# MULTIPLIERS AND ESSENTIAL NORM ON THE DRURY-ARVESON SPACE

QUANLEI FANG AND JINGBO XIA

Abstract. It is well known that for multipliers $f$ of the Drury-Arveson space $H_n^2$, $\|f\|_\infty$ does not dominate the operator norm of $M_f$. We show that in general $\|f\|_\infty$ does not even dominate the essential norm of $M_f$. A consequence of this is that there exist multipliers $f$ of $H_n^2$ for which $M_f$ fails to be essentially hyponormal, i.e., if $K$ is any compact, self-adjoint operator, then the inequality $M_f^*M_f - M_fM_f^* + K \geq 0$ does not hold.

## 1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z : |z| < 1\}$ in $\mathbf{C}^n$. In this paper, the complex dimension $n$ is assumed to be greater than or equal to 2. An analogue of the classic Hardy space is the space $H_n^2$ of analytic functions on $\mathbf{B}$ introduced by Drury [8] and Arveson [2]. Because of its connection to various topics in operator theory, e.g. the von Neumann inequality for commuting row contractions, $H_n^2$ has been the subject of intense recent studies [1-7,9,10,12].

Recall that the Drury-Arveson space $H_n^2$ is a reproducing kernel Hilbert space with

$$K(z,w) = \frac{1}{1-\langle z,w\rangle}, \quad z,w \in \mathbf{B},$$

as its kernel [2, 8]. Note that $K(z,w)$ is a multivariable-generalization of the one-variable Szegö kernel. An orthonormal basis of $H_n^2$ is given by $\{(|\alpha|!/\alpha!)^{1/2}\zeta^\alpha : \alpha \in \mathbf{Z}_+^n\}$, where we use the standard multi-index notation. Thus for functions $f, g \in H_n^2$ with Taylor expansions

$$f(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} c_\alpha\zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} d_\alpha\zeta^\alpha,$$

the inner product is given by

$$\langle f,g\rangle = \sum_{\alpha\in\mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} c_\alpha\overline{d_\alpha}.$$

With the identification of each variable $\zeta_i$ with each multiplication operator $M_{\zeta_i}$, $H_n^2$ is a free Hilbert module over the polynomial ring $\mathbf{C}[\zeta_1,\dots,\zeta_n]$. See [2].

An analytic function $f$ on $\mathbf{B}$ is said to be a *multiplier* of the Drury-Arveson space $H_n^2$ if $fg \in H_n^2$ for every $g \in H_n^2$. Throughout the paper, we denote the collection of multipliers of $H_n^2$ by $\mathcal{M}$. For each $f \in \mathcal{M}$, the multiplication operator $M_f$ defined by $M_fg = fg$ is necessarily bounded on $H_n^2$ [2], and the operator norm $\|M_f\|$ is also called the multiplier norm of $f$.

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Multipliers are an important part of operator theory on $H^2_n$. For example, if $\mathcal{E}$ is a closed linear subspace of $H^2_n$ which is invariant under $M_{\zeta_1}, \dots, M_{\zeta_n}$, then there exist $\{f_1, \dots, f_k, \dots\} \subset \mathcal{M}$ such that the operator

$$M_{f_1}M^*_{f_1} + \cdots + M_{f_k}M^*_{f_k} + \cdots$$

is the orthogonal projection from $H^2_n$ onto $\mathcal{E}$ (see page 191 in [3]).

Among the recent results related to multipliers, we would like to mention the following developments. Interpolation problems for multipliers and model theory related to the Drury-Arveson space have been intensely studied over the past decade or so [4, 5, 10, 12]. Recently, Arcozzi, Rochberg and Sawyer gave a characterization of the multipliers in terms of Carleson measures for $H^2_n$ [1]. In [7], Costea, Sawyer and Wick proved a corona theorem for $\mathcal{M}$. More recently, we showed in [9] that for each $f \in \mathcal{M}$ and each $1 \le i \le n$, the commutator $[M^*_f, M_{\zeta_i}]$ belongs to the Schatten class $\mathcal{C}_p$, $p > 2n$.

Of particular relevance to this paper is the fact that under the assumption $n \ge 2$, $\mathcal{M}$ is strictly smaller than $H^\infty$ [2]. Moreover, Arveson showed in [2] that, even for polynomials $q$, $\|q\|_\infty$ in general does not dominate the operator norm of $M_q$ on $H^2_n$. This naturally brings up the question, what about the essential norm of $M_f$ on $H^2_n$ for general $f \in \mathcal{M}$?

Recall that the *essential norm* of a bounded operator $A$ on a Hilbert space $\mathcal{H}$ is

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \in \mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ is the collection of compact operators on $\mathcal{H}$. Alternately, $\|A\|_{\mathcal{Q}} = \|\pi(A)\|$, where $\pi$ denotes the quotient homomorphism from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Let $\mathcal{T}_n$ be the $C^*$-algebra generated by $M_{\zeta_1}, \cdots, M_{\zeta_n}$ on $H^2_n$, which was introduced by Arveson in [2]. In more ways than one, $\mathcal{T}_n$ is the analogue of the $C^*$-algebra generated by Toeplitz operators with *continuous* symbols. Indeed Arveson showed that there is an exact sequence

$$\{0\} \to \mathcal{K}(H^2_n) \to \mathcal{T}_n \xrightarrow{\tau} C(S) \to \{0\}, \tag{1.1}$$

where the homomorphism $\tau$ is an extension of the map

$$\tau(M_{\zeta_j}) = \zeta_j,$$

$j = 1, \dots, n$, and $S = \{z \in \mathbf{C}^n : |z| = 1\}$. It follows that if $q$ is a *polynomial*, then

$$\|M_q\|_{\mathcal{Q}} = \|q\|_\infty. \tag{1.2}$$

This equality can also be understood from a slightly different point of view. Indeed by Proposition 5.3 in [2], for each polynomial $q$, the operator $M_q$ is *essentially normal*, i.e., $[M^*_q, M_q]$ is compact. On the other hand, by Proposition 2.12 in [2], if $q$ is a polynomial, then the spectral radius of $M_q$ equals $\|q\|_\infty$. Since the norm and the spectral radius of any normal element in any $C^*$-algebra coincide, it follows that $\|M_q\|_{\mathcal{Q}} \le \|q\|_\infty$ whenever $q$ is a polynomial. The reverse inequality, $\|M_q\|_{\mathcal{Q}} \ge \|q\|_\infty$, is easy once $M^*_q$ is applied to the normalized reproducing kernel of $H^2_n$.

Equality (1.2) is particularly interesting in view of the fact that, even for polynomials, $\|q\|_\infty$ in general does not dominate the operator norm of $M_q$. The obvious question is, what happens if we consider a general $f \in \mathcal{M}$?

We report that (1.2) in general fails if we consider multipliers which are not polynomials.

**Theorem 1.1.** *There exists a sequence $\{\psi_k\} \subset \mathcal{M}$ such that*

$$\inf_{k\geq 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|\psi_k\|_\infty = 0.$$

This has implications for other essential properties of multipliers.

Recall that an operator $T$ is said to be *hyponormal* if $T^*T - TT^* \geq 0$. It is well known that the norm of a hyponormal operator coincides with its spectral radius. As we mentioned earlier, by Proposition 2.12 in [2], if $q$ is a polynomial, then the spectral radius of $M_q$ equals $\|q\|_\infty$. Therefore if $q$ is a polynomial such that $\|M_q\| > \|q\|_\infty$, then $M_q$ is not hyponormal. Thus there are plenty of multipliers $f \in \mathcal{M}$ for which $M_f$ fails to be hyponormal on $H^2_n$. This is one phenomenon that sets the Drury-Arveson space apart from the Hardy space and the Bergman space. We will show that this phenomenon persists under compact perturbation.

**Definition.** An operator $T$ is said to be *essentially hyponormal* if there is a compact self-adjoint operator $K$ such that

$$T^*T - TT^* + K \geq 0.$$

Obviously, $T$ is essentially hyponormal if and only if $\pi(T)$ is a hyponormal element in the Calkin algebra $\mathcal{Q}$, i.e., $\pi(T^*)\pi(T) - \pi(T)\pi(T^*) \geq 0$.

**Theorem 1.2.** *There exists a $\psi \in \mathcal{M}$ such that the multiplication operator $M_\psi$ on $H^2_n$ is not essentially hyponormal.*

Having introduced our results, the rest of this short paper consists of their proofs.

## 2. Estimates for Certain Multipliers

The proof of Theorem 1.1 involves Möbius transform. For each $z \in \mathbf{B}\backslash\{0\}$, let

$$\varphi_z(\zeta) = \frac{1}{1-\langle \zeta, z\rangle}\left\{z - \frac{\langle \zeta, z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(\zeta - \frac{\langle \zeta, z\rangle}{|z|^2}z\right)\right\}. \tag{2.1}$$

Then $\varphi_z$ is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$ [13,Theorem 2.2.2]. Recall that the normalized reproducing kernel for $H^2_n$ is given by

$$k_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{1-\langle \zeta, z\rangle}, \qquad z, \zeta \in \mathbf{B}. \tag{2.2}$$

For each $z \in \mathbf{B}\backslash\{0\}$, define the operator $U_z$ by the formula

$$(U_z g)(\zeta) = g(\varphi_z(\zeta))k_z(\zeta), \qquad g \in H^2_n. \tag{2.3}$$

It follows easily from Theorem 2.2.2 in [13] that if $z \in \mathbf{B}\backslash\{0\}$ and $x, y \in \mathbf{B}$, then

$$\langle U_z k_x, U_z k_y\rangle = \frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{1-\langle y, x\rangle} = \langle k_x, k_y\rangle.$$

Hence each $U_z$ is a unitary operator on $H^2_n$. Moreover, we have

$$U_z M_f U_z^* = M_{f\circ\varphi_z} \tag{2.4}$$

for all $z \in \mathbf{B}\backslash\{0\}$ and $f \in \mathcal{M}$.

For each $j \in \mathbf{N}$, let $E_j$ be the linear span of $\{\zeta^\alpha : |\alpha| \leq j\}$ in $H^2_n$, and let $P_j : H^2_n \to E_j$ be the orthogonal projection. Moreover, denote

$$Q_j = 1 - P_j.$$

Obviously, we have the strong convergence $Q_j \to 0$ as $j \to \infty$.

**Lemma 2.1.** *For each $j \in \mathbf{N}$, there is a constant $1 \leq C_j < \infty$ such that*

$$\limsup_{|z|\uparrow 1} \|M_{f\circ\varphi_z} P_j\| \leq C_j \|M_f\|_{\mathcal{Q}}$$

*for every $f \in \mathcal{M}$.*

*Proof.* For each $j \in \mathbf{N}$, since $\dim(E_j) < \infty$, any two norms on $E_j$ are equivalent. Since $E_j$ consists of polynomials, we have $\|M_g\| < \infty$ for each $g \in E_j$. Hence there is a $C_j$ such that

$$\|M_g\| \leq C_j \|g\| \tag{2.5}$$

for every $g \in E_j$. Now let $f \in \mathcal{M}$. Using the unitary $U_z$, we have $\|f \circ \varphi_z\| = \|U_z(f \circ \varphi_z)\| = \|f k_z\|$. Since $k_z \to 0$ weakly as $|z| \uparrow 1$, we have

$$\limsup_{|z|\uparrow 1} \|f \circ \varphi_z\| = \limsup_{|z|\uparrow 1} \|f k_z\| \leq \limsup_{|z|\uparrow 1} \|(M_f + K) k_z\|$$

for every compact operator $K$. Consequently,

$$\limsup_{|z|\uparrow 1} \|f \circ \varphi_z\| \leq \|M_f\|_{\mathcal{Q}}. \tag{2.6}$$

Now if $g \in E_j$, then, using (2.5), we have

$$\|M_{f\circ\varphi_z} g\| = \|M_g(f \circ \varphi_z)\| \leq \|M_g\| \|f \circ \varphi_z\| \leq C_j \|f \circ \varphi_z\| \|g\|.$$

Hence $\|M_{f\circ\varphi_z} P_j\| \leq C_j \|f \circ \varphi_z\|$. Combining this with (2.6), the lemma follows. □

The next lemma is so elementary that its proof will be omitted.

**Lemma 2.2.** *For each bounded operator $A$ on $H^2_n$, we have*

$$\limsup_{j\to\infty} \|Q_j A\| \leq \|A\|_{\mathcal{Q}} \qquad \textit{and} \qquad \limsup_{j\to\infty} \|A Q_j\| \leq \|A\|_{\mathcal{Q}}.$$

*Proof of Theorem* 1.1. By Theorem 3.3 in [2], there is a sequence of polynomials $\{p_i\}$ such that

$$\|M_{p_i}\| = 1 \tag{2.7}$$

for every $i$ and

$$\lim_{i\to\infty} \|p_i\|_\infty = 0. \tag{2.8}$$

We will find a sequence of natural numbers $\{i(j)\}_{j=1}^\infty$ and a sequence $\{z_j\} \subset \mathbf{B}\backslash\{0\}$ such that the desired multipliers $\{\psi_k\}$ will have the form

$$\psi_k = \sum_{j=k}^\infty p_{i(j)} \circ \varphi_{z_j}, \tag{2.9}$$

$k \in \mathbf{N}$. To do this, we note that (2.8) enables us to inductively select an ascending sequence of natural numbers

$$\ell(1) < \ell(2) < \cdots < \ell(m) < \cdots$$

such that

$$C_m\|p_{\ell(m)}\|_\infty \le \frac{1}{2^m} \tag{2.10}$$

for each $m \in \mathbf{N}$, where $C_m$ is the constant provided by Lemma 2.1. Since each $p_i$ is a polynomial, by (1.2) this implies

$$C_m\|M_{p_{\ell(m)}}\|_{\mathcal{Q}} \le \frac{1}{2^m}. \tag{2.11}$$

By (2.11) and Lemma 2.1, for each $m \in \mathbf{N}$ there is a $w_m \in \mathbf{B}\backslash\{0\}$ such that

$$\|M_{p_{\ell(m)}\circ\varphi_{w_m}}P_m\| \le \frac{2}{2^m}. \tag{2.12}$$

It follows from Lemma 2.2 that for each $m \in \mathbf{N}$ there is a natural number $r(m) > m$ such that

$$\|M_{p_{\ell(m)}\circ\varphi_{w_m}}Q_{r(m)}\| \le 2\|M_{p_{\ell(m)}\circ\varphi_{w_m}}\|_{\mathcal{Q}} = 2\|M_{p_{\ell(m)}}\|_{\mathcal{Q}},$$

where the $=$ is a consequence of (2.4). By (2.11) and the fact that $C_m \ge 1$, we have

$$\|M_{p_{\ell(m)}\circ\varphi_{w_m}}Q_{r(m)}\| \le \frac{2}{2^m}. \tag{2.13}$$

By a similar argument, for each $m \in \mathbf{N}$, there is an $s(m) > m$ such that

$$\|Q_{s(m)}M_{p_{\ell(m)}\circ\varphi_{w_m}}\| \le \frac{2}{2^m}. \tag{2.14}$$

Note that for each $m$, the subspace $Q_mH_n^2$ is invariant under $\{M_f : f \in \mathcal{M}\}$. That is,

$$M_fg = Q_mM_fg \quad \text{if} \quad g \in Q_mH_n^2 \quad \text{and} \quad f \in \mathcal{M}.$$

Using this fact and the relation $P_i = 1 - Q_i$, it follows from simple algebra that

$$\begin{aligned} M_{p_{\ell(m)}\circ\varphi_{w_m}} = {} & M_{p_{\ell(m)}\circ\varphi_{w_m}}P_m + Q_{s(m)}M_{p_{\ell(m)}\circ\varphi_{w_m}}(Q_m - Q_{r(m)}) + M_{p_{\ell(m)}\circ\varphi_{w_m}}Q_{r(m)} \\ & + (P_{s(m)} - P_m)M_{p_{\ell(m)}\circ\varphi_{w_m}}(P_{r(m)} - P_m). \end{aligned}$$

Thus if we set

$$Y_m = M_{p_{\ell(m)}\circ\varphi_{w_m}}P_m + Q_{s(m)}M_{p_{\ell(m)}\circ\varphi_{w_m}}(Q_m - Q_{r(m)}) + M_{p_{\ell(m)}\circ\varphi_{w_m}}Q_{r(m)},$$

then

$$M_{p_{\ell(m)}\circ\varphi_{w_m}} = Y_m + (P_{s(m)} - P_m)M_{p_{\ell(m)}\circ\varphi_{w_m}}(P_{r(m)} - P_m), \tag{2.15}$$

$m \in \mathbf{N}$. Note that by (2.12), (2.13) and (2.14), we have

$$\|Y_m\| \le \frac{6}{2^m}. \tag{2.16}$$

Set $m_1 = 5$. We then inductively select a sequence of integers $m_1 < m_2 < \cdots < m_j < \cdots$ such that the inequality

$$m_{j+1} > \max\{r(m_j), s(m_j)\} \tag{2.17}$$

holds for every $j \ge 1$. Now set

$$i(j) = \ell(m_j) \quad \text{and} \quad z_j = w_{m_j} \tag{2.18}$$

for each $j \in \mathbf{N}$. With this notation, from (2.15) we obtain

$$M_{p_{i(j)}\circ\varphi_{z_j}} = Y_{m_j} + (P_{s(m_j)} - P_{m_j})M_{p_{i(j)}\circ\varphi_{z_j}}(P_{r(m_j)} - P_{m_j}). \tag{2.19}$$

With $i(j)$ and $z_j$ determined as above, we now define $\psi_k$ by (2.9) for each $k \ge 1$.

Next we show that the sequence $\{\psi_k\}$ has the desired properties. First we need to show that $\psi_k \in \mathcal{M}$ for each $k$. By the relations $s(m) > m, r(m) > m$ and (2.17), we have

$$P_{s(m_j)} - P_{m_j} \perp P_{s(m_{j'})} - P_{m_{j'}} \quad \text{and} \quad P_{r(m_j)} - P_{m_j} \perp P_{r(m_{j'})} - P_{m_{j'}} \tag{2.20}$$

whenever $j < j'$. Recall that $\|M_{p_i \circ \varphi_z}\| = \|M_{p_i}\|$ by (2.4) and that $\|M_{p_i}\| = 1$ by choice. Combining these facts with (2.20), we see that the norm of the operator

$$B_k = \sum_{j=k}^{\infty} (P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j})$$

does not exceed 1. By (2.16) and the choice that $m_1 = 5$, the norm of the operator

$$A_k = \sum_{j=k}^{\infty} Y_{m_j}$$

does not exceed 1/2. By (2.9) and (2.19), $M_{\psi_k} = A_k + B_k$. Thus the norm of the operator $M_{\psi_k}$ is at most 3/2. That is, $\psi_k \in \mathcal{M}$ for each $k \in \mathbf{N}$. Applying (2.18) and (2.10), we have

$$\|\psi_k\|_\infty \le \sum_{j=k}^{\infty} \|p_{i(j)} \circ \varphi_{z_j}\|_\infty = \sum_{j=k}^{\infty} \|p_{i(j)}\|_\infty \le \sum_{j=k}^{\infty} \frac{1}{2^{m_j}}.$$

Hence

$$\lim_{k\to\infty} \|\psi_k\|_\infty = 0.$$

What remains for the proof is the inequality

$$\inf_{k\ge 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0.$$

Since $M_{\psi_k} = A_k + B_k$ and $\|A_k\| \le 1/2$, it suffices to show that

$$\inf_{k\ge 1} \|B_k\|_{\mathcal{Q}} \ge 1.$$

Since $\|M_{p_{i(j)} \circ \varphi_{z_j}}\| = \|M_{p_{i(j)}}\| = 1$ and $\lim_{j\to\infty} \|Y_{m_j}\| = 0$, by (2.19) we have

$$\lim_{j\to\infty} \|(P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j})\| = 1.$$

Therefore there exists a sequence of unit vectors $\{g_j\}$ such that

$$g_j \in (P_{r(m_j)} - P_{m_j}) H_n^2 \tag{2.21}$$

for every $j$ and

$$\lim_{j\to\infty} \|(P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j}) g_j\| = 1. \tag{2.22}$$

By (2.21) and (2.20), $g_j \to 0$ weakly as $j \to \infty$. By this weak convergence and (2.22), we have

$$\lim_{j\to\infty} \|(B_k + K) g_j\| = 1$$

for every compact operator $K$. This implies $\|B_k\|_{\mathcal{Q}} \ge 1$, completing the proof of Theorem 1.1. □

## 3. Spectral radius

To prove Theorem 1.2, we begin with a simple fact about multipliers.

**Lemma 3.1.** *Let $f \in \mathcal{M}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then $1/f$ is also a multiplier of $H^2_n$.*

*Proof.* This certainly follows from the recently proved corona theorem for $\mathcal{M}$ [7]. But it also follows from an earlier, much simpler result due to Chen [6]. By Theorem 2 in [6], there are constants $0 < A \leq B < \infty$ such that $A\|g\| \leq \|g\|_\# \leq B\|g\|$ for every $g \in H^2_n$, where

$$\|g\|^2_\# = |g(0)|^2 + \iint \frac{|g(z) - g(w)|^2}{|1 - \langle z, w\rangle|^{2n+1}} dv(z)dv(w). \tag{3.1}$$

Let $f \in \mathcal{M}$ be such that $|f| \geq c > 0$ on $\mathbf{B}$. Then for each $g \in H^2_n$,

$$\frac{g(z)}{f(z)} - \frac{g(w)}{f(w)} = \frac{g(z) - g(w)}{f(z)} + \frac{g(z) - g(w)}{f(w)} + \frac{f(w)g(w) - f(z)g(z)}{f(z)f(w)}, \tag{3.2}$$

$z, w \in \mathbf{B}$. From (3.1) and (3.2) we see that $1/f \in \mathcal{M}$. $\square$

From this lemma we immediately obtain

**Proposition 3.2.** *For each $f \in \mathcal{M}$, the spectrum of the operator $M_f$ on $H^2_n$ is contained in the closure of $\{f(z) : z \in \mathbf{B}\}$. Consequently the spectral radius of $M_f$ does not exceed $\|f\|_\infty$.*

**Remark 1.** In the case where $f$ has the property that there is a sequence of polynomials $\{p_k\}$ such that $\lim_{k\to\infty} \|M_f - M_{p_k}\| = 0$, Proposition 3.2 was proved by Arveson. See Proposition 2.12 in [2].

**Remark 2.** It follows from (3.1) and (3.2) that if $f \in \mathcal{M}$ and if $\inf_{z\in\mathbf{B}} |f(z)| > 0$, then

$$\|M_{1/f}\| \leq C(\|1/f\|_\infty + \|1/f\|^2_\infty \|M_f\|).$$

Surprisingly, Chen himself did not seem to notice this fact in [6].

**Proposition 3.3.** *Let $f \in \mathcal{M}$. If $f$ has the property that $\|M_f\|_\mathcal{Q} > \|f\|_\infty$, then the operator $M_f$ on $H^2_n$ is not essentially hyponormal.*

*Proof.* Recall that we denote the quotient map from $\mathcal{B}(H^2_n)$ to the Calkin algebra $\mathcal{Q}$ by $\pi$. Let $\Phi$ be the GNS representation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$. If $M_f$ were essentially hyponormal, then $\{\pi(M_f)\}^*\pi(M_f) - \pi(M_f)\{\pi(M_f)\}^* \geq 0$ in $\mathcal{Q}$. Consequently $\Phi(\pi(M_f))$ would be a hyponormal operator on $\mathcal{H}$.

Write $\mathrm{rad}(T)$ for the spectral radius of any operator $T$. It is well known that if $T$ is a hyponormal operator, then $\|T\| = \mathrm{rad}(T)$. See Problem 205 in [11]. Thus we would have

$$\|\Phi(\pi(M_f))\| = \mathrm{rad}(\Phi(\pi(M_f))).$$

By Proposition 3.2,

$$\mathrm{rad}(\Phi(\pi(M_f))) \leq \mathrm{rad}(M_f) \leq \|f\|_\infty.$$

On the other hand, since $\Phi$ is a faithful representation, we have

$$\|\Phi(\pi(M_f))\| = \|\pi(M_f)\| = \|M_f\|_\mathcal{Q}.$$

These three displayed lines together contradict the assumption $\|M_f\|_\mathcal{Q} > \|f\|_\infty$. $\square$

*Proof of Theorem* 1.2. It follows immediately from Proposition 3.3 and Theorem 1.1. □

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Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260, USA

*E-mail address*: qfang2@buffalo.edu

*E-mail address*: jxia@acsu.buffalo.edu