FOCK SPACE: A BRIDGE BETWEEN FREDHOLM INDEX AND THE QUANTUM HALL EFFECT

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Abstract. We compute the quantized Hall conductance at various Landau levels by using the classic trace. The computations reduce to the single elementary one for the lowest Landau level. By using the theories of Helton-Howe-Carey-Pincus, and Toeplitz operators on the classic Fock space and higher Fock spaces, the Hall conductance is naturally identified with a Fredholm index. This brings new mathematical insights to the extraordinary precision of quantization observed in quantum Hall measurements.

1. Introduction

Consider an electron confined to a plane with a perpendicular magnetic field $\mathbf{B}$ of uniform strength. Under the right choice of orientation, identify the plane with $\mathbf{C}$, the complex plane. In the symmetric gauge, we have the free Hamiltonian

$$H_b = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2$$

representing this system, where $b = e|\mathbf{B}|/\hbar c > 0$. When the Fermi energy E is in a spectral gap of H_b, we have the Fermi projection $P_{\leq E} = \chi_{(-\infty,E]}(H_b)$. Let f_1 and f_2 be *switch functions* in the x and y directions respectively (see Definition 3.5 below). It is known that the expression

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -i\mathrm{tr}(P_{\leq E}[[f_1, P_{\leq E}], [f_2, P_{\leq E}]]) \tag{1.1}$$

is the Kubo formula for the Hall conductance of $P_{\leq E}$, provided that the trace on the right-hand side makes sense. See [7,1].

The spectrum of H_b is well known. In fact, an easy diagonalization of H_b shows that its spectrum consists of the eigenvalues $\{(2\ell + 1)b : \ell = 0, 1, 2, \dots\}$, called Landau levels, each with infinite degeneracy. In the complex coordinates, it is easy to verify that

$$H_b - b = 4(-\partial + (b/4)\bar{z})(\bar{\partial} + (b/4)z),$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$. From this we see that the eigenspace corresponding to the lowest Landau level (LLL) is the closure in $L^2(\mathbf{C})$ of

$$\mathrm{span}\{z^k e^{-(b/4)|z|^2} : k = 0, 1, 2, \dots\}. \tag{1.2}$$

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One immediately recognizes that this is a unitarily equivalent form of the familiar Fock space. Thus in the study of the quantum Hall effect, the wealth of knowledge that has been accumulated in the study of operators on the Fock space can be brought to bear.

Define the measure

$$d\mu(z) = \frac{1}{\pi}e^{-|z|^2}dA(z)$$

on $\mathbf{C}$, where dA is the standard area measure. Recall that the Fock space $\mathcal{F}^2$ is the closure of the analytic polynomials $\mathbf{C}[z]$ in $L^2(\mathbf{C}, d\mu)$. As usual, we write P for the orthogonal projection from $L^2(\mathbf{C}, d\mu)$ onto $\mathcal{F}^2$.

Define

$$(U_b\psi)(z) = (2/b)^{1/2}\psi((2/b)^{1/2}z)e^{|z|^2/2}, \quad \psi \in L^2(\mathbf{C}, dA).$$

Then U_b is a unitary operator that maps $L^2(\mathbf{C}, dA)$ onto $L^2(\mathbf{C}, d\mu)$. Let $\mathcal{L}$ be the eigenspace for H_b corresponding to the eigenvalue b. By (1.2), we have

$$U_b\mathcal{L} = \mathcal{F}^2.$$

Moreover,

$$U_bP_{\le E}U_b^* = P \quad \text{when} \quad b < E < 3b.$$

Combining this with (1.1), we see that when $b < E < 3b$,

$$\sigma_{\mathrm{Hall}}(P_{\le E}) = -i\mathrm{tr}(P[[M_{\hat{f}_1}, P], [M_{\hat{f}_2}, P]]), \tag{1.3}$$

where $\hat{f}_i(z) = f_i((2/b)^{1/2}z)$, and $M_{\hat{f}_i}$ is the operator of multiplication by the function $\hat{f}_i$ on $L^2(\mathbf{C}, d\mu)$, $i = 1, 2$. This realizes the Hall conductance of the lowest Landau level as a trace on the Fock space. We can further rewrite the right-hand side in terms of Toeplitz operators.

Recall that for each $f \in L^\infty(\mathbf{C})$, the Toeplitz operator T_f is defined by the formula

$$T_fh = P(fh), \quad h \in \mathcal{F}^2.$$

For any $f, g \in L^\infty(\mathbf{C})$, we have

$$\begin{aligned}[T_f, T_g] &= PM_fPM_gP - PM_gPM_fP \\ &= P(M_fP - PM_f)(M_gP - PM_g) - P(M_gP - PM_g)(M_fP - PM_f) \\ &= P[[M_f, P], [M_g, P]]. \end{aligned} \tag{1.4}$$

Therefore, renaming $\hat{f}_1$, $\hat{f}_2$ as f_1, f_2, when $b < E < 3b$, (1.3) becomes

$$\sigma_{\mathrm{Hall}}(P_{\le E}) = -i\mathrm{tr}[T_{f_1}, T_{f_2}]. \tag{1.5}$$

In an earlier work [11], the first author showed, by an explicit calculation, that

$$\mathrm{tr}[T_{f_1}, T_{f_2}] = \mathrm{tr}(P[[M_{f_1}, P], [M_{f_2}, P]]) = \frac{1}{2\pi i}, \tag{1.6}$$

independently of the choice of switch functions. This obtains the value $-1/2\pi$ for the Hall conductance $\sigma_{\text{Hall}}(P_{\leq E})$, which gives a mathematical proof to what was a folklore in physics.

In the decades since the initial discovery of the quantum Hall effect, repeated experiments have seen an extraordinary stability for the observed integer values of $2\pi\sigma_{\text{Hall}}$. This phenomenon calls for an explanation in terms of a mathematical quantity that is stable under small perturbations and quantized to integer values. In this regard, the Fredholm index is an obvious candidate.

There were previous attempts along this line, one of the earliest being [1]. In Theorem 6.8 of that paper, Avron, Seiler and Simon used their theory of *relative index of projections* to show that

$$2\pi i \operatorname{Tr}[T_{f_1}, T_{f_2}] = -\operatorname{index}(T_{z/|z|}) \tag{1.7}$$

and checked that the right-hand side was 1. But in [1], the left-hand side of (1.7) was treated as a limit of truncated traces, not the classic trace. (See the comment after [1,(6.6)].) This leaves room for improvement. For example, if the commutator $[T_{f_1}, T_{f_2}]$ is known to be in the trace class, we can directly deduce its stability with respect to, e.g., trace class perturbations of P, without using the right-hand side.

Improvements started with [9], in which Ludewig and the first author showed, using general abstract methods, that the commutator $[T_{f_1}, T_{f_2}]$ is indeed in the trace class. (Later in Proposition 3.7, we will verify this trace-class condition more directly.) Then in [11], the first author did the trace calculation (1.6).

In this paper we take the next step. Our starting point is (1.5). Recall that the values of the switch functions f_1 and f_2 are contained in $[0, 1]$. The fact that $[T_{f_1}, T_{f_2}]$ is in the trace class means that the theories of Helton-Howe [8] and Carey-Pincus [3,4,10] come into play. Define $F = f_1 + if_2$. Using a new technique for constructing Fredholm inverses, we will show in Theorem 4.1 that the essential spectrum of the Toeplitz operator T_F is the boundary ∂S of the square

$$S = \{x + iy : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}.$$

Knowing that the two-dimensional Lebesgue measure of this essential spectrum is 0, we deduce from the Carey-Pincus theory that the *principal function* g for the almost commuting pair T_{f_1}, T_{f_2} is given by the formula

$$g(x, y) = n\chi_S(x + iy),$$

where

$$n = \operatorname{index}(T_F - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

The Carey-Pincus theory also tells us that

$$\operatorname{tr}[T_{f_1}, T_{f_2}] = \frac{-1}{2\pi i} \iint g(x, y)\,dxdy = \frac{-n}{2\pi i}. \tag{1.8}$$

Combining this with (1.5), we find that when $b < E < 3b$,

$$\sigma_{\mathrm{Hall}}(P_{\le E}) = \frac{1}{2\pi}\mathrm{index}(T_F - \lambda) \quad \text{for every} \quad \lambda \in S\backslash\partial S. \tag{1.9}$$

This identifies the Hall conductance with a bona fide Fredholm index in a direct way, distinct from the Avron-Seiler-Simon approach leading to (1.7). Moreover, (1.8) turns (1.6), a trace calculation, into an index calculation:

$$\mathrm{index}(T_F - \lambda) = -1 \quad \text{for every} \quad \lambda \in S\backslash\partial S.$$

Consequently, $g(x,y) = -\chi_S(x+iy)$. Thus the Carey-Pincus theory gives us the formula

$$\mathrm{tr}[p(T_{f_1}, T_{f_2}), q(T_{f_1}, T_{f_2})] = \frac{1}{2\pi i}\iint_S \{p,q\}(x,y)dxdy \tag{1.10}$$

for $p, q \in \mathbf{C}[x,y]$, where $\{p,q\}$ is the Poisson bracket of the pair p, q. In other words, the one trace calculation (1.6), plus the determination of the essential spectrum of T_F, plus the Carey-Pincus theory, give us all the explicit traces above.

It is worth stressing that even something like $\mathrm{tr}([T_{f_1}, T_{f_2}]T_{f_2})$ would be impossible to calculate without the Carey-Pincus theory. But with the Carey-Pincus theory, not only do we know that $\mathrm{tr}([T_{f_1}, T_{f_2}]T_{f_2}) = (4\pi i)^{-1}$, every trace in (1.10) is at our finger tips.

As the main part of our paper, we will prove the analogues of (1.6), (1.9) and (1.10) when E is between *arbitrary* Landau levels $(2\ell+1)b$ and $(2\ell+3)b$, $\ell \ge 0$. We emphasize that our aim is to determine, for the general case of $\ell \ge 0$, the full principal function, which is a lot more than the calculation of an individual trace or index.

After establishing the general version of (1.9), we will take a look at another pair of functions, inspired by (1.7). We consider

$$\varphi_1(z) = \mathrm{Re}(z/|z|) \quad \text{and} \quad \varphi_2(z) = \mathrm{Im}(z/|z|),$$

$z \in \mathbf{C}\backslash\{0\}$, which are not the kind of switch functions suitable for the Kubo formula, but are still worth investigating on mathematical grounds. Define $\Phi = \varphi_1 + i\varphi_2$. That is,

$$\Phi(z) = z/|z|$$

for $z \in \mathbf{C}\backslash\{0\}$. We will show that the Toeplitz operator T_Φ on $\mathcal{F}^2$ is a compact perturbation of the *unilateral shift*. Consequently, the essential spectrum of T_Φ equals the unit circle $\mathbf{T}$. We will also show that $[T_{\varphi_1}, T_{\varphi_2}]$ is in the trace class, and that $\mathrm{tr}[T_{\varphi_1}, T_{\varphi_2}] = (2i)^{-1}$. This leads to the identity

$$\mathrm{tr}(P[[M_{\varphi_1}, P], [M_{\varphi_2}, P]]) = -\frac{1}{2i}\mathrm{index}(T_\Phi - \lambda) \quad \text{for every} \quad \lambda \in D,$$

where $D = \{z \in \mathbf{C} : |z| < 1\}$. Furthermore, we have

$$\mathrm{tr}[p(T_{\varphi_1}, T_{\varphi_2}), q(T_{\varphi_1}, T_{\varphi_2})] = \frac{1}{2\pi i} \iint_D \{p, q\}(x, y) dx dy$$

for all $p, q \in \mathbf{C}[x, y]$.

After that, we will work out a "non-Toeplitz" example in Section 10. In addition to the value of the example itself, this goes some way to demonstrate that our techniques are quite general.

It should be mentioned that in addition to (1.3), there are at least two other versions of "Kubo formula for Hall conductance" studied in the literature. If one uses "trace-per-unit-area",

$$\mathcal{T}(A) = \lim_{\Lambda \to \mathbf{C}} \frac{1}{|\Lambda|} \mathrm{tr}(\chi_\Lambda A \chi_\Lambda),$$

then the quantity

$$\sigma_{\mathrm{Hall}}^{(2)} = -i\mathcal{T}(P[[\hat{x}, P], [\hat{y}, P]]),$$

where $\hat{x}$ and $\hat{y}$ are the coordinate functions in the x and y directions, is a second version of Hall conductance.

The quantity $\sigma_{\mathrm{Hall}}^{(2)}$ fits very nicely in the context of non-commuative geometry [2]. Thus the study of $\sigma_{\mathrm{Hall}}^{(2)}$ benefits from the tremendous progress that has been made in non-commuative geometry in the last few decades. The equality of σ_{Hall} and $\sigma_{\mathrm{Hall}}^{(2)}$ can be deduced once the trace class membership needed for the former is established. See [6, Appendix Lemma 8] for such arguments in the context of discrete models. Otherwise, if only the limit of truncated traces is used in defining σ_{Hall}, then the magnetic translation invariance of H_b, thus of P, needs to be invoked to obtain equality with $\sigma_{\mathrm{Hall}}^{(2)}$, as explained on page 416 of [1].

For a third version, consider a Hamiltonian H that is periodic with respect to the lattice $\mathbf{Z}^2$ in $\mathbf{R}^2$. Under additional assumptions, P decomposes in the form $\int_{\mathbf{T}^2}^{\oplus} P(k) dm(k)$, where $\mathbf{T}^2$ is the dual Brillouin torus to $\mathbf{Z}^2$, and each $P(k)$ is a finite-rank projection matrix depending smoothly on $k \in \mathbf{T}^2$. That is, P is a map from $\mathbf{T}^2$ to $M_{n \times n}$. Then a third version of Hall conductance is given by the formula

$$\sigma_{\mathrm{Hall}}^{(3)} = -i \int_{\mathbf{T}^2} \mathrm{tr}(P[\partial_{k_1} P, \partial_{k_2} P]) dm(k),$$

which may be recognized as a Chern number of the vector bundle over $\mathbf{T}^2$ formed by the ranges of $P(k)$.

We emphasize that the focus of this paper is on the Hall conductance given by the classic trace, as in (1.1). Operator theory is our main tool for analyzing (1.3) and its analogue for higher Landau levels. Like the field of non-commutative geometry, tremendous progress has also been made in operator theory on reproducing-kernel Hilbert spaces. This paper benefits from this progress in the sense that our results are really applications of this particular form of operator theory.

2. Landau levels and eigenspaces

As we already mentioned in the Introduction, the magnetic Hamiltonian

$$H_b = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2$$

admits the factorization

$$H_b - b = 4(-\partial + (b/4)\bar{z})(\bar{\partial} + (b/4)z).$$

Let us denote

$$\tilde{A} = 2\bar{\partial} + (b/2)z \quad \text{and} \quad \tilde{C} = -2\partial + (b/2)\bar{z}.$$

Then

$$H_b - b = \tilde{C}\tilde{A}.$$

This pair of operators $\tilde{A}$, $\tilde{C}$ are called annihilation and creation operators, and gives us an explicit diagonalization of H_b. Indeed this diagonalization is a standard exercise using the canonical commutation relation (CCR).

We begin with the obvious commutation relation $[\tilde{A}, \tilde{C}] = 2b$. Define

$$\mathcal{S}_0 = \operatorname{span}\{z^k e^{-(b/4)|z|^2} : k = 0, 1, 2, \dots\}.$$

Then $\tilde{A}\mathcal{S}_0 = \{0\}$. For each $j \in \mathbf{N}$, define

$$\mathcal{S}_j = \tilde{C}^j \mathcal{S}_0.$$

By the "product rule" for commutators, the relation $[\tilde{A}, \tilde{C}] = 2b$ implies $[\tilde{A}, \tilde{C}^j] = 2bj\tilde{C}^{j-1}$ for $j \geq 1$. Let $\varphi \in \mathcal{S}_j$ for some $j \geq 0$. Then there is a $\psi \in \mathcal{S}_0$ such that $\varphi = \tilde{C}^j\psi$. Therefore

$$(H_b - b)\varphi = \tilde{C}\tilde{A}\varphi = \tilde{C}\tilde{A}\tilde{C}^j\psi = \tilde{C}[\tilde{A}, \tilde{C}^j]\psi = 2bj\tilde{C}^j\psi = 2bj\varphi.$$

Hence

$$\mathcal{S}_j \subset \ker(H_b - (2j+1)b) \tag{2.1}$$

for every $j \geq 0$. Since H_b is self-adjoint, this in particular means that $\mathcal{S}_i \perp \mathcal{S}_j$ for $i \neq j$.

For each $j \geq 0$, let $\mathcal{E}_j$ be the closure of $\mathcal{S}_j$ in $L^2(\mathbf{C}, dA)$.

Recall that the formula

$$(U_b\psi)(z) = (2/b)^{1/2}\psi((2/b)^{1/2}z)e^{|z|^2/2}, \quad \psi \in L^2(\mathbf{C}, dA),$$

defines a unitary operator that maps $L^2(\mathbf{C}, dA)$ onto $L^2(\mathbf{C}, d\mu)$. For each $j \geq 0$, define

$$\mathcal{F}_j = U_b \mathcal{E}_j. \tag{2.2}$$

Obviously, we have $\mathcal{F}_0 = \mathcal{F}^2$, which is the classic Fock space. We will see that the spaces $\mathcal{F}_j$, $j \geq 1$, are in fact *higher Fock spaces.*

Let us now denote

$$A = \bar{\partial} \quad \text{and} \quad C = -\partial + \bar{z},$$

which satisfy the commutation relation $[A, C] = 1$. It is easy to verify that $\langle Cu, v\rangle = \langle u, Av\rangle$ for all $u, v \in \mathbf{C}[z, \bar{z}]$. That is, over the dense subspace $\mathbf{C}[z, \bar{z}]$ of $L^2(\mathbf{C}, d\mu)$, C and A are each-other's adjoint. It is also easy to verify that

$$U_b \tilde{A} U_b^* = \sqrt{2b} A \quad \text{and} \quad U_b \tilde{C} U_b^* = \sqrt{2b} C.$$

Consequently,

$$U_b H_b U_b^* - b = 2bCA.$$

It is easy to see that $U_b \mathcal{S}_0 = \mathbf{C}[z]$ and that $U_b \mathcal{S}_j = C^j \mathbf{C}[z]$ for every $j \geq 1$. Thus by an induction on the power of $\bar{z}$, we have

$$U_b \mathrm{span}\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_k, \dots\} = \mathbf{C}[z, \bar{z}]. \tag{2.3}$$

This shows that $\mathrm{span}\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_k, \dots\}$ is dense in $L^2(\mathbf{C}, dA)$. For each $j \geq 0$, let $E_j : L^2(\mathbf{C}, dA) \to \mathcal{E}_j$ be the orthogonal projection. Combining (2.1) with (2.3), we obtain

$$H_b = \sum_{j=0}^{\infty} (2j+1) b E_j. \tag{2.4}$$

This is an explicit diagonalization of the magnetic Hamiltonian H_b.

To summarize, the spectrum of H_b comprises evenly-spaced Landau levels $(2j+1)b$, $j \geq 0$. The j-th Landau level eigenspace $\mathcal{E}_j$ is identified with the j-th higher Fock space $\mathcal{F}_j$ under the unitary U_b.

For each $j \geq 0$, let $P_j : L^2(\mathbf{C}, d\mu) \to \mathcal{F}_j$ be the orthogonal projection. Then, of course, $P_0 = P$. It follows from (2.2) that

$$P_j = U_b E_j U_b^* \quad \text{for every} \ \ j \geq 0. \tag{2.5}$$

For each $j \geq 0$ and $p \in \mathbf{C}[z]$, we have

$$\|C^j p\|^2 = \langle C^j p, C^j p\rangle = \langle A^j C^j p, p\rangle = j!\langle p, p\rangle = j!\|p\|^2.$$

Hence, for a given $j \geq 0$, if we define

$$V_j u = \frac{1}{\sqrt{j!}} C^j P u \quad \text{for} \ \ u \in \mathbf{C}[z, \bar{z}], \tag{2.6}$$

then V_j extends to a partial isometry on $L^2(\mathbf{C}, d\mu)$ such that

$$V_j^* V_j = P = P_0 \quad \text{and} \quad V_j V_j^* = P_j. \tag{2.7}$$

Inspired by the classic Toeplitz operators on $\mathcal{F}^2$, on each higher Fock space $\mathcal{F}_j$, $j \geq 1$, we define the "Toeplitz operator"

$$T_{f,j} h = P_j(fh), \quad h \in \mathcal{F}_j. \tag{2.8}$$

Given any integer (the Landau level label) $\ell \geq 0$, we define

$$P^{(\ell)} = \sum_{j=0}^{\ell} P_j.$$

We similarly define the "Toeplitz operator"

$$T_f^{(\ell)} h = P^{(\ell)}(fh), \quad h \in \bigoplus_{j=0}^{\ell} \mathcal{F}_j.$$

That is, $T_f^{(\ell)} = P^{(\ell)} M_f P^{(\ell)}$. Similar to (1.4), we have the identity

$$[T_f^{(\ell)}, T_g^{(\ell)}] = P^{(\ell)}[[M_f, P^{(\ell)}], [M_g, P^{(\ell)}]]$$

for all $f, g \in L^\infty(\mathbf{C})$ and $\ell \geq 0$.

Recall the relation (2.5) that binds E_j and P_j. Thus

$$P^{(\ell)} = U_b \sum_{j=0}^{\ell} E_j U_b^* = U_b P_{\leq E} U_b^*$$

when the Fermi energy E is strictly between $(2\ell + 1)b$ and $(2\ell + 3)b$. For a given pair of switch functions f and g, if it happens that both products $[M_f, P^{(\ell)}][M_g, P^{(\ell)}]$ and $[M_g, P^{(\ell)}][M_f, P^{(\ell)}]$ are in the trace class, then the Kubo formula for Hall conductance reads

$$\sigma_{\text{Hall}}(P_{\leq E}) = -i\text{tr}[T_f^{(\ell)}, T_g^{(\ell)}], \tag{2.9}$$

which generalizes (1.5). We have three goals for this paper:

(i) Prove that for a large class of switch functions, the products of commutators $[M_f, P^{(\ell)}][M_g, P^{(\ell)}]$ and $[M_g, P^{(\ell)}][M_f, P^{(\ell)}]$ are indeed in the trace class.

(ii) Compute the trace of $\text{tr}[T_f^{(\ell)}, T_g^{(\ell)}]$.

(iii) Identify the Hall conductance in (2.9) with a Fredholm index.

We will see that (i) follows from Proposition 3.7, while (ii) and (iii) are the last two identities given in Section 8.

3. Trace class products of commutators

The main purpose of this section is to establish Propositions 3.7 and 3.9, which tell us that certain products of commutators are in the trace class. We begin with some of the basic estimates.

For vectors x, y in a Hilbert space $\mathcal{H}$, the notation $x \otimes y$ denotes the operator on $\mathcal{H}$ defined by the formula

$$x \otimes y h = \langle h, y \rangle x, \quad h \in \mathcal{H}.$$

Note that this definition of $x \otimes y$ is the opposite of the usual convention adopted by physicists, but it is the standard practice in operator theory.

Recall that for each $z \in \mathbf{C}$, the function

$$k_z(\zeta) = e^{-|z|^2/2} e^{\zeta \bar{z}}$$

is the normalized reproducing kernel for the Fock space $\mathcal{F}^2$. For arbitrary $\varphi, \psi \in L^2(\mathbf{C}, d\mu)$,

$$\langle P\varphi, \psi \rangle = \frac{1}{\pi} \int_{\mathbf{C}} (P\varphi)(z) \overline{(P\psi)(z)} e^{-|z|^2} dA(z) = \frac{1}{\pi} \int_{\mathbf{C}} \langle \varphi, k_z \rangle \langle k_z, \psi \rangle dA(z).$$

This can be rewritten as the operator identity

$$P = \frac{1}{\pi} \int_{\mathbf{C}} k_z \otimes k_z dA(z) \tag{3.1}$$

on $L^2(\mathbf{C}, d\mu)$. Define

$$\Gamma = \{m + in : m, n \in \mathbf{Z}\} \quad \text{and} \quad Q = \{x + iy : x, y \in [0, 1)\}.$$

Continuing with (3.1), we have

$$P = \sum_{u \in \Gamma} \frac{1}{\pi} \int_{Q+u} k_z \otimes k_z dA(z) = \frac{1}{\pi} \int_Q G_z dA(z), \tag{3.2}$$

where

$$G_z = \sum_{u \in \Gamma} k_{u+z} \otimes k_{u+z}$$

for every $z \in Q$.

Easy calculation shows that

$$(C^j k_z)(\zeta) = (\bar{\zeta} - \bar{z})^j k_z(\zeta) \quad \text{for all} \quad j \geq 0 \quad \text{and} \quad z \in \mathbf{C}.$$

We now define

$$k_z^{(j)}(\zeta) = (C^j k_z)(\zeta) = (\bar{\zeta} - \bar{z})^j k_z(\zeta), \tag{3.3}$$

$j \geq 0$ and $z \in \mathbf{C}$. From (2.7) and (2.6) we see that

$$\langle P_j u, v \rangle = \frac{1}{j!} \langle C^j P A^j u, v \rangle \quad \text{for} \quad u, v \in \mathbf{C}[\zeta, \bar{\zeta}],$$

$j \geq 0$. Combining this identity with (3.2), we see that for each $j \geq 0$,

$$P_j = \frac{1}{j! \pi} \int_Q G_{z,j} dA(z), \tag{3.4}$$

where

$$G_{z,j} = \sum_{u \in \Gamma} k_{u+z}^{(j)} \otimes k_{u+z}^{(j)} \tag{3.5}$$

for every $z \in Q$. Keep in mind that $k_{u+z}^{(j)}$ is defined by (3.3).

For $z \in \mathbf{C}$ and $a > 0$, denote $B(z, a) = \{w \in \mathbf{C} : |z - w| < a\}$.

Lemma 3.1. *Let $j \geq 0$ be given. Then there is a constant $0 < C_{3.1}(j) < \infty$ such that for all $z \in \mathbf{C}$ and $\rho > 0$, we have $\|\chi_{\mathbf{C} \setminus B(z,\rho)} k_z^{(j)}\| \leq C_{3.1}(j) e^{-\rho^2/3}$.*

Proof. Given a $j \geq 0$, there is a $0 < C_1 = C_1(j) < \infty$ such that $x^j e^{-x/3} \leq C_1$ for every $x \geq 0$. Let $z \in \mathbf{C}$ and $\rho > 0$. Then

$$\begin{aligned} \|\chi_{\mathbf{C} \setminus B(z,\rho)} k_z^{(j)}\|^2 &= \frac{1}{\pi} \int_{|\zeta - z| \geq \rho} |k_z^{(j)}(\zeta)|^2 e^{-|\zeta|^2} dA(\zeta) = \frac{1}{\pi} \int_{|\zeta - z| \geq \rho} |\zeta - z|^{2j} e^{-|\zeta - z|^2} dA(\zeta) \\ &\leq \frac{C_1}{\pi} \int_{|\zeta| \geq \rho} e^{-2|\zeta|^2/3} dA(\zeta) = 2C_1 \int_\rho^\infty e^{-2r^2/3} r dr = (3/2) C_1 e^{-2\rho^2/3}. \end{aligned}$$

Thus the constant $C_{3.1}(j) = \{(3/2)C_1\}^{1/2}$ will do for the lemma. $\square$

For any nonempty subset $F \subset \mathbf{C}$ and any $z \in \mathbf{C}$, denote

$$d(z, F) = \inf\{|z - \zeta| : \zeta \in F\}.$$

For $j \geq 0$ and $z \in \mathbf{C}$, from the facts that $A k_z = 0$ and $[A, C] = 1$ we obtain

$$\|k_z^{(j)}\|^2 = \langle C^j k_z, C^j k_z \rangle = \langle A^j C^j k_z, k_z \rangle = j! \langle k_z, k_z \rangle = j!. \tag{3.6}$$

Lemma 3.2. *Given any $j \geq 0$, there is a constant $0 < C_{3.2}(j) < \infty$ such that the following estimate holds: Let $\varphi \in L^\infty(\mathbf{C})$ be such that $|\varphi(z)| \leq 1$ for every $z \in \mathbf{C}$. Furthermore, suppose that there is a closed subset F of $\mathbf{C}$ such that for every disc $B(w, a)$ satisfying the* *condition* $B(w,a) \cap F = \emptyset$ ($w \in \mathbf{C}\backslash F$ *and* $a > 0$), φ *is a constant on* $B(w,a)$. *Then for every* $z \in \mathbf{C}$,

$$\|(\varphi - \varphi(z))k_z^{(j)}\| \le C_{3.2}(j)\exp(-d^2(z,F)/3).$$

Proof. Since $\|k_z^{(j)}\| = \sqrt{j!}$ and $\|\varphi - \varphi(z)\|_\infty \le 2$, it suffices to consider $z \in \mathbf{C}\backslash F$. Let $0 < r < d(z,F)$. By the assumption on φ, we have $\varphi(w) = \varphi(z)$ for every $w \in B(z,r)$. Therefore

$$\|(\varphi - \varphi(z))k_z^{(j)}\| \le 2\|\chi_{\mathbf{C}\backslash B(z,r)}k_z^{(j)}\| \le 2C_{3.1}(j)e^{-r^2/3}$$

by an application of Lemma 3.1. Since this holds for every $0 < r < d(z,F)$, the desired conclusion follows. □

Lemma 3.3. *Let* $0 < a < \infty$ *and define* $V = \{x + iy : -a \le x \le a$ *and* $y \in \mathbf{R}\}$. *Then* $d^2(z, i\mathbf{R}) \le 2a^2 + 2d^2(z,V)$ *for every* $z \in \mathbf{C}$.

Proof. It suffices to observe that $d(z, i\mathbf{R}) \le a + d(z,V)$ for every $z \in \mathbf{C}$. □

Lemma 3.4. *Given any pair of* $j \ge 0$ and $k \ge 0$, *there is a constant* $0 < C_{3.4}(j,k) < \infty$ *such that for all* $z, w \in \mathbf{C}$,

$$\langle |k_z^{(j)}|, |k_w^{(k)}|\rangle \le C_{3.4}(j,k)e^{-|z-w|^2/8}.$$

Proof. We have

$$\begin{aligned}
\langle |k_z^{(j)}|, |k_w^{(k)}|\rangle &= \frac{1}{\pi}\int |\zeta - z|^j e^{\mathrm{Re}(\zeta\bar{z})}e^{-|z|^2/2}|\zeta - w|^k e^{\mathrm{Re}(\zeta\bar{w})}e^{-|w|^2/2}e^{-|\zeta|^2}dA(\zeta) \\
&= \frac{1}{\pi}\int |\zeta - z|^j e^{-|\zeta - z|^2/2}|\zeta - w|^k e^{-|\zeta-w|^2/2}dA(\zeta) \\
&= \frac{1}{\pi}\int e^{-\{|\zeta-z|^2+|\zeta-w|^2\}/4}|\zeta - z|^j|\zeta - w|^k e^{-\{|\zeta-z|^2+|\zeta-w|^2\}/4}dA(\zeta) \\
&\le e^{-|z-w|^2/8}\frac{1}{\pi}\int |\zeta - z|^j|\zeta - w|^k e^{-\{|\zeta-z|^2+|\zeta-w|^2\}/4}dA(\zeta).
\end{aligned}$$

Once $j \ge 0$ and $k \ge 0$ are given, there is obviously a $0 < C_{3.4}(j,k) < \infty$ such that

$$\frac{1}{\pi}\int |\zeta - z|^j|\zeta - w|^k e^{-\{|\zeta-z|^2+|\zeta-w|^2\}/4}dA(\zeta) \le C_{3.4}(j,k)$$

for all $z, w \in \mathbf{C}$. □

Definition 3.5. Let $0 < a < \infty$. Then Σ_a denotes the collection of measurable functions η on $\mathbf{R}$ satisfying the following three conditions:

(1) $0 \le \eta(x) \le 1$ for every $x \in \mathbf{R}$.
(2) $\eta(x) = 1$ if $x > a$.
(3) $\eta(x) = 0$ if $x < -a$.

We will use elements of Σ_a to construct “switch functions” on $\mathbf{C}$ which interpolate between values 0 and 1 within certain strips in $\mathbf{C}$.

We write $\|\cdot\|_1$ for the norm of the trace class.

Lemma 3.6. *Let $\eta, \xi \in \Sigma_a$ for some $0 < a < \infty$, and let $0 < \theta < \pi$. Define the functions f and g on $\mathbf{C}$ by the formulas*

$$f(\zeta) = \eta(\mathrm{Re}(\zeta)) \quad \textit{and} \quad g(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)), \quad \zeta \in \mathbf{C}.$$

Let $j \geq 0$ and $k \geq 0$. Then for all $z, w \in Q$, the product $[M_f, G_{z,j}][M_g, G_{w,k}]$ is in the trace class. Moreover, there is a $0 < C < \infty$ which depends only on a, θ, j and k such that

$$\|[M_f, G_{z,j}][M_g, G_{w,k}]\|_1 \leq C$$

for every pair of $z, w \in Q$.

Proof. Given any pair of $z, w \in Q$, we have

$$[M_f, G_{z,j}][M_g, G_{w,k}] = \sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} [M_f, k_u^{(j)} \otimes k_u^{(j)}][M_g, k_v^{(k)} \otimes k_v^{(k)}] = \sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} H_{u,v},$$

where

$$\begin{aligned}
H_{u,v} &= \{(f - f(u))k_u^{(j)} \otimes k_u^{(j)} - k_u^{(j)} \otimes (f - f(u))k_u^{(j)}\} \\
&\quad \cdot \{(g - g(v))k_v^{(k)} \otimes k_v^{(k)} - k_v^{(k)} \otimes (g - g(v))k_v^{(k)}\} \\
&= \langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle (f - f(u))k_u^{(j)} \otimes k_v^{(k)} - \langle k_v^{(k)}, k_u^{(j)}\rangle (f - f(u))k_u^{(j)} \otimes (g - g(v))k_v^{(k)} \\
&\quad - \langle (g - g(v))k_v^{(k)}, (f - f(u))k_u^{(j)}\rangle k_u^{(j)} \otimes k_v^{(k)} + \langle k_v^{(k)}, (f - f(u))k_u^{(j)}\rangle k_u^{(j)} \otimes (g - g(v))k_v^{(k)}.
\end{aligned}$$

It suffices to estimate the sum $\sum_{u,v} \|H_{u,v}\|_1$.

For every pair of $u \in \Gamma + z$ and $v \in \Gamma + w$, the above along with (3.6) give us

$$\|H_{u,v}\|_1 \leq \sqrt{k!}h_{u,v}^{(1)} + h_{u,v}^{(2)} + \sqrt{j!k!}h_{u,v}^{(3)} + \sqrt{j!}h_{u,v}^{(4)},$$

where

$$\begin{aligned}
h_{u,v}^{(1)} &= |\langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle| \|(f - f(u))k_u^{(j)}\|, \\
h_{u,v}^{(2)} &= |\langle k_v^{(k)}, k_u^{(j)}\rangle| \|(f - f(u))k_u^{(j)}\| \|(g - g(v))k_v^{(k)}\|, \\
h_{u,v}^{(3)} &= |\langle (g - g(v))k_v^{(k)}, (f - f(u))k_u^{(j)}\rangle| \quad \text{and} \\
h_{u,v}^{(4)} &= |\langle k_v^{(k)}, (f - f(u))k_u^{(j)}\rangle| \|(g - g(v))k_v^{(k)}\|.
\end{aligned}$$

To prove the lemma, it suffices to find constants $0 < C_\nu < \infty$ such that $\Sigma_{u,v} h_{u,v}^{(\nu)} \leq C_\nu$ for $\nu = 1, 2, 3, 4$. Below we present the details of the estimates for the sum $\Sigma_{u,v} h_{u,v}^{(1)}$; the other three sums can be handled similarly.

First, note that since

$$|\langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle| = |\langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle|^{1/2} |\langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle|^{1/2},$$

we have

$$h^{(1)}_{u,v} \le \langle |k^{(k)}_v|, |k^{(j)}_u|\rangle^{1/2}\sqrt{j!}\|(g-g(v))k^{(k)}_v\|^{1/2}\|(f-f(u))k^{(j)}_u\|.$$

Applying Lemmas 3.4, 3.2 and 3.3, we obtain

$$\begin{aligned} h^{(1)}_{u,v} &\le C_1\big(e^{-|u-v|^2/8}\big)^{1/2}\big(e^{a^2/3}e^{-d^2(u,i\mathbf{R})/6}\big)^{1/2}e^{a^2/3}e^{-d^2(v,ie^{i\theta}\mathbf{R})/6} \\ &\le C_2\exp\bigg(-\frac{1}{16}\left\{|u-v|^2+d^2(u,i\mathbf{R})+d^2(v,ie^{i\theta}\mathbf{R})\right\}\bigg). \end{aligned} \tag{3.7}$$

For $x \ge 0$ and $y \ge 0$, $x^2+y^2 \ge (1/2)(x+y)^2$. By this and the triangle inequality,

$$\begin{aligned} (1/2)|u-v|^2+(1/2)d^2(v,ie^{i\theta}\mathbf{R}) \ge (1/4)d^2(u,ie^{i\theta}\mathbf{R}) \quad \text{and} \\ (1/2)|u-v|^2+(1/2)d^2(u,i\mathbf{R}) \ge (1/4)d^2(v,i\mathbf{R}). \end{aligned}$$

Therefore from (3.7) we deduce

$$h^{(1)}_{u,v} \le C_2\exp\bigg(-\frac{1}{64}\left\{d^2(u,i\mathbf{R})+d^2(u,ie^{i\theta}\mathbf{R})+d^2(v,i\mathbf{R})+d^2(v,ie^{i\theta}\mathbf{R})\right\}\bigg). \tag{3.8}$$

Denote $\alpha = \min\{\sin(\theta/2), \sin((\pi-\theta)/2)\}$. By simple plane geometry, for every $q \in \mathbf{C}$,

$$\max\{d(q,i\mathbf{R}), d(q,ie^{i\theta}\mathbf{R})\} \ge \alpha|q|.$$

Thus if we write $\beta = \alpha^2/64$, then from (3.8) we obtain

$$h^{(1)}_{u,v} \le C_2 e^{-\beta|u|^2}e^{-\beta|v|^2}.$$

For $x \in \Gamma$, $|z|^2+|x+z|^2 \ge (1/2)|x|^2$. Therefore

$$\begin{aligned} \sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} h^{(1)}_{u,v} &\le C_2\sum_{u\in\Gamma+z}e^{-\beta|u|^2}\sum_{v\in\Gamma+w}e^{-\beta|v|^2} \\ &\le C_2\sum_{x\in\Gamma}e^{-(\beta/2)|x|^2+|z|^2}\sum_{y\in\Gamma}e^{-(\beta/2)|y|^2+|w|^2} \le e^4C_2\bigg(\sum_{x\in\Gamma}e^{-(\beta/2)|x|^2}\bigg)^2. \end{aligned}$$

This completes the proof. □

Proposition 3.7. *Let $\eta, \xi \in \Sigma_a$ for some $0 < a < \infty$. Let $0 < \theta < \pi$. Define the functions f and g on $\mathbf{C}$ by the formulas*

$$f(\zeta) = \eta(\mathrm{Re}(\zeta)) \quad \textit{and} \quad g(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$. Then for all $j \ge 0$ and $k \ge 0$, the operator $[M_f, P_j][M_g, P_k]$ is in the trace class. Consequently, for every $\ell \ge 0$, the commutator $[T^{(\ell)}_f, T^{(\ell)}_g]$ is in the trace class.

Proof. It follows from (3.4) that

$$[M_f, P_j][M_g, P_k] = \frac{1}{j!k!\pi^2} \iint_{Q\times Q} [M_f, G_{z,j}][M_g, G_{w,k}]dA(z)dA(w).$$

Therefore

$$\|[M_f, P_j][M_g, P_k]\|_1 \le \frac{1}{j!k!\pi^2} \iint_{Q\times Q} \|[M_f, G_{z,j}][M_g, G_{w,k}]\|_1 dA(z)dA(w).$$

By Lemma 3.6, the right-hand side is finite. □

In terms of simple plane geometry, we can think of the commutators $[M_f, P_j]$ and $[M_g, P_k]$ in Proposition 3.7 as being "supported" in strips at the angle θ to each other. The intersection of the two strips is a parallelogram, which is why the product $[M_f, P_j][M_g, P_k]$ is in the trace class. Proposition 3.9 below uses a variant of this idea.

Lemma 3.8. *Let $s, t \in \mathbf{R}$ be such that $s < t < s + \pi$. Define the subset*

$$W = \{re^{ix} : s \le x \le t \text{ and } r \ge 0\}$$

of $\mathbf{C}$. Let $\theta \in \mathbf{R}$ satisfy the conditions $ie^{i\theta}\mathbf{R} \cap e^{is}\mathbf{R} = \{0\}$ and $ie^{i\theta}\mathbf{R} \cap e^{it}\mathbf{R} = \{0\}$. Pick a $\xi \in \Sigma_a$ for some $0 < a < \infty$ and define the function

$$g(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$. Given any $j \ge 0$ and $k \ge 0$, there is a $0 < C < \infty$ such that

$$\|[M_{\chi_W}, G_{z,j}][M_g, G_{w,k}]\|_1 \le C$$

for every pair of $z, w \in Q$.

Proof. Similar to what happened in the proof of Lemma 3.6, for any $z, w \in Q$, we have

$$\begin{aligned}
[M_{\chi_W}, G_{z,j}][M_g, G_{w,k}] &= \sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} [M_{\chi_W}, k_u^{(j)} \otimes k_u^{(j)}][M_g, k_v^{(k)} \otimes k_v^{(k)}] \\
&= \sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} H_{u,v},
\end{aligned}$$

where

$$\begin{aligned}
H_{u,v} &= \langle (g - g(v))k_v^{(k)}, k_u^{(j)}\rangle(\chi_W - \chi_W(u))k_u^{(j)} \otimes k_v^{(k)} \\
&\quad - \langle k_v^{(k)}, k_u^{(j)}\rangle(\chi_W - \chi_W(u))k_u^{(j)} \otimes (g - g(v))k_v^{(k)} \\
&\quad - \langle (g - g(v))k_v^{(k)}, (\chi_W - \chi_W(u))k_u^{(j)}\rangle k_u^{(j)} \otimes k_v^{(k)} \\
&\quad + \langle k_v^{(k)}, (\chi_W - \chi_W(u))k_u^{(j)}\rangle k_u^{(j)} \otimes (g - g(v))k_v^{(k)}.
\end{aligned}$$

Again, it suffices to estimate the sum $\sum_{u,v} \|H_{u,v}\|_1$.

For every pair of $u \in \Gamma + z$ and $v \in \Gamma + w$,

$$\|H_{u,v}\|_1 \le \sqrt{k!}h^{(1)}_{u,v} + h^{(2)}_{u,v} + \sqrt{j!k!}h^{(3)}_{u,v} + \sqrt{j!}h^{(4)}_{u,v},$$

where

$$\begin{aligned}
h^{(1)}_{u,v} &= |\langle (g - g(v))k^{(k)}_v, k^{(j)}_u\rangle| \|(\chi_W - \chi_W(u))k^{(j)}_u\|,\\
h^{(2)}_{u,v} &= |\langle k^{(k)}_v, k^{(j)}_u\rangle| \|(\chi_W - \chi_W(u))k^{(j)}_u\| \|(g - g(v))k^{(k)}_v\|,\\
h^{(3)}_{u,v} &= |\langle (g - g(v))k^{(k)}_v, (\chi_W - \chi_W(u))k^{(j)}_u\rangle| \quad \text{and}\\
h^{(4)}_{u,v} &= |\langle k^{(k)}_v, (\chi_W - \chi_W(u))k^{(j)}_u\rangle| \|(g - g(v))k^{(k)}_v\|.
\end{aligned}$$

This time, let us estimate $\Sigma_{u,v} h^{(3)}_{u,v}$; the other three sums are handled similarly.

It is easy to see that

$$h^{(3)}_{u,v} \le \langle |k^{(k)}_v|, |k^{(j)}_u|\rangle^{1/2} \|(g - g(v))k^{(k)}_v\|^{1/2} \|(\chi_W - \chi_W(u))k^{(j)}_u\|^{1/2}.$$

Applying Lemmas 3.4, 3.2 and 3.3, we obtain

$$h^{(3)}_{u,v} \le C_1\big(e^{-|u-v|^2/8} \cdot e^{a^2/3} e^{-d^2(v, ie^{i\theta}\mathbf{R})/6} \cdot e^{-d^2(u,\partial W)/3}\big)^{/1/2}.$$

Obviously, $\partial W \subset e^{is}\mathbf{R} \cup e^{it}\mathbf{R}$. Therefore the above implies

$$h^{(3)}_{u,v} \le C_2 \max\{a_{u,v}, b_{u,v}\} \le C_2(a_{u,v} + b_{u,v}),$$

where

$$\begin{aligned}
a_{u,v} &= \exp\bigg(-\frac{1}{16}\left\{|u - v|^2 + d^2(v, ie^{i\theta}\mathbf{R}) + d^2(u, e^{is}\mathbf{R})\right\}\bigg) \quad \text{and}\\
b_{u,v} &= \exp\bigg(-\frac{1}{16}\left\{|u - v|^2 + d^2(v, ie^{i\theta}\mathbf{R}) + d^2(u, e^{it}\mathbf{R})\right\}\bigg).
\end{aligned}$$

Let us estimate $\Sigma_{u,v} b_{u,v}$; the sum $\Sigma_{u,v} a_{u,v}$ is handled similarly.

By the reasoning before inequality (3.8), we have

$$b_{u,v} \le \exp\bigg(-\frac{1}{64}\left\{d^2(v, e^{it}\mathbf{R}) + d^2(v, ie^{i\theta}\mathbf{R}) + d^2(u, e^{it}\mathbf{R}) + d^2(u, ie^{i\theta}\mathbf{R})\right\}\bigg). \tag{3.9}$$

Let γ be the angle between the lines $e^{it}\mathbf{R}$ and $ie^{i\theta}\mathbf{R}$. By definition, $0 \le \gamma \le \pi$. By our assumption on t and θ, $\gamma \ne 0$ and $\gamma \ne \pi$. For each $q \in \mathbf{C}$, plane geometry gives us the inequality

$$\max\{d(q, e^{it}\mathbf{R}), d(q, ie^{i\theta}\mathbf{R})\} \ge \alpha|q|,$$

where $\alpha = \min\{\sin(\gamma/2), \sin((\pi - \gamma)/2)\}$. Thus if we write $\beta = \alpha^2/64$, then from (3.9) we obtain

$$b_{u,v} \le e^{-\beta|u|^2} e^{-\beta|v|^2}.$$

By the argument at the end of the proof of Lemma 3.6, we have

$$\sum_{u\in\Gamma+z}\sum_{v\in\Gamma+w} b_{u,v} \le e^4\left(\sum_{x\in\Gamma} e^{-(\beta/2)|x|^2}\right)^2.$$

This completes the proof. □

Proposition 3.9. *Let $s, t \in \mathbf{R}$ be such that $s < t < s + \pi$. Define the subset*

$$W = \{re^{ix} : s \le x \le t \textit{ and } r \ge 0\}$$

of $\mathbf{C}$. Let $\theta \in \mathbf{R}$ satisfy the conditions $ie^{i\theta}\mathbf{R} \cap e^{is}\mathbf{R} = \{0\}$ and $ie^{i\theta}\mathbf{R} \cap e^{it}\mathbf{R} = \{0\}$. Pick a $\xi \in \Sigma_a$ for some $0 < a < \infty$ and define the function

$$g(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$. Then for all $j \ge 0$ and $k \ge 0$, the operator $[M_{\chi_W}, P_j][M_g, P_k]$ is in the trace class. Consequently, for every $\ell \ge 0$, the semi-commutators

$$T^{(\ell)}_{\chi_W} T^{(\ell)}_g - T^{(\ell)}_{\chi_W g} \quad \textit{and} \quad T^{(\ell)}_g T^{(\ell)}_{\chi_W} - T^{(\ell)}_{g\chi_W}$$

are in the trace class.

Proof. This proposition is derived from Lemma 3.8 in exactly the same way Proposition 3.7 was derived from Lemma 3.6. □

Proposition 3.10. *Let $\varphi \in L^\infty(\mathbf{C})$. If the support of φ is contained in a bounded set, then for every $j \ge 0$, the operator $M_\varphi P_j$ is in the trace class.*

Proof. Let a $j \ge 0$ be given. By (3.4), it suffices to find a $0 < C < \infty$ such that

$$\|M_\varphi G_{z,j}\|_1 \le C \quad \text{for every} \quad z \in Q. \tag{3.10}$$

By (3.5), for each $z \in Q$ we have

$$\|M_\varphi G_{z,j}\|_1 \le \sqrt{j!}\sum_{u\in\Gamma} \|\varphi k^{(j)}_{u+z}\| \tag{3.11}$$

By assumption, there is a $0 < \rho < \infty$ such that $\varphi = 0$ on $\mathbf{C}\backslash B(0,\rho)$. Thus if $|u| > 2\rho + 4$, then $B(u + z, |u|/2) \cap B(0,\rho) = \emptyset$, i.e., $B(0,\rho) \subset \mathbf{C}\backslash B(u + z, |u|/2)$, and we have

$$\|\varphi k^{(j)}_{u+z}\| \le \|\varphi\|_\infty \|\chi_{\mathbf{C}\backslash B(u+z,|u|/2)} k^{(j)}_{u+z}\| \le C_{3.2}(j)\|\varphi\|_\infty e^{-|u|^2/12},$$

where the second $\le$ follows from Lemma 3.1. Thus

$$\sum_{u\in\Gamma\backslash B(0,2\rho+5)} \|\varphi k^{(j)}_{u+z}\| \le C_{3.2}(j)\|\varphi\|_\infty \sum_{u\in\Gamma\backslash B(0,2\rho+5)} e^{-|u|^2/6}.$$

Combining this bound with (3.11), we obtain (3.10). □

4. Fredholm inverse

In this section we deal with Toeplitz operators whose Fredholm inverse cannot be constructed by known methods in the literature. These Toeplitz operators have symbols that oscillate at infinity.

Take an $a > 0$ and pick $\eta, \xi \in \Sigma_a$. Also, pick a $0 < \theta < \pi$. We now fix the switch functions

$$f_1(\zeta) = \eta(\mathrm{Re}(\zeta)) \quad \text{and} \quad f_2(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)), \tag{4.1}$$

$\zeta \in \mathbf{C}$. Furthermore, we define the function

$$F = f_1 + if_2$$

on $\mathbf{C}$. We remind the reader that

$$S = \{x + iy : x, y \in [0, 1]\}.$$

Recall that (2.8) defines the "Toeplitz operator" $T_{f,j}$ on $\mathcal{F}_j$, $j \geq 0$.

Theorem 4.1. *On the space $\mathcal{F}_j$, $j \geq 0$, the essential spectrum of the Toeplitz operator $T_{F,j}$ is contained in ∂S, the boundary of the square S.*

To prove Theorem 4.1, we first establish a general "product rule" for the spectra of self-adjoint operators:

Lemma 4.2. *Let A, B be bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. Suppose that the spectra of A and B are contained in the intervals $[a, b]$ and $[c, d]$ respectively. Then the spectrum of $T = A + iB$ is contained in the rectangle*

$$\{x + iy : x \in [a, b], y \in [c, d]\}.$$

Proof. Let $t \in \mathbf{R}\backslash[a, b]$ and $y \in \mathbf{R}$. Let us show that $T - (t + iy)$ is invertible. If $t < a$, then $A - t > 0$, and $(A - t)^{1/2}$ has a bounded inverse $(A - t)^{-1/2}$. In this case,

$$T - (t + iy) = (A - t)^{1/2}\{1 + i(A - t)^{-1/2}(B - y)(A - t)^{-1/2}\}(A - t)^{1/2}.$$

Since the operator $(A - t)^{-1/2}(B - y)(A - t)^{-1/2}$ is self-adjoint, the above is invertible. In the case $t > b$, we have $t - A > 0$ and

$$T - (t + iy) = (t - A)^{1/2}\{-1 + i(t - A)^{-1/2}(B - y)(t - A)^{-1/2}\}(t - A)^{1/2},$$

which is also invertible. A similar argument shows that if $x \in \mathbf{R}$ and $t \in \mathbf{R}\backslash[c, d]$, then $T - (x + it)$ is invertible. □

Proof of Theorem 4.1. By Lemma 4.2, it suffices to show that the interior of S does not intersect the essential spectrum of $T_{F,j}$. We begin with the following four wedges in $\mathbf{C}$:

$$\begin{aligned}
\mathcal{A} &= \{re^{ix} : \theta/2 \le x \le (\pi+\theta)/2 \text{ and } r \ge 0\},\\
\mathcal{B} &= \{re^{ix} : (\pi+\theta)/2 \le x \le \pi + (\theta/2) \text{ and } r \ge 0\},\\
\mathcal{C} &= \{re^{ix} : \pi + (\theta/2) \le x \le (3\pi+\theta)/2 \text{ and } r \ge 0\} \quad \text{and}\\
\mathcal{D} &= \{re^{ix} : (3\pi+\theta)/2 \le x \le 2\pi + (\theta/2) \text{ and } r \ge 0\}.
\end{aligned}$$

Thus $\mathcal{B} = i\mathcal{A}$, $\mathcal{C} = -\mathcal{A}$, and $\mathcal{D} = -i\mathcal{A}$. We claim that

$$T_{f_2,j}T_{\chi_{\mathcal{A}},j} = T_{\chi_{\mathcal{A}},j} + K_{\mathcal{A}}, \tag{4.2}$$

$$T_{f_1,j}T_{\chi_{\mathcal{B}},j} = K_{\mathcal{B}}, \tag{4.3}$$

$$T_{f_2,j}T_{\chi_{\mathcal{C}},j} = K_{\mathcal{C}} \quad \text{and} \tag{4.4}$$

$$T_{f_1,j}T_{\chi_{\mathcal{D}},j} = T_{\chi_{\mathcal{D}},j} + K_{\mathcal{D}}, \tag{4.5}$$

where $K_{\mathcal{A}}$, $K_{\mathcal{B}}$, $K_{\mathcal{C}}$, $K_{\mathcal{D}}$ are compact operators. Let us consider (4.2).

By Proposition 3.9, we have $T_{f_2,j}T_{\chi_{\mathcal{A}},j} = T_{f_2\chi_{\mathcal{A}},j} + K_{\mathcal{A}}^{(1)}$, where $K_{\mathcal{A}}^{(1)}$ is compact. Thus

$$T_{f_2,j}T_{\chi_{\mathcal{A}},j} = T_{\chi_{\mathcal{A}},j} - T_{(1-f_2)\chi_{\mathcal{A}},j} + K_{\mathcal{A}}^{(1)}. \tag{4.6}$$

By the definition of f_2, if $\zeta = e^{i\theta}(x+iy)$ for $x > a$ and $y \in \mathbf{R}$, then $f_2(\zeta) = 1$. Thus the support of the function $(1-f_2)\chi_{\mathcal{A}}$ is contained in the set

$$\mathcal{A} \cap \{e^{i\theta}(x+iy) : x \le a \text{ and } y \in \mathbf{R}\}.$$

Obviously, this is the region enclosed by the lines $e^{i\theta/2}\mathbf{R}$, $e^{i(\pi+\theta)/2}\mathbf{R}$ and $\{e^{i\theta}(a+iy) : y \in \mathbf{R}\}$, which is a triangle. By Proposition 3.10, the Toeplitz operator $T_{(1-f_2)\chi_{\mathcal{A}},j}$ is compact. Therefore (4.2) follows from (4.6). The proofs of (4.3), (4.4) and (4.5) are similar and will be omitted.

Let $\lambda \in S\backslash\partial S$. That is,

$$\lambda = \alpha + i\beta, \quad \text{where } \alpha, \beta \in (0,1).$$

Since the operators $T_{f_1,j}$ and $T_{f_2,j}$ are self-adjoint, the operators

$$T_{f_1,j} - \alpha + i(1-\beta), \quad T_{if_2,j} - \alpha - i\beta, \quad T_{f_1,j} - \alpha - i\beta \quad \text{and} \quad T_{if_2,j} + (1-\alpha) - i\beta$$

are invertible on $\mathcal{F}_j$. Let A, B, C, D denote their respective inverses. Then

$$\begin{aligned}
(T_{f_1,j} - \alpha + i(1-\beta))A &= 1,\\
(T_{if_2,j} - \alpha - i\beta)B &= 1,\\
(T_{f_1,j} - \alpha - i\beta)C &= 1 \quad \text{and}\\
(T_{if_2,j} + (1-\alpha) - i\beta)D &= 1.
\end{aligned}$$

Combining these identities with (4.2)-(4.5), and with the fact that the commutators $[T_{f_1,j}, T_{\chi_{\mathcal{E}},j}]$ and $[T_{f_2,j}, T_{\chi_{\mathcal{E}},j}]$ are in the trace class for every $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ (see (1.4) and Proposition 3.9), we find that

$$\begin{aligned}(T_{F,j} - \lambda)(T_{\chi_{\mathcal{A}},j}A + T_{\chi_{\mathcal{B}},j}B + T_{\chi_{\mathcal{C}},j}C + T_{\chi_{\mathcal{D}},j}D) &= T_{\chi_{\mathcal{A}},j} + T_{\chi_{\mathcal{B}},j} + T_{\chi_{\mathcal{C}},j} + T_{\chi_{\mathcal{D}},j} + K \\ &= 1 + K,\end{aligned}$$

where K is a compact operator. Similarly, we have

$$\begin{aligned}(AT_{\chi_{\mathcal{A}},j} + BT_{\chi_{\mathcal{B}},j} + CT_{\chi_{\mathcal{C}},j} + DT_{\chi_{\mathcal{D}},j})(T_{F,j} - \lambda) &= T_{\chi_{\mathcal{A}},j} + T_{\chi_{\mathcal{B}},j} + T_{\chi_{\mathcal{C}},j} + T_{\chi_{\mathcal{D}},j} + L \\ &= 1 + L,\end{aligned}$$

where L is a compact operator. The above two identities show that λ is not in the essential spectrum of T_F. This completes the proof. □

Theorem 4.3. *Let $\ell \geq 0$. Then on the space $\mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_\ell$, the essential spectrum of the Toeplitz operator $T_F^{(\ell)}$ is contained in ∂S, the boundary of the square S.*

Proof. It follows from Proposition 3.9 that the commutator $[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}]$ is in the trace class. Therefore by Lemma 4.2, the essential spectrum of $T_F^{(\ell)}$ is contained in S.

It also follows from Proposition 3.7 that the operators $T_{f_i}^{(\ell)}T_{\chi_{\mathcal{E}}}^{(\ell)} - T_{f_i\chi_{\mathcal{E}}}^{(\ell)}$ and $T_{\chi_{\mathcal{E}}}^{(\ell)}T_{f_i}^{(\ell)} - T_{\chi_{\mathcal{E}}f_i}^{(\ell)}$ are in the trace class for all $i \in \{1, 2\}$ and $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, where the wedges $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ were defined in the previous proof. Therefore the argument in the proof of Theorem 4.1 also works here. By that argument, if $\lambda \in S \backslash \partial S$, then λ is not in the essential spectrum of $T_F^{(\ell)}$. This completes the proof. □

5. Trace calculation

In this section we revisit the trace calculation, equation (1.6), carried out in [11]. Thus in this section we only consider the classic Fock space $\mathcal{F}^2 = \mathcal{F}_0$ and classic Toeplitz operators on it. The calculation presented below differs slightly from the calculation in [11] in the respect that we allow any $0 < \theta < \pi$ in (4.1). Because of this, the trace calculation requires one extra cancellation argument.

As observed in [1, Proposition 6.9], for all $\Lambda \in \Sigma_a$ and $t \in \mathbf{R}$, we have

$$\int_{\mathbf{R}} (\Lambda(x + t) - \Lambda(x))dx = t. \tag{5.1}$$

Proposition 5.1. *For the switch functions f_1, f_2 defined by* (4.1), *we have*

$$\mathrm{tr}[T_{f_1}, T_{f_2}] = \frac{1}{2\pi i}.$$

Proof. By (1.4), we have

$$[T_{f_1}, T_{f_2}] = P[M_{f_1}, P][M_{f_2}, P] - P[M_{f_2}, P][M_{f_1}, P]. \tag{5.2}$$

By Proposition 3.7, both terms on the right-hand side are in the trace class, which is a fact that we will use in the calculation.

We can regard P as an integral operator on $L^2(\mathbf{C}, d\mu)$ with the function $e^{\zeta\bar{z}}$ as its integral kernel. Thus $P[M_{f_1}, P][M_{f_2}, P]$ and $P[M_{f_2}, P][M_{f_1}, P]$ are integral operators on $L^2(\mathbf{C}, d\mu)$ with the functions

$$K_{1,2}(\zeta, z) = \iint e^{\zeta\bar{w}} e^{w\bar{u}} e^{u\bar{z}} (f_1(w) - f_1(u))(f_2(u) - f_2(z)) d\mu(w) d\mu(u) \quad \text{and}$$
$$K_{2,1}(\zeta, z) = \iint e^{\zeta\bar{w}} e^{w\bar{u}} e^{u\bar{z}} (f_2(w) - f_2(u))(f_1(u) - f_1(z)) d\mu(w) d\mu(u)$$

as their respective integral kernels. Therefore

$$\begin{aligned} \operatorname{tr}(P[M_{f_1}, P][M_{f_2}, P]) &= \int K_{1,2}(\zeta, \zeta) d\mu(\zeta) \\ &= \iiint e^{\zeta\bar{w}} e^{w\bar{u}} e^{u\bar{\zeta}} (f_1(w) - f_1(u))(f_2(u) - f_2(\zeta)) d\mu(w) d\mu(u) d\mu(\zeta). \end{aligned}$$

It is easy to verify that the identity

$$(\zeta\bar{w} + w\bar{u} + u\bar{\zeta}) - (|\zeta|^2 + |w|^2 + |u|^2) = (\zeta - u)\overline{(w - u)} - |\zeta - u|^2 - |w - u|^2$$

holds for all $\zeta, w, u \in \mathbf{C}$. Therefore

$$\begin{aligned} &\operatorname{tr}(P[M_{f_1}, P][M_{f_2}, P]) \\ &\quad = \frac{1}{\pi^3} \iiint e^{(\zeta-u)\overline{(w-u)}} e^{-|\zeta-u|^2 - |w-u|^2} (f_1(w) - f_1(u))(f_2(u) - f_2(\zeta)) dA(w) dA(u) dA(\zeta) \\ &\quad = \frac{1}{\pi^3} \iint e^{x\bar{y}} e^{-|x|^2 - |y|^2} \left\{ \int (f_1(u + y) - f_1(u))(f_2(u) - f_2(u + x)) dA(u) \right\} dA(x) dA(y). \end{aligned}$$

Let us write $x = x_1 + ix_2$ and $y = y_1 + iy_2$. Combining (5.1) with easy plane geometry, we find that

$$\int (f_1(u + y) - f_1(u))(f_2(u) - f_2(u + x)) dA(u) = -y_1(x_2 + \cot\theta x_1). \tag{5.3}$$

(Those who need the details, see the Remark below.) Hence

$$\operatorname{tr}(P[M_{f_1}, P][M_{f_2}, P]) = \frac{-1}{\pi^3} \iint e^{x\bar{y}} e^{-|x|^2 - |y|^2} y_1(x_2 + \cot\theta x_1) dA(x) dA(y).$$

Similarly,

$$\operatorname{tr}(P[M_{f_2}, P][M_{f_1}, P]) = \frac{-1}{\pi^3} \iint e^{x\bar{y}} e^{-|x|^2 - |y|^2} (y_2 + \cot\theta y_1) x_1 dA(x) dA(y).$$

Therefore, by (5.2),

$$\operatorname{tr}[T_{f_1}, T_{f_2}] = \frac{1}{\pi^3} \iint e^{x\bar{y}} e^{-|x|^2-|y|^2}(y_2x_1 - y_1x_2)dA(x)dA(y),$$

which is independent of θ. The last step in the trace calculation is exactly the same as in [11]. That is, integration in polar coordinates for both variables x and y yields

$$\operatorname{tr}[T_{f_1}, T_{f_2}] = \frac{1}{2\pi i}.$$

□

Remark. Let L be a line that intersects the x-axis at the angle θ. Let S_1 be a strip perpendicular to the x-axis, and let S_2 be a strip perpendicular to L. Identity (5.3) is obtained by computing the area of the parallelogram $S_1 \cap S_2$.

As we have already mentioned, Proposition 5.1 only covers the setting of the classic Fock space $\mathcal{F}^2 = \mathcal{F}_0$. Its generalization to the higher Fock spaces $\mathcal{F}_j$, $j \geq 1$, requires the Carey-Pincus theory of principal functions, which we review next.

6. Trace and index

To express the quantized Hall conductance in terms of a Fredholm index, in addition to the work we do in this paper, we need the theories of Helton-Howe [8] and Carey-Pincus [3,4,10] for almost commuting pairs of self-adjoint operators.

Suppose that A, B are bounded self-adjoint operators such that the commutator $[A, B]$ is in the trace class. In [8], Helton and Howe showed that there is a compactly-supported, real-valued regular Borel measure dP on $\mathbf{R}^2$ such that

$$\operatorname{tr}([p(A, B), q(A, B)]) = i \int \{p, q\} dP$$

for all $p, q \in \mathbf{C}[x, y]$. Here, $\{p, q\}$ is the Poisson bracket for p, q. That is,

$$\{p, q\}(x, y) = \frac{\partial p}{\partial x}(x, y)\frac{\partial q}{\partial y}(x, y) - \frac{\partial p}{\partial y}(x, y)\frac{\partial q}{\partial x}(x, y).$$

Carey and Pincus [3,4] took this trace formula one step further by showing that there is a boundedly-supported $g_{A,B} \in L^1(\mathbf{R}^2)$, which is called the *principal function* for the pair A, B, such that

$$\operatorname{tr}([p(A, B), q(A, B)]) = \frac{-1}{2\pi i} \iint \{p, q\}(x, y) g_{A,B}(x, y) dx dy \tag{6.1}$$

for all $p, q \in \mathbf{C}[x, y]$. In other words, (6.1) tells us that the measure dP is absolutely continuous with respect to the two-dimensional Lebesgue measure on $\mathbf{R}^2$. By functional calculus, (6.1) extends to a much larger class of functions than $\mathbf{C}[x, y]$. For an irreducible pair A, B with rank$([A, B]) = 1$, the principal function g is a complete unitary invariant.

Let $T = A+iB$. For our purpose, the more important fact is that for each point (x, y) such that $x + iy$ is not in the essential spectrum of T,

$$g_{A,B}(x, y) = \text{index}(T - (x + iy)). \tag{6.2}$$

See [3, Theorem 4], or [4, Theorem 8.1].

By (1.4) and Proposition 3.7, the commutator $[T_{f_1}, T_{f_2}]$ is in the trace class, where f_1 and f_2 were defined by (4.1). This allows us to apply the above theory to the pair $A = T_{f_1}$ and $B = T_{f_2}$. Theorem 4.1 says that the essential spectrum of $A + iB = T_F$ is contained in ∂S. It follows from this fact that

$$\text{index}(T_F - \lambda) = 0 \quad \text{for every} \quad \lambda \in \mathbf{C}\backslash S.$$

Therefore for this pair $A = T_{f_1}$ and $B = T_{f_2}$, we have $g_{A,B} = n\chi_S$, where χ_S is the characteristic function of the square S and

$$n = \text{index}(T_F - \lambda) \quad \text{for each} \quad \lambda \in S\backslash\partial S.$$

Applying (6.1) in the case $p(x, y) = x$ and $q(x, y) = y$, we obtain

$$\text{tr}[T_{f_1}, T_{f_2}] = \frac{-1}{2\pi i} \iint n\chi_S(x, y)dxdy = \frac{-n}{2\pi i}. \tag{6.3}$$

The above two identities imply

$$\text{tr}[T_{f_1}, T_{f_2}] = \frac{-1}{2\pi i}\text{index}(T_F - \lambda) \quad \text{for every} \quad \lambda \in S\backslash\partial S. \tag{6.4}$$

This identifies the quantized Hall conductance with a Fredholm index in the case where the Fermi energy is strictly between the Landau levels $\ell = 0$ and $\ell = 1$.

By Proposition 5.1, we have $\text{tr}[T_{f_1}, T_{f_2}] = (2\pi i)^{-1}$. Compare this with (6.3), we find that $n = -1$, i.e.,

$$\text{index}(T_F - \lambda) = -1 \quad \text{for every} \quad \lambda \in S\backslash\partial S.$$

In other words, for the pair $A = T_{f_1}$ and $B = T_{f_2}$ we have $g_{A,B} = -\chi_S$. Therefore, by (6.1), we have

$$\text{tr}[p(T_{f_1}, T_{f_2}), q(T_{f_1}, T_{f_2})] = \frac{1}{2\pi i} \iint_S \{p, q\}(x, y)dxdy$$

for all polynomials $p, q \in \mathbf{C}[x, y]$.

The fact that $n \neq 0$ implies that no point of ∂S can be missing from the essential spectrum of T_F. This strengthens the case $j = 0$ in Theorem 4.1 to the following extent:

Theorem 6.1. *On the Fock space $\mathcal{F}^2$, the essential spectrum of the Toeplitz operator T_F equals ∂S, the boundary of the square S.*

With the Carey-Pincus theory in hand, we are now ready to generalize both Proposition 5.1 and formula (6.4) to the setting of the higher Fock spaces $\mathcal{F}_j$, $j \geq 1$.

7. Trace calculation in the higher Fock spaces

First of all, Proposition 3.7 implies that for every $j \geq 0$, the commutator $[T_{f_1,j}, T_{f_2,j}]$ is in the trace class. Our task in this section is to compute $\mathrm{tr}[T_{f_1,j}, T_{f_2,j}]$. Given Proposition 5.1, one obviously expects

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i} \quad \text{for every} \quad j \geq 0. \tag{7.1}$$

We will show that this is indeed true.

Because of (2.6), for each $j \geq 0$ we will write $(j!)^{-1/2} C^j P = V_j$. Thus $V_j^* = (j!)^{-1/2} P A^j$. Recall from (2.7) that each V_j is a partial isometry.

Lemma 7.1. *Given any $j \geq 1$, there exist coefficients $c_1^{(j)}, \ldots, c_j^{(j)}$ such that if $f \in C^\infty(\mathbf{C})$ and if f and $\partial\bar{\partial} f, \ldots, \partial^j \bar{\partial}^j f$ are all bounded on $\mathbf{C}$, then*

$$V_j^* T_{f,j} V_j = T_f + \sum_{\nu=1}^{j} c_\nu^{(j)} T_{\partial^\nu \bar{\partial}^\nu f}.$$

Proof. First of all, since $V_j = P_j V_j$ and $V_j^* = V_j^* P_j$, we have

$$V_j^* T_{f,j} V_j = V_j^* P_j M_f P_j V_j = V_j^* M_f V_j.$$

It is elementary that

$$\begin{aligned} A^j M_f &= M_f A^j + [A^j, M_f] \\ &= M_f A^j + M_{\bar{\partial} f} A^{j-1} + A M_{\bar{\partial} f} A^{j-2} + \cdots + A^{j-1} M_{\bar{\partial} f} \\ &= M_f A^j + j M_{\bar{\partial} f} A^{j-1} + [A, M_{\bar{\partial} f}] A^{j-2} + \cdots + [A^{j-1}, M_{\bar{\partial} f}] \\ &= \cdots\cdots \\ &= M_f A^j + \sum_{\nu=1}^{j} b_\nu^{(j)} M_{\bar{\partial}^\nu f} A^{j-\nu}. \end{aligned}$$

By the commutation relation $[A, C] = 1$ and the fact $APu = 0$ for every $u \in \mathbf{C}[z, \bar{z}]$, we have $A^{j-\nu} C^j P = (j!/\nu!) C^\nu P$, $1 \leq \nu \leq j$. Hence

$$\begin{aligned} V_j^* M_f V_j &= \frac{1}{j!} P A^j M_f C^j P = \frac{1}{j!} P M_f A^j C^j P + \sum_{\nu=1}^{j} \frac{b_\nu^{(j)}}{j!} P M_{\bar{\partial}^\nu f} A^{j-\nu} C^j P \\ &= T_f + \sum_{\nu=1}^{j} c_\nu^{(j)} P M_{\bar{\partial}^\nu f} C^\nu P. \end{aligned} \tag{7.2}$$

For $\varphi \in C^\infty(\mathbf{C})$ such that φ and $\partial\varphi$ are bounded on $\mathbf{C}$, we have $[M_\varphi, C] = M_{\partial\varphi}$. Also, note that $PC = (AP)^* = 0$. From these facts we deduce $PM_{\bar\partial^\nu f}C^\nu = PM_{\partial^\nu\bar\partial^\nu f}$ for every $\nu \geq 1$. Substituting this in (7.2), we find that

$$V_j^* M_f V_j = T_f + \sum_{\nu=1}^{j} c_\nu^{(j)} PM_{\partial^\nu\bar\partial^\nu f}P = T_f + \sum_{\nu=1}^{j} c_\nu^{(j)} T_{\partial^\nu\bar\partial^\nu f}$$

as promised. □

Lemma 7.2. *Let u and v be real-valued, bounded measurable functions on $\mathbf{R}$. Define*

$$\varphi(\zeta) = u(\mathrm{Re}(\zeta)) \quad \text{and} \quad \psi(\zeta) = v(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$, *where θ is the same as in* (4.1). *Suppose that there is a $0 < \rho < \infty$ such that $u = 0$ and $v = 0$ on $\mathbf{R}\backslash(-\rho, \rho)$. Then the commutators*

$$[T_{f_1}, T_\psi], \quad [T_\varphi, T_{f_2}] \quad \textit{and} \quad [T_\varphi, T_\psi]$$

are in the trace class with zero trace.

Proof. We have $v = v_+ - v_-$, where v_+ and v_- are non-negative, bounded measurable functions on $\mathbf{R}$ which vanish on the set $\mathbf{R}\backslash(-\rho, \rho)$. Thus there are $\xi_{+,1}, \xi_{+,2}, \xi_{-,1}, \xi_{-,2} \in \Sigma_\rho$ and coefficients c_+, c_- such that $v_+ = c_+(\xi_{+,1} - \xi_{+,2})$ and $v_- = c_-(\xi_{-,1} - \xi_{-,2})$. That is,

$$v = c_+(\xi_{+,1} - \xi_{+,2}) - c_-(\xi_{-,1} - \xi_{-,2}).$$

Since $\xi_{+,1}, \xi_{+,2}, \xi_{-,1}, \xi_{-,2} \in \Sigma_\rho$, Proposition 3.7 implies that $[T_{f_1}, T_\psi]$ is in the trace class, and Proposition 5.1 implies that the trace of $[T_{f_1}, T_\psi]$ is zero. Clearly, u admits a decomposition of the same kind. Therefore the other two commutators are also in the trace class with zero trace. □

Lemma 7.3. *Suppose that the functions η and ξ in* (4.1) *satisfy the condition $\eta, \xi \in \Sigma_a \cap C^\infty(\mathbf{R})$. Then*

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i}$$

for every $j \geq 1$.

Proof. The condition $\eta, \xi \in \Sigma_a \cap C^\infty(\mathbf{R})$ ensures that Lemma 7.1 is applicable to f_1 and f_2. By that lemma, for a given $j \geq 1$, we have

$$V_j^*[T_{f_1,j}, T_{f_2,j}]V_j = [T_{f_1}, T_{f_2}] + Z_1 + Z_2 + Z_3,$$

where

$$Z_1 = \sum_{\nu=1}^{j} c_\nu^{(j)} [T_{f_1}, T_{\partial^\nu\bar\partial^\nu f_2}],$$

$$Z_2 = \sum_{\nu=1}^{j} c_\nu^{(j)} [T_{\partial^\nu\bar\partial^\nu f_1}, T_{f_2}] \quad \text{and}$$

$$Z_3 = \sum_{\nu=1}^{j} \sum_{\nu'=1}^{j} c_\nu^{(j)} c_{\nu'}^{(j)} [T_{\partial^\nu\bar\partial^\nu f_1}, T_{\partial^{\nu'}\bar\partial^{\nu'} f_2}].$$

By Definition 3.5, we have $\eta = 0$ and $\xi = 0$ on $(-\infty, -a)$ and $\eta = 1$ and $\xi = 1$ on (a, ∞). Thus all the derivatives of η and ξ are supported on the interval $[-a, a]$. Therefore each $\partial^\nu \bar{\partial}^\nu f_1$ is a φ in Lemma 7.2, and each $\partial^{\nu'} \bar{\partial}^{\nu'} f_2$ is a ψ in Lemma 7.2. It follows from Lemma 7.2 that Z_1, Z_2 and Z_3 are in the trace class with zero trace. Consequently,

$$\begin{aligned} \operatorname{tr}[T_{f_1,j}, T_{f_2,j}] &= \operatorname{tr}([T_{f_1,j}, T_{f_2,j}]P_j) = \operatorname{tr}([T_{f_1,j}, T_{f_2,j}]V_j V_j^*) = \operatorname{tr}(V_j^*[T_{f_1,j}, T_{f_2,j}]V_j) \\ &= \operatorname{tr}[T_{f_1}, T_{f_2}] + \operatorname{tr}(Z_1 + Z_2 + Z_3) = \operatorname{tr}[T_{f_1}, T_{f_2}]. \end{aligned}$$

Now an application of Proposition 5.1 completes the proof. □

Since Proposition 3.7 tells us that the commutator $[T_{f_1,j}, T_{f_2,j}]$ is in the trace class, there is a Carey-Pincus principal function for the almost commuting pair $T_{f_1,j}$, $T_{f_2,j}$.

Lemma 7.4. *Suppose that the η and ξ in* (4.1) *are arbitrary functions in Σ_a. Given a $j \geq 1$, let g_j denote the Carey-Pincus principal function for the almost commuting self-adjoint operators $T_{f_1,j}$ and $T_{f_2,j}$. Then*

$$g_j = -\chi_S.$$

Proof. First of all, the existence of such a g_j is provided by (6.1). Theorem 4.1 tells us that the essential spectrum of $T_{F,j}$ is contained in ∂S, whose two-dimensional Lebesgue measure is 0. Thus by (6.2), we have

$$g_j = n_j \chi_S,$$

where

$$n_j = \operatorname{index}(T_{F,j} - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

The above holds true for an arbitrary pair of $\eta, \xi \in \Sigma_a$ in (4.1).

Now take a pair of functions $\tilde{\eta}, \tilde{\xi} \in \Sigma_a \cap C^\infty(\mathbf{R})$. Accordingly, we define

$$\tilde{f}_1(\zeta) = \tilde{\eta}(\operatorname{Re}(\zeta)) \quad \text{and} \quad \tilde{f}_2(\zeta) = \tilde{\xi}(\operatorname{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$, where the θ is the same as in (4.1). Furthermore, we define

$$\tilde{F} = \tilde{f}_1 + i\tilde{f}_2.$$

By the preceding paragraph, the almost commuting pair $T_{\tilde{f}_1,j}$, $T_{\tilde{f}_2,j}$ has a principal function $\tilde{g}_j$, and the principal function $\tilde{g}_j$ has the form $\tilde{g}_j = \tilde{n}_j \chi_S$, where

$$\tilde{n}_j = \operatorname{index}(T_{\tilde{F},j} - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

Applying Lemma 7.3 and identity (6.1), we have

$$\frac{1}{2\pi i} = \operatorname{tr}[T_{\tilde{f}_1,j}, T_{\tilde{f}_2,j}] = \frac{-\tilde{n}_j}{2\pi i} \iint \chi_S(x, y) dx dy = \frac{-\tilde{n}_j}{2\pi i}.$$

From this we conclude that $\tilde{n}_j = -1$.

Next we show that $n_j = \tilde{n}_j$, and consequently $n_j = -1$, which proves the lemma. To prove that $n_j = \tilde{n}_j$, we define

$$\eta_t = t\eta + (1-t)\tilde{\eta} \quad \text{and} \quad \xi_t = t\xi + (1-t)\tilde{\xi},$$

$0 \le t \le 1$. We then define, for each $0 \le t \le 1$, the functions

$$f_{1,t}(\zeta) = \eta_t(\mathrm{Re}(\zeta)) \quad \text{and} \quad f_{2,t}(\zeta) = \xi_t(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$, and

$$F_t = f_{1,t} + if_{2,t}.$$

Note that for every $0 \le t \le 1$, we have $\eta_t, \xi_t \in \Sigma_a$. Therefore by Theorem 4.1, the essential spectrum of $T_{F_t,j}$ is contained in ∂S, $0 \le t \le 1$. Moreover, the map $t \mapsto T_{F_t,j}$ is obviously continuous with respect to the operator norm. Therefore for each $\lambda \in S\backslash\partial S$, the map

$$t \mapsto \mathrm{index}(T_{F_t,j} - \lambda)$$

remains constant on the interval $[0,1]$. Since $F_0 = \tilde{F}$ and $F_1 = F$, we have $n_j = \tilde{n}_j$ as promised. This completes the proof. □

Proposition 7.5. *Suppose that the η and ξ in (4.1) are arbitrary functions in Σ_a. Then for every $j \ge 0$ we have*

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i}.$$

Proof. The case $j = 0$ is covered by Proposition 5.1. For $j \ge 1$, applying (6.1) and Lemma 7.4, we have

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{-1}{2\pi i}\iint g_j(x,y)dxdy = \frac{1}{2\pi i}\iint \chi_S(x,y)dxdy = \frac{1}{2\pi i}.$$

□

Remark. Using the method in this section, it is easy to verify that $V_j^* T_{z/|z|,j} V_j$ is a compact perturbation of $T_{z/|z|}$. Since $\mathrm{index}(T_{z/|z|})$ is known to be -1 (see [1] or Theorem 9.1 below), we also have $\mathrm{index}(T_{z/|z|,j}) = -1$. Thus Proposition 7.5 allows us to write

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{-1}{2\pi i}\mathrm{index}(T_{z/|z|,j}),$$

which generalizes (1.7) to each individual Landau level above the lowest one.

8. Hall conductance and Fredholm index

We will now put the results from the previous section together and identify the quantized Hall conductance, for the Landau levels below any E, with a Fredholm index.

There is no obvious reason that σ_{Hall} should be additive with respect to orthogonal sum $P^{(\ell)} = \sum_{j=0}^{\ell} P_j$. So the fact that it is additive, is not a trivial matter:

Theorem 8.1. *For each $\ell \geq 0$, the commutator $[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}]$ is in the trace class with*

$$\mathrm{tr}[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}] = \frac{\ell+1}{2\pi i}.$$

Proof. It follows from Proposition 3.7 that $[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}]$ is in the trace class. Define

$$\begin{aligned}
Z_0 &= \sum_{\substack{0\leq i,j,k\leq \ell\\ i\neq j, j\neq k, k\neq i}} P_i[[M_{f_1}, P_j], [M_{f_2}, P_k]],\\
Z_1 &= \sum_{\substack{0\leq i,j\leq \ell\\ i\neq j}} P_i[[M_{f_1}, P_i], [M_{f_2}, P_j]],\\
Z_2 &= \sum_{\substack{0\leq i,j\leq \ell\\ i\neq j}} P_j[[M_{f_1}, P_i], [M_{f_2}, P_j]] \quad \text{and}\\
Z_3 &= \sum_{\substack{0\leq i,j\leq \ell\\ i\neq j}} P_i[[M_{f_1}, P_j], [M_{f_2}, P_j]].
\end{aligned}$$

Since $P^{(\ell)} = P_0 + \cdots + P_\ell$, we have

$$\begin{aligned}
[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}] &= P^{(\ell)}[[M_{f_1}, P^{(\ell)}], [M_{f_2}, P^{(\ell)}]]\\
&= \sum_{j=0}^{\ell} P_j[[M_{f_1}, P_j], [M_{f_2}, P_j]] + Z_0 + Z_1 + Z_2 + Z_3\\
&= \sum_{j=0}^{\ell} [T_{f_1,j}, T_{f_2,j}] + Z_0 + Z_1 + Z_2 + Z_3,
\end{aligned}$$

By Proposition 3.7, the operators Z_0, Z_1, Z_2, Z_3 are in the trace class. Applying Proposition 7.5, the proof will be complete once we show that $\mathrm{tr}(Z_0 + Z_1 + Z_2 + Z_3) = 0$.

Recall that the P_j's are orthogonal to each other. Thus if $i \neq j$, $j \neq k$ and $k \neq i$, then

$$P_i[[M_{f_1}, P_j], [M_{f_2}, P_k]]P_i = 0.$$

Therefore $\mathrm{tr}(Z_0) = 0$.

For any pair of $i \neq j$, we have

$$\begin{aligned}
P_i[[M_{f_1}, P_j], [M_{f_2}, P_j]]P_i &= P_i[M_{f_1}, P_j][M_{f_2}, P_j]P_i - P_i[M_{f_2}, P_j][M_{f_1}, P_j]P_i\\
&= -P_iM_{f_1}P_jM_{f_2}P_i + P_iM_{f_2}P_jM_{f_1}P_i\\
&= (P_iM_{f_2}P_j)(P_jM_{f_1}P_i) - (P_iM_{f_1}P_j)(P_jM_{f_2}P_i). \qquad (8.1)
\end{aligned}$$

Similarly, if $i \neq j$, then

$$P_j[[M_{f_1}, P_i], [M_{f_2}, P_i]]P_j = (P_j M_{f_2} P_i)(P_i M_{f_1} P_j) - (P_j M_{f_1} P_i)(P_i M_{f_2} P_j). \tag{8.2}$$

By the famous Lidskii theorem, if X and Y are bounded operators such that both XY and YX are in the trace class, then $\mathrm{tr}(XY) = \mathrm{tr}(YX)$. Combining this fact with (8.1) and (8.2), we reach the conclusion $\mathrm{tr}(Z_3) = 0$.

For any pair of $i \neq j$, we have

$$\begin{aligned} P_i[[M_{f_1}, P_i], [M_{f_2}, P_j]]P_i &= P_i[M_{f_1}, P_i][M_{f_2}, P_j]P_i - P_i[M_{f_2}, P_j][M_{f_1}, P_i]P_i \\ &= P_i M_{f_1}(P_i - 1)(-P_j)M_{f_2}P_i - P_i M_{f_2} P_j M_{f_1} P_i \\ &= (P_i M_{f_1} P_j)(P_j M_{f_2} P_i) - (P_i M_{f_2} P_j)(P_j M_{f_1} P_i). \end{aligned} \tag{8.3}$$

Similarly, when $i \neq j$, we have

$$\begin{aligned} P_j[[M_{f_1}, P_i], [M_{f_2}, P_j]]P_j &= P_j[M_{f_1}, P_i][M_{f_2}, P_j]P_j - P_j[M_{f_2}, P_j][M_{f_1}, P_i]P_j \\ &= P_j M_{f_1} P_i M_{f_2} P_j - P_j M_{f_2}(P_j - 1)(-P_i)M_{f_1}P_j \\ &= (P_j M_{f_1} P_i)(P_i M_{f_2} P_j) - (P_j M_{f_2} P_i)(P_i M_{f_1} P_j). \end{aligned} \tag{8.4}$$

Combining (8.3) and (8.4) with the Lidskii theorem, we see that

$$\mathrm{tr}(P_i[[M_{f_1}, P_i], [M_{f_2}, P_j]]P_i + P_j[[M_{f_1}, P_i], [M_{f_2}, P_j]]P_j) = 0$$

whenever $i \neq j$. Hence $\mathrm{tr}(Z_1 + Z_2) = 0$. This completes the proof. $\square$

Given an $\ell \geq 0$, let $g^{(\ell)}$ denote the Carey-Pincus principal function for the almost commuting pair $T^{(\ell)}_{f_1}$, $T^{(\ell)}_{f_2}$. It follows from (6.2) and Theorem 4.3 that

$$g^{(\ell)} = n^{(\ell)} \chi_S,$$

where

$$n^{(\ell)} = \mathrm{index}(T^{(\ell)}_F - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

Applying (6.1), we have

$$\mathrm{tr}[T^{(\ell)}_{f_1}, T^{(\ell)}_{f_2}] = \frac{-n^{(\ell)}}{2\pi i} \iint \chi_S(x, y) dxdy = \frac{-n^{(\ell)}}{2\pi i}.$$

Taking Theorem 8.1 and (2.9) into account, we can express the quantized Hall conductance

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -i\mathrm{tr}(P^{(\ell)}[[M_{f_1}, P^{(\ell)}], [M_{f_2}, P^{(\ell)}]])$$

for the case $(2\ell + 1)b < E < (2\ell + 3)b$, $\ell \geq 0$, in the following two ways:

$$\begin{aligned} \sigma_{\mathrm{Hall}}(P_{\leq E}) &= \frac{1}{2\pi} \mathrm{index}(T^{(\ell)}_{f_1 + i f_2} - \lambda), \quad \lambda \in S \backslash \partial S; \\ \sigma_{\mathrm{Hall}}(P_{\leq E}) &= -\frac{\ell + 1}{2\pi}. \end{aligned}$$

9. Switch functions of a different kind

As was mentioned in the Introduction, we now consider the pair of functions

$$\varphi_1(\zeta) = \mathrm{Re}\left(\frac{\zeta}{|\zeta|}\right) \quad \text{and} \quad \varphi_2(\zeta) = \mathrm{Im}\left(\frac{\zeta}{|\zeta|}\right),$$

$\zeta \in \mathbf{C}\backslash\{0\}$. Furthermore, we define

$$\Phi = \varphi_1 + i\varphi_2.$$

That is, $\Phi(\zeta) = \zeta/|\zeta|$ for $\zeta \in \mathbf{C}\backslash\{0\}$. The Toeplitz operator T_Φ is Fredholm, and its index is interpreted as the charge deficiency of the lowest Landau level when one magnetic flux quantum is introduced through the origin [1].

Here is what we can prove in this situation:

Theorem 9.1. (1) *The Toeplitz operator T_Φ is a compact perturbation of the unilateral shift.*
(2) *The commutator $[T_\Phi^*, T_\Phi]$ is in the trace class. Consequently, the commutator $[T_{\varphi_1}, T_{\varphi_2}]$ is in the trace class.*
(3) *We have $\mathrm{tr}[T_\Phi^*, T_\Phi] = 1$. In other words, $\mathrm{tr}[T_{\varphi_1}, T_{\varphi_2}] = (2i)^{-1}$.*

Proof. We have the standard orthonormal basis $\{e_k : k \geq 0\}$ for the Fock space $\mathcal{F}^2$, where

$$e_k(\zeta) = (k!)^{-1/2}\zeta^k.$$

Obviously, we have $\langle T_\Phi e_k, e_j\rangle = 0$ whenever $j \neq k+1$. On the other hand, for each $k \geq 0$,

$$\begin{aligned}\langle T_\Phi e_k, e_{k+1}\rangle = \langle \Phi e_k, e_{k+1}\rangle &= \frac{1}{\pi\sqrt{k!(k+1)!}}\int \frac{\zeta}{|\zeta|}\zeta^k\overline{\zeta^{k+1}}e^{-|\zeta|^2}dA(\zeta) \\ &= \frac{2}{\sqrt{k!(k+1)!}}\int_0^\infty r^{2k+2}e^{-r^2}dr, \end{aligned} \tag{9.1}$$

which we will denote by a_k.

We claim that the following two statements hold true:
(a) $a_{k+1} > a_k$ for every $k \geq 0$.
(b) $\lim_{k\to\infty} a_k = 1$.
Postponing the proofs of (a), (b) for a moment, we first deduce the conclusions of the theorem from these two statements.

By (9.1) and the fact that $\langle T_\Phi e_k, e_j\rangle = 0$ whenever $j \neq k+1$, we have

$$T_\Phi = \sum_{k=0}^\infty a_k e_{k+1} \otimes e_k. \tag{9.2}$$

Thus (b) implies that T_Φ is a compact perturbation of the unilateral shift

$$V = \sum_{k=0}^{\infty} e_{k+1} \otimes e_k,$$

proving (1).

By (9.2), we have

$$T_\Phi^* T_\Phi = \sum_{k=0}^{\infty} a_k^2 e_k \otimes e_k \quad \text{whereas} \quad T_\Phi T_\Phi^* = \sum_{k=0}^{\infty} a_k^2 e_{k+1} \otimes e_{k+1}.$$

Therefore

$$[T_\Phi^*, T_\Phi] = a_0^2 e_0 \otimes e_0 + \sum_{k=1}^{\infty} \{a_k^2 - a_{k-1}^2\} e_k \otimes e_k. \tag{9.3}$$

For $m \geq 1$, we have

$$a_0^2 + \sum_{k=1}^{m} \{a_k^2 - a_{k-1}^2\} = a_m^2 \leq 1,$$

where the $\leq$ follows from the fact that $\|\Phi\|_\infty = 1$. Combining this bound with the positivity $a_k^2 - a_{k-1}^2 > 0$ for every $k \geq 1$, which is provided by (a), we see that $[T_\Phi^*, T_\Phi]$ is in the trace class. Once we know that $[T_\Phi^*, T_\Phi]$ is in the trace class, we have

$$\mathrm{tr}[T_\Phi^*, T_\Phi] = a_0^2 + \lim_{m\to\infty} \sum_{k=1}^{m} \{a_k^2 - a_{k-1}^2\} = \lim_{m\to\infty} a_m^2.$$

By (b), the right-hand side equals 1. That is, $\mathrm{tr}[T_\Phi^*, T_\Phi] = 1$ as promised in (3).

Let us now prove (a). By (9.1), for every $k \geq 0$,

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\sqrt{k!(k+1)!}}{\sqrt{(k+1)!(k+2)!}} \cdot \frac{\int_0^\infty r^{2k+4} e^{-r^2} dr}{\int_0^\infty r^{2k+2} e^{-r^2} dr} = \frac{k + (3/2)}{\sqrt{(k+1)(k+2)}} \\ &= \left(\frac{(k+(3/2))^2}{(k+1)(k+2)} \right)^{1/2} = \left(\frac{k^2 + 3k + (9/4)}{k^2 + 3k + 2} \right)^{1/2} > 1,\end{aligned}$$

where the second $=$ involves an integration by parts. This proves (a).

To prove (b), first note that by (a), the limit

$$\lim_{k\to\infty} a_k = L$$

exists. Since $\|\Phi\|_\infty = 1$, we have $a_k \leq 1$ for every k. Therefore $L \leq 1$. The proof will be complete once we show that $L \geq 1$.

By the Cauchy-Schwarz inequality, we have

$$\left\{\int_0^\infty r^{2k+3}e^{-r^2}dr\right\}^2 \le \int_0^\infty r^{2k+2}e^{-r^2}dr\int_0^\infty r^{2k+4}e^{-r^2}dr.$$

Combining this with (9.1), we find that

$$\begin{aligned} a_k a_{k+1} &= \frac{2}{\sqrt{k!(k+1)!}}\int_0^\infty r^{2k+2}e^{-r^2}dr\frac{2}{\sqrt{(k+1)!(k+2)!}}\int_0^\infty r^{2k+4}e^{-r^2}dr \\ &\ge \frac{1}{(k+1)!\sqrt{k!(k+2)!}}\left\{2\int_0^\infty r^{2k+3}e^{-r^2}dr\right\}^2 = \frac{\{(k+1)!\}^2}{(k+1)!\sqrt{k!(k+2)!}} \\ &= \frac{k+1}{\sqrt{(k+1)(k+2)}} = \sqrt{\frac{k+1}{k+2}}. \end{aligned}$$

From this it is clear that $L \ge 1$. This completes the proof. $\square$

Remark. By (a) and (9.3), the Toeplitz operator T_Φ on $\mathcal{F}^2$ is *hyponormal*.

By (1.4), Theorem 9.1(3) has the alternative form

$$\text{(9.4)} \qquad \operatorname{tr}(P[[M_{\varphi_1}, P], [M_{\varphi_2}, P]]) = \frac{1}{2i}.$$

Denote $D = \{z \in \mathbf{C} : |z| < 1\}$. It is well known that for every $\lambda \in D$, $\operatorname{index}(V - \lambda) = -1$. Since T_Φ is a compact perturbation of V, we have

$$\text{(9.5)} \qquad \operatorname{index}(T_\Phi - \lambda) = -1 \quad \text{for every} \quad \lambda \in D.$$

Thus we can rewrite (9.4) in the form

$$\text{(9.6)} \qquad \operatorname{tr}(P[[M_{\varphi_1}, P], [M_{\varphi_2}, P]]) = -\frac{1}{2i}\operatorname{index}(T_\Phi - \lambda) \quad \text{for every} \quad \lambda \in D.$$

This is the formula inspired by (1.7).

Now consider the pair of operators $A = T_{\varphi_1}$ and $B = T_{\varphi_2}$. By (6.2) and (9.5) and the fact $\|T_\Phi\| \le 1$, we have

$$g_{A,B} = -\chi_D$$

for the Carey-Pincus principal function for this pair. By (6.1), we have

$$\operatorname{tr}[p(T_{\varphi_1}, T_{\varphi_2}), q(T_{\varphi_1}, T_{\varphi_2})] = \frac{1}{2\pi i}\iint_D \{p, q\}(x, y)dxdy$$

for all $p, q \in \mathbf{C}[x, y]$. If we write the unilateral shift V in the form $V = \tilde{A} + i\tilde{B}$, then the above also gives us the identity

$$\operatorname{tr}[p(T_{\varphi_1}, T_{\varphi_2}), q(T_{\varphi_1}, T_{\varphi_2})] = \operatorname{tr}[p(\tilde{A}, \tilde{B}), q(\tilde{A}, \tilde{B})]$$

for all $p, q \in \mathbf{C}[x, y]$.

Using the notation introduced in Section 2, we can rewrite (9.6) in the form

$$\text{(9.7)} \qquad \mathrm{tr}(P^{(0)}[[M_{\varphi_1}, P^{(0)}], [M_{\varphi_2}, P^{(0)}]]) = -\frac{1}{2i}\mathrm{index}(T_\Phi^{(0)} - \lambda) \quad \text{for every} \quad \lambda \in D.$$

One is, of course, not completely satisfied with this. Given the results in Section 8, the obvious question is, what happens with this pair of φ_1 and φ_2 at higher Landau levels? The obvious guess is that (9.7) also holds for $\ell \geq 1$. But so far we have not been able to prove this. Given (6.1) and (6.2), and given what we know about the Fredholm index, all the mathematical difficulties can be reduced to a single problem:

Problem 9.2. For $\ell \geq 1$, does the commutator $[T_{\varphi_1}^{(\ell)}, T_{\varphi_2}^{(\ell)}]$ belong to the trace class?

For the pair of switch functions f_1 and f_2 defined by (4.1), we obtain the membership of $[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}]$ in trace class from Proposition 3.7. In contrast, for φ_1 and φ_2, the individual terms $[M_{\varphi_1}, P][M_{\varphi_2}, P]$ and $[M_{\varphi_2}, P][M_{\varphi_1}, P]$ are not in the trace class, and Theorem 9.1(2) depends on the cancellation between these terms. If, for $\ell \geq 1$, the commutator $[T_{\varphi_1}^{(\ell)}, T_{\varphi_2}^{(\ell)}]$ is to be in the trace class, the right cancellation between the terms

$$[M_{\varphi_1}, P^{(\ell)}][M_{\varphi_2}, P^{(\ell)}] \quad \text{and} \quad [M_{\varphi_2}, P^{(\ell)}][M_{\varphi_1}, P^{(\ell)}]$$

must take place to bring about this membership. In other words, Problem 9.2 deals with a much more subtle situation.

10. A more classic example

Consider the pair of momentum, position operators D, M that appear in every quantum mechanics textbook. Namely,

$$D = -i\frac{d}{dx},$$

which is a self-adjoint operator on a dense domain in $L^2(\mathbf{R})$, and

$$(Mh)(x) = xh(x),$$

which is also a self-adjoint operator on a dense domain in $L^2(\mathbf{R})$. For $f, g \in C_c^\infty(\mathbf{R})$, the operator $f(M)g(D)$ is well known to be compact. Thus if E, F are bounded Borel sets in $\mathbf{R}$, then the operator

$$\chi_E(M)\chi_F(D)$$

is compact.

Lemma 10.1. *Let E be a bounded Borel set in* $\mathbf{R}$. *Then the essential norm of*

$$\chi_E(D) + \chi_E(M)$$

is at most 1.

Proof. By the above, the difference

$$(\chi_E(D) + \chi_E(M)) - (\chi_E(D) + \chi_E(M))^2$$

is a compact operator. Therefore the essential spectrum of $\chi_E(D) + \chi(M)$ is contained in the two-point set $\{0, 1\}$. □

Define the function

$$\sigma(x) = \frac{x}{\sqrt{1+x^2}}, \quad x \in \mathbf{R}.$$

It was shown in [5] that the commutator $[\sigma(D), \sigma(M)]$ is in the trace class, and that

$$\mathrm{tr}[\sigma(D), \sigma(M)] = \frac{2}{\pi i}. \tag{10.1}$$

See Example 2 in [5]. We will work out the Carey-Pincus theory for the pair $\sigma(D)$, $\sigma(M)$. This shows that our techniques are capable of handling diverse examples.

Let $C_{*,*}$ denote the collection of continuous functions f on $\mathbf{R}$ such that both limits

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

exist and are finite.

Lemma 10.2. *For any $f, g \in C_{*,*}$, the commutator $[f(D), g(M)]$ is compact.*

Proof. The maximal ideal space of $C_{*,*}$ is the obvious two-point compactification of $\mathbf{R}$. The Gelfand transform of σ obviously separates points on this maximal ideal space. Thus by the Stone-Weierstrass approximation theorem, $\{p \circ \sigma : p \in \mathbf{C}[x]\}$ is dense in $C_{*,*}$ with respect to the norm $\|\cdot\|_\infty$. Since $[\sigma(D), \sigma(M)]$ is in the trace class, therefore compact, the compactness of $[f(D), g(M)]$, $f, g \in C_{*,*}$, follows from from this density. □

Let Φ_0 denote the collection of continuous functions φ on $\mathbf{R}$ satisfying the following two conditions

(1) $-1 \le \varphi \le 1$ on $\mathbf{R}$.

(2) There is an $R = R(\varphi) \in (0, \infty)$ such that $\varphi = 1$ on $[R, \infty)$ and $\varphi = -1$ on $(-\infty, -R]$.

Define the square $\Omega = \{x + iy : x, y \in [-1, 1]\}$.

Lemma 10.3. *If $\varphi, \psi \in \Phi_0$, then the essential spectrum of the operator*

$$T = \varphi(D) + i\psi(M)$$

is contained in $\partial\Omega$.

Proof. By Lemma 4.2, it suffices to show that the essential spectrum of T does not intersect the interior of Ω.

Let $\lambda = \alpha + i\beta \in \Omega \backslash \partial\Omega$, i.e., $-1 < \alpha < 1$ and $-1 < \beta < 1$. Then there are operators W, X, Y, Z such that

$$\begin{aligned}\{1 - \alpha + i(\psi(M) - \beta)\}W &= 1,\\ \{-1 - \alpha + i(\psi(M) - \beta)\}X &= 1,\\ \{\varphi(D) - \alpha + i(1 - \beta)\}Y &= 1 \quad \text{and}\\ \{\varphi(D) - \alpha + i(-1 - \beta)\}Z &= 1.\end{aligned}$$

By the membership $\varphi, \psi \in \Phi_0$, there is an $R > 0$ such that $\varphi = -1 = \psi$ on $(-\infty, -R]$ and $\varphi = 1 = \psi$ on $[R, \infty)$. Let η be a continuous function on $\mathbf{R}$ satisfy the following conditions:

(a) $0 \le \eta \le 1$ on $\mathbf{R}$.
(b) $\eta = 0$ on $(-\infty, R + 1]$.
(c) $\eta = 1$ on $[R + 2, \infty)$.

We further define the function $\xi(x) = \eta(-x)$, $x \in \mathbf{R}$.

Since $(\varphi - \alpha)\eta = (1 - \alpha)\eta$ on $\mathbf{R}$, we have

$$\begin{aligned}(T - \lambda)\eta(D)W &= (1 - \alpha)\eta(D)W + i(\psi(M) - \beta)\eta(D)W\\ &= (1 - \alpha)\eta(D)W + i\eta(D)(\psi(M) - \beta)W + K_1\\ &= \eta(D)\{(1 - \alpha) + i(\psi(M) - \beta)\}W + K_1\\ &= \eta(D) + K_1,\end{aligned}$$

where $K_1 = i[\psi(M), \eta(D)]W$, which is a compact operator. Similarly, we have

$$\begin{aligned}(T - \lambda)\xi(D)X &= \xi(D) + K_2,\\ (T - \lambda)\eta(M)Y &= \eta(M) + K_3 \quad \text{and}\\ (T - \lambda)\xi(M)Z &= \xi(M) + K_4,\end{aligned}$$

where K_2, K_3 and K_4 are compact operators. Thus if we define

$$G = \eta(D)W + \xi(D)X + \eta(M)Y + \xi(M)Z,$$

then

$$(T - \lambda)G = \eta(D) + \xi(D) + \eta(M) + \xi(M) + K, \tag{10.2}$$

where K is a compact operator. Define the function

$$u = 1 - \eta - \xi$$

on $\mathbf{R}$. Then (10.2) becomes

$$(T - \lambda)G = 2 - \{u(D) + u(M)\} + K. \tag{10.3}$$

A chase of the definitions of η, ξ tells us that $0 \le u \le 1$ on $\mathbf{R}$ and that $u = 0$ on $(-\infty, -R-2] \cup [R+2, \infty)$. Hence it follows from Lemma 10.1 that the essential norm of $u(D) + u(M)$ is at most 1. Consequently, $2 - \{u(D) + u(M)\} + K$ is a Fredholm operator. Therefore from (10.3) we conclude that $T - \lambda$ has a right Fredholm inverse. A similar argument shows that $T - \lambda$ also has a left Fredholm inverse. □

Let Φ denote the collection of continuous functions φ on $\mathbf{R}$ satisfying the following two conditions:

(1) $-1 \le \varphi \le 1$ on $\mathbf{R}$.
(2) $\lim_{x\to\infty} \varphi(x) = 1$ and $\lim_{x\to-\infty} \varphi(x) = -1$.

Theorem 10.4. *If $\varphi, \psi \in \Phi$, then the essential spectrum of the operator*

$$T = \varphi(D) + i\psi(M)$$

is contained in $\partial\Omega$.

Proof. Again, by Lemma 4.2, it suffices to show that the essential spectrum of T does not intersect the interior of Ω.

Let λ be a point in the interior of Ω. For $\varphi, \psi \in \Phi$, there are sequences $\{\varphi_k\} \subset \Phi_0$ and $\{\psi_k\} \subset \Phi_0$ such that $\lim_{k\to\infty} \|\varphi_k - \varphi\|_\infty = 0$ and $\lim_{k\to\infty} \|\psi_k - \psi\|_\infty = 0$. Define

$$T_k = \varphi_k(D) + i\psi_k(M)$$

for each k. Then $\lim_{k\to\infty} \|T - T_k\| = 0$. By Lemma 10.3, $T_k - \lambda$ is a Fredholm operator. Let F_k be a Fredholm inverse of $T_k - \lambda$. Then

$$T - \lambda = T_k - \lambda + (T - T_k) = (T_k - \lambda)\{1 + F_k(T - T_k)\} + K_k, \tag{10.4}$$

where K_k is a compact operator. If N is a normal operator whose spectrum is contained in $\partial\Omega$, then $\|(N-\lambda)^{-1}\| \le 1/\mathrm{dist}(\lambda, \partial\Omega)$. Lemma 10.2 tells us that T_k is essentially normal. Therefore by applying the GNS representation of the Calkin algebra and Lemma 10.3, we obtain the bound $\|F_k\|_{\mathrm{ess}} \le 1/\mathrm{dist}(\lambda, \partial\Omega)$. Thus by choosing a large k, $\|F_k(T - T_k)\|_{\mathrm{ess}}$ can be arbitrarily small. Hence from (10.4) we see that $T - \lambda$ has a Fredholm inverse. □

We now apply the Carey-Pincus theory to the pair $A = \sigma(D)$ and $B = \sigma(M)$. Theorem 10.4 says that the essential spectrum of $\sigma(D) + i\sigma(M)$ is contained in $\partial\Omega$. Hence

$$\mathrm{index}(\sigma(D) + i\sigma(M) - \lambda) = 0 \quad \text{for every} \quad \lambda \in \mathbf{C}\backslash\Omega.$$

Consequently, for the pair $A = \sigma(D)$ and $B = \sigma(M)$, we have $g_{A,B} = n\chi_\Omega$, where

$$n = \mathrm{index}(\sigma(D) + i\sigma(M) - \lambda) \quad \text{for every} \quad \lambda \in \Omega\backslash\partial\Omega.$$

Applying (6.1) in the case $p(x, y) = x$ and $q(x, y) = y$, we obtain

$$\mathrm{tr}[\sigma(D), \sigma(M)] = \frac{-1}{2\pi i} \iint n\chi_\Omega(x, y)dxdy = \frac{-2n}{\pi i}.$$

Comparing this with (10.1), we find that $n = -1$. In other words,

$$\text{index}(\sigma(D) + i\sigma(M) - \lambda) = -1 \quad \text{for every} \quad \lambda \in \Omega \backslash \partial\Omega. \tag{10.5}$$

Thus $-\chi_\Omega$ is the principal function for the pair $A = \sigma(D)$ and $B = \sigma(M)$. Applying (6.1) in this situation, we obtain the trace formula

$$\text{tr}[p(\sigma(D), \sigma(M)), q(\sigma(D), \sigma(M))] = \frac{1}{2\pi i} \iint_\Omega \{p, q\}(x, y) dx dy \tag{10.6}$$

for all polynomials $p, q \in \mathbf{C}[x, y]$.

From (10.6) we see, for example, that $\text{tr}([\sigma(D), \sigma(M)]\sigma(M)) = 0$. Again, such a calculation would not be possible without the Carey-Pincus theory.

Theorem 10.5. *If $\varphi, \psi \in \Phi$, then for every $\lambda \in \Omega \backslash \partial\Omega$ we have*

$$\text{index}(\varphi(D) + i\psi(M) - \lambda) = -1.$$

Proof. Let $\lambda \in \Omega \backslash \partial\Omega$ be given. For each $0 \leq t \leq 1$, define

$$Y_t = (t\varphi + (1 - t)\sigma)(D) + i(t\psi + (1 - t)\sigma)(M) - \lambda.$$

By Theorem 10.4, for every $0 \leq t \leq 1$, the operator Y_t is Fredholm. Obviously, the map $t \mapsto Y_t$ is continuous with respect to the operator norm. Therefore the map $t \mapsto \text{index}(Y_t)$ is a constant on $[0, 1]$. We have $\text{index}(Y_0) = -1$ by (10.5). Hence $\text{index}(Y_1) = -1$. $\square$

Thus, by the above argument, whenever $\varphi, \psi \in \Phi$ are such that the commutator $[\varphi(D), \psi(M)]$ is in the trace class, the principal function for the pair $\varphi(D)$, $\psi(M)$ has no choice but be equal to $-\chi_\Omega$, which means

$$\text{tr}[p(\varphi(D), \psi(M)), q(\varphi(D), \psi(M))] = \text{tr}[p(\sigma(D), \sigma(M)), q(\sigma(D), \sigma(M))].$$

for all $p, q \in \mathbf{C}[x, y]$.

Data availability

No data was used for the research described in the article.

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