and 1.6 to a family of examples. In this section we assume $n = 2$. That is, we only consider the case of two complex variables. Thus $\mathbf{B} = \{(z,w) \in \mathbf{C}^2 : |z|^2 + |w|^2 < 1\}$ and $S = \{(z,w) \in \mathbf{C}^2 : |z|^2 + |w|^2 = 1\}$ for this section.

We begin with a pair of natural numbers $p \geq 2$ and $q \geq 2$ that are *relatively prime*. For convenience, we assume $p < q$. Define

$$\tilde{M}_{p,q} = \{(z,w) \in \mathbf{C}^2 : z^p - w^q = 0\}.$$

Obviously, $\tilde{M}_{p,q}$ is an analytic subset of $\mathbf{C}^2$ with the point $(0,0)$ as its only singularity. In accordance with Notation 2.3, we write

$$\begin{aligned}&M_{p,q} = \tilde{M}_{p,q} \cap \mathbf{B} \quad \text{and} \\ &\mathcal{Q}_{p,q} = H^2(S) \ominus \{f \in H^2(S) : f = 0 \text{ on } M_{p,q}\}.\end{aligned} \tag{9.1}$$

To apply Theorems 1.5 and 1.6 to this $\mathcal{Q}_{p,q}$, we need to verify one fact:

Lemma 9.1. *For any such pair of p, q, $\tilde{M}_{p,q}$ intersects S transversely.*

Proof. Although this is totally elementary, we will work out the details anyway.

We begin with the function $f(x) = x^2 + x^{2q/p}$ defined on $(0,\infty)$. Obviously, f is strictly increasing and C^∞ on $(0,\infty)$. Moreover, the range of f equals $(0,\infty)$. Therefore f has an inverse function $g : (0,\infty) \to (0,\infty)$, which is also strictly increasing and C^∞. There is a unique $b \in (0,\infty)$, in fact $b \in (0,1)$, such that

$$1 - b^2 - b^{2q/p} = 0. \tag{9.2}$$

Let $\zeta \in \tilde{M}_{p,q} \cap S$. Then there are $\theta, \phi \in \mathbf{R}$ with $e^{ip\theta} = e^{iq\phi}$ such that

$$\zeta = \left((1-b^2)^{1/2}e^{i\theta}, be^{i\phi}\right). \tag{9.3}$$

Obviously, (9.2) means $f(b) = 1$. Therefore $b = g(1)$. Since $0 < b < 1$, we see that there is an $\epsilon > 0$ such that the function $x - g^2(x)$ is positive on $I = (1-\epsilon, 1+\epsilon)$. We now define

$$\gamma(x) = \left((x - g^2(x))^{1/2}e^{i\theta}, g(x)e^{i\phi}\right), \quad x \in I.$$

By the equations $f(g(x)) = x$ and $e^{ip\theta} = e^{iq\phi}$, the range of γ is contained in $\tilde{M}_{p,q}$. Moreover, $\gamma(1) = \zeta$. Therefore $\gamma'(1) \in T_\zeta$. Since ζ is an arbitrary point in $\tilde{M}_{p,q} \cap S$, the lemma will be proved if we can show that $\langle \gamma'(1), \zeta \rangle \neq 0$.

To prove this assertion, note that

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{2g(1) + (2q/p)\{g(1)\}^{(2q/p)-1}} = \frac{1}{2b + (2q/p)b^{(2q/p)-1}}.$$

Moreover,

$$\gamma'(1) = \left(\alpha e^{i\theta}, g'(1)e^{i\phi}\right),$$

where

$$\alpha = \frac{1}{2} \cdot \frac{1 - 2g(1)g'(1)}{(1 - g^2(1))^{1/2}} = \frac{1 - \{1 + (q/p)b^{(2q/p)-2}\}^{-1}}{2(1 - b^2)^{1/2}}.$$

Combining these facts with (9.3), we see that $\langle \gamma'(1), \zeta \rangle \neq 0$. This completes the proof. $\square$

Lemma 9.2. *Denote* $X_{p,q} = \tilde{M}_{p,q} \cap S$. *Then*

$$X_{p,q} = \{((1 - b^2)^{1/2} e^{iqt}, b e^{ipt}) : 0 \le t \le 2\pi\}, \tag{9.4}$$

where $b \in (0, 1)$ *satisfies equation* (9.2). *Moreover, for any* $0 \le t < t' < 2\pi$, *we have*

$$((1 - b^2)^{1/2} e^{iqt}, b e^{ipt}) \neq ((1 - b^2)^{1/2} e^{iqt'}, b e^{ipt'}). \tag{9.5}$$

Proof. This is again elementary.

As we saw in the proof of Lemma 9.1, an arbitrary $\zeta \in X_{p,q}$ is given by (9.3), where $\theta, \phi \in \mathbf{R}$ satisfy the condition $e^{ip\theta} = e^{iq\phi}$. That is, $p\theta - q\phi = 2m\pi$ for some $m \in \mathbf{Z}$. Since p, q are relatively prime, there are $j, k \in \mathbf{Z}$ such that $pj - qk = -m$. Consequently, $p(\theta + 2j\pi) - q(\phi + 2k\pi) = 0$. Thus if we set $t = (\phi + 2k\pi)/p$, then

$$e^{iqt} = e^{i(q/p)(\phi+2k\pi)} = e^{i(\theta+2j\pi)} = e^{i\theta} \quad \text{and} \quad e^{ipt} = e^{i(\phi+2k\pi)} = e^{i\phi}.$$

This obviously implies (9.4).

Note that (9.5) is equivalent to the assertion that $(e^{iqt}, e^{ipt}) \neq (1, 1)$ for every $0 < t < 2\pi$. For any $0 < t < 2\pi$, if it were true that $(e^{iqt}, e^{ipt}) = (1, 1)$, then we would have $pt = 2j\pi$ and $qt = 2k\pi$ for some $j, k \in \mathbf{Z}$. Thus $j/p = k/q$, i.e., $jq = kp$. Since p, q are relatively prime, this implies that j is divisible by p and k is divisible by q, which contradicts the condition $0 < t < 2\pi$. Hence (9.5) holds. $\square$

Lemma 9.2 provides a diffeomorphism $\Phi : \mathbf{T} \to X_{p,q}$ by the formula

$$\Phi(\tau) = \left((1 - b^2)^{1/2} \tau^q, b\tau^p\right), \quad \tau \in \mathbf{T}.$$

We choose the orientation of $X_{p,q}$ to be such that Φ is orientation preserving.

Theorem 9.3. *For any $f, g \in \mathbf{C}[\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2]$, the commutator $[Q_f, Q_g]$ on the quotient module $\mathcal{Q}_{p,q}$ is in the trace class, and we have the explicit formula*

$$\operatorname{tr}[Q_f, Q_g] = \frac{1}{2\pi i}\int_{X_{p,q}} f dg = \frac{1}{2\pi i}\int_0^{2\pi} f((1-b^2)^{1/2}e^{iqt}, be^{ipt})dg((1-b^2)^{1/2}e^{iqt}, be^{ipt}),$$

where $0 < b < 1$ satisfies equation (9.2).

Proof. Obviously, $X_{p,q}$ has only one component. Applying Theorem 1.6, we only need to show that $c_1 = 1$ for the present situation. To prove this fact, we use the range space $\mathcal{P}_{p,q}$ that corresponds to the quotient module $\mathcal{Q}_{p,q}$. Recall from Section 2 that $\mathcal{P}_{p,q}$ is the closure of $\mathbf{C}[\zeta_1, \zeta_2]$ in $L^2(M_{p,q}, d\mu)$.

Note that $\Phi^{-1} : X_{p,q} \to \mathbf{T}$ has winding number 1 about 0. By the Tietze extension theorem, there is a $\psi \in C(\overline{M_{p,q}})$ such that $\Phi^{-1} = \psi|X_{p,q}$. By Corollary 7.8, we have

$$c_1 = -\operatorname{index}(\tilde{T}_\psi).$$

Thus it suffices to show that $\operatorname{index}(\tilde{T}_\psi) = -1$. We know from Proposition 7.9 that $c_1 \neq 0$. Therefore $\operatorname{index}(\tilde{T}_\psi) \neq 0$.

Consider the coordinate function ζ_1 on $\mathbf{C}^2$. By Lemma 9.2, the winding number of ζ_1 on $X_{p,q}$ about 0 is q. Thus it follows from Proposition 7.4 that

$$\operatorname{index}(\tilde{T}_{\zeta_1}) = q \times \operatorname{index}(\tilde{T}_\psi). \tag{9.6}$$

Note that $\tilde{T}_{\zeta_1} = M_{\zeta_1}$ on $\mathcal{P}_{p,q}$. It is obvious that $\ker(M_{\zeta_1}) = \{0\}$. Thus $\operatorname{index}(M_{\zeta_1}) \leq 0$. Combining this with (9.6) and with the fact $\operatorname{index}(\tilde{T}_\psi) \neq 0$, we have $\operatorname{index}(M_{\zeta_1}) \leq -q$.

On the other hand, because $\zeta_1^p - \zeta_2^q$ vanishes on $M_{p,q}$ and because $\operatorname{range}(M_{\zeta_1})$ is a closed linear subspace of $\mathcal{P}_{p,q}$, we have

$$\operatorname{range}(M_{\zeta_1}) + \operatorname{span}\{1, \zeta_2, \dots, \zeta_2^{q-1}\} = \mathcal{P}_{p,q}.$$

This means that $\operatorname{index}(M_{\zeta_1}) \geq -q$. Combining this with the conclusion of the previous paragraph, we obtain the equality $\operatorname{index}(\tilde{T}_{\zeta_1}) = \operatorname{index}(M_{\zeta_1}) = -q$. Substituting this in (9.6), we find that $\operatorname{index}(\tilde{T}_\psi) = -1$ as promised. This completes the proof. □

Data availability

No data was used for the research described in the article.

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