THE HELTON-HOWE TRACE FORMULA FOR THE DRURY-ARVESON SPACE

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Abstract. The famous Helton-Howe trace formula was originally established for antisymmetric sums of Toeplitz operators on the Bergman space of the unit ball. We prove the analogue of this formula on the Drury-Arveson space.

1. Introduction

Given any bounded operators $A_1, \dots, A_k$ on a Hilbert space $\mathcal{H}$, one has the antisymmetric sum

$$[A_1, \dots, A_k] = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)},$$

which naturally generalizes the notion of commutator. This was first introduced by Helton and Howe in [11], and has since become an important part of operator theory and non-commutative geometry [5,7]. As it turns out, operators on reproducing-kernel Hilbert spaces provide some of the particularly interesting examples of antisymmetric sums.

Let $\mathbf{B}$ be the unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. As usual, we write $L^2_a(\mathbf{B})$ for the Bergman space. Given an $f \in L^\infty(\mathbf{B})$, we have the familiar Toeplitz operator T_f defined by the formula

$$T_f h = P(fh), \quad h \in L^2_a(\mathbf{B}), \tag{1.1}$$

where $P : L^2(\mathbf{B}) \to L^2_a(\mathbf{B})$ is the orthogonal projection. We recall the following classic result of Helton and Howe:

Theorem 1.1. [11, Theorem 7.2] *Let $f_1, f_2, \dots, f_{2n}$ be C^∞-functions on an open set containing $\overline{\mathbf{B}}$. Then the antisymmetric sum $[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}. \tag{1.2}$$

We further recommend to the reader the recent article [13], in which this trace formula was re-examined from the modern perspective of non-commutative geometry and quantization. More generally, the study of the Arveson-Douglas conjecture [2,3,6] has brought renewed interest in antisymmetric sums and their traces [10,14].

The purpose of this paper is to prove the analogue of (1.2) for the Drury-Arveson space. It is not surprising that this has not been done before, for the Drury-Arveson space

Key words and phrases. Drury-Arveson space, trace formula.

is notorious for the scarcity of results. In fact, this paper is a good illustration of the difficulties for obtaining results on the Drury-Arveson space.

Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z \rangle}$$

as its reproducing kernel [1,9]. Equivalently, H_n^2 can be described as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle h, g \rangle = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} b_\alpha \zeta^\alpha.$$

Throughout the paper, we follow the usual multi-index convention [12, page 3]. Furthermore, we assume that the complex dimension n is at least 2.

There are, of course, no Toeplitz operators on H_n^2 in general. But there are multiplication operators. An analytic function f on $\mathbf{B}$ is said to be a *multiplier* of H_n^2 if $fh \in H_n^2$ for every $h \in H_n^2$. Recall that the notion multiplier was first introduced by Arveson in [1]. As usual, we write $\mathcal{M}$ for the collection of multipliers of H_n^2. For each $f \in \mathcal{M}$, we have the multiplication operator

$$M_f h = fh, \quad h \in H_n^2,$$

on the Drury-Arveson space, which is necessarily bounded [1]. The operator norm $\|M_f\|$ is called the *multiplier norm* of f, and is commonly denoted by $\|f\|_{\mathcal{M}}$. For $f, g \in \mathcal{M}$, the operator $M_f^* M_g$ is the proper analogue on H_n^2 of the Toeplitz operator $T_{\bar{f}g}$.

Taking the condition in Theorem 1.1 as a guide, to obtain trace-class membership for antisymmetric sums on H_n^2, we need some smoothness for the "symbol functions" involved. Thus we need to focus on "smooth subclasses" of $\mathcal{M}$. The following are some of the convenient "smooth subclasses" of $\mathcal{M}$.

For each $s > 1$, let H_s^∞ denote the collection of analytic functions f on the open ball $B(0, s) = \{z \in \mathbf{C}^n : |z| < s\}$ satisfying the condition $\|f\|_{s,\infty} < \infty$, where

$$\|f\|_{s,\infty} = \sup\{|f(z)| : z \in B(0, s)\}.$$

It is easy to show that $\mathcal{M} \supset H_s^\infty$ for every $s > 1$. (See Proposition 2.5 below.) Thus each H_s^∞, $s > 1$, is a smooth subclass of $\mathcal{M}$.

The key to establishing the analogue of Theorem 1.1 for the Drury-Arveson space H_n^2 is a bound on trace norm $\| \cdot \|_1$, which has obvious significance in its own right:

Theorem 1.2. *Given any $1 < s < \infty$, there is a $0 < C = C(s) < \infty$ such that for all $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in H_s^\infty$, the antisymmetric sum $[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}]$ on H_n^2 satisfies the trace-norm bound*

$$\|[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}]\|_1 \le C \prod_{j=1}^{2n} \|f_j\|_{s,\infty} \|g_j\|_{s,\infty}.$$

Using Theorem 1.2, we can prove the following analogue of trace formula (1.2) for the Drury-Arveson space:

Theorem 1.3. *Let $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in H_s^\infty$ for some $s > 1$. Then on the Drury-Arveson space H_n^2, we have the trace formula*

$$\text{(1.3)} \qquad \operatorname{tr}[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_1 g_1 \wedge d\bar{f}_2 g_2 \wedge \cdots \wedge d\bar{f}_{2n} g_{2n}.$$

The rest of the paper is devoted to the proofs of these two theorems. Due to the nature of the Drury-Arveson space, the proofs require many steps and new ideas. Let us briefly describe how these proofs are organized, together with some of the main ideas.

We begin with some basic preliminaries in Section 2. In Section 3, we first derive a precise integral formula for the norm on H_n^2. This formula gives us a convenient resolution of the identity operator (3.4) for the Drury-Arveson space. Based on (3.4), we introduce the operators T_1, T_2 and $T = T_1 + T_2$. The operator T is invertible on H_n^2 and has the property $1 - T \in \mathcal{C}_1$, which paves the way for the next step.

In Section 4, we introduce the *range space* $\mathcal{P}$ for H_n^2. The idea of range space came from the study [8] of the geometric Arveson-Douglas conjecture, and has proven its usefulness in [14]. In effect, we are treating H_n^2 as a quotient module of itself, with $\{0\}$ serving as the corresponding submodule. The introduction of $\mathcal{P}$ addresses a well-known difficulty in the theory of the Drury-Arveson space: there is no L^2 naturally associated with H_n^2. In contrast, there is a natural Hilbert space $\mathcal{L}$ that contains $\mathcal{P}$. Thus $\mathcal{L}$ gives us a surrogate for an "L^2-space containing H_n^2". The operators T_1, T_2 and T then have their respective representations $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}$ on $\mathcal{L}$. If we write P for the orthogonal projection from $\mathcal{L}$ to $\mathcal{P}$, then $P - \hat{T}$ is in the trace class. Since $\hat{T}$ is given by an explicit formula, this gives us a total control of the projection $P : \mathcal{L} \to \mathcal{P}$. Section 4 reduces the proof of Theorem 1.2 to the bounding of the trace norm of antisymmetric sums of the form

$$\text{(1.4)} \qquad [\tilde{M}_{g_1}^* \tilde{M}_{h_1}, \tilde{M}_{g_2}^* \tilde{M}_{h_2}, \dots, \tilde{M}_{g_{2n}}^* \tilde{M}_{h_{2n}}]$$

on the range space $\mathcal{P}$. Because of our control of P, (1.4) can be handled in a way similar to how antisymmetric sums of Toeplitz operators are handled on the Bergman space or the Hardy space. The difference is that the handling of (1.4) on $\mathcal{P}$ is much more tedious.

The bounding of the trace norm of (1.4) involves numerous estimates of Schatten p-norms of commutators, double commutations, and other kinds of operators. Because of the nature of $\mathcal{L}$, these estimates can be transformed to estimates for integral operators on a bona fide L^2-space, $L^2(\Omega, d\mu)$. Thus familiar techniques such as Schur test and interpolation can be brought to bear. Section 5 takes care of the estimates on $L^2(\Omega, d\mu)$.

Then in Section 6, we use the results in Section 5 to derive Schatten-norm bounds for commutators $[\hat{M}_f, P]$ and double commutators $[\hat{M}_g, [\hat{M}_f, P]]$ on $\mathcal{L}$.

Although $\mathcal{L}$ is a surrogate for an "L^2-space containing H_n^2", it is not an actual L^2-space, which causes the following problem. For a multiplication operator $\hat{M}_f$ on $\mathcal{L}$, its adjoint, $\hat{M}_f^*$, is not a multiplication operator. Thus in general we have $[\hat{M}_f^*, \hat{M}_g] \neq 0$. We deal with this problem in Section 7. In essence, we show that the difference $\hat{M}_f^* - \hat{M}_{\bar{f}}$ is small enough for our purpose.

The pair of spaces $\mathcal{L}$ and $\mathcal{P}$ and the orthogonal projection $P : \mathcal{L} \to \mathcal{P}$ naturally lead to the analogue

$$\tilde{T}_f = P\hat{M}_f|\mathcal{P}$$

of Toeplitz operator on $\mathcal{P}$. The main result in Section 8 is Lemma 8.10, which says that modulo appropriate Schatten class, the f in the commutator $[\tilde{T}_f, \tilde{T}_g]$ can be "locally extracted from commutation" under the condition $f \in H_s^\infty$ for some $s > 1$.

Then, by combining the results from the previous sections with general techniques for handling antisymmetric sums, we complete the proof of Theorem 1.2 in Section 9.

In Section 10 we prove Theorem 1.3, which takes two steps. For the first step, using an old idea of Coburn [4], we show that there is a unitary operator $U : L_a^2(\mathbf{B}) \to H_n^2$ such that if $f_1, \ldots, f_{2n}, g_1, \ldots, g_{2n} \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$, then

$$U^*[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \ldots, M_{f_{2n}}^* M_{g_{2n}}]U = [T_{\bar{f}_1 g_1}, T_{\bar{f}_2 g_2}, \ldots, T_{\bar{f}_{2n} g_{2n}}] + Y,$$

where $Y = Y_{f_1,\ldots,f_{2n},g_1,\ldots,g_{2n}}$ is in the trace class with zero trace. By this identity, the polynomial version of (1.3) follows from Theorem 1.1, i.e., the Bergman-space case. Then Theorem 1.2 allows us to accomplish the second step in the proof of Theorem 1.3, namely, we derive the general version of (1.3) from the polynomial version by approximation.

Acknowledgement. The author is grateful to Yi Wang for communicating a valuable calculation, which led to the discovery of the function ψ_m introduced in Section 3.

2. Preliminaries

For each $1 \leq p < \infty$, let $\mathcal{C}_p$ denote the Schatten p-class. In other words, $\mathcal{C}_p = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p < \infty\}$, where $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$. We recall the following:

Lemma 2.1. [5, Lemma 2.9] *Let $A \in \mathcal{C}_{p_1}$ and $B \in \mathcal{C}_{p_2}$, where $p_1, p_2 \in [1, \infty)$. If $p_1p_2/(p_1 + p_2) \geq 1$, then $AB \in \mathcal{C}_{p_1p_2/(p_1+p_2)}$ with*

$$\|AB\|_{p_1p_2/(p_1+p_2)} \leq \|A\|_{p_1}\|B\|_{p_2}.$$

If $p_1p_2/(p_1 + p_2) < 1$, then $AB \in \mathcal{C}_1$.

By Lemma 2.1 and an obvious induction, we have

Corollary 2.2. *Let $p_1, \dots, p_k \in [1, \infty)$ be such that $(1/p_1) + \cdots + (1/p_k) \geq 1$. If operators $A_1, \dots, A_k$ are such that $A_j \in \mathcal{C}_{p_j}$ for every $j \in \{1, \dots, k\}$, then the product $A_1 \cdots A_k$ is in the trace class with*

$$\|A_1 \cdots A_k\|_1 \leq \|A_1\|_{p_1} \cdots \|A_k\|_{p_k}.$$

Write S for the unit sphere $\{\xi \in \mathbf{C}^n : |\xi| = 1\}$. Also, we write $d\sigma$ for the spherical measure on S with the normalization $\sigma(S) = 1$. Let dv denote the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$.

For each $0 < t < 1$, define $\Omega_t = \{z \in \mathbf{B} : t < |z| < 1\}$. We introduce the operator $R : C^1(\Omega_t) \to C(\Omega_t)$, $0 < t < 1$, as follows. For any $f \in C^1(\Omega_t)$, we define

$$(Rf)(r\xi) = r\frac{d}{dr}f(r\xi) \quad \text{for} \quad t < r < 1 \quad \text{and} \quad \xi \in S.$$

Let $\mathrm{Hol}(\mathbf{B})$ be the collection of analytic functions on $\mathbf{B}$. Then it is obvious that

$$R = z_1\partial_1 + \cdots + z_n\partial_n \quad \text{on} \quad \mathrm{Hol}(\mathbf{B}).$$

In other words, R coincides with the usual radial derivative on $\mathrm{Hol}(\mathbf{B})$. Equivalently, R is a natural extension to $C^1(\Omega_t)$ of the usual radial derivative on $\mathrm{Hol}(\mathbf{B})$. We need the following integral inequality:

Lemma 2.3. *Let $-1 < \gamma < \infty$ and $1/2 < r < 1$ be given. Then there is a constant $0 < C = C(\gamma, r) < \infty$ such that the inequality*

$$\int_{\Omega_{1/2}} |f(z)|^2(1 - |z|^2)^\gamma dv(z) \leq C\int_{\Omega_r} |(Rf)(z)|^2(1 - |z|^2)^{\gamma+2}dv(z) + C\int_{\Sigma_r} |f(z)|^2 dv(z)$$

holds for every function f that is C^1 on an open set containing the closure of $\Omega_{1/2}$, where

$$\Sigma_r = \{z \in \mathbf{B} : 1/2 < |z| < (1 + r)/2\}.$$

Proof. Let $-1 < \gamma < \infty$ and $h \in C^1_c[0, \infty)$. By a classic inequality of Hardy, we have

$$\int_0^\infty |h(x)|^2 x^\gamma dx \leq \left(\frac{2}{\gamma + 1}\right)^2 \int_0^\infty |xh'(x)|^2 x^\gamma dx.$$

See [14, page 10]. Let $1/2 < r < 1$ also be given. Applying a standard argument using a smooth cutoff function, there are $0 < C_1 < \infty$ and $0 < C_2 < \infty$ such that

$$\int_0^{1/2} |h(x)|^2 x^\gamma dx \leq C_1\int_0^{1-r} |xh'(x)|^2 x^\gamma dx + C_2\int_{(1-r)/2}^{1/2} |h(x)|^2 x^\gamma dx \tag{2.1}$$

for every $h \in C^1[0,1/2]$. Let f be C^1 on an open set containing the closure of $\Omega_{1/2}$. Then by the radial-spherical decomposition of the volume measure dv, we have

$$\begin{aligned}
&\int_{\Omega_{1/2}} |f(z)|^2(1-|z|^2)^\gamma dv(z) = 2n\int_S \int_{1/2}^1 |f(tu)|^2(1-t^2)^\gamma t^{2n-1} dt d\sigma(u) \\
&\le C_3 \int_S \int_{1/2}^1 |f(tu)|^2(1-t)^\gamma dt d\sigma(u) = C_3 \int_S \int_0^{1/2} |f((1-x)u)|^2 x^\gamma dx d\sigma(u) \\
&\le C_4 \int_S \int_0^{1-r} \left| x\frac{d}{dx} f((1-x)u)\right|^2 x^\gamma dx d\sigma(u) + C_5 \int_S \int_{(1-r)/2}^{1/2} |f((1-x)u)|^2 x^\gamma dx d\sigma(u),
\end{aligned}$$

where for the second $\le$ we apply (2.1). For $0 < x < 1$ and $u \in S$, we have

$$\frac{d}{dx} f((1-x)u) = \frac{-1}{1-x}(Rf)((1-x)u).$$

Substituting this in the above inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_{1/2}} |f(z)|^2(1-|z|^2)^\gamma dv(z) \\
&\le C_6 \int_S \int_0^{1-r} |x(Rf)((1-x)u)|^2 x^\gamma dx d\sigma(u) + C_5 \int_S \int_{(1-r)/2}^{1/2} |f((1-x)u)|^2 x^\gamma dx d\sigma(u) \\
&= C_6 \int_S \int_r^1 |(Rf)(tu)|^2 (1-t)^{\gamma+2} dt d\sigma(u) + C_5 \int_S \int_{1/2}^{(1+r)/2} |f(tu)|^2(1-t)^\gamma dt d\sigma(u).
\end{aligned}$$

Using the radial-spherical decomposition $dv = 2nt^{2n-1}dtd\sigma$ again, the lemma follows from the above inequality. □

Definition 2.4. Let $0 < s < \infty$.
(a) We denote $B(0,s) = \{z \in \mathbf{C}^n : |z| < s\}$.
(b) For any function f on $B(0,s)$, we write $\|f\|_{s,\infty} = \sup\{|f(z)| : z \in B(0,s)\}$.
(c) Let H^∞_s denote the collection of analytic functions f on $B(0,s)$ satisfying the condition $\|f\|_{s,\infty} < \infty$.

Proposition 2.5. *Let $s > 1$. Then there is a $0 < C = C(s) < \infty$ such that $\|f\|_{\mathcal{M}} \le C\|f\|_{s,\infty}$ for every $f \in H^\infty_s$.*

Proof. Let $s > 1$. Given an $f \in H^\infty_s$, we define $F(\zeta) = f(s\zeta)$ for $\zeta \in \mathbf{B} = B(0,1)$. Then $F \in H^\infty_1$ with $\|F\|_{1,\infty} = \|f\|_{s,\infty}$. By the Cauchy integral for the unit ball, we have

$$f(\zeta) = F(\zeta/s) = \int_S \frac{F(\xi)d\sigma(\xi)}{(1-s^{-1}\langle\zeta,\xi\rangle)^n}, \quad \zeta \in \mathbf{B}. \tag{2.2}$$

This implies that for each $j \in \mathbf{Z}_+$, there is a C_j such that $\|R^j f\|_{1,\infty} \le C_j\|f\|_{s,\infty}$ for every $f \in H^\infty_s$. For $h \in H^2_n$ and $f \in H^\infty_s$, we have $R^n(fh) = \sum_{j=0}^n \frac{n!}{j!(n-j)!} R^j f \cdot R^{n-j}h$ by the Leibniz rule. Combining these facts with another well-known fact,

$$\|g\|^2 \approx |g(0)|^2 + \int_{\mathbf{B}} |(R^n g)(w)|^2(1-|w|^2)^n dv(w) \quad \text{for } g \in H^2_n, \tag{2.3}$$

the conclusion of the proposition follows. $\square$

We write $\zeta_1, \dots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$. For Toeplitz operators on the Bergman space, there is the following well-known result, whose proof will be omitted:

Lemma 2.6. *On the Bergman space $L_a^2(\mathbf{B})$, we have $[T_{\zeta_i}, T_{\zeta_j}^*] \in \mathcal{C}_p$ for all $p > n$ and $i, j \in \{1, \dots, n\}$.*

3. An integral formula for the norm in H_n^2

For each natural number $m \in \mathbf{N}$, we define the function

$$\psi_m(t) = \frac{1}{t^n} \int \cdots \int_{t<t_1<\cdots<t_m<1} \frac{1}{t_1} \cdots \frac{1}{t_m} dt_1 \cdots dt_m, \quad 0 < t < 1.$$

This function solves the following moment problem: for every $k \in \mathbf{N}$, we have

$$\begin{aligned} \int_0^1 t^{k+n-1} \psi_m(t) dt &= \int_0^1 t^{k-1} \int \cdots \int_{t<t_1<\cdots<t_m<1} \frac{1}{t_1} \cdots \frac{1}{t_m} dt_1 \cdots dt_m dt \\ &= \int \cdots \int_{0<t<t_1<\cdots<t_m<1} t^{k-1} \frac{1}{t_1} \cdots \frac{1}{t_m} dt_1 \cdots dt_m dt \\ &= \int_0^1 \frac{1}{t_m} \int_0^{t_m} \frac{1}{t_{m-1}} \cdots \int_0^{t_3} \frac{1}{t_2} \int_0^{t_2} \frac{1}{t_1} \int_0^{t_1} t^{k-1} dt dt_1 dt_2 \cdots dt_{m-1} dt_m \\ &= \frac{1}{k^{m+1}}. \end{aligned} \tag{3.1}$$

Obviously, there are $a_1, \dots, a_{n-1} \in \mathbf{Z}_+$ such that

$$\frac{(k+1)\cdots(k+n-1)}{k^{n-1}} = 1 + \sum_{i=1}^{n-1} \frac{a_i}{k^i} \quad \text{for every} \quad k \in \mathbf{N}. \tag{3.2}$$

We now define the function

$$\varphi(t) = \frac{1}{n!} \left(\psi_n(t) + \sum_{i=1}^{n-1} a_i \psi_{n+i}(t) \right), \quad 0 < t < 1.$$

It follows from (3.1) and (3.2) that

$$\int_0^1 t^{k+n-1} \varphi(t) dt = \frac{(k+1)\cdots(k+n-1)}{n! k^{2n}} \quad \text{for every} \quad k \in \mathbf{N}. \tag{3.3}$$

Recall that for any $\alpha \in \mathbf{Z}_+^n$, the H_n^2-norm of ζ^α is $\sqrt{\alpha!/|\alpha|!}$. Also recall that $R\zeta^\alpha = |\alpha|\zeta^\alpha$. Thus, using (3.3) and [12, Proposition 1.4.9], straightforward integration gives us a precise formula for the norm in the Drury-Arveson space:

Proposition 3.1. *For every $f \in H_n^2$, we have*

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbf{B}} |(R^n f)(w)|^2 \varphi(|w|^2) dv(w).$$

This should be compared with (2.3).

Let $j \in \mathbf{Z}_+$. For each $w \in \mathbf{B}$, we define the kernel function

$$K_w^{(j)} = R^j K_w.$$

If $f \in H_n^2$, then

$$\langle f, K_w^{(j)} \rangle = \langle f, R^j K_w \rangle = \langle R^j f, K_w \rangle = (R^j f)(w),$$

$w \in \mathbf{B}$. Thus $K_w^{(j)}$ is the reproducing kernel for the j-th radial derivative on H_n^2. Using this kernel, we can restate Proposition 3.1 in the form of the operator identity

$$1 = E_0 + \int_{\mathbf{B}} K_w^{(n)} \otimes K_w^{(n)} \varphi(|w|^2) dv(w) \tag{3.4}$$

on H_n^2, where E_0 is the orthogonal projection from H_n^2 onto the subspace $\mathbf{C}$. As we will see, this is a very convenient resolution of the identity operator on H_n^2.

Definition 3.2. (a) We write $\Omega = \{w \in \mathbf{B} : 1/2 < |w| < 1\}$.
(b) Let $d\mu(w)$ denote the restriction of the measure $\varphi(|w|^2)dv(w)$ to Ω.
(c) Denote $\eta(w) = \chi_{[0,3/4]}(|w|)$, $w \in \mathbf{C}^n$.

Definition 3.3. (a) We define the operators T_1 and T_2 on H_n^2 by the formulas

$$T_1 = \int_{\Omega} K_w^{(n)} \otimes K_w^{(n)} d\mu(w) \quad \text{and} \quad T_2 = \sum_{j=0}^{n-1} \int_{\Omega} K_w^{(j)} \otimes K_w^{(j)} \eta(w) d\mu(w).$$

(b) Denote $T = T_1 + T_2$.

Lemma 3.4. *The operators T_2 and*

$$L = \int_{\mathbf{B}\setminus\Omega} K_w^{(n)} \otimes K_w^{(n)} \varphi(|w|^2) dv(w) \tag{3.5}$$

are in the trace class. Moreover, the operator $1 - L$ is positive and invertible on H_n^2.

Proof. From the definition of ψ_m it is obvious that

$$\psi_m(t) \leq \frac{1}{t^n} \int_t^1 \frac{1}{t_1} dt_1 \cdots \int_t^1 \frac{1}{t_m} dt_m = \frac{1}{t^n} \left(\log \frac{1}{t} \right)^m, \quad 0 < t < 1. \tag{3.6}$$

We have, of course, $K_w^{(0)} = K_w$. For $j \in \mathbf{N}$, by straightforward differentiation and an easy induction, there are $b_1^{(j)}, \dots, b_j^{(j)} \in \mathbf{Z}_+$ with $b_j^{(j)} = j!$ such that

$$K_w^{(j)}(z) = \sum_{i=1}^{j} b_i^{(j)} \frac{\langle z, w\rangle^i}{(1-\langle z, w\rangle)^{i+1}}. \tag{3.7}$$

By the definition of η, we have

$$\operatorname{tr}(T_2) = \sum_{j=0}^{n-1} \int_\Omega \operatorname{tr}(K_w^{(j)} \otimes K_w^{(j)})\eta(w)d\mu(w) = \sum_{j=0}^{n-1} \int_\Omega K_w^{(2j)}(w)\eta(w)d\mu(w) < \infty.$$

That is, $T_2 \in \mathcal{C}_1$.

By (3.6), if we set

$$C = \int_{\mathbf{B}\backslash\Omega} |w|^2\varphi(|w|^2)dv(w),$$

then $C < \infty$. Write $e_\alpha(\zeta) = (|\alpha|!/\alpha!)^{1/2}\zeta^\alpha$ for $\alpha \in \mathbf{Z}_+^n$. Then $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ is the standard orthonormal basis for H_n^2. To prove that $L \in \mathcal{C}_1$, we first observe that since $\mathbf{B}\backslash\Omega$ is spherically symmetric, the operator L is diagonal with respect to $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$. Integrating in the radial-spherical coordinates, for $\alpha \in \mathbf{Z}_+^n\backslash\{0\}$ we have

$$\langle Le_\alpha, e_\alpha\rangle = |\alpha|^{2n}\frac{|\alpha|!}{\alpha!}\int_{\mathbf{B}\backslash\Omega} |w^\alpha|^2\varphi(|w|^2)dv(w) \le C(n-1)!|\alpha|^{n+1}2^{-2(|\alpha|-1)}. \tag{3.8}$$

For any $k \in \mathbf{N}$, $\operatorname{card}\{\alpha \in \mathbf{Z}_+^n : |\alpha| = k\} = \frac{(k+n-1)!}{(n-1)!k!} \le C_1k^{n-1}$. Combining this fact with the above inequality, we see that $L \in \mathcal{C}_1$ as promised.

The operator inequality $1 - L \ge 0$ is obvious from (3.4). To prove the invertibility of $1 - L$ on H_n^2, we use the fact that $1 - L$ is diagonal with respect to the orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$. For every $\alpha \in \mathbf{Z}_+^n$ we have

$$\langle (1-L)e_\alpha, e_\alpha\rangle = |e_\alpha(0)|^2 + |\alpha|^{2n}\frac{|\alpha|!}{\alpha!}\int_\Omega |w^\alpha|^2\varphi(|w|^2)dv(w) > 0.$$

By (3.8), $\langle Le_\alpha, e_\alpha\rangle \to 0$ as $|\alpha| \to \infty$. Therefore $\langle (1-L)e_\alpha, e_\alpha\rangle \to 1$ as $|\alpha| \to \infty$. Consequently, $1 - L$ is invertible on H_n^2. □

Proposition 3.5. *We have $1 - T \in \mathcal{C}_1$. Moreover, the operator T is invertible on H_n^2.*

Proof, By (3.4), (3.5) and Definition 3.3, we have $1-T = E_0+L-T_2$. Thus the membership $1 - T \in \mathcal{C}_1$ follows from Lemma 3.4.

Obviously, there is a $0 < c \le 1$ such that $T_2 \ge cE_0$. We can rewrite (3.4) as $1 = E_0 + T_1 + L$. Thus

$$T = T_1 + T_2 = (1-c)T_1 + T_2 - cE_0 + c(T_1 + E_0) = (1-c)T_1 + T_2 - cE_0 + c(1-L).$$

Lemma 3.4 tells us that the operator $c(1-L)$ is positive and invertible on H^2_n. Since both operators $(1-c)T_1$ and $T_2 - cE_0$ are positive, it follows that T is invertible. □

4. The range space

The idea of range space was first introduced in the study [8] of the geometric Arveson-Douglas conjecture, with recent success in [14]. This idea turns out to be a key to the proof of Theorem 1.2. In effect, here we are treating the Drury-Arveson space H^2_n as a *quotient module* of itself, with the corresponding submodule being the trivial one, $\{0\}$.

Let $\mathcal{L}_0$ be the collection of functions f that are C^∞ on an open set containing $\overline{\Omega}$. For $f \in \mathcal{L}_0$, we define

$$\|f\|_\# = \left\{ \int_\Omega |(R^n f)(w)|^2 d\mu(w) + \sum_{j=1}^{n-1} \int_\Omega |R^j f(w)|^2 \eta(w) d\mu(w) \right\}^{1/2}.$$

Obviously, $\|\cdot\|_\#$ is the norm on $\mathcal{L}_0$ induced by the inner product

$$\langle f, g\rangle_\# = \int_\Omega (R^n f)(w)\overline{(R^n g)(w)} d\mu(w) + \sum_{j=1}^{n-1} \int_\Omega (R^j f)(w)\overline{(R^j g)(w)}\eta(w) d\mu(w),$$

$f, g \in \mathcal{L}_0$. Let $\mathcal{L}$ denote the completion of $\mathcal{L}_0$ with respect to the norm $\|\cdot\|_\#$. Then $\mathcal{L}$ is a Hilbert space.

Definition 4.1. (a) Let $\mathcal{P}$ be the closure of the analytic polynomials $\mathbf{C}[\zeta_1, \dots, \zeta_n]$ in $\mathcal{L}$.
(b) Let P denote the orthogonal projection from $\mathcal{L}$ onto $\mathcal{P}$.

Recalling Definition 3.3, if $f \in H^2_n$, then

$$\|f\|^2_\# = \langle Tf, f\rangle = \|T^{1/2} f\|^2. \tag{4.1}$$

Since $\mathbf{C}[\zeta_1, \dots, \zeta_n]$ is dense in H^2_n, every $f \in H^2_n$ is naturally an element in $\mathcal{P}$.

Definition 4.2. Let J denote the operator that takes each $f \in H^2_n$ to the same f in $\mathcal{P}$.

Thus we can rewrite (4.1) in the form of the operator identity

$$J^* J = T. \tag{4.2}$$

Intuitively, we think of J as restricting each $f \in H^2_n$ to the set Ω. We call $\mathcal{P}$ the *range space* for the restriction operator J. By Proposition 3.5, there is an $a > 0$ such that

$$\|Jf\|_\# = \|T^{1/2} f\| \geq a\|f\| \quad \text{for every} \quad f \in H^2_n.$$

Thus J is an invertible operator that maps H^2_n onto $\mathcal{P}$.

Definition 4.3. (a) We define the operators

$$(\hat{T}_1 f)(z) = \int_\Omega K_w^{(n)}(z)(R^n f)(w)d\mu(w) \quad \text{and}$$
$$(\hat{T}_2 f)(z) = \sum_{j=1}^{n-1} \int_\Omega K_w^{(j)}(z)(R^j f)(w)\eta(w)d\mu(w),$$

$f \in \mathcal{L}_0$.
(b) We define $\hat{T} = \hat{T}_1 + \hat{T}_2$.

Lemma 4.4. *The operators $\hat{T}_1$ and $\hat{T}_2$ are bounded on $\mathcal{L}_0$. Therefore $\hat{T}_1$ and $\hat{T}_2$ naturally extend to bounded operators on $\mathcal{L}$.*

Proof. Using (3.7) and the Schur test, the kernels $K_w^{(m)}(z)$, $0 \le m \le 2n$, define bounded operators on $L^2(\Omega, d\mu)$. (See the proof of Lemma 5.1 below for details.) Combining this fact with the definition of $\|\cdot\|_\#$, it is easy to see that $\hat{T}_1$ and $\hat{T}_2$ are bounded on $\mathcal{L}_0$. □

Lemma 4.5. *With respect to the inner product $\langle \cdot, \cdot \rangle_\#$, the operator $\hat{T}$ is self-adjoint.*

Proof. Let $f \in \mathcal{L}_0$. Then

$$\begin{aligned}\langle \hat{T} f, f\rangle_\# = &\int_\Omega \int_\Omega K_w^{(2n)}(z)(R^n f)(w)d\mu(w)\overline{(R^n f)(z)}d\mu(z)\\
&+ \sum_{j=0}^{n-1} \int_\Omega \int_\Omega K_w^{(j+n)}(z)(R^j f)(w)\eta(w)d\mu(w)\overline{(R^n f)(z)}d\mu(z)\\
&+ \sum_{j=0}^{n-1} \int_\Omega \int_\Omega K_w^{(n+j)}(z)(R^n f)(w)d\mu(w)\overline{(R^j f)(z)}\eta(z)d\mu(z)\\
&+ \sum_{0\le i,j\le n-1} \int_\Omega \int_\Omega K_w^{(j+i)}(z)(R^j f)(w)\eta(w)d\mu(w)\overline{(R^i f)(z)}\eta(z)d\mu(z). \end{aligned} \tag{4.3}$$

It follows from (3.7) that $K_w^{(m)}(z) = \overline{K_z^{(m)}(w)}$, $m \in \mathbf{Z}_+$. Substituting this in (4.3), we see that $\langle \hat{T} f, f\rangle_\#$ is a real number. Therefore $\hat{T}$ is self-adjoint. □

Proposition 4.6. (a) *$\hat{T}$ maps $\mathcal{L}$ into $\mathcal{P}$.*
(b) *Let $\tilde{T}$ denote the restriction of $\hat{T}$ to the subspace $\mathcal{P}$. Then $\tilde{T} = JJ^*$. In particular, $\tilde{T}$ is invertible on $\mathcal{P}$.*
(c) *With respect to the orthogonal decomposition $\mathcal{L} = \mathcal{P} \oplus \mathcal{P}^\perp$, we have $\hat{T} = \tilde{T} \oplus 0$.*

Proof. (a) Recall that we write $\Omega_t = \{z \in \mathbf{B} : t < |z| < 1\}$ for $0 < t < 1$. As we mentioned in the proof of Lemma 4.4, the kernels $K_w^{(m)}(z)$, $0 \le m \le 2n$, define bounded operators on $L^2(\Omega, d\mu)$. Therefore for any $f \in \mathcal{L}_0$, we have

$$\lim_{t\uparrow 1} \left\| \int_{\Omega_t} (R^n f)(w) J K_w^{(n)} d\mu(w)\right\|_\# = 0.$$

Since we already know that $JH_n^2 \subset \mathcal{P}$, we have

$$\int_{\Omega \backslash \Omega_t} (R^n f)(w) J K_w^{(n)} d\mu(w) \in \mathcal{P}$$

for every $1/2 < t < 1$. Therefore $\hat{T}_1 f \in \mathcal{P}$. It is obvious that $\hat{T}_2 f \in \mathcal{P}$ for $f \in \mathcal{L}_0$. Thus $\hat{T}\mathcal{L}_0 \subset \mathcal{P}$. Since $\mathcal{L}_0$ is dense in $\mathcal{L}$ and since Lemma 4.4 tells us that $\hat{T}$ is a bounded operator, it follows that $\hat{T}\mathcal{L} \subset \mathcal{P}$.

(b) For each $f \in H_n^2$, it is easy to see that $\tilde{T}Jf = JTf$. Combining this with (4.2), we have $\tilde{T}Jf = JTf = JJ^*Jf$. Since $JH_n^2 = \mathcal{P}$, this implies $\tilde{T} = JJ^*$. Since $J : H_n^2 \to \mathcal{P}$ and $J^* : \mathcal{P} \to H_n^2$ are invertible, so is $\tilde{T}$.

(c) This follows from (a) and the self-adjointness of $\hat{T}$ provided by Lemma 4.5. □

Definition 4.7. For each $\xi \in C^n(\Omega)$, $\hat{M}_\xi$ denotes the operator of multiplication by the function ξ on $\mathcal{L}$.

Proposition 4.8. *We have $JM_h = \hat{M}_h J$ for every $h \in \mathcal{M}$.*

Proof. If $h \in \mathcal{M}$ and $f \in H_n^2$, then $JM_h f = J(hf) = hf = \hat{M}_h Jf$. □

Corollary 4.9. *If $h \in \mathcal{M}$, then $\mathcal{P}$ is an invariant subspace for $\hat{M}_h$.*

Corollary 4.9 makes it possible for us to introduce

Definition 4.10. For each $h \in \mathcal{M}$, let $\tilde{M}_h$ denote the restriction of the operator $\hat{M}_h$ to the invariant subspace $\mathcal{P}$.

Accordingly, we can restate Proposition 4.8 as

Proposition 4.11. *We have $JM_h = \tilde{M}_h J$ for every $h \in \mathcal{M}$. Consequently, there is a constant $0 < C < \infty$ such that $\|\tilde{M}_h\| \leq C\|M_h\| = C\|h\|_{\mathcal{M}}$ for every $h \in \mathcal{M}$.*

Proposition 4.12. *Given any $k \in \mathbf{N}$, there is a constant $0 < C = C(k) < \infty$ such that*

$$\|M_{g_1}^* M_{h_1} \cdots M_{g_k}^* M_{h_k} - J^* \tilde{M}_{g_1}^* \tilde{M}_{h_1} \cdots \tilde{M}_{g_k}^* \tilde{M}_{h_k} J\|_1 \leq C \prod_{j=1}^{k} \|g_j\|_{\mathcal{M}} \|h_j\|_{\mathcal{M}}$$

for all $g_1, h_1, \ldots, g_k, h_k \in \mathcal{M}$.

Proof. Let us denote $K = E_0 + L - T_2$. Recall from Lemma 3.4 that $K \in \mathcal{C}_1$. By (3.4), (3.5) and Definition 3.3, we have $T = 1 - K$. Let

$$J = U|J|$$

be the polar decomposition of J. Since $J : H_n^2 \to \mathcal{P}$ is invertible, U is a unitary operator that maps H_n^2 onto $\mathcal{P}$. By (4.2), we have

$$JJ^* = U|J|^2U^* = UTU^* = 1 - UKU^*. \tag{4.4}$$

For any $h \in \mathcal{M}$, it follows from (4.2) and Proposition 4.11 that

$$M_h - KM_h = TM_h = J^*JM_h = J^*\tilde{M}_h J.$$

Thus for any $h, g \in \mathcal{M}$,

$$M_g^* M_h = (J^*\tilde{M}_g^* J + M_g^* K)(J^*\tilde{M}_h J + KM_h) = J^*\tilde{M}_g^* JJ^*\tilde{M}_h J + K_1(g,h),$$

where $K_1(g,h) \in \mathcal{C}_1$ with $\|K_1(g,h)\|_1 \le C_1\|g\|_{\mathcal{M}}\|h\|_{\mathcal{M}}$. Consequently, for any $g_1, h_1, \dots, g_k, h_k \in \mathcal{M}$, we have

$$\begin{aligned} M_{g_1}^* M_{h_1} \cdots M_{g_k}^* M_{h_k} = {} & J^*\tilde{M}_{g_1}^* JJ^*\tilde{M}_{h_1} J \cdots J^*\tilde{M}_{g_k}^* JJ^*\tilde{M}_{h_k} J \\ & + K_k(g_1, h_1, \dots, g_k, h_k), \end{aligned} \tag{4.5}$$

where $K_k(g_1, h_1, \dots, g_k, h_k) \in \mathcal{C}_1$ with

$$\|K_k(g_1, h_1, \dots, g_k, h_k)\|_1 \le C_k \prod_{j=1}^{k} \|g_j\|_{\mathcal{M}}\|h_j\|_{\mathcal{M}}.$$

Substituting (4.4) in (4.5), we obtain the desired conclusion. □

Applying Proposition 4.12 to antisymmetric sums, we immediately obtain

Corollary 4.13. *There is a constant* $0 < C < \infty$ *such that*

$$\|[M_{g_1}^* M_{h_1}, \dots, M_{g_{2n}}^* M_{h_{2n}}] - J^*[\tilde{M}_{g_1}^*\tilde{M}_{h_1}, \dots, \tilde{M}_{g_{2n}}^*\tilde{M}_{h_{2n}}]J\|_1 \le C \prod_{j=1}^{2n} \|g_j\|_{\mathcal{M}}\|h_j\|_{\mathcal{M}}$$

for all $g_1, h_1, \dots, g_{2n}, h_{2n} \in \mathcal{M}$.

This tells us that, to prove Theorem 1.2, it suffices to consider antisymmetric sums of the form $[\tilde{M}_{g_1}^*\tilde{M}_{h_1}, \dots, \tilde{M}_{g_{2n}}^*\tilde{M}_{h_{2n}}]$ on the range space $\mathcal{P}$.

Proposition 4.14. *The operator* $P - \hat{T}$ *is in the trace class.*

Proof. By Proposition 4.6(c), it suffices to show that $1 - \tilde{T} \in \mathcal{C}_1$ on $\mathcal{P}$. By Proposition 4.6(b), we have $\tilde{T} = JJ^*$. Thus from (4.4) we obtain $1 - \tilde{T} = UKU^* \in \mathcal{C}_1$. □

5. Integral operators on L^2

In this section we focus on integral operators on $L^2(\Omega, d\mu)$. For any $m \in \mathbf{N}$, a review of the definition of ψ_m in Section 3 gives us the inequality

$$\begin{aligned} \psi_m(t) &\ge \int \cdots \int_{t < t_1 < \cdots < t_m < 1} 1 dt_1 \cdots dt_m \\ &= \int_t^1 \int_{t_1}^1 \cdots \int_{t_{m-2}}^1 \int_{t_{m-1}}^1 1 dt_m dt_{m-1} \cdots dt_2 dt_1 = \frac{1}{m!}(1-t)^m, \end{aligned}$$

$0 < t < 1$. Combining this with (3.6), there are $0 < c < C < \infty$ such that

$$c(1-|z|^2)^n \le \varphi(|z|^2) \le C(1-|z|^2)^n \quad \text{for every} \;\; z \in \Omega. \tag{5.1}$$

That is, $d\mu$ is comparable to the weighted volume measure $(1-|z|^2)^n dv(z)$ on Ω.

Lemma 5.1. *If $G(z,w)$ is a bounded Borel function on $\Omega \times \Omega$, then the operator*

$$(A_G f)(z) = \int_\Omega \frac{G(z,w)}{(1-\langle z,w\rangle)^{2n+1}} f(w) d\mu(w), \quad f \in L^2(\Omega, d\mu), \tag{5.2}$$

is bounded on $L^2(\Omega, d\mu)$. Moreover, there is a constant C_1 such that $\|A_G\| \le C_1\|G\|_\infty$.

Proof. Consider the function $h(w) = (1-|w|^2)^{-1/2}$ on Ω. By (5.1), we have

$$\begin{aligned}\int_\Omega h(w)\left|\frac{G(z,w)}{(1-\langle z,w\rangle)^{2n+1}}\right| d\mu(w) &\le C\|G\|_\infty \int_\Omega \frac{(1-|w|^2)^{n-(1/2)}}{|1-\langle z,w\rangle|^{n+1+n-(1/2)+(1/2)}} dv(w)\\ &\le C_1\|G\|_\infty h(z),\end{aligned}$$

where the second $\le$ follows from [12, Proposition 1.4.10]. Similarly,

$$\int_\Omega h(z)\left|\frac{G(z,w)}{(1-\langle z,w\rangle)^{2n+1}}\right| d\mu(z) \le C_1\|G\|_\infty h(w).$$

Thus the Schur test gives us $\|A_G\| \le C_1\|G\|_\infty$. □

It is also a consequence of [12, Proposition 1.4.10] that if $c > 0$, then

$$\iint_{\mathbf{B}\times\mathbf{B}} \frac{1}{|1-\langle z,w\rangle|^{n+1+1-c}} dv(w)dv(z) < \infty. \tag{5.3}$$

Lemma 5.2. *Let $G(z,w)$ be a Borel function on $\Omega \times \Omega$. If G satisfies the condition*

$$\iint_{\Omega\times\Omega} \frac{|G(z,w)|^p}{|1-\langle z,w\rangle|^{4n+2}} d\mu(w)d\mu(z) < \infty \tag{5.4}$$

for some $2 \le p < \infty$, then the operator A_G defined by (5.2) *belongs to the Schatten class $\mathcal{C}_p$. Moreover, for each $2 \le p < \infty$, there is a constant $0 < C = C(p) < \infty$ such that*

$$\|A_G\|_p^p \le C \iint_{\Omega\times\Omega} \frac{|G(z,w)|^p}{|1-\langle z,w\rangle|^{4n+2}} d\mu(w)d\mu(z)$$

for every G satisfying (5.4).

Proof. The case $p = 2$ is obvious. By Lemma 5.1, we have $\|A_G\| \le C_1\|G\|_\infty$. Thus the case $2 < p < \infty$ follows from the standard interpolation. □

Recall that we write $K_w^{(m)} = R^m K_w$, $w \in \mathbf{B}$. In particular, $K_w^{(0)}$ means K_w itself.

Definition 5.3. (a) For each $0 \le j \le 2n$, X_j denotes the operator on $L^2(\Omega, d\mu)$ defined by the formula

$$(X_j f)(z) = \int_\Omega \frac{\langle z, w\rangle^j}{(1 - \langle z, w\rangle)^{j+1}} f(w) d\mu(w), \quad f \in L^2(\Omega, d\mu).$$

(b) For each $0 \le j \le 2n$, Y_j denotes the operator on $L^2(\Omega, d\mu)$ defined by the formula

$$(Y_j f)(z) = \int_\Omega K_w^{(j)}(z) f(w) d\mu(w), \quad f \in L^2(\Omega, d\mu).$$

Lemma 5.4. (a) *If there is a $0 < C < \infty$ such that $|G(z, w)| \le C|1 - \langle z, w\rangle|$ for all $z, w \in \Omega$, then $A_G \in \mathcal{C}_p$ for every $p > n$.*
(b) *If $j \le 2n - 1$, then $X_j, Y_j \in \mathcal{C}_p$ for every $p > n$.*

Proof. By (5.1) and the assumption on G,

$$\iint_{\Omega\times\Omega} \frac{|G(z, w)|^p}{|1 - \langle z, w\rangle|^{4n+2}} d\mu(w)d\mu(z) \le C_1 \iint_{\Omega\times\Omega} \frac{1}{|1 - \langle z, w\rangle|^{n+1+1-(p-n)}} dv(w)dv(z).$$

If $p > n$, then by (5.3) this is finite. Thus Lemma 5.2 provides the membership $A_G \in \mathcal{C}_p$ for $p > n$, proving (a). By Definition 5.3, it follows from (a) that if $j \le 2n - 1$, then $X_j \in \mathcal{C}_p$ for every $p > n$. Recalling (3.7), we see that if $j \le 2n - 1$, then $Y_j \in \mathcal{C}_p$ for every $p > n$. $\square$

If f is a Lipschitz function on Ω, we write $L(f)$ for its Lipschitz constant.

Lemma 5.5. (a) *For each $p > 2n$, there is a constant $0 < B_p < \infty$ such that $\|[M_f, X_{2n}]\|_p \le B_p L(f)$ for every Lipschitz function f* on Ω.
(b) *For each $p > n$, there is a constant $0 < C_p < \infty$ such that $\|[M_f, [M_g, X_{2n}]]\|_p \le C_p L(f)L(g)$ for every pair of Lipschitz functions f, g on Ω.*

Proof. (a) We have $[M_f, X_{2n}] = A_G$ with $G(z, w) = (f(z) - f(w))\langle z, w\rangle^{2n}$. Since

$$|G(z, w)| \le L(f)|z - w| \le \sqrt{2}L(f)|1 - \langle z, w\rangle|^{1/2},$$

for $p > 2n$, the bound $\|[M_f, X_{2n}]\|_p \le B_p L(f)$ follows from Lemma 5.2.

(b) We have $[M_f, [M_g, X_{2n}]] = A_H$ with $H(z, w) = (f(z)-f(w))(g(z)-g(w))\langle z, w\rangle^{2n}$. Since

$$|H(z, w)| \le L(f)L(g)|z - w|^2 \le 2L(f)L(g)|1 - \langle z, w\rangle|,$$

for $p > n$, the bound $\|[M_f, [M_g, X_{2n}]]\|_p \le C_p L(f)L(g)$ again follows from Lemma 5.2. $\square$

Lemma 5.6. (a) *For each $p > 2n$, there is a constant $0 < B'_p < \infty$ such that $\|[M_f, Y_{2n}]\|_p \le B'_p L(f)$ for every Lipschitz function f* on Ω.
(b) *For each $p > n$, there is a constant $0 < C'_p < \infty$ such that $\|[M_f, [M_g, Y_{2n}]]\|_p \le C'_p L(f)L(g)$ for every pair of Lipschitz functions f, g on Ω.*

Proof. Applying (3.7), (a) and (b) are obtained by combining the corresponding parts in Lemma 5.5 with Lemma 5.4. □

By the definition of the norm $\|\cdot\|_\#$, inequality (5.1) and Lemma 2.3, it is clear that for each integer $0 \le j \le n$, there is a $0 < C_j < \infty$ such that

$$\int_\Omega |(R^j f)(w)|^2 d\mu(w) \le C_j \|f\|_\#^2 \tag{5.5}$$

for every $f \in \mathcal{L}_0$. This allows us to introduce

Definition 5.7. For each $0 \le j \le n$, $\mathcal{D}_j$ denotes the operator that maps each $f \in \mathcal{L}_0$ to the function $R^j f$ in $L^2(\Omega, d\mu)$.

By (5.5), each $\mathcal{D}_j : \mathcal{L}_0 \to L^2(\Omega, d\mu)$ is a bounded operator, $0 \le j \le n$. Consequently, $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n$ naturally extend to bounded operators that map $\mathcal{L}$ into $L^2(\Omega, d\mu)$. With these operators, we can rewrite the inner product $\langle \cdot, \cdot \rangle_\#$ in the form

$$\langle f, g \rangle_\# = \langle \mathcal{D}_n f, \mathcal{D}_n g \rangle + \sum_{j=0}^{n-1} \langle M_\eta \mathcal{D}_j f, \mathcal{D}_j g \rangle, \tag{5.6}$$

$f, g \in \mathcal{L}_0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\Omega, d\mu)$, and M_η is the operator of multiplication by η on $L^2(\Omega, d\mu)$.

We now introduce the function $\rho(z) = 1 - |z|^2$ on $\mathbf{C}^n$.

Lemma 5.8. (a) *For every $0 \le j \le n-1$, the operator $M_{\rho^{-1}} \mathcal{D}_j$ is bounded.*
(b) *Let $a < 1/2$. Then for every $0 \le i \le n-2$, the operator $M_{\rho^{-1-a}} \mathcal{D}_i$ is bounded.*

Proof. This follows immediately from Lemma 2.3 and (5.1). □

Lemma 5.9. *For every pair of $0 \le j \le n-1$ and $p > n$, there is a constant $0 < C = C(j, p) < \infty$ such that*

$$\|Y_{2n} M_\xi \mathcal{D}_j\|_p \le C \|\xi\|_\infty$$

for every $\xi \in L^\infty(\Omega, d\mu)$.

Proof. Since $Y_{2n} M_\xi \mathcal{D}_j = Y_{2n} M_\rho \cdot M_\xi \cdot M_{\rho^{-1}} \mathcal{D}_j$, and since Lemma 5.8 tells us that $M_{\rho^{-1}} \mathcal{D}_j$ is bounded if $j \le n-1$, the lemma will follow if we can show that $Y_{2n} M_\rho \in \mathcal{C}_p$ for every $p > n$. By (3.7) and Lemma 5.4, it suffices to show that $X_{2n} M_\rho \in \mathcal{C}_p$ for every $p > n$.

In terms of (5.2), $X_{2n} M_\rho = A_G$, where $G(z, w) = \langle z, w \rangle^{2n} (1 - |w|^2)$. We have

$$\iint_{\Omega \times \Omega} \frac{|G(z,w)|^p}{|1 - \langle z, w \rangle|^{4n+2}} d\mu(w) d\mu(z) \le \iint_{\Omega \times \Omega} \frac{C_1}{|1 - \langle z, w \rangle|^{n+1+1-(p-n)}} dv(w) dv(z).$$

By (5.3), this is finite if $p > n$. Applying Lemma 5.2, we have $X_{2n} M_\rho \in \mathcal{C}_p$ for $p > n$. □

Definition 5.10. (a) Let $\mathcal{E}$ denote the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(\Omega, d\mu)$.
(b) Let E denote the orthogonal projection from $L^2(\Omega, d\mu)$ onto $\mathcal{E}$.

Definition 5.11. Define the integral operator Z on $L^2(\Omega, d\mu)$ by the formula

$$(Zf)(z) = \int_\Omega \frac{1}{(1 - \langle z, w\rangle)^{2n+1}} f(w) d\mu(w), \quad f \in L^2(\Omega, d\mu).$$

Proposition 5.12. *The operator Z is self-adjoint and positive on $L^2(\Omega, d\mu)$. There is a $\gamma > 0$ such that the spectrum of Z does not intersect the interval $(0, \gamma)$. Moreover, the range of Z equals $\mathcal{E}$.*

Proof. It is obvious that Z is self-adjoint on $L^2(\Omega, d\mu)$. Since

$$\tag{5.7} \frac{1}{(1 - \langle z, w\rangle)^{2n+1}} = \sum_{j=0}^{\infty} \frac{(j+2n)!}{j!(2n)!} \sum_{|\alpha|=j} \frac{j!}{\alpha!} z^\alpha \overline{w^\alpha}, \quad z, w \in \Omega,$$

we see that Z is positive on $L^2(\Omega, d\mu)$. Obviously, the range of Z is contained in $\mathcal{E}$. Thus, to complete the proof, it suffices to find a $\gamma > 0$ such that $\langle Zf, f\rangle \geq \gamma \|f\|^2$ for every $f \in \mathbf{C}[z_1, \dots, z_n]$. Since both Ω and $d\mu$ are invariant under spherical rotation, we have both $\langle z^\alpha, z^\beta\rangle = 0$ and $\langle Zz^\alpha, z^\beta\rangle = 0$ for all $\alpha \neq \beta$ in $\mathbf{Z}_+^n$. Hence it suffices to find a $\gamma > 0$ such that $\langle Zz^\alpha, z^\alpha\rangle \geq \gamma \|z^\alpha\|^2$ for every $\alpha \in \mathbf{Z}_+^n$.

First of all, it is obvious that $\langle Zz^\alpha, z^\alpha\rangle > 0$ for every $\alpha \in \mathbf{Z}_+^n$. Let $\alpha \in \mathbf{Z}_+^n$ be such that $|\alpha| = k$. Then by (5.7) and the spherical symmetry of Ω and $d\mu$, we have

$$\langle Zz^\alpha, z^\alpha\rangle = \frac{(k+2n)!}{\alpha!(2n)!} \left(\int_\Omega |w^\alpha|^2 d\mu(w) \right)^2 \geq c_1 \frac{(k+2n)!}{\alpha!(2n)!} \left(\int_\Omega |w^\alpha|^2 (1 - |w|^2)^n dv(w) \right)^2,$$

where the $\geq$ follows from (5.1). Integrating in the radial-spherical coordinates, we have

$$\tag{5.8} \begin{aligned} \langle Zz^\alpha, z^\alpha\rangle &\geq c_2 \frac{(k+2n)!}{\alpha!(2n)!} \left(\frac{\alpha!(n-1)!}{(k+n-1)!} \right)^2 \left(\int_{1/4}^1 t^{k+n-1}(1-t)^n dt \right)^2 \\ &= c_3 \frac{(k+2n)!\alpha!}{\{(k+n-1)!\}^2} \left(\frac{(k+n-1)!n!}{(k+2n)!} - a_k \right)^2, \end{aligned}$$

where

$$a_k = \int_0^{1/4} t^{k+n-1}(1-t)^n dt.$$

Also by (5.1), we have

$$\tag{5.9} \|z^\alpha\|^2 \leq C \int_\Omega |w^\alpha|^2 (1 - |w|^2)^n dv(w) \leq C \frac{\alpha!(n!)^2}{(k+2n)!}.$$

Since $a_k \leq 4^{-k}$, from (5.8) and (5.9) we see that there is a $\gamma > 0$ such that $\langle Zz^\alpha, z^\alpha\rangle \geq \gamma \|z^\alpha\|^2$ for every $\alpha \in \mathbf{Z}_+^n$. This completes the proof. $\square$

Lemma 5.13. (a) *We have $EM_\rho \in \mathcal{C}_p$ for every $p > n$.*

(b) *Let $p > 2n/3$. If $a < 1/2$ satisfies the condition $p(1+a) > n$, then $EM_{\rho^{1+a}} \in \mathcal{C}_p$.*

Proof. Let $\tilde{Z}$ be the restriction of Z to the subspace $\mathcal{E}$. Then it follows from Proposition 5.12 that $\tilde{Z}$ is invertible on $\mathcal{E}$. Consequently, $E = (\tilde{Z}^{-1} \oplus 0)Z$. Therefore it suffices consider ZM_ρ and $ZM_{\rho^{1+a}}$ instead of EM_ρ and $EM_{\rho^{1+a}}$.

(a) In terms of (5.2), we have $ZM_\rho = A_G$, where $G(z,w) = 1 - |w|^2$. Thus by an estimate similar to the one in the proof of Lemma 5.9, we have $ZM_\rho \in \mathcal{C}_p$ for every $p > n$.

(b) Suppose that $n \geq 3$. Then $2n/3 \geq 2$, and Lemma 5.2 applies to every $p > 2n/3$. We can write $ZM_{\rho^{1+a}} = A_H$, where $H(z,w) = (1-|w|^2)^{1+a}$. By the conditions $p > 2n/3$ and $p(1+a) > n$, an estimate similar to the one in the proof of Lemma 5.9 gives us the membership $ZM_{\rho^{1+a}} \in \mathcal{C}_p$.

Suppose that $n = 2$. If $p \geq 2$, then Lemma 5.2 still applies, and the argument in the above paragraph holds. Thus let us assume that $n = 2$ and that $4/3 < p < 2$.

In this case, set $t = 2p(2-p)^{-1}$. Then $t > 4$ and $1/p = (1/t) + (1/2)$. Let $a < 1/2$ be such that $p(1+a) > 2$. Then this means $p(2-p)^{-1} > a^{-1}$. Thus $t/2 > a^{-1}$, or, equivalently, $2a > 4/t$. This allows us to pick an r such that $4/t < r < 2a$.

We first show that $EM_{\rho^{r/2}} \in \mathcal{C}_t$. Equivalently, it suffices to show that $ZM_{\rho^{r/2}} \in \mathcal{C}_t$. We have $ZM_{\rho^{r/2}} = A_L$ with $L(z,w) = (1-|w|^2)^{r/2}$. Since $n = 2$, we have

$$\begin{aligned}\iint \frac{|L(z,w)|^t}{|1-\langle z,w\rangle|^{10}} d\mu(w)d\mu(z) &\leq C_1 \iint \frac{(1-|w|^2)^{rt/2}}{|1-\langle z,w\rangle|^6} dv(w)dv(z) \\ &\leq C_2 \iint \frac{1}{|1-\langle z,w\rangle|^{2+1+\{3-(rt/2)\}}} dv(w)dv(z).\end{aligned}$$

Since $4/t < r$, i.e., $rt/2 > 2$, we have $3 - (rt/2) < 1$. Thus, by (5.3), the above is finite. By Lemma 5.2, we have $ZM_{\rho^{r/2}} \in \mathcal{C}_t$.

Next we show that $M_{\rho^{-r/2}} ZM_{\rho^{1+a}} \in \mathcal{C}_2$. Indeed $M_{\rho^{-r/2}} ZM_{\rho^{1+a}} = A_Q$, where

$$Q(z,w) = (1-|z|^2)^{-r/2}(1-|w|^2)^{1+a}.$$

Thus

$$\|A_Q\|_2^2 \leq C_3 \iint \frac{(1-|w|^2)^{2+2a}}{(1-|z|^2)^r|1-\langle z,w\rangle|^6} dv(w)dv(z) \leq C_4 \int \frac{dv(z)}{(1-|z|^2)^{r+1-2a}},$$

where the second $\leq$ follows from [12, Proposition 1.4.10]. Since $r < 2a$, the above is finite. Hence $M_{\rho^{-r/2}} ZM_{\rho^{1+a}} \in \mathcal{C}_2$.

We have

$$ZM_{\rho^{1+a}} = EZM_{\rho^{1+a}} = EM_{\rho^{r/2}} \cdot M_{\rho^{-r/2}} ZM_{\rho^{1+a}}.$$

Since $1/p = (1/t) + (1/2)$, it follows from the last two paragraphs that $ZM_{\rho^{1+a}} \in \mathcal{C}_p$. $\square$

Lemma 5.14. (a) *For every pair of $0 \leq j \leq n-1$ and $p > n$, we have $\mathcal{D}_j P \in \mathcal{C}_p$.*

(b) *For every pair of* $0 \le i \le n-2$ *and* $p > 2n/3$, *we have* $\mathcal{D}_i P \in \mathcal{C}_p$.

Proof. (a) Note that $\mathcal{D}_j P = E\mathcal{D}_j P$. Therefore we have the factorization

$$\mathcal{D}_j P = EM_\rho \cdot M_{\rho^{-1}}\mathcal{D}_j P.$$

Thus the desired conclusion follows from Lemmas 5.8(a) and 5.13(a).

(b) Given any $p > 2n/3$, i.e., $p(3/2) > n$, we can pick an $a < 1/2$ such that $p(1+a) > n$. This time, we take the factorization

$$\mathcal{D}_i P = EM_{\rho^{1+a}} \cdot M_{\rho^{-1-a}}\mathcal{D}_i P.$$

Thus the desired conclusion follows from Lemmas 5.8(b) and 5.13(b). □

Lemma 5.15. *For every pair of* $0 \le j \le n-1$ *and* $p > n$, *there is a constant* $0 < C = C(j,p) < \infty$ *such that*

$$\|P\mathcal{D}_n^* M_\xi \mathcal{D}_j\|_p \le C\|\xi\|_\infty$$

for every $\xi \in L^\infty(\Omega, d\mu)$.

Proof. Since $\mathcal{D}_n P = E\mathcal{D}_n P$, we have $P\mathcal{D}_n^* = (\mathcal{D}_n P)^* = (E\mathcal{D}_n P)^* = P\mathcal{D}_n^* E$. Hence

$$P\mathcal{D}_n^* M_\xi \mathcal{D}_j = P\mathcal{D}_n^* E M_\xi \mathcal{D}_j = P\mathcal{D}_n^* \cdot EM_\rho \cdot M_\xi \cdot M_{\rho^{-1}}\mathcal{D}_j.$$

Applying Lemmas 5.8(a) and 5.13(a), the lemma follows. □

Lemma 5.16. *For every triple of* $0 \le j \le n-1$, $0 \le k \le n-1$ *and* $p > n/2$, *there is a constant* $0 < C = C(j,k,p) < \infty$ *such that*

$$\|P\mathcal{D}_k^* M_\xi \mathcal{D}_j\|_p \le C\|\xi\|_\infty$$

for every $\xi \in L^\infty(\Omega, d\mu)$.

Proof. This time, we have

$$P\mathcal{D}_k^* M_\xi \mathcal{D}_j = P\mathcal{D}_k^* E^2 M_\xi \mathcal{D}_j = P(M_{\rho^{-1}}\mathcal{D}_k)^* \cdot (EM_\rho)^* EM_\rho \cdot M_\xi \cdot M_{\rho^{-1}}\mathcal{D}_j,$$

and the desired conclusion again follows from Lemmas 5.8(a) and 5.13(a). □

Lemma 5.17. *We have* $\hat{T}_2 \in \mathcal{C}_p$ *for every* $p > n$.

Proof. Let $f, g \in \mathcal{L}_0$. By the definitions of $\hat{T}_2$ and $\langle \cdot, \cdot \rangle_\#$, we have

$$\begin{aligned}
\langle \hat{T}_2 f, g\rangle_\# &= \sum_{i=0}^{n-1} \int_\Omega \int_\Omega K_w^{(i+n)}(z)(R^i f)(w)\eta(w)d\mu(w)\overline{(R^n g)(z)}d\mu(z) \\
&+ \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \int_\Omega \int_\Omega K_w^{(i+j)}(z)(R^i f)(w)\eta(w)d\mu(w)\overline{(R^j g)(z)}\eta(z)d\mu(z) \\
&= \sum_{i=0}^{n-1} \langle Y_{i+n} M_\eta \mathcal{D}_i f, \mathcal{D}_n g\rangle + \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \langle Y_{i+j} M_\eta \mathcal{D}_i f, M_\eta \mathcal{D}_j g\rangle.
\end{aligned}$$

That is,

$$\hat{T}_2 = \sum_{i=0}^{n-1} \mathcal{D}_n^* Y_{i+n} M_\eta \mathcal{D}_i + \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \mathcal{D}_j^* M_\eta Y_{i+j} M_\eta \mathcal{D}_i. \tag{5.10}$$

By Lemma 5.4, we have $\hat{T}_2 \in \mathcal{C}_p$ for every $p > n$. □

6. Commutators and double commutators

We will now use the results in Section 5 to deal with commutators and double commutators on $\mathcal{L}$.

Definition 6.1. Let $C_*(\Omega)$ be the collection of the functions $f \in C^\infty(\Omega)$ satisfying the condition $\|f\|_* < \infty$, where

$$\|f\|_* = \sum_{0\le|\alpha|+|\beta|\le n+2} \sup_{z\in\Omega} |(\partial^\alpha \bar{\partial}^\beta f)(z)|.$$

Lemma 6.2. *For every $p > n$, there is a $0 < C = C(p) < \infty$ such that*

$$\|[\hat{M}_f, \hat{T}_1] - \mathcal{D}_n^*[M_f, Y_{2n}]\mathcal{D}_n\|_p \le C\|f\|_*$$

for every $f \in C_*(\Omega)$.

Proof. For $f \in C_*(\Omega)$ and $h \in \mathcal{L}_0$, the Leibniz rule gives us $R^k(fh) = \sum_{j=0}^k C_j^k R^j f \cdot R^{k-j} h$, where the C_j^k are the binomial coefficients. Recalling Definition 4.3, we have

$$\begin{aligned}([\hat{M}_f, \hat{T}_1]h)(z) &= \int_\Omega (f(z) - f(w)) K_w^{(n)}(z)(R^n h)(w) d\mu(w) \\ &\quad - \int_\Omega K_w^{(n)}(z) \sum_{j=1}^n C_j^n (R^j f)(w)(R^{n-j} h)(w) d\mu(w).\end{aligned} \tag{6.1}$$

Similar to the derivation of (5.10), a computation of $\langle [\hat{M}_f, \hat{T}_1]h, g\rangle_\#$, $g \in \mathcal{L}_0$, gives us

$$\begin{aligned}[\hat{M}_f,\hat{T}_1] &= \mathcal{D}_n^*[M_f, Y_{2n}]\mathcal{D}_n + \sum_{i=1}^n C_i^n \mathcal{D}_n^* M_{R^i f} Y_{2n-i} \mathcal{D}_n \\ &\quad + \sum_{s=0}^{n-1} \mathcal{D}_s^* M_\eta \Big([M_f, Y_{n+s}]\mathcal{D}_n + \sum_{r=1}^s C_r^s M_{R^r f} Y_{n+s-r} \mathcal{D}_n \Big) \\ &\quad - \sum_{j=1}^n C_j^n \mathcal{D}_n^* Y_{2n} M_{R^j f} \mathcal{D}_{n-j} - \sum_{j=1}^n C_j^n \sum_{s=0}^{n-1} \mathcal{D}_s^* M_\eta Y_{n+s} M_{R^j f} \mathcal{D}_{n-j}.\end{aligned} \tag{6.2}$$

Let $p > n$ be given. By Lemma 5.4, the Schatten p-norm of every term in (6.2) with a factor Y_k, $k \leq 2n-1$, is dominated by $C\|f\|_*$. By Lemma 5.9, the Schatten p-norm of every term in

$$\sum_{j=1}^{n} C_j^n \mathcal{D}_n^* Y_{2n} M_{R^j f} \mathcal{D}_{n-j}.$$

is also dominated by $C\|f\|_*$. This completes the proof. □

Proposition 6.3. *For every $p > 2n$, there is a $0 < C = C(p) < \infty$ such that*

$$\|[\hat{M}_f, P]\|_p \leq C\|f\|_*$$

for every $f \in C_*(\Omega)$.

Proof. By Proposition 4.14 and Lemma 5.17, we have $P - \hat{T}_1 \in \mathcal{C}_p$ for every $p > n$. Thus it suffices to consider commutators of the form $[\hat{M}_f, \hat{T}_1]$. By Lemmas 5.5(a) and 5.4, for every $p > 2n$ there is a $0 < C = C(p) < \infty$ such that

$$\|\mathcal{D}_n^*[M_f, Y_{2n}]\mathcal{D}_n\|_p \leq CL(f) \leq C_1\|f\|_* \quad \text{for} \quad f \in C_*(\Omega).$$

Combining this inequality with Lemma 6.2, the proposition is proved. □

Proposition 6.4. *For every $p > n$, there is a $0 < C = C(p) < \infty$ such that*

$$\|[\hat{M}_g, [\hat{M}_f, P]]\|_p \leq C\|f\|_*\|g\|_*$$

for all $f, g \in C_*(\Omega)$.

Proof. Again, since $P - \hat{T}_1 \in \mathcal{C}_p$ for every $p > n$, it suffices to consider double commutators of the form $[\hat{M}_g, [\hat{M}_f, \hat{T}_1]]$. Let $f, g \in C_*(\Omega)$ and $h \in \mathcal{L}_0$. Continuing with (6.1), we have

$$([\hat{M}_g, [\hat{M}_f, \hat{T}_1]]h)(z) = (Ah)(z) - (Bh)(z), \tag{6.3}$$

where

$$(Ah)(z) = \int_\Omega (g(z) - g(w))(f(z) - f(w))K_w^{(n)}(z)(R^n h)(w)d\mu(w)$$

and

$$\begin{aligned}(Bh)(z) = &\sum_{i=1}^{n} C_i^n \int_\Omega (f(z) - f(w))K_w^{(n)}(z)(R^i g)(w)(R^{n-i}h)(w)d\mu(w) \\ &+ \sum_{j=1}^{n} C_j^n \int_\Omega (g(z) - g(w))K_w^{(n)}(z)(R^j f)(w)(R^{n-j}h)(w)d\mu(w) \\ &- \sum_{j=1}^{n}\sum_{\nu=1}^{n-j} C_j^n C_\nu^{n-j} \int_\Omega K_w^{(n)}(z)(R^j f)(w)(R^\nu g)(w)(R^{n-j-\nu}h)(w)d\mu(w).\end{aligned}$$

A computation of $\langle Bh, h'\rangle_{\#}$, $h' \in \mathcal{L}_0$, tells us that B is a linear combination of operators of the form $\cdots Y_m M_\xi \mathcal{D}_k$, where $0 \le m \le 2n$ and $0 \le k \le n-1$. Thus for $p > n$, it follows from Lemmas 5.4 and 5.9 that

$$\|B\|_p \le C_1 \|f\|_* \|g\|_*. \tag{6.4}$$

On the other hand, a computation of $\langle Ah, h'\rangle_{\#}$, $h' \in \mathcal{L}_0$, tells us that

$$A = \mathcal{D}_n^*[M_g, [M_f, Y_{2n}]]\mathcal{D}_n + A',$$

where A' is a linear combination of terms of the form $\cdots Y_j \cdots$ with $n \le j \le 2n-1$. Hence for $p > n$, it follows from Lemmas 5.4 and 5.5 that

$$\|A\|_p \le C_2 \|f\|_* \|g\|_*. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5), the proposition is proved. □

7. Adjoints

Due to the nature of the inner product $\langle \cdot, \cdot \rangle_{\#}$, in general $\hat{M}_f^*$ is not a multiplication operator on $\mathcal{L}$. This causes additional difficulties for the proof of Theorem 1.2. We take care of these additional difficulties in this section.

Lemma 7.1. *For any $f \in C_*(\Omega)$, we have*

$$\begin{aligned} \hat{M}_f^* - \hat{M}_{\bar f} = &\sum_{j=1}^{n} C_j^n (\mathcal{D}_{n-j}^* M_{R^j \bar f} \mathcal{D}_n - \mathcal{D}_n^* M_{R^j \bar f} \mathcal{D}_{n-j}) \\ &+ \sum_{i=1}^{n-1}\sum_{\nu=1}^{i} C_\nu^i (\mathcal{D}_{i-\nu}^* M_{R^\nu \bar f} M_\eta \mathcal{D}_i - \mathcal{D}_i^* M_{R^\nu \bar f} M_\eta \mathcal{D}_{i-\nu}). \end{aligned} \tag{7.1}$$

Proof. Let $g, h \in \mathcal{L}_0$. By (5.6) and the Leibniz rule,

$$\begin{aligned} \langle \hat{M}_f^* g, h\rangle_{\#} = \langle g, \hat{M}_f h\rangle_{\#} &= \langle \mathcal{D}_n g, \mathcal{D}_n(fh)\rangle + \sum_{i=0}^{n-1} \langle M_\eta \mathcal{D}_i g, \mathcal{D}_i(fh)\rangle \\ &= \sum_{j=0}^{n} C_j^n \langle \mathcal{D}_n g, M_{R^j f} \mathcal{D}_{n-j} h\rangle + \sum_{i=0}^{n-1}\sum_{\nu=0}^{i} C_\nu^i \langle M_\eta \mathcal{D}_i g, M_{R^\nu f} \mathcal{D}_{i-\nu} h\rangle. \end{aligned} \tag{7.2}$$

On the other hand,

$$\begin{aligned} \langle \hat{M}_{\bar f} g, h\rangle_{\#} &= \langle \mathcal{D}_n(\bar f g), \mathcal{D}_n h\rangle + \sum_{i=0}^{n-1} \langle M_\eta \mathcal{D}_i(\bar f g), \mathcal{D}_i h\rangle \\ &= \sum_{j=0}^{n} C_j^n \langle M_{R^j \bar f} \mathcal{D}_{n-j} g, \mathcal{D}_n h\rangle + \sum_{i=0}^{n-1}\sum_{\nu=0}^{i} C_\nu^i \langle M_\eta M_{R^\nu \bar f} \mathcal{D}_{i-\nu} g, \mathcal{D}_i h\rangle. \end{aligned} \tag{7.3}$$

Subtracting (7.3) from (7.2), we obtain (7.1). □

Lemma 7.2. *For each $p > n$, there is a constant $0 < C = C(p) < \infty$ such that*

$$\|P(\hat{M}_f^* - \hat{M}_{\bar{f}})\|_p \le C\|f\|_* \quad \textit{and} \quad \|(\hat{M}_f^* - \hat{M}_{\bar{f}})P\|_p \le C\|f\|_*$$

for every $f \in C_*(\Omega)$.

Proof. This is an immediate consequence of Lemmas 7.1, 5.14(a) and 5.15. □

Definition 7.3. For each $f \in C_*(\Omega)$, let $\tilde{T}_f$ be the operator on $\mathcal{P}$ defined by the formula

$$\tilde{T}_f h = P(fh), \quad h \in \mathcal{P}.$$

In other words, $\tilde{T}_f$ is the analogue of a Toeplitz operator on $\mathcal{P}$. We also consider $\tilde{T}_f$ as an operator on $\mathcal{L}$, with the equivalent formula $\tilde{T}_f = P\hat{M}_f P$.

Proposition 7.4. *For each $p > 2n/3$, there is a constant $0 < C = C(p) < \infty$ such that*

$$\|[\tilde{T}_f^* - \tilde{T}_{\bar{f}}, \tilde{T}_g]\|_p \le C\|f\|_*\|g\|_*$$

for all $f, g \in C_*(\Omega)$.

Proof. Since $\tilde{T}_f^* = P\hat{M}_f^* P$, we have

$$[\tilde{T}_f^* - \tilde{T}_{\bar{f}}, \tilde{T}_g] = [P(\hat{M}_f^* - \hat{M}_{\bar{f}})P, P\hat{M}_g P] = F - G + H,$$

where

$$\begin{aligned} F &= P(\hat{M}_f^* - \hat{M}_{\bar{f}})(P-1)\hat{M}_g P, \\ G &= P\hat{M}_g(P-1)(\hat{M}_f^* - \hat{M}_{\bar{f}})P \quad \text{and} \\ H &= P[(\hat{M}_f^* - \hat{M}_{\bar{f}}), \hat{M}_g]P. \end{aligned}$$

Obviously, $F = P(\hat{M}_f^* - \hat{M}_{\bar{f}})[P, \hat{M}_g]P$. Thus it follows from Proposition 6.3 and Lemma 7.2 that for each $p > 2n/3$, there is a constant $0 < C_1 = C_1(p) < \infty$ such that

$$\|F\|_p \le C_1\|f\|_*\|g\|_*.$$

A similar bound holds for $\|G\|_p$. What remains is to bound $\|H\|_p$.

First, we show that for any $p > 2n/3$, there is a $0 < C_2 = C_2(p) < \infty$ such that

$$\|P[\mathcal{D}_k^* M_\xi \mathcal{D}_n, \hat{M}_g]P\|_p \le C_2\|\xi\|_\infty\|g\|_* \quad \text{and} \tag{7.4}$$

$$\|P[\mathcal{D}_n^* M_\xi \mathcal{D}_k, \hat{M}_g]P\|_p \le C_2\|\xi\|_\infty\|g\|_* \tag{7.5}$$

for $0 \le k \le n-1$ and $\xi \in L^\infty(\Omega, d\mu)$.

Let $0 \le k \le n-1$. By the Leibniz rule,

$$[\mathcal{D}_k^* M_\xi \mathcal{D}_n, \hat{M}_g] = (\mathcal{D}_k^* M_g - \hat{M}_g \mathcal{D}_k^*) M_\xi \mathcal{D}_n + \sum_{j=1}^{n} C_j^n \mathcal{D}_k^* M_\xi M_{R^j g} \mathcal{D}_{n-j}. \tag{7.6}$$

It follows from Lemma 5.14(a) that if $p > n/2$, then

$$\left\| P\left(\sum_{j=1}^{n} C_j^n \mathcal{D}_k^* M_\xi M_{R^j g} \mathcal{D}_{n-j} \right) P \right\|_p \le C_3 \|\xi\|_\infty \|g\|_*. \tag{7.7}$$

On the other hand,

$$\begin{aligned} \mathcal{D}_k^* M_g - \hat{M}_g \mathcal{D}_k^* &= (M_{\bar{g}} \mathcal{D}_k - \mathcal{D}_k \hat{M}_g^*)^* = (M_{\bar{g}} \mathcal{D}_k - \mathcal{D}_k \hat{M}_{\bar{g}})^* + \{\mathcal{D}_k (\hat{M}_{\bar{g}} - \hat{M}_g^*)\}^* \\ &= -\sum_{i=1}^{k} C_i^k \{M_{R^i \bar{g}} \mathcal{D}_{k-i}\}^* + (\hat{M}_{\bar{g}}^* - \hat{M}_g) \mathcal{D}_k^* = -\sum_{i=1}^{k} C_i^k \mathcal{D}_{k-i}^* M_{R^i g} + (\hat{M}_{\bar{g}}^* - \hat{M}_g) \mathcal{D}_k^*. \end{aligned}$$

Consequently,

$$\begin{aligned} P(\mathcal{D}_k^* M_g - \hat{M}_g \mathcal{D}_k^*) M_\xi \mathcal{D}_n P = &-\sum_{i=1}^{k} C_i^k P \mathcal{D}_{k-i}^* M_{R^i g} M_\xi \mathcal{D}_n P \\ &+ P(\hat{M}_{\bar{g}}^* - \hat{M}_g) \mathcal{D}_k^* M_\xi \mathcal{D}_n P. \end{aligned} \tag{7.8}$$

It follows from Lemma 5.14(b) that if $p > 2n/3$, then

$$\left\| \sum_{i=1}^{k} C_i^k P \mathcal{D}_{k-i}^* M_{R^i g} M_\xi \mathcal{D}_n P \right\|_p \le C_4 \|\xi\|_\infty \|g\|_*. \tag{7.9}$$

For each $p > n/2$, we also have

$$\|P(\hat{M}_{\bar{g}}^* - \hat{M}_g) \mathcal{D}_k^* M_\xi \mathcal{D}_n P\|_p \le \|P(\hat{M}_{\bar{g}}^* - \hat{M}_g)\|_{2p} \|\mathcal{D}_k^* M_\xi \mathcal{D}_n P\|_{2p} \le C_5 \|g\|_* \cdot C_6 \|\xi\|_\infty,$$

where the second $\le$ follows from Lemmas 7.2 and 5.15. Combining this with (7.8) and (7.9), we find that

$$\|P(\mathcal{D}_k^* M_g - \hat{M}_g \mathcal{D}_k^*) M_\xi \mathcal{D}_n P\|_p \le C_7 \|\xi\|_\infty \|g\|_* \tag{7.10}$$

if $p > 2n/3$. Inequality (7.4) now follows from (7.6), (7.7) and (7.10).

To prove (7.5), note that

$$\begin{aligned} (P[\mathcal{D}_n^* M_\xi \mathcal{D}_k, \hat{M}_g] P)^* &= P[\hat{M}_g^*, \mathcal{D}_k^* M_{\bar{\xi}} \mathcal{D}_n] P \\ &= P[\hat{M}_{\bar{g}}, \mathcal{D}_k^* M_{\bar{\xi}} \mathcal{D}_n] P + P[\hat{M}_g^* - \hat{M}_{\bar{g}}, \mathcal{D}_k^* M_{\bar{\xi}} \mathcal{D}_n] P. \end{aligned} \tag{7.11}$$

By Lemmas 7.2, 5.14(a) and 5.15, we have

$$\|P[\hat{M}_g^* - \hat{M}_{\bar{g}}, \mathcal{D}_k^* M_{\bar{\xi}} \mathcal{D}_n]P\|_p \le C_8 \|\xi\|_\infty \|g\|_*$$

if $p > n/2$. An application of (7.4) to the term $P[\hat{M}_{\bar{g}}, \mathcal{D}_k^* M_{\bar{\xi}} \mathcal{D}_n]P$ in (7.11) then completes the proof of (7.5).

If $0 \le k \le n-1$, $0 \le m \le n-1$ and $p > n/2$, then it follows from Lemma 5.16 that

$$\|P[\mathcal{D}_k^* M_\xi \mathcal{D}_m, \hat{M}_g]P\|_p \le C_9 \|\xi\|_\infty \|g\|_*. \tag{7.12}$$

Since $H = P[(\hat{M}_f^* - \hat{M}_{\bar{f}}), \hat{M}_g]P$, from Lemma 7.1, (7.4), (7.5) and (7.12) we obtain

$$\|H\|_p \le C_{10} \|f\|_* \|g\|_*.$$

when $p > 2n/3$. This completes the proof. $\square$

8. Local frame

Define $e_1(w) = w/|w|$ for $w \in \mathbf{C}^n \backslash \{0\}$. Then e_1 is a $\mathbf{C}^n$-valued C^∞ function on $\mathbf{C}^n \backslash \{0\}$. Moreover,

$$F_1(w) = 1 - e_1(w) \otimes e_1(w)$$

is a projection-valued C^∞ function on $\mathbf{C}^n \backslash \{0\}$.

Let an $a \in \overline{\Omega}$ be given. We pick a nonzero vector $v_2 \in F_1(a)\mathbf{C}^n$. Since $F_1(a)v_2 = v_2$, there is an open set N_2 containing a such that $F_1(w)v_2 \ne 0$ for every $w \in N_2$. We define

$$e_2(w) = F_1(w)v_2/|F_1(w)v_2|, \quad w \in N_2.$$

Then e_2 is a $\mathbf{C}^n$-valued C^∞ function on N_2. Define

$$F_2(w) = 1 - e_1(w) \otimes e_1(w) - e_2(w) \otimes e_2(w),$$

which is a projection-valued C^∞ function on N_2. If $n > 2$, then $F_2(a)\mathbf{C}^n \ne \{0\}$, and we can pick a nonzero vector $v_3 \in F_2(a)\mathbf{C}^n$ and repeat the above process. Thus we have

Proposition 8.1. *Given an $a \in \overline{\Omega}$, there exist a positive number $0 < \delta = \delta(a) < 1/2$ and vectors $\{e_2(w), \dots, e_n(w)\} \subset \mathbf{C}^n$, $w \in B(a, \delta)$, which have the following properties:*

(1) *For each $2 \le i \le n$, the map $w \mapsto e_i(w)$ is C^∞ on $B(a, \delta)$.*

(2) *For every $w \in B(a, \delta)$, $\{e_1(w), e_2(w), \dots, e_n(w)\}$ is an orthonormal basis for $\mathbf{C}^n$.*

Obviously, the above construction of local frame is just a smoothly parametrized version of the Gram-Schmidt process with $e_1(w) = w/|w|$. If f is an analytic function on $B(0, s)$ for some $s > 1$, then Proposition 8.1 provides the representation

$$\langle (\partial f)(w), \overline{z - w} \rangle = \sum_{i=1}^n \langle z - w, e_i(w) \rangle \langle (\partial f)(w), \overline{e_i(w)} \rangle \tag{8.1}$$

for $w \in B(a,\delta) \cap B(0,s)$ and $z \in B(0,s)$.

Definition 8.2. With the number $0 < \delta = \delta(a) < 1/2$ provided by Proposition 8.1, we let γ_a be a C^∞-function on $\mathbf{C}^n$ satisfying the conditions $0 \le \gamma_a \le 1$ on $\mathbf{C}^n$, $\gamma_a = 1$ on $B(a,\delta/2)$, and $\gamma_a = 0$ on $\mathbf{C}^n \backslash B(a, 2\delta/3)$.

Definition 8.3. (1) We extend the $e_2, \dots, e_n$ in Proposition 8.1 to vector-valued functions on the entire $\mathbf{C}^n$ by setting $e_i = 0$ on $\mathbf{C}^n \backslash B(a,\delta)$, $2 \le i \le n$.
(2) With the definition of $e_2, \dots, e_n$ extended as in (1), we define the vector-valued functions $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ on $\mathbf{C}^n$ by the formula $\epsilon_i = \gamma_a e_i$ for $1 \le i \le n$.

Definition 8.3 ensures that the vector-valued functions $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are C^∞ on $\mathbf{C}^n$.

Definition 8.4. For any analytic function f on $B(0,s)$, $s > 1$, we define the functions $D_1 f, \dots, D_n f$ by the formula

$$(D_i f)(w) = \langle (\partial f)(w), \overline{\epsilon_i(w)} \rangle$$

for $w \in B(0,s)$ and $1 \le i \le n$.

Recall that we write $\zeta_1, \dots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$.

Definition 8.5. (1) Let A be a bounded operator on $\mathcal{L}$. For each $1 \le i \le n$, we write

$$C_i(A) = \sum_{j=1}^n [\hat{M}_{\zeta_j}, A] \hat{M}_{\bar{\epsilon}_{i,j}},$$

where $\epsilon_{i,1}, \dots, \epsilon_{i,n}$ are the components of the vector-valued function ϵ_i.
(2) Let B be a bounded operator on $L^2(\Omega, d\mu)$. For each $1 \le i \le n$, we write

$$C_i(B) = \sum_{j=1}^n [M_{\zeta_j}, B] M_{\bar{\epsilon}_{i,j}},$$

where $\epsilon_{i,1}, \dots, \epsilon_{i,n}$ are the components of the vector-valued function ϵ_i.

Lemma 8.6. *Given any $s > s' > 1$, there is a constant $0 < C = C(s,s') < \infty$ such that if $f \in H_s^\infty$ and $z, w \in B(0,s')$, then*

$$|f(z) - f(w) - \langle (\partial f)(w), \overline{z-w} \rangle| \le C \|f\|_{s,\infty} |z-w|^2.$$

Proof. This is immediate from the first-order Taylor expansion

$$f(z) - f(w) = \langle (\partial f)(w), \overline{z-w} \rangle + \int_0^1 \langle (\partial f)(w + t(z-w)) - (\partial f)(w), \overline{z-w} \rangle dt.$$

□

Proposition 8.7. *Given $p > n$ and $s > 1$, there is a $0 < C = C(s,p) < \infty$ such that*

$$\left\| [M_f, Y_{2n}]M_{\gamma_a^2} - \sum_{i=2}^{n} C_i(Y_{2n})M_{D_i f} \right\|_p \le C\|f\|_{s,\infty}$$

for every $f \in H_s^\infty$.

Proof. Obviously, $[M_f, Y_{2n}]M_{\gamma_a^2}$ is the operator on $L^2(\Omega, d\mu)$ with the function

$$(f(z) - f(w))K_w^{(2n)}(z)\gamma_a^2(w) \tag{8.2}$$

as its integral kernel. Similarly, $\sum_{i=1}^{n} C_i(Y_{2n})M_{D_i f}$ is the operator on $L^2(\Omega, d\mu)$ with the function

$$\sum_{i=1}^{n} \langle z - w, \epsilon_i(w)\rangle K_w^{(2n)}(z)(D_i f)(w) \tag{8.3}$$

as its integral kernel. If we write the difference of (8.2) and (8.3) as $u(z,w)K_w^{(2n)}(z)$, then it follows from (8.1) that

$$u(z,w) = (f(z) - f(w) - \langle (\partial f)(w), \overline{z - w}\rangle)\gamma_a^2(w).$$

By Lemma 8.6, $|u(z,w)| \le C\|f\|_{s,\infty}|z - w|^2$. For $z, w \in \mathbf{B}$, $|z - w|^2 \le 2|1 - \langle z, w\rangle|$. Thus it follows from Lemma 5.2 and (5.3) that if $p > n$, then

$$\left\| [M_f, Y_{2n}]M_{\gamma_a^2} - \sum_{i=1}^{n} C_i(Y_{2n})M_{D_i f} \right\|_p \le C_1\|f\|_{s,\infty} \tag{8.4}$$

for every $f \in H_s^\infty$, $s > 1$. Since $e_1(w) = w/|w|$, we have

$$|\langle z - w, \epsilon_1(w)\rangle| \le 2|\langle z - w, w\rangle| \le 2(1 - |w|^2) + 2|1 - \langle z, w\rangle|$$

for $z, w \in \Omega$. Thus by Lemma 5.4, $C_1(Y_{2n}) \in \mathcal{C}_p$ for every $p > n$. Combining this fact with (8.4), the proposition is proved. □

Lemma 8.8. *Let $p > n$. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[\hat{M}_f, \hat{T}_1]\hat{M}_u - \mathcal{D}_n^*[M_f, Y_{2n}]M_u\mathcal{D}_n\|_p \le C\|f\|_*\|u\|_*$$

for all $f, u \in C_*(\Omega)$.

Proof. In view of Lemma 6.2, it suffices to show that for each $p > n$, there is a $0 < C = C(p) < \infty$ such that

$$\|\mathcal{D}_n^*[M_f, Y_{2n}]\mathcal{D}_n\hat{M}_u - \mathcal{D}_n^*[M_f, Y_{2n}]M_u\mathcal{D}_n\|_p \le C\|f\|_*\|u\|_* \tag{8.5}$$

for $f, u \in C_*(\Omega)$. Applying the Leibniz rule to $\mathcal{D}_n \hat{M}_u h$, $h \in \mathcal{L}_0$, we have

$$\mathcal{D}_n^*[M_f, Y_{2n}]\mathcal{D}_n \hat{M}_u - \mathcal{D}_n^*[M_f, Y_{2n}]M_u\mathcal{D}_n = \sum_{j=1}^{n} C_j^n \mathcal{D}_n^*[M_f, Y_{2n}]M_{R^j u}\mathcal{D}_{n-j}.$$

Thus an application of Lemma 5.9 proves (8.5). □

Lemma 8.9. *Given $p > n$ and $s > 1$, there is a $0 < C = C(s,p) < \infty$ such that*

$$\left\|[\hat{M}_f, P]\hat{M}_{\gamma_a^2} - \sum_{i=2}^{n} C_i(P)\hat{M}_{D_i f}\right\|_p \le C\|f\|_{s,\infty}$$

for every $f \in H_s^\infty$.

Proof. By Proposition 4.14 and Lemma 5.17, we have $P - \hat{T}_1 \in \mathcal{C}_p$ for every $p > n$. Thus, given any $p > n$ and $s > 1$, it suffices to find a $0 < C = C(s,p) < \infty$ such that

$$\left\|[\hat{M}_f, \hat{T}_1]\hat{M}_{\gamma_a^2} - \sum_{i=2}^{n} C_i(\hat{T}_1)\hat{M}_{D_i f}\right\|_p \le C\|f\|_* \tag{8.6}$$

for every $f \in H_s^\infty$. First of all, by Lemma 8.8 we have

$$\|[\hat{M}_f, \hat{T}_1]\hat{M}_{\gamma_a^2} - \mathcal{D}_n^*[M_f, Y_{2n}]M_{\gamma_a^2}\mathcal{D}_n\|_p \le C_1\|f\|_* \tag{8.7}$$

for $f \in C_*(\Omega)$. Since $C_i(\hat{T}_1) = \sum_{j=1}^{n}[\hat{M}_{\zeta_j}, \hat{T}_1]\hat{M}_{\bar{\epsilon}_{i,j}}$ and $C_i(Y_{2n}) = \sum_{j=1}^{n}[M_{\zeta_j}, Y_{2n}]M_{\bar{\epsilon}_{i,j}}$, it also follows from Lemma 8.8 that

$$\left\|\sum_{i=2}^{n} C_i(\hat{T}_1)\hat{M}_{D_i f} - \sum_{i=2}^{n} \mathcal{D}_n^* C_i(Y_{2n})M_{D_i f}\mathcal{D}_n\right\|_p \le C_2\|f\|_*, \tag{8.8}$$

$f \in H_s^\infty$, $s > 1$. Therefore (8.6) follows from (8.7), (8.8) and Proposition 8.7. □

Recall that the "Toeplitz operator" $\tilde{T}_f$ on $\mathcal{P}$ was defined in Definition 7.3. Moreover, we identify each $\tilde{T}_f$ with the operator $P\hat{M}_f P$ on $\mathcal{L}$.

Lemma 8.10. *Given $p > 2n/3$ and $s > 1$, there is a $0 < C = C(s,p) < \infty$ such that*

$$\left\|[\tilde{T}_f, \tilde{T}_g]\hat{M}_{\gamma_a^2} - \sum_{i=2}^{n} P[C_i(P), [\hat{M}_g, P]]P\hat{M}_{D_i f}\right\|_p \le C\|f\|_{s,\infty}\|g\|_*$$

for all $f \in H_s^\infty$ and $g \in C_*(\Omega)$.

Proof. Since $[\hat{M}_f, \hat{M}_g] = 0$, we have

$$[\tilde{T}_f, \tilde{T}_g] = P[\hat{M}_f, P][\hat{M}_g, P]P - P[\hat{M}_g, P][\hat{M}_f, P]P. \tag{8.9}$$

Given $p > 2n/3$ and $s > 1$, the notation $A \sim_p B$ in this proof means that $\|A - B\|_p \leq C\|f\|_{s,\infty}\|g\|_*$. For $f \in H_s^\infty$ and $g \in C_*(\Omega)$, it follows from Propositions 6.3 and 6.4 that

$$[\tilde{T}_f, \tilde{T}_g]\hat{M}_{\gamma_a^2} \sim_p P[\hat{M}_f, P]\hat{M}_{\gamma_a^2}[\hat{M}_g, P]P - P[\hat{M}_g, P][\hat{M}_f, P]\hat{M}_{\gamma_a^2}P.$$

Then, by Lemma 8.9 and Propositions 6.3 and 6.4, we have

$$\begin{aligned}
[\tilde{T}_f, \tilde{T}_g]\hat{M}_{\gamma_a^2} &\sim_p \sum_{i=2}^{n}\{PC_i(P)\hat{M}_{D_i f}[\hat{M}_g, P]P - P[\hat{M}_g, P]C_i(P)\hat{M}_{D_i f}P\} \\
&\sim_p \sum_{i=2}^{n} P[C_i(P), [\hat{M}_g, P]]P\hat{M}_{D_i f}
\end{aligned}$$

as promised. □

Lemma 8.11. *Let $p > n$ be given. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[\tilde{T}_f, \tilde{T}_g]\|_p \leq C\|f\|_*\|g\|_*$$

for all $f, g \in C_*(\Omega)$.

Proof. This follows immediately from (8.9) and Proposition 6.3. □

Lemma 8.12. *Let $p > 2n/3$ be given. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[[\tilde{T}_f, \tilde{T}_g], \hat{M}_h]\|_p \leq C\|f\|_*\|g\|_*\|h\|_* \tag{8.10}$$

for all $f, g, h \in C_*(\Omega)$.

Proof. Continuing with (8.9), we have

$$\begin{aligned}
&[[\tilde{T}_f, \tilde{T}_g], \hat{M}_h] \\
&\quad = [P, \hat{M}_h][\hat{M}_f, P][\hat{M}_g, P]P + P[[\hat{M}_f, P][\hat{M}_g, P], \hat{M}_h]P + P[\hat{M}_f, P][\hat{M}_g, P][P, \hat{M}_h] \\
&\quad - [P, \hat{M}_h][\hat{M}_g, P][\hat{M}_f, P]P - P[[\hat{M}_g, P][\hat{M}_f, P], \hat{M}_h]P - P[\hat{M}_g, P][\hat{M}_f, P][P, \hat{M}_h].
\end{aligned}$$

Applying Propositions 6.3 and 6.4 in this identity, we obtain (8.10). □

Lemma 8.13. *Let $p > 2n/3$ be given. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[[\tilde{T}_f, \tilde{T}_g], \tilde{T}_h]\|_p \leq C\|f\|_*\|g\|_*\|h\|_*$$

for all $f, g, h \in C_*(\Omega)$.

Proof. Since $[[\tilde{T}_f, \tilde{T}_g], \tilde{T}_h] = P[[\tilde{T}_f, \tilde{T}_g], \hat{M}_h]P$, this is an immediate consequence of Lemma 8.12. □

Lemma 8.14. *Let $p > 2n/3$ be given. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[[\tilde{T}_f, \tilde{T}_g], \tilde{T}_h^*]\|_p \leq C\|f\|_*\|g\|_*\|h\|_* \quad \textit{and} \quad \|[[\tilde{T}_f, \tilde{T}_g^*], \tilde{T}_h]\|_p \leq C\|f\|_*\|g\|_*\|h\|_*$$

for all $f, g, h \in C_*(\Omega)$.

Proof. This follows immediately from Proposition 7.4 and Lemma 8.13. □.

Lemma 8.15. *For each* $s > 1$, *there is a* $0 < C = C(s) < \infty$ *such that* $\|f\|_* \le C\|f\|_{s,\infty}$ *for every* $f \in H_s^\infty$.

Proof. This is an obvious consequence of (2.2). □

9. Antisymmetric sums on $\mathcal{P}$

After the preparations in Sections 6-8, we now consider antisymmetric sums on the range space.

Lemma 9.1. *Let* $a \in \overline{\Omega}$, δ, γ_a, *etc, be the same as in Proposition* 8.1 *and Definition* 8.2. *Given an* $s > 1$, *there is a* $0 < C = C(s, a) < \infty$ *such that*

$$\left\| \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)[\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_n}]\hat{M}_{\gamma_a^{2n}} \right\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_* \cdots \|f_n\|_{s,\infty}\|g_n\|_*$$

for all $f_1, \dots, f_n \in H_s^\infty$ *and* $g_1, \dots, g_n \in C_*(\Omega)$.

Proof. For this proof, the notation $A \sim_1 B$ means

$$\|A - B\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_* \cdots \|f_n\|_{s,\infty}\|g_n\|_*.$$

Let $f_1, \dots, f_n \in H_s^\infty$ and let $g_1, \dots, g_n \in C_*(\Omega)$. Then we have

$$\begin{aligned}
&[\tilde{T}_{f_1}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_n}, \tilde{T}_{g_n}]\hat{M}_{\gamma_a^{2n}} \sim_1 [\tilde{T}_{f_1}, \tilde{T}_{g_1}]\hat{M}_{\gamma_a^2} \cdots [\tilde{T}_{f_n}, \tilde{T}_{g_n}]\hat{M}_{\gamma_a^2} \\
&\qquad \sim_1 \sum_{i_1=2}^{n} \cdots \sum_{i_n=2}^{n} P[C_{i_1}(P), [\hat{M}_{g_1}, P]]P\hat{M}_{D_{i_1}f_1} \cdots P[C_{i_n}(P), [\hat{M}_{g_n}, P]]P\hat{M}_{D_{i_n}f_n} \\
&\qquad \sim_1 \sum_{i_1=2}^{n} \cdots \sum_{i_n=2}^{n} P[C_{i_1}(P), [\hat{M}_{g_1}, P]]P \cdots P[C_{i_n}(P), [\hat{M}_{g_n}, P]]P\hat{M}_{D_{i_1}f_1} \cdots \hat{M}_{D_{i_n}f_n},
\end{aligned} \tag{9.1}$$

where the first $\sim_1$ follows from Lemmas 8.11 and 8.12, the second $\sim_1$ follows from Lemmas 8.10, 8.11 and Proposition 6.3, and the third $\sim_1$ follows from Propositions 6.3 and 6.4. Writing

$$K_{i_1,\dots,i_n} = P[C_{i_1}(P), [\hat{M}_{g_1}, P]]P \cdots P[C_{i_n}(P), [\hat{M}_{g_n}, P]]P$$

for $i_1, \dots, i_n \in \{2, \dots, n\}$, from (9.1) we obtain

$$\begin{aligned}
&\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)[\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_n}]\hat{M}_{\gamma_a^{2n}} \\
&\qquad \sim_1 \sum_{i_1=2}^{n} \cdots \sum_{i_n=2}^{n} K_{i_1,\dots,i_n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\hat{M}_{D_{i_1}f_{\sigma(1)}} \cdots \hat{M}_{D_{i_n}f_{\sigma(n)}}.
\end{aligned} \tag{9.2}$$

For each choice of $i_1, \dots, i_n \in \{2, \dots, n\}$, there are $j \neq k$ in $\{1, \dots, n\}$ such that $i_j = i_k$, i.e., $D_{i_j} = D_{i_k}$. Therefore

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \hat{M}_{D_{i_1} f_{\sigma(1)}} \cdots \hat{M}_{D_{i_n} f_{\sigma(n)}} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \hat{M}_{D_{i_1} f_{\sigma(1)} \cdots D_{i_n} f_{\sigma(n)}} = 0.$$

Thus (9.2) actually says

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) [\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_n}] \hat{M}_{\gamma_a^{2n}} \sim_1 0,$$

which proves the lemma. □

Lemma 9.2. *Given an $s > 1$, there is a $0 < C = C(s) < \infty$ such that*

$$\left\| \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) [\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_n}] \right\|_1 \leq C \|f_1\|_{s,\infty} \|g_1\|_* \cdots \|f_n\|_{s,\infty} \|g_n\|_*$$

for all $f_1, \dots, f_n \in H_s^\infty$ and $g_1, \dots, g_n \in C_*(\Omega)$.

Proof. For each $a \in \overline{\Omega}$, we have the function γ_a and the open ball $B(a, \delta(a)/2)$ given in Definition 8.2. Since $a \in B(a, \delta(a)/2)$ and since $\overline{\Omega}$ is compact, there is a finite subset F of $\overline{\Omega}$ such that $\cup_{a \in F} B(a, \delta(a)/2) \supset \overline{\Omega}$.

We now apply Lemma 9.1 to each $a \in F$. Since $\operatorname{card}(F) < \infty$, this gives us

$$\left\| \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) [\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_1}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_n}] \sum_{a \in F} \hat{M}_{\gamma_a^{2n}} \right\|_1 \leq C \prod_{j=1}^{n} \|f_j\|_{s,\infty} \|g_j\|_* \tag{9.3}$$

for $f_1, \dots, f_n \in H_s^\infty$ and $g_1, \dots, g_n \in C_*(\Omega)$. For each $a \in F$, $\gamma_a = 1$ on $B(a, \delta(a)/2)$ by Definition 8.2. If we define the function

$$u = \left(\sum_{a \in F} \gamma_a^{2n} \right)^{-1}$$

on Ω, then it belongs to $C_*(\Omega)$. In particular, $\hat{M}_u$ is bounded on $\mathcal{L}$. Since $\sum_{a \in F} \hat{M}_{\gamma_a^{2n}} \hat{M}_u = 1$, the lemma follows from (9.3). □

Lemma 9.3. [15, Lemma 4.4] *Let $G_1, \dots, G_k$ be operators such that $[G_i, G_j] = 0$ for all $i, j \in \{1, \dots, k\}$. Then for any operators $H_1, \dots, H_k$,*

$$[G_1, H_1, \dots, G_k, H_k] = \sum_{\sigma \in S_k} \sum_{\lambda \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\lambda) [G_{\sigma(1)}, H_{\lambda(1)}] \cdots [G_{\sigma(k)}, H_{\lambda(k)}].$$

Also, for all $k \in \mathbf{N}$ and operators $B_1, B_2, \dots, B_{2k}$, the identity

$$\sum_{\sigma \in S_{2k}} \operatorname{sgn}(\sigma) [B_{\sigma(1)}, B_{\sigma(2)}] \cdots [B_{\sigma(2k-1)}, B_{\sigma(2k)}] = 2^k [B_1, B_2, \dots, B_{2k}] \tag{9.4}$$

holds. See [15, (4.9)].

Proposition 9.4. *Given an $s > 1$, there is a $0 < C = C(s) < \infty$ such that*

$$\|[\tilde{T}_{f_1}, \tilde{T}_{g_1}, \dots, \tilde{T}_{f_n}, \tilde{T}_{g_n}]\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_* \cdots \|f_n\|_{s,\infty}\|g_n\|_*$$

for all $f_1, \dots, f_n \in H_s^\infty$ and $g_1, \dots, g_n \in C_*(\Omega)$.

Proof. Let $f_1, \dots, f_n \in H_s^\infty$. Then for all $1 \le i, j \le n$ we have $[\tilde{T}_{f_i}, \tilde{T}_{f_j}] = [\tilde{M}_{f_i}, \tilde{M}_{f_j}] = 0$. Thus this proposition is an immediate consequence of Lemmas 9.2 and 9.3. □

Lemma 9.5. *Let $p > n$ be given. Then there is a $0 < C = C(p) < \infty$ such that*

$$\|[\tilde{T}_f, \tilde{T}_g^*]\|_p \le C\|f\|_*\|g\|_*$$

for all $f, g \in C_*(\Omega)$.

Proof. This follows immediately from Lemma 8.11 and Proposition 7.4. □

Proposition 9.6. *Given an $s > 1$, there is a $0 < C = C(s) < \infty$ such that*

$$\|[\tilde{T}_{f_1}, \tilde{T}_{g_1}^*, \dots, \tilde{T}_{f_n}, \tilde{T}_{g_n}^*]\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_* \cdots \|f_n\|_{s,\infty}\|g_n\|_* \tag{9.5}$$

for all $f_1, \dots, f_n \in H_s^\infty$ and $g_1, \dots, g_n \in C_*(\Omega)$.

Proof. Again, for this proof the notation $A \sim_1 B$ means

$$\|A - B\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_* \cdots \|f_n\|_{s,\infty}\|g_n\|_*.$$

Let $f_1, \dots, f_n \in H_s^\infty$ and $g_1, \dots, g_n \in C_*(\Omega)$. Since $[\tilde{T}_{f_i}, \tilde{T}_{f_j}] = 0$ for all $1 \le i, j \le n$, starting with Lemma 9.3, we have

$$\begin{aligned}
[\tilde{T}_{f_1}, \tilde{T}_{g_1}^*, \dots, \tilde{T}_{f_n}, \tilde{T}_{g_n}^*] &= \sum_{\sigma \in S_n}\sum_{\lambda \in S_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\lambda)[\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{g_{\lambda(1)}}^*] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{g_{\lambda(n)}}^*] \\
&\sim_1 \sum_{\sigma \in S_n}\sum_{\lambda \in S_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\lambda)[\tilde{T}_{f_{\sigma(1)}}, \tilde{T}_{\bar{g}_{\lambda(1)}}] \cdots [\tilde{T}_{f_{\sigma(n)}}, \tilde{T}_{\bar{g}_{\lambda(n)}}] \\
&= [\tilde{T}_{f_1}, \tilde{T}_{\bar{g}_1}, \dots, \tilde{T}_{f_n}, \tilde{T}_{\bar{g}_n}],
\end{aligned}$$

where the $\sim_1$ follows from Proposition 7.4, Lemma 8.11 and Lemma 9.5. Applying Proposition 9.4 to this last antisymmetric sum, we obtain (9.5). □

Proposition 9.7. *For each $s > 1$, there is a $0 < C = C(s) < \infty$ such that*

$$\|[\tilde{M}_{f_1}^*\tilde{M}_{g_1}, \tilde{M}_{f_2}^*\tilde{M}_{g_2}, \dots, \tilde{M}_{f_{2n}}^*\tilde{M}_{g_{2n}}]\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_{s,\infty} \cdots \|f_{2n}\|_{s,\infty}\|g_{2n}\|_{s,\infty} \tag{9.6}$$

for all $f_1, g_1, \dots, f_{2n}, g_{2n} \in H_s^\infty$.

Proof. For this proof, the notation $A \sim_1 B$ means

$$\|A - B\|_1 \le C\|f_1\|_{s,\infty}\|g_1\|_{s,\infty} \cdots \|f_{2n}\|_{s,\infty}\|g_{2n}\|_{s,\infty}.$$

Let $f_1, g_1, \ldots, f_{2n}, g_{2n} \in H_s^\infty$ be given. For each $1 \le i \le 2n$, we denote

$$A_i = \tilde{M}_{f_i}^* \tilde{M}_{g_i}, \quad A_{i,1} = \tilde{M}_{f_i}^*, \quad A_{i,2} = \tilde{M}_{g_i} \quad B_{i,1} = \tilde{M}_{g_i}, \quad \text{and} \quad B_{i,2} = \tilde{M}_{f_i}^*.$$

Thus for each $1 \le i \le 2n$,

$$A_{i,1}A_{i,2} = A_i, \quad A_{i,1}B_{i,1} = A_i, \quad \text{and} \quad B_{i,2}A_{i,2} = A_i.$$

Applying the "product rule" for commutators, it follows from Lemmas 9.5 and 8.14 that

$$\begin{aligned}
&[A_1, A_2] \cdots [A_{2n-1}, A_{2n}] \\
&\qquad = [A_{1,1}A_{1,2}, A_{2,1}A_{2,2}] \cdots [A_{2n-1,1}A_{2n-1,2}, A_{2n,1}A_{2d,n}] \\
&\qquad \sim_1 \sum_{j_1,\ldots,j_{2n}=1}^{2} [A_{1,j_1}, A_{2,j_2}] \cdots [A_{2n-1,j_{2n-1}}, A_{2n,j_{2n}}] B_{1,j_1} B_{2,j_2} \cdots B_{2n,j_{2n}}.
\end{aligned}$$

Let $\sigma \in S_{2n}$. Then the map $(j_1, \ldots, j_{2n}) \mapsto (j_{\sigma(1)}, \ldots, j_{\sigma(2n)})$ is injective on the product set $\{1,2\}^{2n}$, hence surjective also. Therefore

$$\begin{aligned}
&[A_{\sigma(1)}, A_{\sigma(2)}] \cdots [A_{\sigma(2n-1)}, A_{\sigma(2n)}] \\
&\qquad \sim_1 \sum_{j_1,\ldots,j_{2n}=1}^{2} [A_{\sigma(1),j_1}, A_{\sigma(2),j_2}] \cdots [A_{\sigma(2n-1),j_{2n-1}}, A_{\sigma(2n),j_{2n}}] \\
&\qquad \times B_{\sigma(1),j_1} B_{\sigma(2),j_2} \cdots B_{\sigma(2n),j_{2n}} \\
&\qquad = \sum_{j_1,\ldots,j_{2n}=1}^{2} [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2n-1),j_{\sigma(2n-1)}}, A_{\sigma(2n),j_{\sigma(2n)}}] \\
&\qquad \times B_{\sigma(1),j_{\sigma(1)}} B_{\sigma(2),j_{\sigma(2)}} \cdots B_{\sigma(2n),j_{\sigma(2n)}} \\
&\qquad \sim_1 \sum_{j_1,\ldots,j_{2n}=1}^{2} [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2n-1),j_{\sigma(2n-1)}}, A_{\sigma(2n),j_{\sigma(2n)}}] \\
&\qquad \times B_{1,j_1} B_{2,j_2} \cdots B_{2n,j_{2n}},
\end{aligned}$$

where the second $\sim_1$ follows from Lemma 9.5. By (9.4), we have

$$\begin{aligned}
&[A_1, A_2, \ldots, A_{2n}] = 2^{-n} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) [A_{\sigma(1)}, A_{\sigma(2)}] \cdots [A_{\sigma(2n-1)}, A_{\sigma(2n)}] \\
&\sim_1 2^{-n} \sum_{j_1,\ldots,j_{2n}=1}^{2} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) [A_{\sigma(1),j_{\sigma(1)}}, A_{\sigma(2),j_{\sigma(2)}}] \cdots [A_{\sigma(2n-1),j_{\sigma(2n-1)}}, A_{\sigma(2n),j_{\sigma(2n)}}] \\
&\times B_{1,j_1} B_{2,j_2} \cdots B_{2n,j_{2n}} \\
&= \sum_{j_1,\ldots,j_{2n}=1}^{2} [A_{1,j_1}, A_{2,j_2}, \ldots, A_{2n,j_{2n}}] B_{1,j_1} B_{2,j_2} \cdots B_{2n,j_{2n}}.
\end{aligned}$$

Applying Proposition 9.6 to each nonzero $[A_{1,j_1}, A_{2,j_2}, \dots, A_{2n,j_{2n}}]$, we obtain (9.6). □

Proof of Theorem 1.2. This follows immediately from Corollary 4.13, Proposition 2.5, and Proposition 9.7. □

10. Proof of Theorem 1.3

Our proof of Theorem 1.3 requires two steps. First, we will show that trace formula (1.3) holds for analytic polynomials $f_1, g_1, \dots, f_{2n}, g_{2n} \in \mathbf{C}[\zeta_1, , \dots, \zeta_{2n}]$. This step takes up most of this section. Once we have the polynomial version of (1.3), the trace-norm bound in Theorem 1.2 allows us to deduce the general case of (1.3) for $f_1, g_1, \dots, f_{2n}, g_{2n} \in H^\infty_s$, $s > 1$, by approximation.

The proof of (1.3) for $f_1, g_1, \dots, f_{2n}, g_{2n} \in \mathbf{C}[\zeta_1, \dots, \zeta_{2n}]$ is based on a fifty-year old idea due to Coburn [4], which predated the discovery of the Drury-Arveson space in [1,9]. As we will see, this idea allows us to transfer the problem from the Drury-Arveson space H^2_n back to the Bergman space $L^2_a(\mathbf{B})$, so that Theorem 1.1 can be applied.

Recall that we write T_f for the Toeplitz operator with symbol f on $L^2_a(\mathbf{B})$.

For $\alpha \in \mathbf{Z}^n_+$ and $1 \le i \le n$, we write α_i for the i-th component of α. That is, $\alpha = (\alpha_1, \dots, \alpha_n)$. For each $1 \le j \le n$, let ϵ_j denote the element in $\mathbf{Z}^n_+$ whose j-th component is 1 and whose other components are 0.

Let $\{e_\alpha : \alpha \in \mathbf{Z}^n_+\}$ be the standard orthonormal basis for the Drury-Arveson space H^2_n. As we already mentioned in Section 3, it is well known that

$$e_\alpha(\zeta) = (|\alpha|!/\alpha!)^{1/2}\zeta^\alpha, \quad \alpha \in \mathbf{Z}^n_+.$$

Therefore for any $1 \le j \le n$ and $\alpha \in \mathbf{Z}^n_+$, we have

$$M_{\zeta_j} e_\alpha = \{\|\zeta^{\alpha+\epsilon_j}\|_{H^2_n}/\|\zeta^\alpha\|_{H^2_n}\} e_{\alpha+\epsilon_j} = \left(\frac{\alpha_j + 1}{|\alpha| + 1}\right)^{1/2} e_{\alpha+\epsilon_j}. \tag{10.1}$$

Let $\{u_\alpha : \alpha \in \mathbf{Z}^n_+\}$ be the standard orthonormal basis for $L^2_a(\mathbf{B})$. It is well known that

$$u_\alpha(\zeta) = ((|\alpha| + n)!/\alpha!n!)^{1/2}\zeta^\alpha, \quad \alpha \in \mathbf{Z}^n_+.$$

For any $1 \le j \le n$ and $\alpha \in \mathbf{Z}^n_+$, we have

$$T_{\zeta_j} u_\alpha = \{\|\zeta^{\alpha+\epsilon_j}\|_{L^2_a(\mathbf{B})}/\|\zeta^\alpha\|_{L^2_a(\mathbf{B})}\} u_{\alpha+\epsilon_j} = \left(\frac{\alpha_j + 1}{|\alpha| + 1 + n}\right)^{1/2} u_{\alpha+\epsilon_j}. \tag{10.2}$$

Let $U : L^2_a(\mathbf{B}) \to H^2_n$ be the unitary operator such that

$$U u_\alpha = e_\alpha \quad \text{for every} \quad \alpha \in \mathbf{Z}^n_+. \tag{10.3}$$

Furthermore, we define the diagonal operator D on $L_a^2(\mathbf{B})$ by the formula

$$D = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{n}{\sqrt{|\alpha|+1}(\sqrt{|\alpha|+1} + \sqrt{|\alpha|+1+n})} u_\alpha \otimes u_\alpha. \tag{10.4}$$

By (10.1), (10.2) and elementary algebra, we find that

$$U^* M_{\zeta_j} U = T_{\zeta_j}(1 + D) \tag{10.5}$$

for every $1 \le j \le n$.

For convenience, we denote

$$d_\alpha = \frac{n}{\sqrt{|\alpha|+1}(\sqrt{|\alpha|+1} + \sqrt{|\alpha|+1+n})}, \quad \alpha \in \mathbf{Z}_+^n. \tag{10.6}$$

For each $k \in \mathbf{N}$, we define

$$E_k = \sum_{|\alpha|<k} d_\alpha u_\alpha \otimes u_\alpha \quad \text{and} \quad F_k = \sum_{|\alpha| \ge k} d_\alpha u_\alpha \otimes u_\alpha. \tag{10.7}$$

Then, of course, $E_k + F_k = D$, and $\operatorname{rank}(E_k) < \infty$.

Lemma 10.1. *For each $p > n$, we have*

$$\lim_{k \to \infty} \|F_k\|_p = 0.$$

Proof. Write m_n for the n-dimensional Lebesgue measure on $\mathbf{R}^n$. Then

$$\begin{aligned} \|F_k\|_p^p &= \sum_{|\alpha| \ge k} d_\alpha^p \le \sum_{|\alpha| \ge k} \frac{n^p}{(|\alpha|+1)^p} \\ &\le C \int_{(x_1^2 + \cdots + x_n^2)^{1/2} \ge k/\sqrt{n}} \frac{dm_n(x_1, \ldots, x_n)}{(x_1^2 + \cdots + x_n^2)^{p/2}} = C_1 \int_{k/\sqrt{n}}^\infty \frac{r^{n-1} dr}{r^p}. \end{aligned}$$

For $p > n$, the right-hand side tends to 0 as $k \to \infty$. □

Lemma 10.2. *For $t > n-1$ and $1 \le j \le n$, we have*

$$\lim_{k \to \infty} \|[T_{\zeta_j}, F_k]\|_t = 0.$$

Proof. By (10.7), for any $1 \le j \le n$ we have

$$[T_{\zeta_j}, F_k] = \sum_{|\alpha| \ge k} d_\alpha [T_{\zeta_j}, u_\alpha \otimes u_\alpha]. \tag{10.8}$$

For $\alpha \in \mathbf{Z}_+^n$, we have $T^*_{\zeta_j} u_\alpha = 0$ if $\alpha_j = 0$ and $T^*_{\zeta_j} u_\alpha = \alpha_j^{1/2}(|\alpha| + n)^{-1/2} u_{\alpha - \epsilon_j}$ if $\alpha_j \geq 1$. Combining this fact with (10.2) and with (10.8), we find that $[T_{\zeta_j}, F_k] = A_k - B_k$, where

$$A_k = \sum_{|\alpha| \geq k} (d_\alpha - d_{\alpha+\epsilon_j}) \left(\frac{\alpha_j + 1}{|\alpha| + 1 + n} \right)^{1/2} u_{\alpha+\epsilon_j} \otimes u_\alpha \quad \text{and}$$

$$B_k = \sum_{|\alpha| = k-1} d_{\alpha+\epsilon_j} \left(\frac{\alpha_j + 1}{|\alpha| + 1 + n} \right)^{1/2} u_{\alpha+\epsilon_j} \otimes u_\alpha .$$

Given $t > n - 1$, it suffices to show that $\|B_k\|_t \to 0$ and $\|A_k\|_t \to 0$ as $k \to \infty$.

Note that

$$\operatorname{rank}(B_k) = \operatorname{card}\{\alpha \in \mathbf{Z}_+^n : |\alpha| = k - 1\} = \frac{(k - 1 + n - 1)!}{(k-1)!(n-1)!} = O(k^{n-1}).$$

By (10.6), we have $d_{\alpha+\epsilon_j} = O(k^{-1})$ when $|\alpha| = k - 1$. Since $t > n - 1$, we have $\|B_k\|_t \to 0$ as $k \to \infty$. Also by (10.6), we have $d_\alpha - d_{\alpha+\epsilon_j} \leq C(|\alpha| + 1)^{-2}$. Therefore

$$\|A_k\|_t^t \leq C_1 \sum_{|\alpha| \geq k} \frac{1}{(|\alpha| + 1)^{2t}} \leq C_2 \int_{k/\sqrt{n}}^\infty \frac{r^{n-1} dr}{r^{2t}} .$$

Since $t > n - 1$ and $n \geq 2$, we have $2t > 2n - 2 \geq n$. Thus $\|A_k\|_t \to 0$ as $k \to \infty$. $\square$

Definition 10.3. Let $\mathcal{A}$ denote the unital algebra generated by the operators

$$D, \quad T_{\zeta_1}, \ldots, T_{\zeta_n}, \quad T^*_{\zeta_1}, \ldots, T^*_{\zeta_n} .$$

Lemma 10.4. *Let $t > n - 1$. Then for every $A \in \mathcal{A}$ we have*

$$\lim_{k \to \infty} \|[A, F_k]\|_t = 0. \tag{10.9}$$

Proof. Given $t > n - 1$, (10.9) is a consequence of the following three statements:
(1) $\lim_{k\to\infty} \|[T_{\zeta_j}, F_k]\|_t = 0$ for every $1 \leq j \leq n$.
(2) $\lim_{k\to\infty} \|[T^*_{\zeta_j}, F_k]\|_t = 0$ for every $1 \leq j \leq n$.
(3) $\lim_{k\to\infty} \|[D, F_k]\|_t = 0$.
Obviously, (1) is provided by Lemma 10.2. Since F_k is self-adjoint, (2) follows from (1). Since we assume $n \geq 2$, we have $n/2 \leq n - 1$. Note that Lemma 10.1 implies $D \in \mathcal{C}_p$ for every $p > n$. Therefore it follows from Lemmas 10.1 and 2.1 that $\lim_{k\to\infty} \|[D, F_k]\|_{p/2} = 0$ for every $p > n$. This obviously implies (3). $\square$

Lemma 10.5. *For any $A, B \in \mathcal{A}$, we have $[A, B] \in \mathcal{C}_p$ for every $n > p$.*

Proof, This is an obvious consequence of Lemma 2.6 and the fact that $D \in \mathcal{C}_p$ for every $p > n$. $\square$

Lemma 10.6. *Let $t > n-1$ be given. Then for all $A, B, C \in \mathcal{A}$ we have*

$$\lim_{k\to\infty} \|[A, BF_kC]\|_t = 0.$$

Proof. Applying Lemma 10.4, what remains to be shown is that

$$\lim_{k\to\infty} \|[A, B]F_k\|_t = 0 \quad \text{and} \quad \lim_{k\to\infty} \|F_k[A, C]\|_t = 0.$$

Set $p = 2t$. Then $p > 2n-2 \geq n$. By Lemma 2.1, we have

$$\|[A, B]F_k\|_t \leq \|[A, B]\|_p \|F_k\|_p.$$

Thus it follows from Lemmas 10.1 and 10.5 that $\|[A, B]F_k\|_t \to 0$ as $k \to \infty$. A similar argument shows that $\|F_k[A, C]\|_t \to 0$ as $k \to \infty$. This completes the proof. $\square$

Next we consider antisymmetric sums. It is obvious that if $E \in \mathcal{C}_1$, then

$$\text{tr}[E, Z_2, Z_3, \ldots, Z_{2n}] = 0 \tag{10.10}$$

for all operators $Z_2, Z_3, \ldots, Z_{2n}$.

Lemma 10.7. *Let $A, B, A_2, A_3, \ldots, A_{2n} \in \mathcal{A}$. Then the antisymmetric sum*

$$[ADB, A_2, A_3, \ldots, A_{2n}]$$

is in the trace class with zero trace.

Proof. Since $D = E_k + F_k$ and since E_k is a finite-rank operator, in view of (10.10), it suffices to show that

$$\lim_{k\to\infty} \|[AF_kB, A_2, A_3, \ldots, A_{2n}]\|_1 = 0.$$

To prove this, we define $A_1^{(k)} = AF_kB$ and $A_j^{(k)} = A_j$ for $2 \leq j \leq 2n$. By (9.4), we have

$$\begin{aligned}[AF_kB, A_2, A_3, \ldots, A_{2n}] &= [A_1^{(k)}, A_2^{(k)}, \ldots, A_{2n}^{(k)}] \\ &= 2^{-n} \sum_{\sigma\in S_{2n}} \text{sgn}(\sigma)[A_{\sigma(1)}^{(k)}, A_{\sigma(2)}^{(k)}] \cdots [A_{\sigma(2n-1)}^{(k)}, A_{\sigma(2n)}^{(k)}].\end{aligned}$$

Thus it suffices to show that for every $\sigma \in S_{2n}$, we have

$$\lim_{k\to\infty} \|[A_{\sigma(1)}^{(k)}, A_{\sigma(2)}^{(k)}] \cdots [A_{\sigma(2n-1)}^{(k)}, A_{\sigma(2n)}^{(k)}]\|_1 = 0. \tag{10.11}$$

Let $\sigma \in S_{2n}$ be given. Then there is an $i = i(\sigma) \in \{1, \ldots, n\}$ such that either $\sigma(2i-1) = 1$ or $\sigma(2i) = 1$. Hence there is a $\nu = \nu(\sigma) \in \{\sigma(2i-1), \sigma(2i)\}$ such that

$$[A_{\sigma(2i-1)}^{(k)}, A_{\sigma(2i)}^{(k)}] = \pm[AF_kB, A_\nu]. \tag{10.12}$$

We obviously have

$$(10.13)\qquad [A^{(k)}_{\sigma(2j-1)}, A^{(k)}_{\sigma(2j)}] = [A_{\sigma(2j-1)}, A_{\sigma(2j)}] \quad \text{for } j \in \{1, \dots, n\}\backslash\{i\}.$$

Pick a t satisfying the condition $n - 1 < t < n$. Then there is a $p > n$ such that $(n-1)/p = 1 - (1/t)$. It follows from (10.12), (10.13) and Lemma 2.1 that

$$(10.14)\quad \|[A^{(k)}_{\sigma(1)}, A^{(k)}_{\sigma(2)}] \cdots [A^{(k)}_{\sigma(2n-1)}, A^{(k)}_{\sigma(2n)}]\|_1 \le \|[AF_kB, A_\nu]\|_t \prod_{j \ne i} \|[A_{\sigma(2j-1)}, A_{\sigma(2j)}]\|_p.$$

By Lemma 10.5, we have $\|[A_{\sigma(2j-1)}, A_{\sigma(2j)}]\|_p < \infty$ for every $j \ne i$. Therefore (10.11) follows from (10.14) and Lemma 10.6. This completes the proof. □

Proposition 10.8. *Given any $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, there is a trace-class operator Y with $\mathrm{tr}(Y) = 0$ such that*

$$U^*[M^*_{f_1}M_{g_1}, M^*_{f_2}M_{g_2}, \dots, M^*_{f_{2n}}M_{g_{2n}}]U = [T_{\bar{f}_1 g_1}, T_{\bar{f}_2 g_2}, \dots, T_{\bar{f}_{2n} g_{2n}}] + Y.$$

Proof. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_+$, it follows from (10.5) that

$$\begin{aligned} U^*M_{\zeta^\alpha}U = U^*M^{\alpha_1}_{\zeta_1} \cdots M^{\alpha_n}_{\zeta_n}U &= \{T_{\zeta_1}(1 + D)\}^{\alpha_1} \cdots \{T_{\zeta_n}(1 + D)\}^{\alpha_n} \\ &= T_{\zeta^\alpha} + \sum_{\nu=1}^{2^{|\alpha|}-1} X_\nu D Y_\nu, \end{aligned}$$

where $X_1, Y_1, \dots, X_{2^{|\alpha|}-1}, Y_{2^{|\alpha|}-1} \in \mathcal{A}$. Therefore, given any analytic polynomials $f_1, g_1, \dots, f_{2n}, g_{2n} \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, there exist an $m \in \mathbf{N}$ and $A_{j,i}, B_{j,i} \in \mathcal{A}$, $1 \le j \le 2n$ and $1 \le i \le m$, such that

$$U^*M^*_{f_j}M_{g_j}U = T^*_{f_j}T_{g_j} + \sum_{i=1}^m A_{j,i}DB_{j,i} = T_{\bar{f}_j g_j} + \sum_{i=1}^m A_{j,i}DB_{j,i}$$

for every $1 \le j \le 2n$. Combining this identity with the linearity for each slot in an antisymmetric sum, we have

$$\begin{aligned} &U^*[M^*_{f_1}M_{g_1}, M^*_{f_2}M_{g_2}, \dots, M^*_{f_{2n}}M_{g_{2n}}]U \\ &\qquad = [T_{\bar{f}_1 g_1}, T_{\bar{f}_2 g_2}, \dots, T_{\bar{f}_{2n} g_{2n}}] + \sum_{r=1}^{(m+1)^{2n}-1} [X_{1,r}, X_{2,r}, \dots, X_{2n,r}], \end{aligned}$$

where the operators $X_{j,r}$ satisfy the following two conditions:

(1) $X_{j,r} \in \mathcal{A}$ for all $1 \le j \le 2n$ and $1 \le r \le (m + 1)^{2n} - 1$.

(2) For each $1 \le r \le (m + 1)^{2n} - 1$, there is a $j(r) \in \{1, \dots, 2n\}$ such that $X_{j(r),r} = ADB$ with $A, B \in \mathcal{A}$.

By these two conditions and Lemma 10.7, for each $1 \le r \le (m+1)^{2n}-1$, the antisymmetric sum $[X_{1,r}, X_{2,r}, \dots, X_{2n,r}]$ is in the trace class with zero trace. This completes the proof. $\square$

We can now prove (1.3) for analytic polynomials:

Proposition 10.9. *For $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$ we have*

$$\operatorname{tr}[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_1 g_1 \wedge d\bar{f}_2 g_2 \wedge \cdots \wedge d\bar{f}_{2n} g_{2n}.$$

Proof. This follows immediately from Proposition 10.8 and Theorem 1.1. $\square$

Lemma 10.10. *Let $0 < r < s < \infty$ and let $f \in H_s^\infty$. Then there is a sequence $\{f_k\}$ of analytic polynomials such that*

$$\lim_{k \to \infty} \|f - f_k\|_{r,\infty} = 0.$$

Proof. Even though this is obvious, a proof is included here for completeness.

Given an $f \in H_s^\infty$, define $g(\zeta) = f(s\zeta)$, $\zeta \in \mathbf{B}$. Then g is in the H^∞ of the unit ball. By the Cauchy formula for $\mathbf{B}$, we have the expansion

$$g(\zeta) = \sum_{j=0}^{\infty} u_j(\zeta), \quad \zeta \in \mathbf{B},$$

where, for each $j \in \mathbf{Z}_+$, u_j is a homogeneous polynomial of degree j. More precisely,

$$u_j(\zeta) = C_j^{j+n-1} \int_S \langle \zeta, \xi \rangle^j g(\xi) d\sigma(\xi)$$

for $j \in \mathbf{Z}_+$, where $C_j^{j+n-1} = \frac{(j+n-1)!}{j!(n-1)!}$. In particular, there is a $0 < C < \infty$ such that

$$|u_j(\zeta)| \le C C_j^{j+n-1} |\zeta|^j \quad \text{for all} \quad \zeta \in \mathbf{C}^n \quad \text{and} \quad j \in \mathbf{Z}_+.$$

For each $k \in \mathbf{N}$, we define

$$f_k(\zeta) = \sum_{j=0}^{k} \frac{1}{s^j} u_j(\zeta), \quad \zeta \in \mathbf{C}^n.$$

Note that $f(\zeta) = g(\zeta/s)$ for $\zeta \in B(0,s)$. Thus for $\zeta \in B(0,r)$ and $k \in \mathbf{N}$, we have

$$|f(\zeta) - f_k(\zeta)| = \left| \sum_{j=k+1}^{\infty} \frac{1}{s^j} u_j(\zeta) \right| \le C \sum_{j=k+1}^{\infty} C_j^{j+n-1} \left(\frac{r}{s} \right)^j,$$

which proves the lemma. □

Proof of Theorem 1.3. Let $f_1, g_1, \dots, f_{2n}, g_{2n} \in H^\infty_s$ for some $s > 1$. We pick an r such that $1 < r < s$. By Lemma 10.10, there are sequences of analytic polynomials $\{f_{j,k}\}$ and $\{g_{j,k}\}$, $1 \le j \le 2n$, such that

$$\lim_{k\to\infty} \|f_j - f_{j,k}\|_{r,\infty} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|g_j - g_{j,k}\|_{r,\infty} = 0, \tag{10.15}$$

$1 \le j \le 2n$. By Theorem 1.2 and the linearity of antisymmetric sums, (10.15) implies

$$\lim_{k\to\infty} \|[M^*_{f_1} M_{g_1}, \dots, M^*_{f_{2n}} M_{g_{2n}}] - [M^*_{f_{1,k}} M_{g_{1,k}}, \dots, M^*_{f_{2n,k}} M_{g_{2n,k}}]\|_1 = 0.$$

Therefore

$$\mathrm{tr}[M^*_{f_1} M_{g_1}, \dots, M^*_{f_{2n}} M_{g_{2n}}] = \lim_{k\to\infty} \mathrm{tr}[M^*_{f_{1,k}} M_{g_{1,k}}, \dots, M^*_{f_{2n,k}} M_{g_{2n,k}}]. \tag{10.16}$$

Since $r > 1$, (10.15) also implies

$$\lim_{k\to\infty} \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_{1,k} g_{1,k} \wedge \cdots \wedge d\bar{f}_{2n,k} g_{2n,k} = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_1 g_1 \wedge \cdots \wedge d\bar{f}_{2n} g_{2n}. \tag{10.17}$$

For each $k \in \mathbf{N}$, Proposition 10.9 gives us the identity

$$\mathrm{tr}[M^*_{f_{1,k}} M_{g_{1,k}}, \dots, M^*_{f_{2n,k}} M_{g_{2n,k}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_{1,k} g_{1,k} \wedge \cdots \wedge d\bar{f}_{2n,k} g_{2n,k}. \tag{10.18}$$

Combining (10.16), (10.17) and (10.18), we obtain (1.3). This completes the proof. □

Data availability

No data was used for the research described in the article.

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