ON THE CONTINUOUS SPECTRA OF TOEPLITZ OPERATORS WITH PLURIHARMONIC SYMBOLS

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Abstract. We show that certain self-adjoint Toeplitz operators on the Hardy space of the unit sphere have purely absolutely continuous spectrum. For a subclass of these operators, we show that the spectral multiplicity function is not locally L^1.

1. Introduction

In this paper, all Hilbert spaces we consider are assumed to be separable. Suppose that A is a bounded, self-adjoint operator on a Hilbert space H. Then it is well known that the space H admits an orthogonal decomposition

$$H = H_{\mathrm{ac}}(A) \oplus H_{\mathrm{s}}(A)$$

such that both subspaces $H_{\mathrm{ac}}(A)$ and $H_{\mathrm{s}}(A)$ are invariant under A, the spectral measure of the restricted operator $A|H_{\mathrm{ac}}(A)$ is absolutely continuous with respect to the Lebesgue measure, and the spectral measure of $A|H_{\mathrm{s}}(A)$ is singular to the Lebesgue measure. Recall that $A|H_{\mathrm{ac}}(A)$ and $A|H_{\mathrm{s}}(A)$ are respectively called the absolutely continuous part and the singular part of A. Accordingly, we say that A is *purely absolutely continuous* if $H_{\mathrm{s}}(A) = \{0\}$, and we say that A is *purely singular* if $H_{\mathrm{ac}}(A) = \{0\}$.

Among the most interesting examples of self-adjoint operators with purely absolutely continuous spectrum, we would like to mention Toeplitz operators on the unit circle [6,11] and various singular integral operators [1,7,8,13]. In particular, what motivates this paper is the classic result that if f is a bounded, real-valued, non-constant function on the unit circle $\mathbf{T}$, then the corresponding Toeplitz operator T_f on the one-variable Hardy space H^2 has purely absolutely continuous spectrum [6,11]. In fact, even the spectral multiplicity function of T_f was explicitly determined in [12].

Because of the importance of absolute continuity, with the success for one-variable Toeplitz operators, it is natural to ask, what happens if one considers Toeplitz operators in the context of several variables? Rosenblum considered this question in the case of the n-dimensional torus. He showed in [14] that if a non-constant symbol is the real part of a bounded analytic function, then the corresponding Toeplitz operator is purely absolutely continuous. Moreover, he gave an example showing that on the n-dimensional torus, $n \geq 2$, Toeplitz operator with a real-valued, non-constant symbol can have an eigenvalue.

In this paper we will examine Toeplitz operators on the unit sphere in $\mathbf{C}^n$. Note that both torus and sphere are natural generalizations of the classic case of the unit circle $\mathbf{T}$, albeit in different directions.

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Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. We write $d\sigma$ for the spherical measure on S with the normalization $\sigma(S) = 1$. The Hardy space $H^2(S)$ is defined to be the collection of analytic functions f on the unit ball $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$ satisfying the condition $\|f\| < \infty$, where the norm $\|\cdot\|$ is given by the formula

$$\|f\| = \sup_{0<r<1} \left(\int_S |f(rz)|^2 d\sigma(z) \right)^{1/2}.$$

Each $f \in H^2(S)$ has the so-called K-limit f^* σ-a.e. on S. Moreover, $f^* \in L^2(S, d\sigma)$ with $\|f^*\| = \|f\|$. See [15, Section 5.6] for these facts. Naturally, we identify each $f \in H^2(S)$ with its boundary value f^*. This identifies $H^2(S)$ with a closed linear subspace of $L^2(S, d\sigma)$. It is well known that, under this identification, $H^2(S)$ is just the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Thus by the calculation in [15, Section 1.4], the standard orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ for $H^2(S)$ is given by the formula

$$e_\alpha(z) = \left(\frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \right)^{1/2} z^\alpha,$$

$\alpha \in \mathbf{Z}_+^n$, where we follow the standard multi-index notation [15, page 3].

For $f \in L^\infty(S, d\sigma)$, we define the Toeplitz operator T_f by the formula

$$T_f h = P(fh), \quad h \in H^2(S),$$

where P is the orthogonal projection from $L^2(S, d\sigma)$ onto $H^2(S)$.

In the case $n \geq 2$, in sharp contrast to the classic result mentioned above, there are plenty of real-valued, non-constant $f \in L^\infty(S, d\sigma)$ for which the corresponding Toeplitz operator T_f has *purely singular* spectrum. For example, if f depends only on $|z_1|, \dots, |z_n|$, then by the rotation invariance of $d\sigma$ we have

$$\langle T_f e_\alpha, e_\beta \rangle = \int_S f e_\alpha \bar{e}_\beta d\sigma = 0 \quad \text{for all } \alpha \neq \beta \text{ in } \mathbf{Z}_+^n.$$

Thus T_f is a diagonal operator with respect to the orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ in $H^2(S)$ and, therefore, has a pure point spectrum.

Despite such examples, we will show that there is a modified analogue of the classic absolute continuity result in the case $n \geq 2$. That is, we will show that there is a subclass of real-valued, non-constant $f \in L^\infty(S, d\sigma)$ for which the spectrum of the corresponding Toeplitz operator T_f is purely absolutely continuous.

Recall that a C^∞-function u on $\mathbf{B}$ is said to be *pluriharmonic* if the equation

$$\partial_j \bar{\partial}_k u = 0$$

holds for every pair of $j, k \in \{1, \dots, n\}$. We cite [15, Section 4.4] as a general reference for pluriharmonic functions. The involvement of pluriharmonicity in this paper will be limited to its well-known characterization [15, Theorem 4.4.9]. That is, for a real-valued C^∞-function u on $\mathbf{B}$, the following conditions are equivalent:

(1) u is pluriharmonic on $\mathbf{B}$.

(2) u is the real part of an analytic function on $\mathbf{B}$.

(3) For every $\zeta \in S$, the function $z \mapsto u(z\zeta)$ is harmonic on $\{z \in \mathbf{C} : |z| < 1\}$.

(4) For every $\psi \in \mathrm{Aut}(\mathbf{B})$, the function $u \circ \psi$ is harmonic on $\mathbf{B}$.

Recall from [15] that for any $f \in L^1(S, d\sigma)$, its Poisson extension is the function $[f]$ on $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$ given by the formula

$$[f](z) = \int \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}} f(\xi) d\sigma(\xi), \quad z \in \mathbf{B}.$$

Definition 1.1. Let BPH $= \{f \in L^\infty(S, d\sigma) : [f]$ is pluriharmonic on $\mathbf{B}\}$.

If $f \in$ BPH and if f is real valued, then, as we explained above, there is an analytic function φ on $\mathbf{B}$ such that $[f] = \mathrm{Re}(\varphi)$. We will call $\mathrm{Im}(\varphi)$ a *pluriharmonic conjugate* of f. Of course, any two pluriharmonic conjugates of f differ at most by a constant. Note that although the membership $f \in$ BPH assumes the boundedness of f, it does not imply the boundedness of any pluriharmonic conjugate of f. Here is what we can prove in terms of absolute continuity:

Theorem 1.2. *Let f be a real-valued, non-constant function in* BPH. *If the pluriharmonic conjugate of f is either bounded from above or bounded from below, then the Toeplitz operator T_f on $H^2(S)$ is purely absolutely continuous.*

Note that if $n = 1$, then BPH is just the L^∞ on the unit circle. Therefore if one could remove the semi-boundedness assumption in Theorem 1.2 about the pluriharmonic conjugate of f, then it could be reasonably regarded as the right analogue for complex dimensions $n \geq 2$ of the classic absolute continuity result. But unfortunately, at the moment we are only able to prove Theorem 1.2 under the semi-boundedness assumption.

On the other hand, we can prove the absence of point spectrum without any assumption on the pluriharmonic conjugate:

Theorem 1.3. *If f is a real-valued, non-constant function in* BPH, *then the Toeplitz operator T_f on $H^2(S)$ has no eigenvalues.*

Remark. In the case where the pluriharmonic conjugate of f is bounded, i.e., in the case where $f = \mathrm{Re}(\varphi)$ for some $\varphi \in H^\infty(S)$, Theorem 1.2 can be deduced from [14, Theorem 2]. However, our intention in Theorem 1.2 is to cover as large a class of symbol functions as we can, which makes unbounded operators unavoidable. Consequently, a significant part of the proof of Theorem 1.2 is the handling of unbounded pluriharmonic conjugate of f.

Next, let us consider the spectral multiplicity functions for Toeplitz operators. To do that, we first review the definition of spectral multiplicity function.

For any Borel set Δ in $\mathbf{R}$, we write $M^{(\Delta)}$ for the operator of multiplication by the coordinate function x on the Hilbert space $L^2(\Delta, dm)$, where dm is the Lebesgue measure on $\mathbf{R}$. Let A be a self-adjoint operator on a Hilbert space H, and suppose that $H_s(A) = \{0\}$, i.e., A is purely absolutely continuous. Then there exists a countable family of Borel sets $\{\Delta_i : i \in I\}$ in $\mathbf{R}$ such that A is unitarily equivalent to the operator

$$\bigoplus_{i \in I} M^{(\Delta_i)}.$$

The spectral multiplicity function of A is defined by the formula

$$m_A(x) = \sum_{i \in I} \chi_{\Delta_i}(x), \quad x \in \mathbf{R}.$$

It is well known that this m_A is a complete unitary invariant for A.

Let $H^\infty(S)$ denote the collection of bounded analytic functions on the unit ball $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$. Let $\varphi \in H^\infty(S)$ and suppose that φ is not a constant. For $f = \mathrm{Re}(\varphi)$, it follows from Theorem 1.2 that the Toeplitz operator T_f on $H^2(S)$ is purely absolutely continuous. We have the following result about its spectral multiplicity function:

Theorem 1.4. *Consider any complex dimension $n \geq 2$. Let φ be any non-constant function in $H^\infty(S)$, and write $f = \mathrm{Re}(\varphi)$. Then the spectral multiplicity function m_{T_f} of the Toeplitz operator T_f has the property*

$$\int_{\mathbf{R}} m_{T_f}(x)dx = \infty.$$

This exhibits an aspect of the spectral theory for self-adjoint Toeplitz operators in the case $n \geq 2$ that is quite different from the spectral theory in the case $n = 1$. When $n = 1$, if f is any non-constant, real-valued function in $L^\infty(\mathbf{T})$, then the spectral multiplicity function is completely known for the Toeplitz operator T_f on H^2. See [12]. In particular, by the method of determining spectral multiplicity function given in [12], there are plenty of non-constant, real-valued $f \in C^1(\mathbf{T})$ such that the spectral multiplicity function m_{T_f} is bounded. Thus there are plenty of non-constant, real-valued C^1-functions f on the unit circle $\mathbf{T}$ such that for the corresponding Toeplitz operator T_f on H^2, we have

$$\int_{\mathbf{R}} m_{T_f}(x)dx < \infty.$$

If f is a real-valued function in $C^1(\mathbf{T})$, then, of course, there is a $\varphi \in H^\infty$ such that $f = \mathrm{Re}(\varphi)$. Thus we see that Theorem 1.4 is a sharp contrast to the case $n = 1$.

Given Theorem 1.4, one wonders if there is more that can be said about the spectral multiplicity function. Specifically, it seems reasonable to consider

Problem 1.5. Suppose that $n \geq 2$. Let φ be a non-constant function in $H^\infty(S)$, and let $f = \mathrm{Re}(\varphi)$. Is it true that the Lebesgue measure of the set

$$\{x \in \mathbf{R} : 0 < m_{T_f}(x) < \infty\}$$

is zero? In other words, is there a dichotomy $m_{T_f}(x) \in \{0, \infty\}$ for a.e. $x \in \mathbf{R}$? Stated in yet another equivalent way, is it true that

$$\int_G m_{T_f}(x)dx \in \{0, \infty\}$$

for every Borel set $G \subset \mathbf{R}$?

The main result of the paper is a local version of Theorem 1.4, which can certainly be viewed as a strong piece of evidence in support of the above-mentioned dichotomy:

Theorem 1.6. *Consider any complex dimension $n \geq 3$. Let f be a real-valued, non-constant function in* BPH $\cap\, C^1(S)$. *Then for every open interval (a, b) satisfying the condition*

$$\min_{z \in S} f(z) \leq a < b \leq \max_{z \in S} f(z),$$

we have

$$\int_a^b m_{T_f}(x)dx = \infty.$$

Obviously, Theorem 1.6 leaves out a single complex dimension, namely the case $n = 2$. Our proof of Theorem 1.6 does not work in the case $n = 2$. In fact, the case $n = 2$ seems to be much more delicate than the complex dimensions $n \geq 3$.

The reader will see that the proofs of the above theorems all involve an interplay between Toeplitz operators and Hankel operators.

The rest of the paper is organized as follows. First of all, Sections 2 and 3 contain preparations for the proofs of Theorems 1.2 and 1.3. Specifically, in Section 2 we revisit a classic theorem of Putnam, and Section 3 collects a number of Hardy-space facts that are needed in the proofs of Theorems 1.2 and 1.3.

After treatment of unbounded Toeplitz operators, the proofs of Theorems 1.2 and 1.3 are presented in Section 4.

We then prove Theorem 1.4 in Section 5. The proof of Theorem 1.6 is quite long, which will be presented in Section 6.

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2. Putnam's theorem revisited

In the classic case where $n = 1$, the absolute continuity of the spectrum of a self-adjoint joint Toeplitz operator was obtained by an explicit diagonalization of the operator [6,11,12]. At the very core of this classic proof is the fact that if f is a real-valued L^∞-function on the unit circle $\mathbf{T}$, then for every $\lambda < -\|f\|_\infty$ there is an outer function h_λ such that $f - \lambda = \bar{h}_\lambda h_\lambda$. Consequently, for the resolvent $(T_f - \lambda)^{-1}$ we have the factorization

$$(T_f - \lambda)^{-1} = T_{h_\lambda^{-1}} T_{\bar{h}_\lambda^{-1}}$$

in terms of Toeplitz operators. This, of course, is not something that one can hope to mimic for the case of complex dimensions $n \geq 2$.

Since explicit diagonalization is not possible in the case $n \geq 2$, the only other possible approach is to use a theorem of Putnam that relates spectral absolute continuity to the positivity of commutators. This is where the pluriharmonicity in Theorem 1.2 comes in.

Theorem 2.1. [10, Theorem 2.2.4] *Suppose that A, B are bounded, self-adjoint operators on a Hilbert space H and let $C = -i[A, B]$. If $C \geq 0$ or $C \leq 0$, then $H_{\mathrm{ac}}(A) \supset \mathcal{L}$, where $\mathcal{L}$ denotes the smallest subspace of H reducing both A, B and containing the range of C.*

Below we present a variant of this theorem, in which the description of $\mathcal{L}$ is significantly simplified for the purpose of application. This variant is based on two ideas, the first which is a general fact:

Proposition 2.2. *Suppose that A, B are bounded, self-adjoint operators on a Hilbert space H and denote $C = -i[A, B]$. Let $\mathcal{E}$ be the spectral measure for A. That is,*

$$A = \int_{-\|A\|}^{\|A\|} \lambda d\mathcal{E}(\lambda).$$

If G is a Borel subset of $I = [-\|A\|, \|A\|]$ such that $C\mathcal{E}(G) = 0$, then the subspace $\mathcal{E}(G)H$ is invariant under B.

Proof. If K is a compact subset of G and L is a compact subset of $I \backslash G$, then $\mathcal{E}(L)C\mathcal{E}(K) = \mathcal{E}(L)C\mathcal{E}(G)\mathcal{E}(K) = 0$. From this we deduce

$$[A, \mathcal{E}(L)B\mathcal{E}(K)] = 0.$$

This obviously implies that

$$[f(A), \mathcal{E}(L)B\mathcal{E}(K)] = 0 \quad \text{for every} \quad f \in C(I). \tag{2.1}$$

Since K and L are disjoint compact subsets of I, there is a $g \in C(I)$ such that $g = 1$ on L and $g = 0$ on K. Thus from (2.1) we see that $\mathcal{E}(L)B\mathcal{E}(K) = 0$ for all compact subsets $K \subset G$ and $L \subset I \backslash G$. By the regularity of the spectral measure $\mathcal{E}$, this implies $\mathcal{E}(I \backslash G)B\mathcal{E}(G) = 0$, i.e., $(1 - \mathcal{E}(G))B\mathcal{E}(G) = 0$, which means that the subspace $\mathcal{E}(G)H$ is invariant under B. □

The second idea is the use of the following fact which is by now well known: If T is a *purely singular* self-adjoint operator on a Hilbert space H, then there is a sequence of finite-rank orthogonal projections $\{F_k\}$ such that $F_k \to 1$ strongly and $\|[T, F_k]\|_1 \to 0$ as $k \to \infty$, where $\| \cdot \|_1$ denotes the norm of the trace class. This fact is usually used in the context of diagonalization modulo the trace class [2,16,17,19]. But it is an easy exercise to produce such a sequence $\{F_k\}$. Indeed it is quite obvious what the F_k's are if $T = M_x$ on some $L^2(d\mu)$, where $d\mu$ has a compact support K with $m(K) = 0$. Then note that a general purely singular T is unitarily equivalent to the orthogonal sum of countably many multiplication operators of this kind.

Theorem 2.3. *Suppose that A, B are bounded, self-adjoint operators on a Hilbert space H and let $C = -i[A, B]$. If either $C \geq 0$ or $C \leq 0$, then $CH_{\mathrm{s}}(A) = \{0\}$ and $H_{\mathrm{s}}(A)$ is an invariant subspace for B.*

Proof. Let E be the orthogonal projection from H onto $H_{\mathrm{s}}(A)$ and write $A' = AE$, $B' = EBE$, and $C' = -i[A', B']$. Then we have $C' = ECE$. Note that the self-adjoint operator A' is purely singular. Hence, as we explained above, there is a sequence of finite-rank orthogonal projection $\{F_k\}$ on H such that

$$\text{s-}\lim_{k\to\infty} F_k = 1 \quad \text{and} \quad \lim_{k\to\infty} \|[A', F_k]\|_1 = 0.$$

On the other hand,

$$\mathrm{tr}(C'F_k) = i\mathrm{tr}(B'A'F_k - A'B'F_k) = i\mathrm{tr}(B'F_kA' - A'B'F_k + B'[A', F_k]) = i\mathrm{tr}(B'[A', F_k]).$$

Thus $|\mathrm{tr}(C'F_k)| \leq \|B'\|\|[A', F_k]\|_1$, and consequently

$$\lim_{k\to\infty} \mathrm{tr}(C'F_k) = 0.$$

It suffices to consider the case where $C \geq 0$. Then $C' \geq 0$. Since the sequence $\{F_k\}$ strongly converges to 1 on H and since $C' \geq 0$, the above limit implies $C' = 0$. Hence we have $(C^{1/2}E)^*C^{1/2}E = EC^{1/2}C^{1/2}E = C' = 0$, which implies $C^{1/2}E = 0$. Thus we also have $CE = C^{1/2}C^{1/2}E = 0$, i.e., $CH_{\mathrm{s}}(A) = \{0\}$.

To prove that $H_{\mathrm{s}}(A)$ is invariant under B, write $\mathcal{E}$ for the spectral measure for A, i.e.,

$$A = \int_{-\|A\|}^{\|A\|} \lambda d\mathcal{E}(\lambda).$$

Then there is a Borel set $\Sigma \subset [-\|A\|, \|A\|]$ with $m(\Sigma) = 0$ such that $E = \mathcal{E}(\Sigma)$. The fact $CE = 0$ proved above now translates to $C\mathcal{E}(\Sigma) = 0$. Applying Proposition 2.2, we see that the subspace $\mathcal{E}(\Sigma)H = EH = H_{\mathrm{s}}(A)$ is invariant under B. □

3. Some facts about the Hardy space

Here we collect a few facts concerning the Hardy space that will be needed later.

Proposition 3.1. [15, Theorem 5.5.9] *Let $h \in H^2(S)$. If $\sigma(\{\xi \in S : h(\xi) = 0\}) > 0$, then $h = 0$.*

Recall that for $f \in L^\infty(S, d\sigma)$, the *Hankel operator* H_f is defined by the formula

$$H_f h = (1 - P)(fh), \quad h \in H^2(S).$$

As we will see, Hankel operators play an essential role in this paper.

Proposition 3.2. *Let f be a real-valued non-constant function in $L^\infty(S, d\sigma)$. Then the Hankel operator H_f faithfully detects the invariant subspaces of T_f in the following sense:*

If E is the orthogonal projection from $H^2(S)$ onto an invariant subspace of T_f, then the condition $H_f E = 0$ implies $E = 0$.

Proof. Let M be an invariant subspace of T_f, and suppose that for the orthogonal projection $E : H^2(S) \to M$ we have $H_f E = 0$. Consider any $h \in M$. Since $(1-P)(fh) = 0$, we have $fh = T_f h \in M$. Combining this with a simple induction, we see that if p is any polynomial, then $p(f)h \in M$.

Suppose that the essential range of f is contained in $[a, b]$. Since f is not a constant, there are $a \le \alpha < \beta \le b$ such that $\sigma(f^{-1}[a, \alpha]) > 0$ and $\sigma(f^{-1}[\beta, b]) > 0$. There is a continuous function ψ on $[a, b]$ such that $\psi = 0$ on $[a, \alpha]$ and $\psi = 1$ on $[\beta, b]$. Since $p(f)h \in M$ for every polynomial p, by the Weierstrass approximation theorem, we also have $\psi(f)h \in M$. Note that $\psi(f)h$ vanishes on the set $f^{-1}[a, \alpha]$. Since $\psi(f)h \in M \subset H^2(S)$ and since $\sigma(f^{-1}[a, \alpha]) > 0$, by Proposition 3.1 we have $\psi(f)h = 0$. Since $\psi(f)$ equals 1 on the set $f^{-1}[\beta, b]$, the condition $\psi(f)h = 0$ implies that h vanishes on $f^{-1}[\beta, b]$. Again, by Proposition 3.1 and the fact $\sigma(f^{-1}[\beta, b]) > 0$, we have $h = 0$. Since this is true for every $h \in M$, it follows that $M = \{0\}$, i.e., $E = 0$. □

For any function $f \in L^1(S, d\sigma)$, we define

$$\|f\|_{\mathrm{BMO}} = \sup\left\{\frac{1}{\sigma(B(\zeta, r))}\int_{B(\zeta,r)} |f - f_{B(\zeta,r)}| d\sigma : \zeta \in S \text{ and } r > 0\right\}, \tag{3.2}$$

where $B(\zeta, r) = \{x \in S : |1 - \langle x, \zeta\rangle|^{1/2} < r\}$. Recall that a function f is said to have *bounded mean oscillation* if $\|f\|_{\mathrm{BMO}} < \infty$. Write BMO for the collection of functions of bounded mean oscillation on the sphere S.

It is well known that $\|[M_f, P]\| \le C\|f\|_{\mathrm{BMO}}$ for $f \in \mathrm{BMO}$. See [3,22].

Proposition 3.3. *If f is a real-valued function in* BPH, *then its pluriharmonic conjugate belongs to* BMO.

Proof. Let φ be an analytic function on $\mathbf{B}$ such that $\mathrm{Re}(\varphi) = [f]$. For each $0 \le r < 1$, we define the functions

$$g_r(\xi) = [f](r\xi) \quad \text{and} \quad h_r(\xi) = \varphi(r\xi),$$

$\xi \in S$. Clearly, the relation $\mathrm{Re}(\varphi) = [f]$ on $\mathbf{B}$ implies $h_r + \bar{h}_r = 2g_r$ on S. From this it is easy to see that $2Pg_r = P(h_r + \bar{h}_r) = h_r + \bar{\varphi}(0)$. Since $\|g_r\|_\infty \le \|f\|_\infty$ for every $0 \le r < 1$, this implies $\varphi \in H^2(S)$, and consequently $\|h_r - \varphi\| \to 0$ as $r \uparrow 1$.

By [21, Proposition 2.2], there is a $0 < C < \infty$ such that $\|P\psi\|_{\mathrm{BMO}} \le C\|\psi\|_{\mathrm{BMO}}$ for every $\psi \in \mathrm{BMO}$ (also see [3]). Hence for $0 \le r < 1$, the relation $2Pg_r = h_r + \bar{\varphi}(0)$ implies

$$\|h_r\|_{\mathrm{BMO}} = 2\|Pg_r\|_{\mathrm{BMO}} \le 2C\|g_r\|_{\mathrm{BMO}} \le 4C\|g_r\|_\infty \le 4C\|f\|_\infty.$$

Combining this with the fact $\lim_{r\uparrow 1}\|h_r - \varphi\| = 0$ and with (3.2), the definition of the BMO-norm, we obtain $\|\varphi\|_{\mathrm{BMO}} \le 4C\|f\|_\infty$. Therefore $\|\mathrm{Im}(\varphi)\|_{\mathrm{BMO}} \le \|\varphi\|_{\mathrm{BMO}} \le 4C\|f\|_\infty < \infty$ as promised. □

In view of Proposition 3.3, for every real-valued function $f \in \mathrm{BPH}$, if φ is an analytic function on $\mathbf{B}$ such that $[f] = \mathrm{Re}(\varphi)$, then the Hankel operator $H_{\bar{\varphi}}$ is bounded.

4. Unbounded Toeplitz operators and self-adjointness

First of all, we cite [4, Chapter X] as a reference for the general theory of unbounded symmetric operators and unbounded self-adjoint operators.

Suppose that g is a real-valued function in $L^2(S, d\sigma)$. Then obviously the formula

$$T_g h = P(gh), \quad h \in H^\infty(S), \tag{4.1}$$

defines a symmetric operator in $H^2(S)$ with $\mathcal{D} = H^\infty(S)$ as its initial domain. We regard the closure of this symmetric operator as a Toeplitz operator, albeit possibly unbounded. A natural and important question is, when is this closure a self-adjoint operator? While we do not know the answer to this general question, below is a partial result that gives us what we need for the purpose of this paper:

Proposition 4.1. *Suppose that f is a real-valued function in* BPH *and that $\varphi = [f] + ig$ is analytic on $\mathbf{B}$, where g is real valued. Then the closure of the symmetric operator T_g defined by* (4.1) *is a self-adjoint operator.*

Proof. Let us show that $\{T_g h - ih : h \in H^\infty(S)\}$ is dense in $H^2(S)$. Suppose that $\psi \in H^2(S)$ is orthogonal to $\{T_g h - ih : h \in H^\infty(S)\}$. Then we have

$$\int g\psi\bar{h}d\sigma = \langle \psi, T_g h\rangle = \langle \psi, ih\rangle = -i\langle \psi, h\rangle \quad \text{for every} \quad h \in H^\infty(S). \tag{4.2}$$

Since f is bounded, it follows that

$$\int \varphi\psi\bar{h}d\sigma = \langle T_f\psi, h\rangle + \langle \psi, h\rangle, \qquad h \in H^\infty(S).$$

Replacing h by the reproducing kernel $K_w(\zeta) = (1 - \langle \zeta, w\rangle)^{-n}$ in the above, we find that

$$\varphi(w)\psi(w) = (T_f\psi)(w) + \psi(w) \quad \text{for every} \quad w \in \mathbf{B}.$$

Clearly, this tells us that $\varphi\psi \in H^2(S)$. Since f is bounded, this implies $g\psi \in L^2(S, d\sigma)$. Once this is established, from (4.2) we deduce

$$\int g|\psi|^2 d\sigma = -i \int |\psi|^2 d\sigma.$$

Since g is real valued, this is possible only if $\psi = 0$. Therefore $\{T_g h - ih : h \in H^\infty(S)\}$ is dense in $H^2(S)$. A similar argument shows that $\{T_g h + ih : h \in H^\infty(S)\}$ is also dense in $H^2(S)$. Thus the deficiency indices of the symmetric operator T_g on $H^\infty(S)$ are 0 and 0. Therefore its closure is a self-adjoint operator (see, e.g., [4, Theorem X.2.20]). □

Lemma 4.2. *Let g be the same as in Proposition* 4.1 *and let $\mathcal{D}(T_g)$ denote the domain of the self-adjoint operator T_g, as provided in that proposition. Then $\mathcal{D}(T_g) \supset H^p(S)$ for every $p > 2$. Moreover, for every pair of $b \in L^\infty(S, d\sigma)$ and $h \in H^\infty(S)$, we have $T_b h \in \mathcal{D}(T_g)$.*

Proof. Proposition 3.3 tells us that $g \in \mathrm{BMO}$. By the well-known John-Nirenberg theorem, $\mathrm{BMO} \subset L^t(S, d\sigma)$ for every $1 \le t < \infty$. Hence $g \in L^t(S, d\sigma)$ for every $1 \le t < \infty$. Let $2 < p < \infty$ and let $h \in H^p(S)$. For each $0 \le r < 1$, we define the function $h_r(\zeta) = h(r\zeta)$ on $\mathbf{B}$. Then $\|h - h_r\|_p \to 0$ as $r \uparrow 1$ [15, Theorem 5.6.6]. By Hölder's inequality,

$$\|gh - gh_r\|_2 \le \|g\|_{2p/(p-2)} \|h - h_r\|_p.$$

Thus we have both $\|P(gh) - T_g h_r\|_2 = \|P(gh - gh_r)\|_2 \to 0$ and $\|h - h_r\|_2 \to 0$ as $r \uparrow 1$. By the definition of the closure of an operator, we have $h \in \mathcal{D}(T_g)$ with $T_g h = P(gh)$. This proves the assertion that $\mathcal{D}(T_g) \supset H^p(S)$ for every $2 < p < \infty$.

Suppose that $b \in L^\infty(S, d\sigma)$ and that $h \in H^\infty(S)$. Then $bh \in L^\infty(S, d\sigma)$. Since P maps BMO into BMO, we have $T_b h \in \mathrm{BMO} \cap H^2(S) \subset H^p(S)$ for every $2 < p < \infty$. By what we proved above, $T_b h \in \mathcal{D}(T_g)$. $\square$

Lemma 4.3. *Suppose that f is a real-valued function in* BPH *and that $\varphi = [f] + ig$ is analytic on $\mathbf{B}$, where g is real valued. Let $\mathcal{D}(T_g)$ be the domain of the self-adjoint operator T_g, as provided by Proposition* 4.1. *Then for every $\eta \in \mathcal{D}(T_g)$, we have $T_f \eta \in \mathcal{D}(T_g)$ and*

$$T_f T_g \eta - T_g T_f \eta = -2i H_f^* H_f \eta. \tag{4.3}$$

Proof. First consider any $h \in H^\infty(S)$. Then Lemma 4.2 gives us $T_f h \in \mathcal{D}(T_g)$. Furthermore, since $g, \varphi \in \mathrm{BMO}$ and $\mathrm{BMO} \subset L^t(S, d\sigma)$, $1 \le t < \infty$, we have

$$\begin{aligned} T_f T_g h - T_g T_f h &= PM_f PM_g h - PM_g PM_f h = -(i/2)(PM_{\bar\varphi} PM_\varphi h - PM_\varphi PM_{\bar\varphi} h) \\ &= -(i/2) H_{\bar\varphi}^* H_{\bar\varphi} h. \end{aligned}$$

On the other hand, since φ is analytic, $H_{\bar\varphi} = H_{\bar\varphi + \varphi} = 2H_f$. Therefore

$$T_g T_f h = T_f T_g h + 2i H_f^* H_f h, \quad h \in H^\infty(S).$$

By definition, for each $\eta \in \mathcal{D}(T_g)$ there is a sequence $\{h_k\} \subset H^\infty(S)$ such that $\|\eta - h_k\| \to 0$ and $\|T_g \eta - T_g h_k\| \to 0$ as $k \to \infty$. From the identity

$$T_g T_f h_k = T_f T_g h_k + 2i H_f^* H_f h_k$$

we see that $\{T_g T_f h_k\}$ is a Cauchy sequence in $H^2(S)$. Since $\{T_f h_k\} \subset \mathcal{D}(T_g)$ and since the self-adjoint operator T_g is closed, we conclude that $T_f \eta \in \mathcal{D}(T_g)$ and that

$$T_g T_f \eta = \lim_{k\to\infty} T_g T_f h_k = \lim_{k\to\infty} \left(T_f T_g h_k + 2i H_f^* H_f h_k\right) = T_f T_g \eta + 2i H_f^* H_f \eta,$$

where the limit is taken in the norm topology of $H^2(S)$. This obviously implies (4.3). $\square$

Proposition 4.4. *Suppose that f is a real-valued function in* BPH *and that $\varphi = [f] + ig$ is analytic on* $\mathbf{B}$, *where g is real valued. Then for every $z \in \mathbf{C}\backslash\mathbf{R}$ we have*

$$T_f(T_g - z)^{-1} - (T_g - z)^{-1}T_f = 2i(T_g - z)^{-1}H_f^*H_f(T_g - z)^{-1}. \tag{4.4}$$

Proof. Let $z \in \mathbf{C}\backslash\mathbf{R}$ and $h \in H^2(S)$. If we set $\eta = (T_g - z)^{-1}h$, then $\eta \in \mathcal{D}(T_g)$ and $(T_g - z)\eta = h$. Applying Lemma 4.3, we have $T_f\eta \in \mathcal{D}(T_g)$ and

$$T_f h = T_f(T_g - z)\eta = (T_g - z)T_f\eta - 2iH_f^*H_f\eta.$$

Hence

$$\begin{aligned}(T_g - z)^{-1}T_f h &= T_f\eta - 2i(T_g - z)^{-1}H_f^*H_f\eta \\ &= T_f(T_g - z)^{-1}h - 2i(T_g - z)^{-1}H_f^*H_f(T_g - z)^{-1}h.\end{aligned}$$

Since this holds for every $h \in H^2(S)$, we obtain (4.4). □

Proof of Theorem 1.2. Let f be a real-valued non-constant function in BPH. Furthermore, suppose that $\varphi = [f] + ig$ is analytic on $\mathbf{B}$, where g is real valued and semi-bounded. Obviously, it suffices to consider the case where g is bounded from below. That is, we assume that there is an $L \in \mathbf{R}$ such that $g \geq L$ on S. Then the spectrum of the self-adjoint operator T_g is obviously contained in $[L, \infty)$. Let $\lambda \in (-\infty, L)$. Now apply Proposition 4.4: if we set $z = \lambda + \epsilon i$ in (4.4) and then take the limit $\epsilon \to 0$, we obtain

$$T_f B_\lambda - B_\lambda T_f = iC_\lambda,$$

where $B_\lambda = (T_g - \lambda)^{-1}$ and $C_\lambda = 2(T_g - \lambda)^{-1}H_f^*H_f(T_g - \lambda)^{-1}$. Obviously, $C_\lambda \geq 0$. Let E be the orthogonal projection from $H^2(S)$ onto the subspace $H_{\mathrm{s}}(T_f)$. Then it follows from Theorem 2.3 that $C_\lambda E = 0$. Note that this is true for every $\lambda \in (-\infty, L)$ and that

$$\lim_{\lambda \to -\infty} \lambda(T_g - \lambda)^{-1} = -1$$

in the strong operator topology. Hence

$$2H_f^*H_f E = \lim_{\lambda \to -\infty} \lambda^2 C_\lambda E = 0.$$

This implies that $H_f E = 0$. Since $H_{\mathrm{s}}(T_f)$ is an invariant subspace for T_f, Proposition 3.2 tells us that $E = 0$. That is, T_f is purely absolutely continuous. □

We now turn to the proof of Theorem 1.3, which involves the more conventional representation of the commutator $[T_f, T_g]$:

Lemma 4.5. *Suppose that f is a real-valued function in* BPH *and that $\varphi = [f] + ig$ is analytic on* $\mathbf{B}$, *where g is real valued. Let $\mathcal{D}(T_g)$ be the domain of the self-adjoint operator T_g, as provided by Proposition* 4.1. *Then for every $\eta \in \mathcal{D}(T_g)$, we have*

$$T_f T_g\eta - T_g T_f\eta = (H_g^*H_f - H_f^*H_g)\eta. \tag{4.5}$$

Proof. Since $g \in \mathrm{BMO} \subset \cap_{t>1} L^t(S, d\sigma)$, it is easy to see that the identity

$$T_f T_g h - T_g T_f h = (H_g^* H_f - H_f^* H_g)h$$

holds for every $h \in H^\infty(S)$. Since $\eta \in \mathcal{D}(T_g)$, by definition, there is a sequence $\{h_k\} \subset H^\infty(S)$ such that $\|\eta - h_k\| \to 0$ and $\|T_g\eta - T_g h_k\| \to 0$ as $k \to \infty$. In particular,

$$T_f T_g h_k - T_g T_f h_k = (H_g^* H_f - H_f^* H_g)h_k \tag{4.6}$$

for every k. We showed in the proof of Lemma 4.3 that $\{T_g T_f h_k\}$ is a Cauchy sequence. Therefore, as $k \to \infty$, the limit of the left-hand side of (4.6) equals $T_f T_g \eta - T_g T_f \eta$. On the other hand, since $g \in \mathrm{BMO}$, the Hankel operator H_g is bounded. Thus, as $k \to \infty$, the limit of the right-hand side of (4.6) equals $(H_g^* H_f - H_f^* H_g)\eta$. Hence (4.5) holds. □

Corollary 4.6. *Suppose that f is a real-valued function in* BPH *and that $\varphi = [f] + ig$ is analytic on* $\mathbf{B}$, *where g is real valued. Then we have the identity*

$$-2iH_f^* H_f = H_g^* H_f - H_f^* H_g.$$

Proof. This follows from Lemmas 4.3, 4.5 and the fact that $\mathcal{D}(T_g)$ is dense in $H^2(S)$. □

Proof of Theorem 1.3. Given any real number $\lambda \in \mathbf{R}$, define the subspace $\mathcal{V} = \{h \in H^2(S) : T_f h = \lambda h\}$ of $H^2(S)$. Our goal is to show that $\mathcal{V} = \{0\}$. To do that, consider the orthogonal projection $E : H^2(S) \to \mathcal{V}$. By Proposition 3.2, it suffices to show that $H_f E = 0$. Equivalently, it suffices to show that $H_f h = 0$ for every $h \in \mathcal{V}$.

Let g be a real-valued pluriharmonic conjugate of f. For each $k \geq 1$, define

$$g_k(\zeta) = \begin{cases} k & \text{if} \quad g(\zeta) \geq k \\ g(\zeta) & \text{if} \quad -k < g(\zeta) < k \\ -k & \text{if} \quad g(\zeta) \leq -k \end{cases}.$$

That is, $g_k = \max\{-k, \min\{g, k\}\}$. Thus there is a constant C such that $\|g_k\|_{\mathrm{BMO}} \leq C\|g\|_{\mathrm{BMO}}$ for every k. Consequently, there is a $C_1 < \infty$ such that $\|H_{g_k}\| \leq C_1$ for every k. It is obvious that if $\psi \in H^\infty(S)$, then $\|H_{g_k}\psi - H_g\psi\| \to 0$ as $k \to \infty$. Combining this with the bound $\|H_{g_k}\| \leq C_1$, we obtain the convergence

$$\lim_{k\to\infty} H_{g_k} = H_g \tag{4.7}$$

in the strong operator topology.

Let $h \in \mathcal{V}$. For any k, since $g_k \in L^\infty(S, d\sigma)$, we have

$$\begin{aligned}\langle (H_{g_k}^* H_f - H_f^* H_{g_k})h, h\rangle &= \langle (T_f T_{g_k} - T_{g_k} T_f)h, h\rangle = \langle T_{g_k} h, T_f h\rangle - \langle T_{g_k} T_f h, h\rangle \\ &= \langle T_{g_k} h, \lambda h\rangle - \langle T_{g_k} \lambda h, h\rangle = 0.\end{aligned}$$

Therefore, by (4.7),

$$\langle (H_g^* H_f - H_f^* H_g)h, h\rangle = \lim_{k\to\infty} \langle (H_{g_k}^* H_f - H_f^* H_{g_k})h, h\rangle = 0.$$

Applying Corollary 4.6, we obtain

$$\|H_f h\|^2 = \langle H_f^* H_f h, h\rangle = (i/2)\langle (H_g^* H_f - H_f^* H_g)h, h\rangle = 0.$$

That is, $H_f h = 0$ for every $h \in \mathcal{V}$. This completes the proof. □

5. Spectral multiplicity

The classic spectral multiplicity function is a complete unitary invariant for any self-adjoint operator. But here, we are narrowly focused on self-adjoint operators that are purely absolutely continuous, and so we will only consider the spectral multiplicity of these operators.

Definition 5.1. Let Δ be any Borel set in $\mathbf{R}$. Then $M^{(\Delta)}$ denotes the operator of multiplication by the coordinate function x on the Hilbert space $L^2(\Delta, dm)$, where dm is the Lebesgue measure on $\mathbf{R}$.

Let A be a self-adjoint operator on a Hilbert space H, and suppose that $H_{\mathrm{s}}(A) = \{0\}$, i.e., A is purely absolutely continuous. Then there exists a countable family of Borel sets $\{\Delta_i : i \in I\}$ in $\mathbf{R}$ such that A is unitarily equivalent to the operator

$$\bigoplus_{i\in I} M^{(\Delta_i)}.$$

As we recall, the spectral multiplicity function of A is defined by the formula

$$m_A(x) = \sum_{i\in I} \chi_{\Delta_i}(x), \quad x \in \mathbf{R}.$$

It is well known that this m_A is a complete unitary invariant for A.

Proposition 5.2. *Let A be a bounded self-adjoint operator on a Hilbert space H, and suppose that A is purely absolutely continuous. Furthermore, suppose that B is a bounded self-adjoint operator on H such that $i[A, B] \geq 0$. Then*

$$\|[A, B]\|_1 \leq \frac{1}{\pi}\|B\| \int_{\mathbf{R}} m_A(x)dx, \tag{5.1}$$

where $\|\cdot\|_1$ denotes the norm of the trace class, and for the right-hand side we adopt the convention $0 \cdot \infty = 0$.

Remark. In [17], inequality (5.1) was attributed to Kato. Moreover, (5.1) follows from the combination of [17, Proposition 2.1] and [16, Theorem 4.5]. For the convenience of the reader, a self-contained proof of (5.1) is presented below.

Proof of Proposition 5.2. We begin with the identity

$$\frac{d}{dt}\langle e^{itA}Be^{-itA}h,h\rangle = i\langle e^{itA}[A,B]e^{-itA}h,h\rangle,$$

$h \in H$, which is well known and obvious. Let E be any finite-rank self-adjoint operator satisfying the condition $0 \le E \le i[A,B]$. By the above identity, for any $0 < R < \infty$,

$$\begin{aligned}\int_{-R}^{R}\langle e^{itA}Ee^{-itA}h,h\rangle dt &\le \int_{-R}^{R} i\langle e^{itA}[A,B]e^{-itA}h,h\rangle dt = \int_{-R}^{R}\frac{d}{dt}\langle e^{itA}Be^{-itA}h,h\rangle dt \\ &= \langle e^{iRA}Be^{-iRA}h,h\rangle - \langle e^{-iRA}Be^{iRA}h,h\rangle \le 2\|B\|\|h\|^2. \end{aligned} \tag{5.2}$$

We can write E in the form

$$E = \sum_{j=1}^{\nu}\varphi_j \otimes \varphi_j,$$

where $\varphi_1, \ldots, \varphi_\nu \in H$. As we explained above, there exist a countable family of Borel sets $\{\Delta_i : i \in I\}$ in $\mathbf{R}$ and a unitary operator $U : H \to \tilde{H} = \oplus_{i\in I}L^2(\Delta_i, dm)$ such that

$$UAU^* = \bigoplus_{i\in I} M^{(\Delta_i)}.$$

For each $1 \le j \le \nu$, we have $U\varphi_j = (\psi_{j,i})_{i\in I}$, where $\psi_{j,i} \in L^2(\Delta_i, dm)$, $i \in I$. For each $r \in I$, define the element $\eta_r = (\eta_{r,i})_{i\in I}$ in $\tilde{H}$ by the formula

$$\eta_{r,i} = \begin{cases}\chi_{\Delta_r} & \text{if} \quad i = r \\ 0 & \text{if} \quad i \neq r\end{cases}.$$

Then it follows from (5.2) that for each $r \in I$ and for any $0 < R < \infty$, we have

$$\begin{aligned}\sum_{j=1}^{\nu}\int_{-R}^{R}\left|\int_{\Delta_r} e^{-itx}\overline{\psi_{j,r}(x)}dx\right|^2 dt &= \int_{-R}^{R}\langle e^{itA}Ee^{-itA}U^*\eta_r, U^*\eta_r\rangle dt \\ &\le 2\|B\|\|U^*\eta_r\|^2 = 2\|B\|\|\eta_r\|^2 = 2\|B\|m(\Delta_r).\end{aligned}$$

Since Fourier transform preserves norm on $L^2(\mathbf{R}, dm)$, letting $R \to \infty$ on the left-hand side, we obtain the inequality

$$2\pi\sum_{j=1}^{\nu}\|\psi_{j,r}\|^2 \le 2\|B\|m(\Delta_r)$$

for every $r \in I$. Summing over $r \in I$, we see that

$$2\pi\mathrm{tr}(E) = 2\pi\sum_{j=1}^{\nu}\|\varphi_j\|^2 = 2\pi\sum_{j=1}^{\nu}\sum_{r\in I}\|\psi_{j,r}\|^2 \le 2\|B\|\sum_{r\in I}m(\Delta_r).$$

Since this holds for every finite-rank self-adjoint operator E satisfying the condition $0 \le E \le i[A,B]$, it follows that

$$2\pi\|[A,B]\|_1 \le 2\|B\| \sum_{r \in I} m(\Delta_r) = 2\|B\| \int_{\mathbf{R}} m_A(x)dx.$$

Dividing both sides by 2π, we obtain (5.1). □

Proof of Theorem 1.4. Let φ be a non-constant function in $H^\infty(S)$. Then $\varphi = f + ig$, where f and g are non-constant, real-valued functions. Consider the self-adjoint operators

$$A = T_f \quad \text{and} \quad B = T_g.$$

By Theorem 1.2, A is purely absolutely continuous. Since $\varphi \in H^\infty(S)$ and φ is not a constant, we have $H_{\bar\varphi} \ne 0$. Recall that we assume $n \ge 2$ for Theorem 1.4. Thus we can apply [5, Theorem 1.5], which says that the Hankel operator $H_{\bar\varphi}$ is not in the Schatten class $\mathcal{C}_{2n}$. Since

$$[A,B] = -(i/2)(T_{\bar\varphi}T_\varphi - T_\varphi T_{\bar\varphi}) = -(i/2)H_{\bar\varphi}^* H_{\bar\varphi},$$

the commutator $[A,B]$ is not in the trace class. We also have $i[A,B] = (1/2)H_{\bar\varphi}^* H_{\bar\varphi} \ge 0$. Thus if it were true that

$$\int_{\mathbf{R}} m_{T_f}(x)dx < \infty,$$

then, by Proposition 5.2, we would have the contradiction that $[A,B]$ is in the trace class. This completes the proof. □

There is a localized version of Proposition 5.2, which will be needed in Section 6.

Proposition 5.3. *Let A be a bounded self-adjoint operator on a Hilbert space H, and suppose that A is purely absolutely continuous. Furthermore, suppose that B is a bounded self-adjoint operator on H such that $i[A,B] \ge 0$. Then for every Borel set $\Lambda \subset \mathbf{R}$, we have*

$$\|\mathcal{E}(\Lambda)[A,B]\mathcal{E}(\Lambda)\|_1 \le \frac{1}{\pi}\|B\| \int_\Lambda m_A(x)dx, \tag{5.3}$$

where $\mathcal{E}$ is the spectral measure for A.

Proof. Given any Borel set $\Lambda \subset \mathbf{R}$, denote $H' = \mathcal{E}(\Lambda)H$. Let A' be the restriction of A to its invariant subspace H', and let B' be the compression of B to H'. Note that A' is purely absolutely continuous, and that

$$m_{A'}(x) = \begin{cases} m_A(x) & \text{if} \quad x \in \Lambda \\ 0 & \text{if} \quad x \notin \Lambda \end{cases}. \tag{5.4}$$

Since $[A',B']$ is the compression of $[A,B]$ to H', we have $i[A',B'] \ge 0$. Thus by Proposition 5.2,

$$\|[A',B']\|_1 \le \frac{1}{\pi}\|B'\| \int_{\mathbf{R}} m_{A'}(x)dx. \tag{5.5}$$

We have

$$\int_{\mathbf{R}} m_{A'}(x)dx = \int_{\Lambda} m_A(x)dx$$

by (5.4). Furthermore, $\|[A', B']\|_1 = \|\mathcal{E}(\Lambda)[A, B]\mathcal{E}(\Lambda)\|_1$ and $\|B'\| \le \|B\|$. Thus (5.3) follows from (5.5). □

Note that in the proof of Theorem 1.4, we used the fact that $H_{\bar{\varphi}} \notin \mathcal{C}_{2n}$, which is far stronger a statement than what is needed: the operator $H_{\bar{\varphi}}^* H_{\bar{\varphi}}$ is not in the trace class. This leads to the suspicion that there is a far stronger statement to be made about the spectral multiplicity function m_{T_f} than Theorem 1.4. This suspicion was what led to Theorem 1.6, and its proof, presented below, can be viewed as a further exploitation of fact that $H_{\bar{\varphi}} \notin \mathcal{C}_{2n}$.

6. Spectral multiplicity in an interval

We now turn to the proof of Theorem 1.6, and most of the work in this proof is done in the form of proving

Proposition 6.1. *Consider any complex dimension $n \ge 3$. Let f be a real-valued, non-constant function in* $\mathrm{BPH} \cap C^1(S)$. *Let $\mathcal{E}$ denote the spectral measure for the Toeplitz operator T_f. Then for every open interval (a, b) satisfying the condition*

$$\min_{z \in S} f(z) \le a < b \le \max_{z \in S} f(z), \tag{6.1}$$

the operator

$$H_f \mathcal{E}(a, b) H_f^*$$

is not in the trace class.

The proof of Proposition 6.1 will take many steps. We begin with the obvious:

Lemma 6.2. *If f is a real-valued function in* $\mathrm{BPH} \cap C^1(S)$, *then there is a $\varphi \in H^\infty(S)$ such that $f = \mathrm{Re}(\varphi)$.*

Proof. By Proposition 3.3, there is a $\varphi \in H^2(S) \cap \mathrm{BMO}$ such that $f = \mathrm{Re}(\varphi)$. Thus

$$\varphi + \bar{\varphi}(0) = 2Pf = 2(f + [P, M_f]1).$$

Since $f \in C^1(S)$, it is Lipschitz on S. Therefore it follows from [15, Proposition 1.4.10] that $[P, M_f]1$ is a bounded function on S. Consequently, φ is bounded. □

Although Proposition 6.1 assumes $n \ge 3$, many of the steps below only require the condition $n \ge 2$.

Lemma 6.3. *Let f be the same as in Proposition* 6.1, *and let a, b satisfy* (6.1). *There exist $x, z \in S$ and $0 \le r < s \le \pi/2$ satisfying the following conditions*:

(1) $\langle x, z \rangle = 0$.
(2) *If $t \in [r, s]$, then $a < f(\cos tx + \sin tz) < b$.*
(3) *The function $t \mapsto f(\cos tx + \sin tz)$ is not a constant on the interval $[r, s]$.*

Proof. We claim that there exist $x, z \in S$ satisfying (1) and the conditions

(2′) $a < f(x) < b$.

(3′) The function $t \mapsto f(\cos tx + \sin tz)$ is not a constant on the interval $[0, \pi/2]$.

To prove this claim, we pick a $\xi \in S$ such that $a < f(\xi) < b$, which exists by the intermediate value theorem. Define $W = \{w \in S : \langle \xi, w\rangle = 0\}$. Then

$$S = \{\cos t e^{i\alpha}\xi + \sin t w : w \in W,\ t \in [0, \pi/2],\ \alpha \in [0, 2\pi]\}. \tag{6.2}$$

There are the following two possibilities.

(A) Suppose that the function $\alpha \mapsto f(e^{i\alpha}\xi)$ is a constant on $[0, 2\pi]$. Since f is not a constant on S, by (6.2), there are $\tau \in [0, 2\pi]$ and $w \in W$ such that the function $t \mapsto f(\cos t e^{i\tau}\xi + \sin t w)$ is not a constant on the interval $[0, \pi/2]$. In this case, since $f(e^{i\tau}\xi) = f(\xi)$, we can take $x = e^{i\tau}\xi$ and $z = w$.

(B) Suppose that the function $\alpha \mapsto f(e^{i\alpha}\xi)$ is not a constant on $[0, 2\pi]$. Then by the intermediate value theorem, there is a $\beta \in [0, 2\pi]$ such that $f(e^{i\beta}\xi) \neq f(\xi)$ and $a < f(e^{i\beta}\xi) < b$. Take any $w \in W$ and define the functions

$$g(t) = f(\cos t\xi + \sin t w) \quad \text{and} \quad h(t) = f(\cos t e^{i\beta}\xi + \sin t w), \quad 0 \leq t \leq \pi/2.$$

Since $g(\pi/2) = h(\pi/2)$ while $g(0) \neq h(0)$, the functions g and h cannot both be constants on the interval $[0, \pi/2]$. Thus one of the pairs ξ, w and $e^{i\beta}\xi, w$ is the desired pair x, z.

Hence there indeed exists a pair of $x, z \in S$ satisfying conditions (1), (2′) and (3′). Take such a pair of $x, z \in S$, and define

$$r = \sup\{\rho \in [0, \pi/2] : f(\cos tx + \sin tz) = f(x) \text{ for every } 0 \leq t \leq \rho\}.$$

Then $f(\cos tx + \sin tz) = f(x)$ for every $0 \leq t \leq r$. By condition (3′), we have $r < \pi/2$. Since $f(\cos rx + \sin rz) = f(x) \in (a, b)$, by the continuity of f, there is an $r < s \leq \pi/2$ such that $f(\cos tx + \sin tz) \in (a, b)$ for every $t \in [r, s]$, i.e., (2) holds for this pair of r, s. Finally, the definition of r ensures that (3) also holds. This completes the proof. □

Lemma 6.4. *Let f be the same as in Proposition* 6.1, *and let a, b satisfy* (6.1). *There exist $y, y^\perp \in S$ satisfying the conditions $\langle y, y^\perp\rangle = 0$, $a < f(y) < b$, and*

$$\frac{d}{dt} f(\cos ty + \sin ty^\perp)\Big|_{t=0} \neq 0. \tag{6.3}$$

Proof. Let $x, z \in S$ and $0 \leq r < s \leq \pi/2$ be the same as in Lemma 6.3. Denote $\mathcal{L} = \operatorname{span}\{x, z\}$. For each $t \in \mathbf{R}$, let V_t be the unitary transformation on $\mathbf{C}^n$ such that

$$\begin{cases} V_t x &= \cos tx + \sin tz \\ V_t z &= -\sin tx + \cos tz \\ V_t &= 1 \text{ on } \mathbf{C}^n \ominus \mathcal{L} \end{cases}.$$

By Lemma 6.3, the function $t \mapsto f(V_t x)$ is not a constant on $[r, s]$. Hence there is a $\theta \in (r, s)$ such that

$$\frac{d}{dt} f(V_t x)\bigg|_{t=\theta} \neq 0.$$

For all $t, t' \in \mathbf{R}$, we have $V_{t'+t} = V_{t'} V_t$. Thus the above translates to

$$\frac{d}{dt} f(V_\theta V_t x)\bigg|_{t=0} \neq 0. \tag{6.4}$$

We define $y = V_\theta x$ and $y^\perp = V_\theta z$. Then the condition $\langle x, z\rangle = 0$ implies $\langle y, y^\perp\rangle = 0$. Since $y = \cos\theta x + \sin\theta z$ and $\theta \in (r, s)$, by Lemma 6.3, we have $a < f(y) < b$. Finally, (6.3) follows from (6.4) and the obvious identity

$$\frac{d}{dt} f(V_\theta V_t x)\bigg|_{t=0} = \frac{d}{dt} f(V_\theta(\cos tx + \sin tz))\bigg|_{t=0} = \frac{d}{dt} f(\cos ty + \sin ty^\perp)\bigg|_{t=0}.$$

This completes the proof. □

It is well known that the formula

$$d(\zeta, \xi) = |1 - \langle\zeta, \xi\rangle|^{1/2}, \qquad \zeta, \xi \in S,$$

defines a metric on S [15, page 66]. For the rest of the section, we denote

$$B(\zeta, r) = \{x \in S : |1 - \langle x, \zeta\rangle|^{1/2} < r\}$$

for $\zeta \in S$ and $r > 0$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} r^{2n} \leq \sigma(B(\zeta, r)) \leq A_0 r^{2n} \tag{6.5}$$

for all $\zeta \in S$ and $0 < r \leq \sqrt{2}$ [15, Proposition 5.1.4]. Note that the upper bound actually holds when $r > \sqrt{2}$.

Lemma 6.5. *There are positive numbers $0 < c < (b - a)/12$ and $0 < \rho < 1$ such that $f(u) \in (a + 3c, b - 3c)$ for every $u \in B(y, \rho)$, where y is the same as in Lemma 6.4.*

Proof. By Lemma 6.4, $f(y) \in (a, b)$. Thus the number

$$c = (1/12) \min\{f(y) - a, b - f(y)\}$$

satisfies the condition $0 < c < (b - a)/12$. Since $f(y) \in (a + 3c, b - 3c)$, the existence of the desired $0 < \rho < 1$ follows from the continuity of f. □

Lemma 6.6. *Let ρ be the same as in Lemma 6.5. There exist positive numbers $\delta > 0$, $0 < \tau < \rho/2$ and $0 < \rho_0 < \rho/2$ such that if $u \in B(y, \rho_0)$ and $0 < t < \tau$, then*

$$\sup\{|f(v) - f(u)| : v \in B(u, t)\} \geq \delta t.$$

Proof. Let $\mathcal{U}$ denote the collection of unitary transformations U on $\mathbf{C}^n$. Since f is C^1, the function

$$G(U) = \frac{d}{dt} f(\cos tUy + \sin tUy^{\perp})\bigg|_{t=0}$$

is continuous on $\mathcal{U}$, and the convergence

$$\frac{1}{t}(f(\cos tUy + \sin tUy^{\perp}) - f(Uy)) \to G(U) \quad \text{as} \quad t \to 0$$

is uniform with respect to $U \in \mathcal{U}$. Lemma 6.4 tells us that $G(1) \neq 0$. Thus if we take $\delta = |G(1)|/2$, then there exist an open neighborhood $\mathcal{N}$ of the identity transformation 1 in $\mathcal{U}$ and a positive number $0 < \tau < \rho/2$ such that

$$(6.6) \qquad |f(\cos tUy + \sin tUy^{\perp}) - f(Uy)|/|t| \geq \delta \quad \text{whenever} \quad U \in \mathcal{N} \quad \text{and} \quad 0 < |t| < \tau.$$

There is a $0 < \rho_0 < \rho/2$ such that $\{Uy : U \in \mathcal{N}\} \supset B(y, \rho_0)$. Thus given any $u \in B(y, \rho_0)$, there is a $U_u \in \mathcal{N}$ such that $u = U_u y$. Let $0 < t < \tau$ also be given. Then we define $v = \cos tU_u y + \sin tU_u y^{\perp}$. It follows from (6.6) that

$$|f(v) - f(u)| \geq \delta t.$$

Since $\langle v, u\rangle = \cos t$, we have $d(v, u) = \sqrt{1 - \cos t} < \sin t < t$. That is, $v \in B(u, t)$. This completes the proof. □

Recall that the normalized reproducing kernel for the Hardy space $H^2(S)$ is given by the formula

$$k_w(\zeta) = \frac{(1 - |w|^2)^{n/2}}{(1 - \langle\zeta, w\rangle)^n}, \quad w \in \mathbf{B} \text{ and } \zeta \in S.$$

Lemma 6.7. *Let τ and ρ_0 be the same as in Lemma* 6.6. *There exists a $c_1 > 0$ such that if $u \in B(y, \rho_0)$ and $0 < t < \tau$, and if we set*

$$w = (1 - t^2)^{1/2} u,$$

then $\|H_f k_w\| \geq c_1 t$.

Proof. This is essentially the same as the proof of [5, Lemma 8.12]. Since f is C^1, it is Lipschitz on S. Therefore there is an $L > \delta$, where δ is given in Lemma 6.6, such that

$$(6.7) \qquad |f(\zeta) - f(\xi)| \leq (L/\sqrt{2})|\zeta - \xi| \leq Ld(\zeta, \xi) \quad \text{for all} \quad \zeta, \xi \in S.$$

Let u, t and w be given as in the statement of the lemma. By Lemma 6.6, there is a $v \in B(u, t)$ such that $|f(v) - f(u)| \geq \delta t/2$. Combining this with (6.7), we have

$$|f(\zeta) - f(\xi)| \geq \delta t/6 \quad \text{if} \quad \zeta \in B(v, \delta t/6L) \quad \text{and} \quad \xi \in B(u, \delta t/6L).$$

Note that $B(v, \delta t/6L) \subset B(v,t) \subset B(u,2t)$. Thus for any $\gamma \in \mathbf{C}$, we have

$$\begin{aligned}\sigma(\{\zeta \in B(u,2t) : |f(\zeta)-\gamma| \geq \delta t/12\}) &\geq \min\{\sigma(B(u,\delta t/6L)), \sigma(B(v,\delta t/6L))\} \\ &= \sigma(B(u,\delta t/6L)).\end{aligned}$$

Consequently, there is an $a_1 > 0$ which depends only on δ, L and n such that

$$\frac{1}{\sigma(B(u,2t))}\int_{B(u,2t)} |f-\gamma|^2 d\sigma \geq \frac{\sigma(B(u,\delta t/6L))}{\sigma(B(u,2t))}(\delta t/12)^2 \geq a_1 t^2. \tag{6.8}$$

For a real-valued $h \in L^2(S, d\sigma)$, it is well known that $2\|h - Ph\|^2 \geq \|h - \langle h, 1\rangle\|^2$. Applying Möbius transform, we have

$$2\|H_f k_w\|^2 \geq \|(f - \langle f k_w, k_w\rangle)k_w\|^2. \tag{6.9}$$

If $\zeta \in B(u,2t)$, then $|1-\langle \zeta, w\rangle| \leq 1-|w| + |1-\langle \zeta, u\rangle| \leq t^2 + (2t)^2 = 5t^2$. Thus

$$|k_w(\zeta)|^2 \geq \frac{t^{2n}}{(5t^2)^{2n}} \geq \frac{a_2}{\sigma(B(u,2t))} \quad \text{for } \zeta \in B(u,2t),$$

where $a_2 > 0$ depends only on n. Combining this inequality with (6.8) and (6.9), we see that $2\|H_f k_w\|^2 \geq a_2 a_1 t^2$, which proves the lemma. ☐

Lemma 6.8. [18, Lemma 7.1] *Given any* $0 < \eta < 1$, *there is a constant* $0 < C_{6.8} < \infty$ *such that*

$$\int_S |k_z(\zeta)||k_w(\zeta)| d\sigma(\zeta) \leq C_{6.8}\frac{(1-|z|^2)^{(n/2)-\eta}(1-|w|^2)^{(n/2)-\eta}}{|1-\langle z, w\rangle|^{n-2\eta}}$$

for all $z, w \in \mathbf{B}$.

With the constants ρ_0 and τ given in Lemma 6.6, we now proceed with the following construction. Given a $0 < t < \tau$, we let Γ_t be a subset of $B(y, \rho_0)$ that is *maximal* with respect to the property

$$B(u, t^{1/2}) \cap B(v, t^{1/2}) = \emptyset \quad \text{for all } u \neq v \text{ in } \Gamma_t. \tag{6.10}$$

The maximality of Γ_t implies that

$$\bigcup_{u \in \Gamma_t} B(u, 2t^{1/2}) \supset B(y, \rho_0). \tag{6.11}$$

For each $u \in \Gamma_t$, define

$$w(u) = (1-t^2)^{1/2} u.$$

Let $\varphi \in H^\infty(S)$ be such that $\mathrm{Re}(\varphi) = f$, as provided by Lemma 6.3. Since $H_\varphi = 0$, we have $H_f = (1/2)H_{\bar{\varphi}}$. For each $u \in \Gamma_t$, $0 < t < \tau$, let us denote

$$G_u = H_f k_{w(u)} = (1/2)H_{\bar{\varphi}} k_{w(u)}.$$

By the reproducing property of $k_{w(u)}$, we have $(\bar{\varphi} - \bar{\varphi}(w(u)))k_{w(u)} \perp H^2(S)$. Hence

$$G_u = (1/2)(\bar{\varphi} - \bar{\varphi}(w(u)))k_{w(u)} \quad \text{for every} \quad u \in \Gamma_t. \tag{6.12}$$

By Lemma 6.7, we also have

$$\|G_u\| \geq c_1 t \quad \text{for every} \quad u \in \Gamma_t. \tag{6.13}$$

Mimicking the practice of normalizing the reproducing kernel, we further define

$$g_u = G_u/\|G_u\| \tag{6.14}$$

for $u \in \Gamma_t$, $0 < t < \tau$.

Lemma 6.9. *Given any $\epsilon > 0$, there is a constant $0 < C_{6.9} = C_{6.9}(\epsilon) < \infty$ such that if $u, v \in \Gamma_t$, $0 < t < \tau$, and $u \neq v$, then*

$$|\langle g_u, g_v \rangle| \leq C_{6.9} \frac{t^{2n-2-\epsilon}}{\{d(u,v)\}^{2n}}.$$

Proof. By (6.12) and Lemma 6.8, for all $u \neq v$ in Γ_t, $0 < t < \tau$, we have

$$|\langle G_u, G_v \rangle| \leq \|\varphi\|_\infty^2 \langle |k_{w(u)}|, |k_{w(v)}| \rangle \leq \frac{\|\varphi\|_\infty^2 C_{6.8} t^{2n-\epsilon}}{|1 - \langle w(u), w(v) \rangle|^{n-(\epsilon/2)}} \leq \frac{2^n \|\varphi\|_\infty^2 C_{6.8} t^{2n-\epsilon}}{|1 - \langle u, v \rangle|^n},$$

where the last $\leq$ uses the elementary inequality given at the beginning of Section 2 in [5]. Combining the above inequality with (6.13), the lemma is proved. □

Lemma 6.10. [20, Lemma 4.1] *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\text{card}\{y \in X : (x,y) \in E\} \leq m \quad \text{and} \quad \text{card}\{y \in X : (y,x) \in E\} \leq m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup \ldots \cup E_{2m}$$

and such that for each $1 \leq j \leq 2m$, the conditions $(x,y), (x',y') \in E_j$ and $(x,y) \neq (x',y')$ imply both $x \neq x'$ and $y \neq y'$.

For each $0 < t < \tau$, we define the finite-rank operator

$$F_t = \sum_{u \in \Gamma_t} g_u \otimes g_u.$$

Lemma 6.11. *Under the condition $n \geq 3$, there is a constant $0 < C_{6.11} < \infty$ such that $\|F_t\| \leq C_{6.11}$ for every $0 < t < \tau$.*

Proof. To estimate $\|F_t\|$, we pick an orthonormal set $\{e_u : u \in \Gamma_t\}$ and factor F_t in the form $F_t = A^*A$, where

$$A = \sum_{u\in\Gamma_t} e_u \otimes g_u.$$

Since $\|A^*A\| = \|AA^*\|$, it suffices to estimate the latter. By (6.10), we have

$$AA^* = \sum_{u,v\in\Gamma_t} \langle g_u, g_v\rangle e_v \otimes e_u = \sum_{u\in\Gamma_t} e_u \otimes e_u + \sum_{k=1}^{\infty} B_k, \tag{6.15}$$

where

$$B_k = \sum_{u,v\in E_k} \langle g_u, g_v\rangle e_v \otimes e_u \quad \text{and}$$

$$E_k = \{(u,v) \in \Gamma_t \times \Gamma_t : 2^{k-1}t^{1/2} \le d(u,v) < 2^k t^{1/2}\}$$

for each $k \in \mathbf{N}$. It follows from (6.5) and (6.10) that there is an $N \in \mathbf{N}$ such that

$$\text{card}\{v \in \Gamma_t : d(u,v) < 2^k t^{1/2}\} \le N2^{2nk}$$

for $u \in \Gamma_t$, $0 < t < \tau$, and $k \in \mathbf{N}$. By Lemma 6.10, this means that for each $k \in \mathbf{N}$, there is a partition

$$E_k = E_k^{(1)} \cup E_k^{(2)} \cup \cdots \cup E_k^{(2N2^{2nk})}$$

such that for each $1 \le j \le 2N2^{2nk}$, the conditions $(u,v), (u',v') \in E_k^{(j)}$ and $(u,v) \ne (u',v')$ imply both $u \ne u'$ and $v \ne v'$. Accordingly, we have

$$B_k = B_k^{(1)} + B_k^{(2)} + \cdots + B_k^{(2N2^{2nk})},$$

where

$$B_k^{(j)} = \sum_{u,v\in E_k^{(j)}} \langle g_u, g_v\rangle e_v \otimes e_u$$

for each $1 \le j \le 2N2^{2nk}$. For any non-empty $E_k^{(j)}$, it follows from above-mentioned property of $E_k^{(j)}$ that

$$\|B_k^{(j)}\| = \max\{|\langle g_u, g_v\rangle| : (u,v) \in E_k^{(j)}\}.$$

Applying Lemma 6.9 with $\epsilon = 1/2$ and recalling the definition of E_k, we have

$$\|B_k^{(j)}\| \le C_{6.9} \frac{t^{2n-2-(1/2)}}{\{2^{k-1}t^{1/2}\}^{2n}} = C_1 2^{-2nk} t^{n-2-(1/2)}.$$

Writing $C_2 = 2NC_1$ and using the condition $n \geq 3$, we now have $\|B_k\| \leq C_2 t^{1/2}$ for every k. Then note that $B_k = 0$ for every k such that $E_k = \emptyset$. Thus if $k(t)$ is the smallest natural number such that $2^{k(t)-1}t^{1/2} > 2$, then $\|B_k\| = 0$ for every $k \geq k(t)$. Hence

$$\sum_{k=1}^{\infty} \|B_k\| \leq C_2 t^{1/2} k(t) \leq C_3 t^{1/2}\{1 + \log(1/t)\} \leq C_4.$$

Recalling (6.15), we now have $\|F_t\| = \|AA^*\| \leq 1 + C_4$. This completes the proof. □

Let $\psi \in C_c(\mathbf{R})$ be a function satisfying the following three conditions:
(i) $0 \leq \psi \leq 1$ on $\mathbf{R}$.
(ii) $\psi = 1$ on $[a + c, b - c]$, where c is the same as in Lemma 6.5.
(iii) $\psi = 0$ on $\mathbf{R} \backslash (a, b)$.
Thus $\psi^2(x) \leq \chi_{(a,b)}(x)$ for every $x \in \mathbf{R}$. Accordingly, we have the operator inequality

$$\{\psi(T_f)\}^2 \leq \mathcal{E}(a, b) \tag{6.16}$$

on $H^2(S)$, where, as we recall, $\mathcal{E}$ is the spectral measure for T_f. Since f is continuous, it is well known that

$$\psi(T_f) = T_{\psi\circ f} + K, \tag{6.17}$$

where the operator K is compact and self-adjoint.

For $u \in \Gamma_t$, $0 < t < \tau$, we write

$$h_u = H_f^* g_u.$$

Recalling the definitions of g_u and G_u, we have

$$\|h_u\| \geq |\langle h_u, k_{w(u)}\rangle| = |\langle H_f^* H_f k_{w(u)}, k_{w(u)}\rangle| / \|G_u\| = \|H_f k_{w(u)}\|. \tag{6.18}$$

Lemma 6.12. *There is a* $0 < \tau_0 < \tau$ *such that if* $0 < t < \tau_0$ *and* $u \in \Gamma_t$, *then*

$$\|T_{\psi\circ f} h_u\| \geq (2/3)\|h_u\|.$$

Proof. Define the sets

$$\begin{aligned} A_3 &= \{\zeta \in S : a + 3c \leq f(\zeta) \leq b - 3c\}, \\ A_2 &= \{\zeta \in S : a + 2c < f(\zeta) < b - 2c\} \quad \text{and} \\ B &= \{\zeta \in S : f(\zeta) \notin (a + c, b - c)\}. \end{aligned}$$

It is easy to see that there is a $d_0 > 0$ such that $d(\zeta, \xi) \geq d_0$ if $\zeta \in A_3$ and $\xi \in S \backslash A_2$, and such that $d(\zeta, \xi) \geq d_0$ if $\zeta \in A_2$ and $\xi \in B$.

By the choice of ψ, we have $\psi \circ f = 1$ on $S \backslash B$. Combining this with the fact that $0 \leq \psi \circ f \leq 1$ on S, we have

$$\|T_{\psi \circ f} h_u\| = \|h_u - T_{1-\psi \circ f} h_u\| \geq \|h_u\| - \|\chi_B h_u\|.$$

Thus it suffices to find a $0 < \tau_0 < \tau$ such that

$$\|\chi_B h_u\| \leq (1/3)\|h_u\| \tag{6.19}$$

when $u \in \Gamma_t$ and $0 < t < \tau_0$.

To find such a τ_0, we begin with (6.12) and (6.14), which give us

$$h_u = (2\|G_u\|)^{-1} P M_f M_{\bar{\varphi}-\bar{\varphi}(w(u))} k_{w(u)},$$

$u \in \Gamma_t$ and $0 < t < \tau$. Thus

$$\chi_B h_u = (2\|G_u\|)^{-1}\{p_u + q_u\}, \tag{6.20}$$

where

$$\begin{aligned} p_u &= M_{\chi_B} P M_{\chi_{A_2}} M_{(\bar{\varphi}-\bar{\varphi}(w(u)))f} k_{w(u)} \quad \text{and} \\ q_u &= M_{\chi_B} P M_{\chi_{S \backslash A_2}} M_{(\bar{\varphi}-\bar{\varphi}(w(u)))f} k_{w(u)}, \end{aligned}$$

$u \in \Gamma_t$ and $0 < t < \tau$. We estimate the norms of p_u and q_u separately.

To estimate $\|p_u\|$, denote $X = M_{\chi_B} P M_{\chi_{A_2}}$. Then

$$(Xg)(\xi) = \int \frac{\chi_B(\xi)\chi_{A_2}(\zeta)}{(1 - \langle \xi, \zeta \rangle)^n} g(\zeta) d\sigma(\zeta)$$

for $g \in L^2(S, d\sigma)$. As we mentioned before, $d(\zeta, \xi) \geq d_0$ if $\zeta \in A_2$ and $\xi \in B$. Therefore

$$\begin{aligned} |p_u(\xi)| &= |(X M_{(\bar{\varphi}-\bar{\varphi}(w(u)))f} k_{w(u)})(\xi)| \\ &\leq 2\|\varphi\|_\infty \|f\|_\infty d_0^{-2n} \int |k_{w(u)}(\zeta)| d\sigma(\zeta) = 2\|\varphi\|_\infty \|f\|_\infty d_0^{-2n} \int \frac{t^n}{|1 - \langle \zeta, w(u) \rangle|^n} d\sigma(\zeta) \\ &\leq 2\|\varphi\|_\infty \|f\|_\infty d_0^{-2n} \frac{C_1 t^n}{(1 - |w(u)|^2)^{1/4}} = C_2 t^{n-(1/2)} \end{aligned}$$

for every $\xi \in B$, where the second $\leq$ is an application of [15, Proposition 1.4.10]. Also, $p_u = 0$ on $S \backslash B$. Hence $\|p_u\| \leq C_2 t^{n-(1/2)}$ for $u \in \Gamma_t$, $0 < t < \tau$.

For q_u, note that

$$\|q_u\| \leq 2\|\varphi\|_\infty \|f\|_\infty \|\chi_{S \backslash A_2} k_{w(u)}\|.$$

Recall that $\Gamma_t \subset B(y, \rho_0) \subset B(y, \rho)$ for every $0 < t < \tau$. By Lemma 6.5, we have $\Gamma_t \subset A_3$ for every $0 < t < \tau$. Thus if $u \in \Gamma_t$ for some $0 < t < \tau$ and $\xi \in S \backslash A_2$, then $d(\xi, u) \geq d_0$, and consequently

$$|k_{w(u)}(\xi)| = \frac{t^n}{|1 - \langle \xi, w(u) \rangle|^n} \leq \frac{2^n t^n}{|1 - \langle \xi, u \rangle|^n} \leq 2^n d_0^{-2n} t^n.$$

Therefore if we set $C_3 = 2^{n+1} \|\varphi\|_\infty \|f\|_\infty d_0^{-2n}$, then $\|q_u\| \leq C_3 t^n$ for $u \in \Gamma_t$, $0 < t < \tau$.

Combining the conclusions of the last two paragraphs with (6.20), we find that

$$\|\chi_B h_u\| \leq (2\|G_u\|)^{-1} \{C_2 t^{n-(1/2)} + C_3 t^n\}$$

if $u \in \Gamma_t$ and $0 < t < \tau$. Recalling (6.13), we now have

$$\|\chi_B h_u\| \leq (2c_1)^{-1} \{C_2 t^{n-(3/2)} + C_3 t^{n-1}\} \tag{6.21}$$

if $u \in \Gamma_t$ and $0 < t < \tau$. On the other hand, it follows from (6.18) and Lemma 6.7 that

$$\|h_u\| \geq c_1 t \tag{6.22}$$

if $u \in \Gamma_t$ and $0 < t < \tau$. By the condition $n \geq 3$, there is a $0 < \tau_0 < \tau$ such that

$$(2c_1)^{-1} \{C_2 \tau_0^{n-(5/2)} + C_3 \tau_0^{n-2}\} \leq c_1/3. \tag{6.23}$$

From (6.21), (6.22) and (6.23) we see that (6.19) holds when $u \in \Gamma_t$ and $0 < t < \tau_0$. This completes the proof. □

We are now ready to prove the main technical result of the section.

Proof of Proposition 6.1. Our goal is to show that

$$\lim_{t \downarrow 0} \operatorname{tr}(H_f \mathcal{E}(a, b) H_f^* F_t) = \infty. \tag{6.24}$$

Since Lemma 6.11 tells us that $\|F_t\| \leq C_{6.11}$ for every $0 < t < \tau$, the above limit means that $H_f \mathcal{E}(a, b) H_f^*$ is not in the trace class.

To prove (6.24), we decompose the compact, self-adjoint operator K in (6.17) in the form $K = K_1 + K_2$, where K_1 and K_2 are self-adjoint, $\operatorname{rank}(K_1) < \infty$, and K_2 has the property $\|K_2\| \leq 1/3$. Applying Lemma 6.12, if $0 < t < \tau_0$, then we have

$$\|(T_{\psi \circ f} + K_2) h_u\| \geq (1/3) \|h_u\| \quad \text{for every} \quad u \in \Gamma_t. \tag{6.25}$$

By (6.17) and the property $\operatorname{rank}(K_1) < \infty$, we have

$$\{\psi(T_f)\}^2 = (T_{\psi \circ f} + K)^2 = (T_{\psi \circ f} + K_2)^2 + Y,$$

where Y is a finite-rank operator. Taking (6.16) into account, we have

$$H_f\mathcal{E}(a,b)H_f^* \geq H_f\{\psi(T_f)\}^2H_f^* = H_f(T_{\psi\circ f}+K_2)^2H_f^* + H_fYH_f^*. \tag{6.26}$$

Let us consider the two terms on the right-hand side.

For each $0 < t < \tau_0$, we have

$$\begin{aligned}\operatorname{tr}(H_f(T_{\psi\circ f}+K_2)^2H_f^*F_t) &= \sum_{u\in\Gamma_t}\langle H_f(T_{\psi\circ f}+K_2)^2H_f^*g_u, g_u\rangle = \sum_{u\in\Gamma_t}\|(T_{\psi\circ f}+K_2)H_f^*g_u\|^2 \\ &= \sum_{u\in\Gamma_t}\|(T_{\psi\circ f}+K_2)h_u\|^2 \geq \frac{1}{9}\sum_{u\in\Gamma_t}\|h_u\|^2,\end{aligned}$$

where the $\geq$ follows from (6.25). Recalling (6.18) and Lemma 6.7, we now have

$$\operatorname{tr}(H_f(T_{\psi\circ f}+K_2)^2H_f^*F_t) \geq (c_1/3)^2\operatorname{card}(\Gamma_t)t^2 \tag{6.27}$$

for $0 < t < \tau_0$. By (6.11) and (6.5), we have

$$\sigma(B(y,\rho_0)) \leq \sum_{u\in\Gamma_t}\sigma(B(u,2t^{1/2})) \leq A_0\operatorname{card}(\Gamma_t)(2t^{1/2})^{2n}.$$

That is, $\operatorname{card}(\Gamma_t) \geq 2^{-2n}A_0^{-1}\sigma(B(y,\rho_0))t^{-n}$. Substituting this in (6.27), we find that

$$\operatorname{tr}(H_f(T_{\psi\circ f}+K_2)^2H_f^*F_t) \geq (c_1/3)^2 2^{-2n}A_0^{-1}\sigma(B(y,\rho_0))t^{2-n} \tag{6.28}$$

for $0 < t < \tau_0$.

On the other hand, by Lemma 6.11, we have

$$|\operatorname{tr}(H_fYH_f^*F_t)| \leq \|H_fYH_f^*\|_1\|F_t\| \leq C_{6.11}\|H_fYH_f^*\|_1 \tag{6.29}$$

for every $0 < t < \tau$, where $\|\cdot\|_1$ denotes the norm of the trace class. Since $\operatorname{rank}(Y) < \infty$, we have $\|H_fYH_f^*\|_1 < \infty$. Thus the desired conclusion (6.24) follows from (6.26), (6.28), (6.29) and the condition $n \geq 3$. This proves Proposition 6.1. □

Proof of Theorem 1.6. Define $Z = H_f\mathcal{E}(a,b)$. Proposition 6.1 tells us that the operator

$$ZZ^* = H_f\mathcal{E}(a,b)H_f^*$$

is not in the trace class. Since ZZ^* and Z^*Z have identical s-numbers, the operator

$$Z^*Z = \mathcal{E}(a,b)H_f^*H_f\mathcal{E}(a,b)$$

is not in the trace class.

By Lemma 6.2, there is a $\varphi \in H^\infty(S)$ such that $\varphi = f + ig$, where g is real valued. As in the proof of Theorem 1.4, consider the self-adjoint operators

$$A = T_f \quad \text{and} \quad B = T_g.$$

We know from Theorem 1.2 that A is purely absolutely continuous. Moreover,

$$[A, B] = -(i/2)[T_{\bar{\varphi}}, T_\varphi] = -(i/2)H^*_{\bar{\varphi}}H_{\bar{\varphi}} = -2iH^*_f H_f.$$

This gives us $i[A, B] \geq 0$ as in the proof of Theorem 1.4. Furthermore,

$$\mathcal{E}(a, b)[A, B]\mathcal{E}(a, b) = -2i\mathcal{E}(a, b)H^*_f H_f \mathcal{E}(a, b),$$

which, as we showed above, is not in the trace class. Thus it follows from Proposition 5.3 that

$$\int_a^b m_{T_f}(x)dx = \int_a^b m_A(x)dx = \infty.$$

This completes the proof of Theorem 1.6. □

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