SINGULAR INTEGRAL OPERATORS AND ESSENTIAL COMMUTATIVITY ON THE SPHERE

Jingbo Xia

Abstract. Let $\mathcal{T}$ be the C^*-algebra generated by the Toeplitz operators $\{T_\varphi : \varphi \in L^\infty(S, d\sigma)\}$ on the Hardy space $H^2(S)$ of the unit sphere in $\mathbf{C}^n$. It is well known that $\mathcal{T}$ is contained in the essential commutant of $\{T_\varphi : \varphi \in \mathrm{VMO} \cap L^\infty(S, d\sigma)\}$. We show that the essential commutant of $\{T_\varphi : \varphi \in \mathrm{VMO} \cap L^\infty(S, d\sigma)\}$ is strictly larger than $\mathcal{T}$.

1. Introduction

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. Furthermore we normalize σ such that $\sigma(S) = 1$. The *Cauchy projection* P is defined by the integral formula

$$(Pf)(z) = \int \frac{f(v)}{(1 - \langle z, v\rangle)^n} d\sigma(v), \qquad |z| < 1.$$

See [16,page 39]. Recall that P is the orthogonal projection from $L^2(S, d\sigma)$ onto the Hardy space $H^2(S)$. For each $\varphi \in L^\infty(S, d\sigma)$, the Toeplitz operator T_φ is the operator on $H^2(S)$ defined by the formula

$$T_\varphi g = P\varphi g, \qquad g \in H^2(S).$$

We will write

$$\mathcal{T} = \text{the } C^*\text{-algebra generated by } \{T_\varphi : \varphi \in L^\infty(S, d\sigma)\}.$$

Recall that the formula

$$d(u, v) = |1 - \langle u, v\rangle|^{1/2}, \qquad u, v \in S, \tag{1.1}$$

defines a metric on S [16,page 66]. Throughout the paper, $B(u, a)$ denotes an open ball with respect to the metric d given in (1.1). That is, for any $u \in S$ and $a > 0$, we write

$$B(u, a) = \{v \in S : |1 - \langle u, v\rangle|^{1/2} < a\}.$$

A function $f \in L^1(S, d\sigma)$ is said to have *bounded mean oscillation* if

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{\sigma(B)} \int_B |f - f_B| d\sigma < \infty,$$

2000 *Mathematics Subject Classification.* Primary 32A55, 46L05, 47L80.

This work was supported in part by National Science Foundation grant DMS-0456448.

where $f_B = \int_B f d\sigma/\sigma(B)$ and the supremum is take over all $B = B(u, a)$, $u \in S$ and $a > 0$. A function $f \in L^1(S, d\sigma)$ is said to have *vanishing mean oscillation* if

$$\lim_{\delta \downarrow 0} \sup_{\substack{u \in S \\ 0 < a \leq \delta}} \frac{1}{\sigma(B(u,a))} \int_{B(u,a)} |f - f_{B(u,a)}| d\sigma = 0.$$

We denote the collection of functions of bounded mean oscillation on S by BMO. Similarly, let VMO be the collection of functions of vanishing mean oscillation on S. We define

$$\mathrm{VMO}_{\mathrm{bdd}} = \mathrm{VMO} \cap L^\infty(S, d\sigma)$$

and

$$\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}) = \text{the } C^*\text{-algebra generated by } \{T_\varphi : \varphi \in \mathrm{VMO}_{\mathrm{bdd}}\}.$$

For any separable, infinite-dimensional Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the collection of bounded operators on $\mathcal{H}$. The *essential commutant* of a subset $\mathcal{G}$ of $\mathcal{B}(\mathcal{H})$ is defined to be

$$\mathrm{EssCom}(\mathcal{G}) = \{X \in \mathcal{B}(\mathcal{H}) : [A, X] \in \mathcal{K}(\mathcal{H}) \ \text{ for every } A \in \mathcal{G}\},$$

where $\mathcal{K}(\mathcal{H})$ denotes the collection of compact operators on $\mathcal{H}$. Let π be the quotient map from $\mathcal{B}(\mathcal{H})$ into the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then $\pi(\mathrm{EssCom}(\mathcal{G}))$ is the commutant of $\pi(\mathcal{G})$ in $\mathcal{Q}$. That is, $\{\pi(\mathcal{G})\}' = \pi(\mathrm{EssCom}(\mathcal{G}))$.

When $n = 1$, i.e., in the case of unit circle, $\mathrm{VMO}_{\mathrm{bdd}}$ is better known as QC and has an alternate description [9,Section IX.2]. A famous result due to Davidson [6] asserts that $\mathcal{T}(\mathrm{QC})$ is the essential commutant of $\mathcal{T}$. This result was later generalized to the case $n \geq 2$ by Ding, Guo and Sun [7,10]. That is, for whatever complex dimension n, $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ is always the essential commutant of $\mathcal{T}$. This naturally motivates the question, what is the essential commutant of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$? In particular, does the essential commutant of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ coincide with $\mathcal{T}$? Given the results of [6] and [7,10], this is equivalent to asking, does $\pi(\mathcal{T})$ satisfy the double commutant relation in the Calkin algebra $\mathcal{Q}$?

In our previous investigation [21], we showed that in the case $n = 1$, the essential commutant of $\mathcal{T}(\mathrm{QC})$ is strictly larger than $\mathcal{T}$. In other words, in the unit circle case $\pi(\mathcal{T})$ does not satisfy the double commutant relation. The purpose of this paper is to report that the same assertion holds true in all complex dimensions. That is, we will prove

Theorem 1.1. *For every $n \geq 2$, the essential commutant of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ is also strictly larger than $\mathcal{T}$.*

As we explained in [21], although the essential-commutant problem of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ is motivated by C^*-algebraic considerations [11,15,18,19], its solution relies heavily on harmonic analysis. It is even more so in the case $n \geq 2$, as we will see.

To prove Theorem 1.1, we obviously need to construct an operator which belongs to $\mathrm{EssCom}(\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}))$ and which does not belong to $\mathcal{T}$. But if an operator essentially commutes with $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$, how does one show that it does *not* belong to $\mathcal{T}$?

In the case $n = 1$, we used a criterion based on the canonical commutation relation, which we could take advantage of because the unit disc is conformally equivalent to the upper-half plane.

Let $D = -id/dx$. Then $\chi_{(0,\infty)}(D)$ is the orthogonal projection from $L^2(\mathbf{R})$ to the Hardy space $H^2(\mathbf{R})$ of the upper-half plane. For each $\lambda \in \mathbf{R}$, define the unitary operator

$$(V_\lambda g)(x) = e^{i\lambda x} g(x), \qquad g \in L^2(\mathbf{R}).$$

Obviously,

$$V_\lambda^* D V_\lambda = D + \lambda.$$

Thus

$$V_\lambda^* \chi_{(0,\infty)}(D) V_\lambda = \chi_{(0,\infty)}(D + \lambda) = \chi_{(-\lambda,\infty)}(D).$$

Consequently,

$$\text{s-}\lim_{\lambda\to\infty} V_\lambda^* \chi_{(0,\infty)}(D) V_\lambda = 1.$$

Let $\tilde{V}_\lambda$ be the compression of V_λ to the subspace $H^2(\mathbf{R})$. Then the above limit implies that the strong limit

$$s(A) = \text{s-}\lim_{\lambda\to\infty} \tilde{V}_\lambda^* A \tilde{V}_\lambda$$

exists for every operator A in the Toeplitz algebra on $H^2(\mathbf{R})$. This was the membership criterion for the Toeplitz algebra that we used in [21]. Obviously, this is not something that we can hope to mimic in the case of sphere with $n \geq 2$.

What the above limit recovers is in fact the *symbol* of the operator A, as the notation $s(A)$ indicates. In the case $n \geq 2$, we will also use the fact that every operator in $\mathcal{T}$ has a symbol, which is proved in Proposition 4.13 below. But the difference is that here we recover the symbol through the *normalized reproducing kernel* for $H^2(S)$. Note that the method of recovering symbols through the normalized reproducing kernel was discovered by Engliš [8] in the case of the unit circle.

Guided by Proposition 4.13, we construct an operator $\tilde{F}$ (see (4.3) and (4.2) below) which essentially commutes with $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ and which has no symbol. The latter fact ensures, of course, that $\tilde{F} \notin \mathcal{T}$. Although the proof for the fact $\tilde{F} \in \mathrm{EssCom}(\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}))$ uses techniques which are standard in the theory of Calderón-Zygmund operators on $\mathbf{R}^k$ [2,3,17], there are no results in the literature for us to cite directly to cover the case of the sphere S. This forces us to produce the necessary details here.

This paper is organized as follows. Sections 2 and 3 deal with the singular integral operators, culminating in Proposition 3.11, the main technical step. In Section 4 we construct the operator $\tilde{F}$, which is quite involved and requires results from [12,14].

For the rest of the paper, we will assume $n \geq 2$. We conclude this section with an inequality which will be used frequently. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$2^{-n} a^{2n} \leq \sigma(B(u,a)) \leq A_0 a^{2n} \tag{1.2}$$

for all $u \in S$ and $0 < a \leq \sqrt{2}$ [16,Proposition 5.1.4].

2. Singular Integrals on the Sphere

For the rest of the paper, let ω be a C^1 function which maps $(0, \infty)$ into $\mathbf{C}$. Let

$$K(u,v) = \frac{\omega(|1 - \langle u, v\rangle|)}{(1 - \langle u, v\rangle)^n}, \qquad u \neq v \;\text{ and }\; u, v \in S.$$

For $f \in L^1(S, d\sigma)$ and $\epsilon > 0$, define

$$(T_\epsilon f)(u) = \int_{S \setminus B(u,\epsilon)} K(u,v) f(v) d\sigma(v), \qquad u \in S.$$

We assume that ω and T_ϵ satisfy the following three conditions:

(i) $\|\omega\|_\infty = \sup_{t>0} |\omega(t)| < \infty$.

(ii) There is a constant C such that $|\omega'(t)| \leq C/t$ for $0 < t \leq 3$.

(iii) There exist a bounded operator T on $L^2(S, d\sigma)$ and a sequence of positive numbers $\{\epsilon_k\}$ with

$$\lim_{k\to\infty} \epsilon_k = 0$$

such that

$$\lim_{k\to\infty} \|T_{\epsilon_k} f - Tf\|_2 = 0 \tag{2.1}$$

for every $f \in L^2(S, d\sigma)$.

Recall that the Hardy-Littlewood maximal function is defined by the formula

$$(Mf)(u) = \sup_{r>0} \frac{1}{\sigma(B(u,r))} \int_{B(u,r)} |f| d\sigma, \qquad u \in S.$$

Lemma 2.1. *For all $f \in L^1(S, d\sigma)$, $u \in S$, $\alpha > 0$, and $\rho > 0$, we have*

$$\int_{|1-\langle u,v\rangle| \geq \rho} \frac{\rho^\alpha}{|1 - \langle u, v\rangle|^{n+\alpha}} |f(v)| d\sigma(v) \leq \frac{2^{n+\alpha} A_0}{2^\alpha - 1} (Mf)(u)$$

where A_0 is the constant in (1.2).

Proof. Given $u \in S$ and $\rho > 0$, define $B_k = \{v \in S : |1 - \langle u, v\rangle| < 2^k \rho\}$, $k = 0, 1, 2, \ldots$. For $v \in B_{k+1} \setminus B_k$, we have $(\rho/|1 - \langle u, v\rangle|)^\alpha \leq 2^{-\alpha k}$ and $|1 - \langle u, v\rangle|^n \geq (2^k\rho)^n \geq A_0^{-1} 2^{-n} \sigma(B_{k+1})$, where the second $\geq$ follows from (1.2). Hence

$$\frac{\rho^\alpha}{|1 - \langle u, v\rangle|^{n+\alpha}} \leq 2^n A_0 \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k} \sigma(B_{k+1})} \chi_{B_{k+1} \setminus B_k}(v)$$

for $v \in S \backslash B_0$. The lemma follows from this inequality. $\square$

Lemma 2.2. *There is a constant $C_{2.2}$ such that for any $u \in S$ and $r > 0$, if $x, z \in B(u, r)$ and $y \in S \backslash B(u, 2r)$, then*

$$|K(x, y) - K(z, y)| \leq C_{2.2} \frac{|1 - \langle x, z\rangle|^{1/2}}{|1 - \langle x, y\rangle|^{n+(1/2)}}.$$

Proof. For $x, z \in B(u, r)$ and $y \in S \backslash B(u, 2r)$, we have

$$|K(x, y) - K(z, y)| \leq \frac{|a(y; x, z)|}{|1 - \langle x, y\rangle|^n} + \|\omega\|_\infty |b(y; x, z)|, \tag{2.2}$$

where

$$a(y; x, z) = \omega(|1 - \langle x, y\rangle|) - \omega(|1 - \langle z, y\rangle|) \quad \text{and}$$
$$b(y; x, z) = \frac{1}{(1 - \langle x, y\rangle)^n} - \frac{1}{(1 - \langle z, y\rangle)^n}.$$

We will estimate the two terms in (2.2) separately.

To begin, we observe that the conditions $x, z \in B(u, r)$ and $y \in S \backslash B(u, 2r)$ imply

$$d(x, y) \leq 3d(z, y). \tag{2.3}$$

Hence $|1 - \langle z, y\rangle| \geq |1 - \langle x, y\rangle|/9$ and, by the fundamental theorem of calculus,

$$|a(y; x, z)| = \left| \int_{|1-\langle x,y\rangle|}^{|1-\langle z,y\rangle|} \omega'(t)dt \right| \leq \left| \int_{|1-\langle x,y\rangle|}^{|1-\langle z,y\rangle|} \frac{C}{t} dt \right| \leq \frac{9C|\langle z - x, y\rangle|}{|1 - \langle x, y\rangle|}. \tag{2.4}$$

To estimate $|\langle z - x, y\rangle|$, we write $y = \langle y, x\rangle x + y^\perp$ and $z = \langle z, x\rangle x + z^\perp$, where $\langle y^\perp, x\rangle = 0 = \langle z^\perp, x\rangle$. Thus $\langle z - x, y\rangle = (\langle z, x\rangle - 1)\langle x, y\rangle + \langle z^\perp, y^\perp\rangle$. Therefore

$$\begin{aligned} |\langle z - x, y\rangle| &\leq |1 - \langle z, x\rangle| + |z^\perp||y^\perp| = |1 - \langle z, x\rangle| + (1 - |\langle z, x\rangle|^2)^{1/2}(1 - |\langle y, x\rangle|^2)^{1/2} \\ &\leq |1 - \langle z, x\rangle| + 2|1 - \langle z, x\rangle|^{1/2}|1 - \langle y, x\rangle|^{1/2}. \end{aligned}$$

Since $d(x, z) < 2r$ whereas $d(x, y) \geq r$, the above leads to the estimate

$$|\langle z - x, y\rangle| \leq 4|1 - \langle z, x\rangle|^{1/2}|1 - \langle y, x\rangle|^{1/2}. \tag{2.5}$$

Substituting this in (2.4), we obtain

$$|a(y; x, z)| \leq 36C \frac{|1 - \langle x, z\rangle|^{1/2}}{|1 - \langle x, y\rangle|^{1/2}}. \tag{2.6}$$

To estimate $|b(y;x,z)|$, note that it follows from (2.5) and (2.3) that

$$\left|\frac{1}{1-\langle x,y\rangle}-\frac{1}{1-\langle z,y\rangle}\right|=\frac{|\langle x-z,y\rangle|}{|1-\langle x,y\rangle||1-\langle z,y\rangle|}\le 36\frac{|1-\langle x,z\rangle|^{1/2}}{|1-\langle x,y\rangle|^{3/2}}.$$

By simple algebra and another application of (2.3), we have

$$|b(y;x,z)|\le n\cdot\frac{9^{n-1}}{|1-\langle x,y\rangle|^{n-1}}\cdot 36\frac{|1-\langle x,z\rangle|^{1/2}}{|1-\langle x,y\rangle|^{3/2}}.$$

Combining this with (2.2) and (2.6), the lemma follows. □

Lemma 2.3. *For each $1<t\le 2$, there is a constant $C_{2.3}(t)$ such that $\|Tf\|_t\le C_{2.3}(t)\|f\|_t$ for every $f\in L^2(S,d\sigma)$. Therefore T uniquely extends to a bounded operator on $L^t(S,d\sigma)$.*

Proof. As usual, we will establish the weak-type (1,1) estimate

$$\sigma(\{u\in S:|(Tf)(u)|>\lambda\})\le(A/\lambda)\|f\|_1. \tag{2.7}$$

The lemma will then follow from the L^2-boundedness of T, (2.7), and the interpolation theorem of Marcinkiewicz [9,page 26].

To prove (2.7), we only need to consider the case where $\lambda>\|f\|_1$. We use the Calderón-Zygmund decomposition of f. Denote $A_4=\sup_{r>0}\sigma(B(u,4r))/\sigma(B(u,r))$. According to [16,Lemma 6.2.1], there exists a family of open d-balls $\{B_i\}$ in S and a family of pairwise disjoint Borel sets $\{V_i\}$, where $V_i\subset B_i$ for every i, such that

(a) $\{u\in S:(Mf)(u)>\lambda\}\subset\cup_i B_i=\cup V_i$;
(b) $\sum_i\sigma(B_i)\le(A_4/\lambda)\|f\|_1$;
(c) $\int_{V_i}|f|d\sigma<A_4\lambda\sigma(V_i)$.

As in the proof of [16,Theorem 6.2.2], set $c_i=\int_{V_i}fd\sigma/\sigma(V_i)$ for each i and define

$$g=f\chi_{S\backslash(\cup_i V_i)}+\sum_i c_i\chi_{V_i}\quad\text{ and }$$
$$b_i=(f-c_i)\chi_{V_i}.$$

Then

$$f=g+b,\qquad\text{where }\ b=\sum_i b_i. \tag{2.8}$$

Since the set of Lebesgue points for $|f|$ has measure 1 with respect to σ [16,Theorem 5.3.1], (a) implies that $|f(u)|\le\lambda$ for σ-a.e. $u\in S\backslash(\cup_i V_i)$. Thus

$$\int_{S\backslash(\cup_i V_i)}|g|^2d\sigma=\int_{S\backslash(\cup_i V_i)}|f|^2d\sigma\le\lambda\int_{S\backslash(\cup_i V_i)}|f|d\sigma\le\lambda\|f\|_1.$$

On the other hand, it follows from (c) and (b) that

$$\int_{\cup_i V_i} |g|^2 d\sigma = \sum_i |c_i|^2 \sigma(V_i) \le (A_4\lambda)^2 \sum_i \sigma(V_i) \le (A_4\lambda)^2 (A_4/\lambda)\|f\|_1 = A_4^3 \lambda \|f\|_1.$$

Hence $\|g\|_2^2 \le (1 + A_4^3)\lambda\|f\|_1$ and

$$\begin{aligned} \sigma(\{u \in S : |(Tg)(u)| > \lambda/2\}) &\le (2/\lambda)^2 \|Tg\|_2^2 \le (2/\lambda)^2 \|T\|^2 \|g\|_2^2 \\ &\le (4/\lambda)\|T\|^2 (1 + A_4^3)\|f\|_1. \end{aligned} \tag{2.9}$$

To estimate Tb, we switch to the argument given on page 21 of [17].

For each i, we suppose that $B_i = B(v_i, r_i)$ and define $B_i' = B(v_i, 2r_i)$. Then $S\backslash B_i' = \{y \in S : |1 - \langle y, v_i\rangle| \ge (2r_i)^2\}$. It follows from Lemmas 2.2 and 2.1 that if $v \in B_i$, then

$$\int_{S\backslash B_i'} |K(y,v) - K(y,v_i)| d\sigma(y) \le C_{2.2} \cdot \frac{2^{n+(1/2)} A_0}{\sqrt{2} - 1} = \tilde{C}. \tag{2.10}$$

On the set $S\backslash(\cup_j B_j')$, each Tb_i can be represented by the obvious integral formula. Thus for $y \in S\backslash(\cup_j B_j')$ we have

$$\begin{aligned} |(Tb)(y)| \le \sum_i |(Tb_i)(y)| &= \sum_i \left| \int_{V_i} K(y,v) b_i(v) d\sigma(v) \right| \\ &= \sum_i \left| \int_{V_i} (K(y,v) - K(y,v_i)) b_i(v) d\sigma(v) \right|, \end{aligned}$$

where the second $=$ is follows from the fact that $\int_{V_i} b_i d\sigma = 0$. Hence

$$\begin{aligned} \int_{S\backslash(\cup_j B_j')} |Tb| d\sigma &\le \sum_i \int_{S\backslash(\cup_j B_j')} \left| \int_{V_i} (K(y,v) - K(y,v_i)) b_i(v) d\sigma(v) \right| d\sigma(y) \\ &\le \sum_i \int_{V_i} \left\{ \int_{S\backslash B_i'} |K(y,v) - K(y,v_i)| d\sigma(y) \right\} |b_i(v)| d\sigma(v) \\ &\le \tilde{C} \sum_i \int_{V_i} |b_i| d\sigma, \end{aligned}$$

where the last $\le$ follows from (2.10). But $\int_{V_i} |b_i| d\sigma \le 2 \int_{V_i} |f| d\sigma$ and the Borel sets $\{V_i\}$ are pairwise disjoint. Therefore

$$\int_{S\backslash(\cup_j B_j')} |Tb| d\sigma \le 2\tilde{C}\|f\|_1,$$

which implies

$$\sigma(\{u \in S : |(Tb)(u)| > \lambda/2\} \backslash \{\cup_j B_j'\}) \le (4\tilde{C}/\lambda)\|f\|_1. \tag{2.11}$$

On the other hand, by the definition of B_i' and (b), we have

$$\sigma(\cup_j B_j') \le \sum_j \sigma(B_j') \le A_4 \sum_j \sigma(B_j) \le (A_4^2/\lambda)\|f\|_1.$$

Combining this with (2.11) and (2.9), we obtain (2.7). This completes the proof. □

For each $f \in L^1(S, d\sigma)$, define

$$(T_* f)(u) = \sup_{\epsilon>0} |(T_\epsilon f)(u)|, \qquad u \in S.$$

Lemma 2.4. *There exists a constant $C_{2.4}$ such that the inequality*

$$T_* f \le C_{2.4}\{M(Tf) + Mf\}$$

holds on S for every $f \in L^t(S, d\sigma)$, $1 < t \le 2$.

Proof. We follow the proof on page 35 of [17], making the obvious modifications to suit the present setting. Consider any $u \in S$ and any $\epsilon > 0$. We have $f = f_1 + f_2$, where $f_1 = f\chi_{B(u,\epsilon)}$ and $f_2 = f\chi_{S\backslash B(u,\epsilon)}$. For $z \in B(u, \epsilon/2)$ we have

$$(T_\epsilon f)(u) - (Tf_2)(z) = \int_{S\backslash B(u,\epsilon)} (K(u, y) - K(z, y))f(y)d\sigma(y).$$

Thus it follows from Lemmas 2.2 and 2.1 that if $z \in B(u, \epsilon/2)$, then

$$|(T_\epsilon f)(u) - (Tf_2)(z)| \le \int_{S\backslash B(u,\epsilon)} |K(u, y) - K(z, y)||f(y)|d\sigma(y) \le \tilde{C}(Mf)(u),$$

where $\tilde{C} = (\sqrt{2} - 1)^{-1}2^{n+(1/2)}A_0C_{2.2}$. Since $Tf_2 = Tf - Tf_1$, we conclude that

$$|(T_\epsilon f)(u)| \le |(Tf)(z)| + |(Tf_1)(z)| + \tilde{C}(Mf)(u) \quad \text{for } \sigma\text{-a.e. } z \in B(u, \epsilon/2). \tag{2.12}$$

By (1.2), we have $\sigma(B(v, r)) \le 2^{3n}A_0\sigma(B(v, r/2))$ for all $v \in S$ and $r > 0$. Now set $\lambda_0 = 4\{(M(Tf))(u) + 2^{3n}A_0A(Mf)(u)\}$, where A is the constant in (2.7). Then

$$\begin{aligned}\sigma(\{z \in B(u, \epsilon/2) : |(Tf)(z)| > \lambda_0\}) &\le \frac{1}{\lambda_0}\int_{B(u,\epsilon/2)} |Tf|d\sigma \\ &\le \frac{1}{\lambda_0}(M(Tf))(u)\sigma(B(u, \epsilon/2)) \le \frac{1}{4}\sigma(B(u, \epsilon/2)).\end{aligned} \tag{2.13}$$

By (2.7) and the definition of f_1,

$$\sigma(\{z \in S : |(Tf_1)(z)| > \lambda_0\}) \le \frac{A}{\lambda_0}\|f_1\|_1 \le \frac{A}{\lambda_0}(Mf)(u)\sigma(B(u, \epsilon)) \le \frac{1}{4}\sigma(B(u, \epsilon/2)). \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\sigma(\{z \in B(u, \epsilon/2) : |(Tf)(z)| \le \lambda_0 \text{ and } |(Tf_1)(z)| \le \lambda_0\}) \ge (1/2)\sigma(B(u, \epsilon/2)).$$

Recalling (2.12) and the definition of λ_0, we now have

$$|(T_\epsilon f)(u)| \le 2\lambda_0 + \tilde{C}(Mf)(u) \le (8 + 2^{3n+3}A_0A + \tilde{C})\{(M(Tf))(u) + (Mf)(u)\}.$$

This completes the proof. □

Corollary 2.5. *For each $1 < t \le 2$, there is a constant $C_{2.5}(t)$ such that $\|T_* f\|_t \le C_{2.5}(t)\|f\|_t$ for every $f \in L^t(S, d\sigma)$.*

Proof. This follows from Lemmas 2.4 and 2.3, and the fact that if $t > 1$, then the maximal operator is bounded on $L^t(S, d\sigma)$. □

Lemma 2.6. *There exists a constant $C_{2.6}$ such that if $f \in L^1(S, d\sigma)$ and if the d-ball $B = \{\zeta \in S : |1 - \langle a, \zeta\rangle| < \rho\}$ contains η, v such that $(Mf)(\eta) \le \lambda$ and $(T_*f)(v) \le \lambda$, then we have $(T_*\chi_{S\backslash Q}f)(u) \le C_{2.6}\lambda$ for every $u \in B$, where $Q = \{\zeta \in S : |1 - \langle a, \zeta\rangle| < 25\rho\}$.*

Proof. Let $\epsilon \ge 9\rho$. Given a $u \in B$, define $E = \{y \in S : |1 - \langle u, y\rangle| \ge \epsilon \text{ and } |1 - \langle v, y\rangle| < \epsilon\}$. If $y \in E$, then $d(u, y) \le d(u, v) + d(v, y) < 2\sqrt{\rho} + \sqrt{\epsilon} < 2\sqrt{\epsilon}$. Since $d(u, \eta) < 2\sqrt{\rho} < \sqrt{\epsilon}$, we have $B(u, 2\sqrt{\epsilon}) \subset B(\eta, 3\sqrt{\epsilon})$. Thus

$$\int_E |K(u, y)||f(y)|d\sigma(y) \le \frac{\|\omega\|_\infty}{\epsilon^n}\int_{B(u,2\sqrt{\epsilon})} |f(y)|d\sigma(y) \le C_1(Mf)(\eta) \le C_1\lambda.$$

Similarly, if we set $F = \{y \in S : |1 - \langle u, y\rangle| < \epsilon \text{ and } |1 - \langle v, y\rangle| \ge \epsilon\}$, then

$$\int_F |K(v, y)||f(y)|d\sigma(y) \le C_1\lambda.$$

Let $G = \{y \in S : |1 - \langle u, y\rangle| \ge \epsilon \text{ and } |1 - \langle v, y\rangle| \ge \epsilon\}$. Then by these estimates we have

$$\left|\int_{|1-\langle u,y\rangle|\ge\epsilon} K(u, y)\chi_{S\backslash Q}(y)f(y)d\sigma(y) - \int_{|1-\langle v,y\rangle|\ge\epsilon} K(v, y)\chi_{S\backslash Q}(y)f(y)d\sigma(y)\right|$$
$$\le J + 2C_1\lambda,$$

where

$$J = \left|\int_{G\backslash Q}(K(u, y) - K(v, y))f(y)d\sigma(y)\right|$$
$$\le \int_{G\backslash Q} |K(u, y) - K(\eta, y)||f(y)|d\sigma(y) + \int_{G\backslash Q} |K(v, y) - K(\eta, y)||f(y)|d\sigma(y).$$

Since $u,\ v \in B(\eta, 2\sqrt{\rho})$ and $Q \supset B(\eta, 4\sqrt{\rho})$, it follows from Lemmas 2.2 and 2.1 that $J \le 2\tilde{C}(Mf)(\eta) \le 2\tilde{C}\lambda$. Hence

(2.15)
$$\left| \int_{|1-\langle u,y\rangle| \ge \epsilon} K(u,y)\chi_{S\backslash Q}(y)f(y)d\sigma(y) - \int_{|1-\langle v,y\rangle| \ge \epsilon} K(v,y)\chi_{S\backslash Q}(y)f(y)d\sigma(y) \right| \le C_2\lambda$$

for all $u \in B$ and $\epsilon \ge 9\rho$. Let $W = \{y \in Q : |1 - \langle v, y\rangle| \ge \epsilon\}$. Then

$$\int_{|1-\langle v,y\rangle| \ge \epsilon} K(v,y)\chi_{S\backslash Q}(y)f(y)d\sigma(y) = (T_{\sqrt{\epsilon}}f)(v) - \int_W K(v,y)f(y)d\sigma(y). \tag{2.16}$$

Because $|K(v,y)| \le \|\omega\|_\infty \epsilon^{-n}$ for $y \in W$, $\epsilon \ge 9\rho$, and $\eta \in Q$, we have

$$\int_W |K(v,y)||f(y)|d\sigma(y) \le \frac{C_3}{\sigma(Q)} \int_Q |f|d\sigma \le C_4(Mf)(\eta) \le C_4\lambda. \tag{2.17}$$

Since $|(T_*f)(v)| \le \lambda$, from (2.15-17) we obtain

$$|(T_{\sqrt{\epsilon}}\chi_{S\backslash Q}f)(u)| = \left| \int_{|1-\langle u,y\rangle| \ge \epsilon} K(u,y)\chi_{S\backslash Q}(y)f(y)d\sigma(y) \right| \le (C_2 + 1 + C_4)\lambda \tag{2.18}$$

for all $u \in B$ and $\epsilon \ge 9\rho$.

On the other hand, if $u \in B$ and $0 < \epsilon < 9\rho$, then $\{y \in S\backslash Q : \epsilon \le |1 - \langle u, y\rangle| < 9\rho\} = \emptyset$. Hence $\{y \in S\backslash Q : |1 - \langle u, y\rangle| \ge \epsilon\} = \{y \in S\backslash Q : |1 - \langle u, y\rangle| \ge 9\rho\}$ if $u \in B$ and $0 < \epsilon < 9\rho$. Thus (2.18) actually holds for all $\epsilon > 0$. Consequently, $C_{2.6} = C_2 + 1 + C_4$ will do. $\square$

For each $1 \le t < \infty$, we define the maximal function

$$(M_tf)(u) = \sup_{r>0} \left(\frac{1}{\sigma(B(u,r))} \int_{B(u,r)} |f|^t d\sigma \right)^{1/t}, \qquad u \in S.$$

But we will continue to write Mf for M_1f.

Proposition 2.7. *For each $1 < t \le 2$, there exists a constant $C_{2.7}(t)$ such that the following estimate holds: Let $f \in L^1(S, d\sigma)$. If $B = \{\zeta \in S : |1 - \langle a, \zeta\rangle| < \rho\}$ and $\lambda > 0$ satisfy the condition $B \cap \{v \in S : (T_*f)(v) \le \lambda\} \ne \emptyset$, then*

$$\sigma(\{u \in B : (T_*f)(u) > (1 + C_{2.6})\lambda \text{ and } (M_tf)(u) \le \alpha\lambda\}) \le \alpha C_{2.7}(t)\sigma(B)$$

for every $0 < \alpha \le 1$, where $C_{2.6}$ is the constant in Lemma 2.6.

Proof. If $(M_tf)(u) > \lambda$ for every $u \in B$, then the conclusion is trivial. Thus we may assume that there is an $\eta \in B$ such that $(M_tf)(\eta) \le \lambda$. Then $(Mf)(\eta) \le \lambda$. Define $Q = \{\zeta \in S : |1 - \langle a, \zeta\rangle| < 25\rho\}$ as in Lemma 2.6. Also define $g = \chi_Q f$ and $h = \chi_{S\backslash Q}f$. Then

$f = g + h$. Since $\{v \in B : (T_*f)(v) \le \lambda\} \neq \emptyset$, Lemma 2.6 tells us that $(T_*h)(u) \le C_{2.6}\lambda$ for every $u \in B$. By the subadditivity of T_*, this gives us

$$\{u \in B : (T_*f)(u) > (1 + C_{2.6})\lambda\} \subset \{u \in B : (T_*g)(u) > \lambda\}. \tag{2.19}$$

For a given $0 < \alpha \le 1$, let $Y = \{u \in B : (T_*g)(u) > \alpha^{-1}(M_tf)(u)\}$. Then, by (2.19),

$$Y \supset \{u \in B : (T_*f)(u) > (1 + C_{2.6})\lambda \text{ and } (M_tf)(u) \le \alpha\lambda\}.$$

Since $\|g\|_t^t = \int_Q |f|^t d\sigma$ and $Q \supset B$, there is a constant $c > 0$ such that $c(\|g\|_t^t/\sigma(Q))^{1/t} \le (M_tf)(u)$ for every $u \in B$. Thus if we let

$$X = \{u \in B : (T_*g)(u) > \alpha^{-1}c(\|g\|_t^t/\sigma(Q))^{1/t}\},$$

then $X \supset Y$. To prove the lemma, it suffices to estimate $\sigma(X)$. We have

$$\begin{aligned}
\sigma(X) &\le \alpha c^{-1}(\|g\|_t^t/\sigma(Q))^{-1/t}\|\chi_B T_*g\|_1 \\
&\le \alpha c^{-1}(\|g\|_t^t/\sigma(Q))^{-1/t}(\sigma(B))^{(t-1)/t}\|T_*g\|_t \\
&\le \alpha c^{-1}(\|g\|_t^t/\sigma(Q))^{-1/t}(\sigma(B))^{(t-1)/t}C_{2.5}(t)\|g\|_t \\
&= \alpha c^{-1}C_{2.5}(t)(\sigma(Q))^{1/t}(\sigma(B))^{(t-1)/t} \le \alpha c^{-1}C_{2.5}(t)C_1\sigma(B),
\end{aligned}$$

where the second $\le$ follows from Hölder's inequality, the third $\le$ is an application of Corollary 2.5, and the last $\le$ is due to (1.2). Thus $C_{2.7}(t) = c^{-1}C_{2.5}(t)C_1$ will do. $\square$

The final lemma of the section is the metric-space version of the Whitney decomposition [17]. For more general forms of such decomposition, see [4].

Lemma 2.8. *Let U be a non-empty open subset of S such that $S\backslash U$ is also non-empty. Then there exists a family of open d-balls $\{B(u_i, r_i) : i \in I\}$ with the following properties*:

(a) $B(u_i, r_i) \cap B(u_j, r_j) = \emptyset$ *if* $i \neq j$;
(b) $\cup_{i \in I} B(u_i, r_i) \subset U$;
(c) $B(u_i, 2r_i) \cap (S\backslash U) \neq \emptyset$ *for every* $i \in I$;
(d) $U \subset \cup_{i \in I} B(u_i, 2r_i)$.

Proof. For integers $k = -1, 0, 1, 2, \ldots$, let $E_k = \{u \in U : B(u, 2^{-k}) \subset U\}$. Since $S\backslash U \neq \emptyset$, we have $E_{-1} = \emptyset$. We set $F_{-1} = \emptyset$. Suppose that $k \ge 0$ and that we have defined the subset F_j of E_j for $-1 \le j \le k-1$. We let F_k be a subset of $E_k\backslash\{\cup_{j=-1}^{k-1} \cup_{u \in F_j} B(u, 2^{-j+1})\}$ which is *maximal* with respect to the property that

$$B(u, 2^{-k}) \cap B(v, 2^{-k}) = \emptyset \quad \text{if } u, v \in F_k \text{ and } u \neq v. \tag{2.20}$$

The maximality of F_k implies that for every $z \in E_k\backslash\{\cup_{j=-1}^{k-1} \cup_{u \in F_j} B(u, 2^{-j+1})\}$ there is a $u(z) \in F_k$ such that $B(z, 2^{-k}) \cap B(u(z), 2^{-k}) \neq \emptyset$. Therefore

$$\cup_{u \in F_k} B(u, 2^{-k+1}) \supset E_k\backslash\{\cup_{j=-1}^{k-1} \cup_{u \in F_j} B(u, 2^{-j+1})\}. \tag{2.21}$$

Since $F_k \subset E_k$, by the definition of E_k we have

$$B(u, 2^{-k}) \subset U \quad \text{ if } u \in F_k. \tag{2.22}$$

Thus we have inductively defined F_{-1}, F_0, F_1, ..., F_k ... such that (2.20-22) hold for every k. Let $\{B(u_i, r_i) : i \in I\}$ be a re-enumeration of the balls in the families $\{B(u, 2^{-k}) : u \in F_k\}$, $k \geq 0$. Then (b) follows from (2.22).

If $k < \ell$, $u \in F_k$ and $v \in F_\ell$, then the definition of F_ℓ ensures that $v \notin B(u, 2^{-k+1})$, which implies $d(u, v) \geq 2^{-k+1} > 2^{-k} + 2^{-\ell}$. Therefore

$$B(u, 2^{-k}) \cap B(v, 2^{-\ell}) = \emptyset \quad \text{ if } u \in F_k, \ v \in F_\ell, \text{ and } k < \ell. \tag{2.23}$$

Thus (a) follows from (2.20) and (2.23). Note that (2.21) implies

$$E_{k-1} \subset \cup_{j=-1}^{k-1} \cup_{u \in F_j} B(u, 2^{-j+1}). \tag{2.24}$$

Since $F_k \subset E_k \backslash \{\cup_{j=-1}^{k-1} \cup_{u \in F_j} B(u, 2^{-j+1})\}$, we have $F_k \cap E_{k-1} = \emptyset$ for all $k \geq 0$. By the definition of E_{k-1}, if $u \in F_k$, then U does not contain $B(u, 2^{-(k-1)}) = B(u, 2^{-k+1})$, which proves (c). Finally, (d) follows from (2.24) and the fact that $U = \cup_{k=0}^{\infty} E_k$. $\square$

3. Condition (A_p) and Commutators

The well-known (A_p)-condition, $1 < p < \infty$, was introduced by Muckenhoupt [13] for Euclidian spaces and by Calderón [1] for metric spaces in general.

Definition 3.1. [1] *A weight function w on S is said to satisfy condition (A_p) if*

$$\sup_B \left(\frac{1}{\sigma(B)} \int_B w d\sigma \right) \left(\frac{1}{\sigma(B)} \int_B w^{-1/(p-1)} d\sigma \right)^{p-1} < \infty,$$

where the supremum is taken over all $B = \{u \in S : |1 - \langle u, a \rangle| < r\}$, $a \in S$, $r > 0$.

Moreover, specializing Calderón's result to the sphere, we have

Theorem 3.2. [1] *If w satisfies condition (A_p) for some $1 < p < \infty$, then:*
(a) *There is a $p_0 \in (1, p)$ such that w satisfies condition (A_r) for every $p_0 < r \leq p$.*
(b) *The maximal operator is bounded on $L^p(S, w d\sigma)$.*

Corollary 3.3. *Suppose that $1 < p < \infty$. If w satisfies condition (A_p), then there exists a $t \in (1, 2]$ such that M_t is also bounded on $L^p(S, w d\sigma)$.*

Proof. By Theorem 3.2(a), there is an $r \in (\max\{1, p/2\}, p)$ such that w satisfies condition (A_r). Let $t = p/r$. Then $1 < t < 2$. If $f \in L^p(S, w d\sigma)$, then $\{M_t(f)\}^p = \{M(|f|^t)\}^{p/t} = \{M(|f|^t)\}^r$. Applying Theorem 3.2(b) to condition (A_r), we have

$$\int \{M_t(f)\}^p w d\sigma = \int \{M(|f|^t)\}^r w d\sigma \leq C \int |f|^{tr} w d\sigma = C \int |f|^p w d\sigma,$$

which completes the proof. $\square$

Proposition 3.4. *Suppose that w satisfies condition (A_p) for some $1 < p < \infty$ and let $d\mu = w d\sigma$. Then there exist positive constants δ and C such that $\mu(E)/\mu(B) \leq C\{\sigma(E)/\sigma(B)\}^{\delta}$ for every open d-ball B in S and every Borel set E contained in B.*

Proof. Calderón showed that the metric-space version of (A_p) also implies

$$\left(\frac{1}{\sigma(B)}\int_B w^{1+\epsilon} d\sigma\right)^{1/(1+\epsilon)} \leq C_1 \frac{1}{\sigma(B)}\int_B w d\sigma$$

[1,page 298]. Given this "reverse Hölder's inequality", the proposition follows from a standard argument. See, for example, page 264 in [9]. $\square$

Lemma 3.5. *Suppose that w satisfies condition (A_p) for some $1 < p < \infty$ and let $d\mu = w d\sigma$. Then there exists a positive constant C such that $\mu(B(u, 2r)) \leq C\mu(B(u, r))$ for all $u \in S$ and $r > 0$.*

Proof. Define $d\nu = w^{-1/(p-1)} d\sigma$. For any d-ball B and any Borel set $E \subset B$, it follows from Hölder's inequality that

$$\frac{\sigma(E)}{\sigma(B)} \leq \left(\frac{\mu(E)}{\sigma(B)}\right)^{1/p}\left(\frac{\nu(E)}{\sigma(B)}\right)^{(p-1)/p} \leq \left(\frac{\mu(E)}{\mu(B)}\right)^{1/p}\left\{\frac{\mu(B)}{\sigma(B)}\left(\frac{\nu(B)}{\sigma(B)}\right)^{p-1}\right\}^{1/p}.$$

By the (A_p)-condition for w, the factor $\{...\}^{1/p}$ is dominated by a constant C_1. Hence

$$\frac{\sigma(E)}{\sigma(B)} \leq C_1 \left(\frac{\mu(E)}{\mu(B)}\right)^{1/p}.$$

Letting $B = B(u, 2r)$ and $E = B(u, r)$, and applying (1.2), the lemma follows. $\square$

Lemma 3.6. *Suppose that w satisfies condition (A_p) for some $1 < p < \infty$ and define $d\mu = w d\sigma$. Let $1 < t \leq 2$ be given. Then there exist positive constants A and δ such that*

$$\mu(\{u \in S : (T_* f)(u) > (1 + C_{2.6})\lambda \text{ and } (M_t f)(u) \leq \alpha\lambda\}) \leq \alpha^{\delta} A \mu(\{u \in S : (T_* f)(u) > \lambda\})$$

for all $f \in L^1(S, d\sigma)$, $\lambda > \inf_{u\in S}(T_* f)(u)$ and $0 < \alpha \leq 1$, where $C_{2.6}$ is the constant in Lemma 2.6.

Proof. Let $U = \{u \in S : (T_* f)(u) > \lambda\}$, which is an open set by the nature of T_*. The condition $\lambda > \inf_{u\in S}(T_* f)(u)$ ensures that $S\backslash U \neq \emptyset$. Suppose that $U \neq \emptyset$. By Lemma 2.8, there exists a family of open balls $\{B(u_i, r_i) : i \in I\}$ such that

(a) $B(u_i, r_i) \cap B(u_j, r_j) = \emptyset$ if $i \neq j$;
(b) $\cup_{i\in I} B(u_i, r_i) \subset U$;
(c) $B(u_i, 2r_i) \cap (S\backslash U) \neq \emptyset$ for every $i \in I$;
(d) $U \subset \cup_{i\in I} B(u_i, 2r_i)$.

Denote $Z = \{u \in S : (T_*f)(u) > (1 + C_{2.6})\lambda$ and $(M_t f)(u) \le \alpha\lambda\}$. For each $i \in I$, write $B_i = B(u_i, 2r_i)$. Condition (c) allows us to apply Proposition 2.7 to obtain

$$\sigma(Z \cap B_i) \le \alpha C_{2.7}(t)\sigma(B_i), \quad i \in I.$$

By Proposition 3.4, there are positive constants δ and A' such that

$$\mu(Z \cap B_i)/\mu(B_i) \le A'\{\sigma(Z \cap B_i)/\sigma(B_i)\}^\delta \le A'(\alpha C_{2.7}(t))^\delta, \quad i \in I.$$

Set $A'' = C_{2.7}^\delta(t)A'$. Then $\mu(Z \cap B_i) \le \alpha^\delta A''\mu(B_i)$. By (d) and the fact $Z \subset U$, we have

$$\mu(Z) \le \sum_{i\in I} \mu(Z \cap B_i) \le \alpha^\delta A'' \sum_{i\in I} \mu(B_i).$$

Lemma 3.5 provides a constant C such that $\mu(B_i) \le C\mu(B(u_i, r_i))$. Hence

$$\mu(Z) \le \alpha^\delta A''C \sum_{i\in I} \mu(B(u_i, r_i)) \le \alpha^\delta A''C\mu(U),$$

where the second $\le$ follows from (a) and (b). This proves the lemma. $\square$

Proposition 3.7. *Let $1 < p < \infty$ and suppose that w satisfies condition (A_p). Denote $d\mu = wd\sigma$. Let $1 < t \le 2$ be given. Then there exists a constant C which depends on n, ω, w, p, and t such that*

$$\int \{T_*f\}^p d\mu \le C \int \{M_t f\}^p d\mu \tag{3.1}$$

for every $f \in L^p(S, d\mu)$.

Proof. We can decompose S as the union of disjoint hemispheres S^+ and S^-. Since $f = f\chi_{S^+} + f\chi_{S^-}$ and since T_* is subadditive, it suffices to prove (3.1) under the additional assumption that f identically vanishes on either S^+ or S^-.

For such an f we have $\inf_{u\in S}(T_*f)(u) \le \|\omega\|_\infty\|f\|_1$. Set $m = (1+C_{2.6})\|\omega\|_\infty\|f\|_1$. If $\lambda > m/(1 + C_{2.6})$, then $\lambda > \inf_{u\in S}(T_*f)(u)$. By Lemma 3.6, if $\lambda > \inf_{u\in S}(T_*f)(u)$, then

$$\mu(\{T_*f > (1 + C_{2.6})\lambda\}) \le \mu(\{M_t f > \alpha\lambda\}) + \alpha^\delta A\mu(\{T_*f > \lambda\}),$$

$0 < \alpha \le 1$. Therefore for all $0 < \alpha \le 1$ and $m < L < \infty$ we have

$$\begin{aligned} p\int_m^L x^{p-1}\mu(\{T_*f > x\})dx &= (1 + C_{2.6})^p p \int_{m/(1+C_{2.6})}^{L/(1+C_{2.6})} \lambda^{p-1}\mu(\{T_*f > (1 + C_{2.6})\lambda\})d\lambda \\ &\le (1 + C_{2.6})^p p \int_{m/(1+C_{2.6})}^{L/(1+C_{2.6})} \lambda^{p-1}(\mu(\{M_t f > \alpha\lambda\}) + \alpha^\delta A\mu(\{T_*f > \lambda\}))d\lambda \\ &\le (1 + C_{2.6})^p \alpha^{-p} \int (M_t f)^p d\mu + (1 + C_{2.6})^p \alpha^\delta Ap \int_0^L \lambda^{p-1}\mu(\{T_*f > \lambda\})d\lambda. \end{aligned}$$

Since $\delta > 0$, we can set α to be such that $(1 + C_{2.6})^p \alpha^\delta A \le 1/2$. With such an α, after the obvious cancellations we have

$$p \int_m^L x^{p-1} \mu(\{T_* f > x\}) dx \le 2(1 + C_{2.6})^p \alpha^{-p} \int (M_t f)^p d\mu + m^p \mu(S).$$

Therefore

$$\begin{aligned} \int (T_* f)^p d\mu &= p \int_0^\infty x^{p-1} \mu(\{T_* f > x\}) dx = p \int_0^m + \lim_{L \to \infty} p \int_m^L \\ &\le m^p \mu(S) + 2(1 + C_{2.6})^p \alpha^{-p} \int (M_t f)^p d\mu + m^p \mu(S). \end{aligned}$$

Since $m = (1 + C_{2.6}) \|\omega\|_\infty \|f\|_1$ and $\|f\|_1^p \mu(S) \le \int (M_t f)^p d\mu$, this completes the proof. □

Corollary 3.8. *Suppose that w satisfies condition (A_p) for some $1 < p < \infty$ and let $d\mu = w d\sigma$. Then T uniquely extends to a bounded operator on $L^p(S, d\mu)$.*

Proof. This follows immediately from Proposition 3.7 and Corollary 3.3. □

As usual, we will write M_φ for the operator of multiplication by the function φ.

Proposition 3.9. *If $\varphi \in$ BMO, then $[M_\varphi, T]$ is a bounded operator on $L^2(S, d\sigma)$.*

Proof. This follows from Corollary 3.8 and a standard argument, which we reproduce below. By the John-Nirenberg Theorem, there are positive constants C_1 and C_2 such that

$$\sigma(\{u \in B : |\varphi(u) - \varphi_B| > \lambda\}) \le C_1 \exp\left(\frac{-C_2 \lambda}{\|\varphi\|_{\mathrm{BMO}}}\right) \sigma(B)$$

for all $\lambda > 0$ and open d-balls B in S. We only need to consider real-valued $\varphi \in$ BMO. For real-valued φ, if we set $a = C_2(2\|\varphi\|_{\mathrm{BMO}})^{-1}$, then

$$\frac{1}{\sigma(B)} \int_B e^{a\varphi} d\sigma \frac{1}{\sigma(B)} \int_B e^{-a\varphi} d\sigma \le \left(\frac{1}{\sigma(B)} \int_B e^{a|\varphi - \varphi_B|} d\sigma\right)^2 \le (1 + C_1)^2$$

for every open d-ball B in S. Hence the function $w = e^{a\varphi}$ satisfies condition (A_2). By Corollary 3.8, T is bounded on $L^2(S, w d\sigma)$. This is equivalent to saying that the operator $M_{w^{1/2}} T M_{w^{-1/2}}$ is bounded on $L^2(S, d\sigma)$. Because w^{-1} also satisfies condition (A_2), the operator $M_{w^{-1/2}} T M_{w^{1/2}}$ is also bounded on $L^2(S, d\sigma)$.

Now, for each complex number z in the strip $V = \{z \in \mathbf{C} : -1 \le \mathrm{Re}(z) \le 1\}$, write

$$w_z^{1/2} = \exp(az\varphi/2) \quad \text{and} \quad w_z^{-1/2} = \exp(-az\varphi/2).$$

Obviously, $\|M_{w_z^{1/2}} T M_{w_z^{-1/2}}\| = \|M_{w^{1/2}} T M_{w^{-1/2}}\|$ if $\mathrm{Re}(z) = 1$ and $\|M_{w_z^{1/2}} T M_{w_z^{-1/2}}\| = \|M_{w^{-1/2}} T M_{w^{1/2}}\|$ if $\mathrm{Re}(z) = -1$. For the given φ, there is an obvious dense subset $\mathcal{D}$ of $L^2(S, d\sigma)$ such that if $f, g \in \mathcal{D}$, then the function

$$z \mapsto \langle M_{w_z^{1/2}} T M_{w_z^{-1/2}} f, g\rangle$$

is bounded on V. By a well-known result in complex analysis (see, e.g., [5,Corollary VI.3.9]), this implies that

$$\|M_{w_z^{1/2}}TM_{w_z^{-1/2}}\| \le \max\{\|M_{w^{1/2}}TM_{w^{-1/2}}\|, \|M_{w^{-1/2}}TM_{w^{1/2}}\|\}, \qquad z \in V.$$

Therefore the operator

$$\frac{a}{2}[M_\varphi, T] = \frac{d}{dz}M_{w_z^{1/2}}TM_{w_z^{-1/2}}\Big|_{z=0} = \frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^2}M_{w_z^{1/2}}TM_{w_z^{-1/2}}dz$$

is bounded on $L^2(S, d\sigma)$. $\square$

For our main application (Proposition 3.11), the result of Proposition 3.9 needs to be strengthened to Proposition 3.10 below. Proposition 3.10 can be proved either by a more careful tracking of all the constants in the results mentioned in this section or by using Proposition 3.9 plus the closed graph theorem. We will take the latter approach for expediency.

Proposition 3.10. *There is a constant $C_{3.10}$ such that*

$$\|[M_\varphi, T]g\|_2 \le C_{3.10}\|\varphi\|_{\mathrm{BMO}}\|g\|_2$$

for all $g \in L^2(S, d\sigma)$ and $\varphi \in$ BMO.

Proof. Consider the linear map $Y : \varphi \mapsto [M_\varphi, T]$, $\varphi \in$ BMO. Proposition 3.9 tells us that the range of Y is contained in the Banach space $\mathcal{B}(L^2(S, d\sigma))$. By the closed graph theorem, to prove the proposition, it suffices to show that the graph of Y is closed.

Let $\{\varphi_k\}$ be a sequence in BMO such that $\lim_{k\to\infty}\|\varphi_k\|_{\mathrm{BMO}} = 0$ and such that

$$\lim_{k\to\infty}\|[M_{\varphi_k}, T] - A\| = 0$$

for some $A \in \mathcal{B}(L^2(S, d\sigma))$. For $f, g \in L^\infty(S, d\sigma)$, by the condition $\lim_{k\to\infty}\|\varphi_k\|_{\mathrm{BMO}} = 0$ and the fact $[M_{\varphi_k}, T] = [M_{\varphi_k - c}, T]$ for any $c \in \mathbf{C}$ we have

$$\lim_{k\to\infty}\langle[M_{\varphi_k}, T]f, g\rangle = 0.$$

Thus $\langle Af, g\rangle = 0$ for all $f, g \in L^\infty(S, d\sigma)$. Since $A \in \mathcal{B}(L^2(S, d\sigma))$, this means $A = 0$. This proves that the graph of Y is closed and completes the proof of the proposition. $\square$

Proposition 3.11. *If $f \in$ VMO, then $[M_f, T]$ is a compact operator on $L^2(S, d\sigma)$.*

Proof. We first consider the case where f satisfies a Lipschitz condition $|f(u) - f(v)| \le L|u - v|$ on S. Let $\epsilon > 0$. For such an f we can write $[M_f, T] = A_\epsilon + B_\epsilon$, where

$$\begin{aligned}
(A_\epsilon g)(u) &= \int_{|1-\langle u,v\rangle|<\epsilon} J(u, v)g(v)d\sigma(v), \\
(B_\epsilon g)(u) &= \int_{|1-\langle u,v\rangle|\ge\epsilon} J(u, v)g(v)d\sigma(v), \qquad \text{and} \\
J(u, v) &= \frac{f(u) - f(v)}{(1 - \langle u, v\rangle)^n}\omega(|1 - \langle u, v\rangle|).
\end{aligned}$$

Since $|u-v| \le \sqrt{2}|1-\langle u,v\rangle|^{1/2}$, we have $|J(u,v)| \le \sqrt{2}L\|\omega\|_\infty |1-\langle u,v\rangle|^{-n+(1/2)}$. Since

$$\int \frac{1}{|1-\langle u,v\rangle|^{n-(1/2)}} d\sigma(v) < \infty$$

[16,Proposition 1.4.10], by a well-known estimate we have $\lim_{\epsilon\downarrow 0} \|A_\epsilon\| = 0$. Obviously, B_ϵ is compact. Therefore $[M_f, T]$ is compact if $f \in \mathrm{Lip}(S)$.

By the usual approximation, it follows from the preceding paragraph that $[M_f, T]$ is also compact if $f \in C(S)$. Finally, suppose that $f \in$ VMO. Then there exists a sequence $\{f_k\}$ in $C(S)$ such that $\lim_{k\to\infty} \|f - f_k\|_{\mathrm{BMO}} = 0$ [20,Proposition 4.1]. Since each $[M_{f_k}, T]$ is compact, it follows from Proposition 3.10 that $[M_f, T]$ is also compact. □

4. The Construction

We will now construct the operator promised in Section 1. The technical steps of construction are presented in the form of the first ten lemmas of the section. In order to better understand the construction, we suggest that the reader read the statements of Lemmas 4.1-10 first and save the proofs for later.

Lemma 4.1. *We have*

$$\lim_{\epsilon\downarrow 0} \int_{|1-\langle u,v\rangle|\ge\epsilon} \frac{1}{(1-\langle u,v\rangle)^n} d\sigma(v) = \frac{1}{2}$$

for every $u \in S$.

Proof. This is very close to [12,Lemma 7.2]. However, since [12,Lemma 7.2] was proved for the "gauge" $\gamma(u,v)$ defined by (7.1) on page 619 of [12], which is somewhat different from the $|1-\langle u,v\rangle|$ used in this paper, we would like to verify the details.

Let dA denote the natural Lebesgue measure on $\mathbf{C}$. In other words, the 1×1 square has measure 1. By formula 1.4.5(2) on page 15 of [16], we have

$$\int_{|1-\langle u,v\rangle|\ge\epsilon} \frac{1}{(1-\langle u,v\rangle)^n} d\sigma(v) = \frac{n-1}{\pi}\int_{D_\epsilon} \frac{(1-|z|^2)^{n-2}}{(1-\bar{z})^n} dA(z),$$

where $D_\epsilon = \{z \in \mathbf{C} : |z| < 1 \text{ and } |1-z| \ge \epsilon\}$. Performing the substitutions $\zeta = \epsilon/(1-z)$ and $w = \zeta - (\epsilon/2)$, we find that

$$\begin{aligned}\int_{|1-\langle u,v\rangle|\ge\epsilon} \frac{1}{(1-\langle u,v\rangle)^n} d\sigma(v) &= \frac{n-1}{\pi}\int_{E_\epsilon} \frac{\{2\mathrm{Re}(\zeta)-\epsilon\}^{n-2}}{\zeta^n} dA(\zeta) \\ &= \frac{2^{n-2}(n-1)}{\pi}\int_{\Lambda_\epsilon} \frac{\{\mathrm{Re}(w)\}^{n-2}}{(w+(\epsilon/2))^n} dA(w),\end{aligned}$$

where $E_\epsilon = \{\zeta \in \mathbf{C} : |\zeta| \le 1 \text{ and } \mathrm{Re}(\zeta) > \epsilon/2\}$ and $\Lambda_\epsilon = \{\zeta - (\epsilon/2) : \zeta \in E_\epsilon\}$. Denote $D_+ = \{w \in \mathbf{C} : |w| \le 1, \mathrm{Re}(w) > 0\}$. It is easy to see that $\Lambda_\epsilon \subset D_+$, that $A(D_+\backslash\Lambda_\epsilon) \le 2(\epsilon/2) = \epsilon$, and that if ϵ is sufficiently small, then $|w+(\epsilon/2)| \ge 1/2$ for $w \in D_+\backslash\Lambda_\epsilon$. Hence

$$\int_{|1-\langle u,v\rangle|>\epsilon} \frac{1}{(1-\langle u,v\rangle)^n} d\sigma(v) = \frac{2^{n-2}(n-1)}{\pi}\int_{D_+} \frac{\{\mathrm{Re}(w)\}^{n-2}}{(w+(\epsilon/2))^n} dA(w) + \eta(\epsilon) \tag{4.1}$$

with $\eta(\epsilon) \to 0$ as $\epsilon \downarrow 0$. For $0 < \delta < 1$, we have

$$\begin{aligned} \int_{D_+} \frac{\{\mathrm{Re}(w)\}^{n-2}}{(\delta + w)^n} dA(w) &= \int_{-\pi/2}^{\pi/2} \int_0^1 \frac{r^{n-2}\cos^{n-2}\theta}{(\delta + re^{i\theta})^n} r dr d\theta \\ &= \int_0^{\pi} \int_0^1 \frac{\sin^{n-2} t}{r((\delta/r) - ie^{it})^n} dr dt = \int_0^{\pi} \int_{\delta}^{\infty} \frac{\sin^{n-2} t}{x(x - ie^{it})^n} dx dt, \end{aligned}$$

where we made the substitutions $t = \theta + (\pi/2)$ and $x = \delta/r$. By [12,Lemma 6.2],

$$\lim_{\delta \downarrow 0} \int_0^{\pi} \int_{\delta}^{\infty} \frac{\sin^{n-2} t}{x(x - ie^{it})^n} dx dt = \frac{\pi}{2^{n-1}(n-1)}.$$

Combining this with (4.1), the lemma follows. □

For each $\epsilon > 0$, define the operator H_ϵ on $L^2(S, d\sigma)$ by the formula

$$(H_\epsilon f)(u) = \int_{|1 - \langle u, v\rangle| \ge \epsilon} \frac{f(v)}{(1 - \langle u, v\rangle)^n} d\sigma(v).$$

We also define the maximal singular integral

$$(H_* f)(u) = \sup_{\epsilon > 0} |(H_\epsilon f)(u)|.$$

Lemma 4.2. *There are constants C_1 and C_2 which depend only on the complex dimension n such that the inequality $H_* f \le C_1 Mf + C_2 M(Pf)$ holds on S for every $f \in L^2(S, d\sigma)$.*

Proof. It is elementary that $2|1 - \rho c| \ge |1 - c|$ if $|c| \le 1$ and $0 \le \rho \le 1$. Thus

$$\begin{aligned} \left| \frac{1}{(1 - \langle u, v\rangle)^n} - \frac{1}{(1 - \langle (1-\epsilon)u, v\rangle)^n} \right| &= \left| \sum_{j=0}^{n-1} \frac{\epsilon \langle u, v\rangle}{(1 - \langle u, v\rangle)^{n-j}(1 - \langle (1-\epsilon)u, v\rangle)^{j+1}} \right| \\ &\le \frac{2^n n \epsilon}{|1 - \langle u, v\rangle|^{n+1}} \end{aligned}$$

for all $0 < \epsilon \le 1$ and $u \ne v$ in S. It follows from Lemma 2.1 that

$$\left| \int_{|1 - \langle u, v\rangle| \ge \epsilon} \left(\frac{f(v)}{(1 - \langle u, v\rangle)^n} - \frac{f(v)}{(1 - \langle (1-\epsilon)u, v\rangle)^n} \right) d\sigma(v) \right| \le C(Mf)(u)$$

for all $0 < \epsilon \le 1$ and $u \in S$. On the other hand, by (1.2),

$$\left| \int_{|1 - \langle u, v\rangle| < \epsilon} \frac{f(v)}{(1 - \langle (1-\epsilon)u, v\rangle)^n} d\sigma(v) \right| \le \frac{1}{\epsilon^n} \int_{|1 - \langle u, v\rangle| < \epsilon} |f(v)| d\sigma(v) \le A_0 (Mf)(u).$$

Hence

$$|(H_\epsilon f)(u) - (Pf)((1-\epsilon)u)| \le (C + A_0)(Mf)(u).$$

The lemma follows from this inequality and the well-known fact that $|(Pf)((1-\epsilon)u)| \le C_2(M(Pf))(u)$ [16,page 75]. □

Lemma 4.3. (i) $\sup_{\epsilon>0} \|H_\epsilon\| < \infty$.
(ii) *The limit* $H = \lim_{\epsilon \downarrow 0} H_\epsilon$ *exists in the strong operator topology.*
(iii) $H = P - (1/2)$.

Proof. (i) is an immediate consequence of Lemma 4.2.

(ii) For $f \in L^2(S, d\sigma)$,

$$f(u)H_\epsilon 1 - (H_\epsilon f)(u) = \int_{|1-\langle u,v\rangle|\ge\epsilon} \frac{f(u) - f(v)}{(1 - \langle u, v\rangle)^n} d\sigma(v).$$

If f is Lipschitz (with respect to the Euclidian metric) on S, then $|f(u)-f(v)| \le L|u-v| \le \sqrt{2}L|1 - \langle u, v\rangle|^{1/2}$. For each $u \in S$, the function $\Phi_u(v) = |1 - \langle u, v\rangle|^{-n+(1/2)}$ belongs to $L^1(S, d\sigma)$ [16,Proposition 1.4.10], and $\|\Phi_u\|_1$ is independent of $u \in S$. Applying the dominated convergence theorem twice, we see that if $f \in \mathrm{Lip}(S)$, then the limit

$$\lim_{\epsilon\downarrow 0}(fH_\epsilon 1 - H_\epsilon f)$$

exists in the norm topology of $L^2(S, d\sigma)$. Combining this with Lemma 4.1, the limit $\lim_{\epsilon\downarrow 0} H_\epsilon f$ exists in the norm topology of $L^2(S, d\sigma)$ for every $f \in \mathrm{Lip}(S)$. By (i) and the fact that $\mathrm{Lip}(S)$ is dense in $L^2(S, d\sigma)$, the strong limit $H = \lim_{\epsilon\downarrow 0} H_\epsilon$ exists.

(iii) Again, this is just a slight variation of [12,Theorem 7.1]. Let φ be a polynomial in $z_1, ..., z_n, \bar{z}_1, ..., \bar{z}_n$. Then it follows from the above argument and Lemma 4.1 that

$$\frac{1}{2}\varphi(u) - (H\varphi)(u) = \varphi(u)H1 - (H\varphi)(u) = \int \frac{\varphi(u) - \varphi(v)}{(1 - \langle u, v\rangle)^n} d\sigma(v).$$

Recall that $2|1 - rc| \ge |1 - c|$ if $0 < r < 1$ and $|c| \le 1$. Thus it follows from the dominated convergence theorem that

$$\frac{1}{2}\varphi(u) - (H\varphi)(u) = \lim_{r\uparrow 1} \int \frac{\varphi(u) - \varphi(v)}{(1 - \langle ru, v\rangle)^n} d\sigma(v) = \lim_{r\uparrow 1}(\varphi(u) - (P\varphi)(ru)).$$

Since such φ's are dense in $L^2(S, d\sigma)$, this completes the proof. □

For the rest of the paper, let ξ be a real-valued, non-decreasing, C^∞ function on $(0, \infty)$ satisfying the conditions $\xi = 0$ on $(0, 1/2]$ and $\xi = 1$ on $[1, \infty)$. The reason that we require ξ to be *non-decreasing* will become clear in the proof of our next lemma.

With this ξ given, for each $a > 0$ we defined the operator

$$(G_a f)(u) = \int \frac{\xi(a^{-1}|1 - \langle u, v\rangle|)}{(1 - \langle u, v\rangle)^n} f(v)d\sigma(v)$$

on the Hilbert space $L^2(S, d\sigma)$. Obviously, each G_a is a compact, self-adjoint operator.

Lemma 4.4. (i) $\sup_{a>0} \|G_a\| < \infty$.
(ii) $\lim_{a\downarrow 0} G_a = H$ *in the strong operator topology.*
(iii) $\lim_{a\downarrow 0} \|G_a g - (g/2)\|_2 = 0$ *for every* $g \in H^2(S)$.

Proof. For each $a > 0$, consider the function $\xi_a(t) = \xi(t/a)$ on $(0, \infty)$. Because ξ_a is non-decreasing and continuous, and because $\xi_a = 0$ on $(0, a/2]$ and $\xi_a = 1$ on $[a, \infty)$, ξ_a can be uniformly approximated on $(0, \infty)$ by convex combinations of functions in the family $\{\chi_{[\epsilon,\infty)} : a/2 \le \epsilon \le a\}$. Hence G_a is in the operator-norm closure of the convex hull of $\{H_\epsilon : a/2 \le \epsilon \le a\}$. Thus this lemma follow from Lemma 4.3. □

As usual, we write k_z for the normalized reproducing kernel function for $H^2(S)$. That is, for each $z \in \mathbf{C}^n$ with $|z| < 1$, we write

$$k_z(w) = \frac{(1 - |z|^2)^{n/2}}{(1 - \langle w, z\rangle)^n}, \qquad |w| \le 1.$$

Lemma 4.5. *For all* $a > 0$, $b > 0$ *and* $0 < r < 1$, *the values of* $\|G_a k_{ru}\|_2$, $\|(G_a - G_b)k_{ru}\|_2$ *and* $\langle G_a k_{ru}, k_{ru}\rangle$ *are independent of* $u \in S$.

Proof. Let $U : \mathbf{C}^n \to \mathbf{C}^n$ be any unitary transformation. Then the formula

$$(\mathcal{U}_U f)(u) = f(Uu)$$

defines a unitary operator on $L^2(S, d\sigma)$. Clearly, $\mathcal{U}_U^* = \mathcal{U}_{U^*}$. Hence

$$\mathcal{U}_U^* G_a \mathcal{U}_U = G_a$$

for every $a > 0$. Also, $\mathcal{U}_U k_z = k_{U^* z}$. The lemma follows from these two facts. □

Lemma 4.6. *There exists a constant* $C_{4.6}$ *such that for all* $u \in S$, $0 < r < 1$ and $b \ge (1-r)^{1/3}$, *we have* $\|G_b k_{ru}\|_2 \le C_{4.6}(1-r)^{1/12}$.

Proof. For $b \ge (1-r)^{1/3}$, we have

$$\begin{aligned}|(G_b k_{ru})(v)| &\le \left(\frac{2}{b}\right)^n \int |k_{ru}(\zeta)| d\sigma(\zeta) \le 2^{3n/2} \int \frac{(1-r)^{(n/2)-(n/3)}}{|1 - r\langle \zeta, u\rangle|^n} d\sigma(\zeta) \\ &\le 2^{3n/2}(1-r)^{1/12} \int \frac{1}{|1 - r\langle \zeta, u\rangle|^{n-(1/12)}} d\sigma(\zeta)\end{aligned}$$

for every $v \in S$. By [16,Proposition 1.4.10], there is a constant C such that

$$\int \frac{1}{|1 - r\langle \zeta, u\rangle|^{n-(1/12)}} d\sigma(\zeta) \le C$$

for all $u \in S$ and $0 < r < 1$. This completes the proof. □

Lemma 4.7. *There exist sequences* $\{r(j)\}$, $\{a(j)\}$ *and* $\{b(j)\}$ *of positive numbers which have the following properties*:

(i) $0 < r(j) < 1$ *for every* $j \in \mathbf{N}$ *and* $\lim_{j\to\infty} r(j) = 1$;

(ii) $0 < a(j) < b(j)$ *for every* $j \in \mathbf{N}$ *and* $\lim_{j\to\infty} b(j) = 0$;

(iii) $b(j+1) \le a(j)/8$ *for every* $j \in \mathbf{N}$;

(iv) $\langle (G_{a(j)} - G_{b(j)})k_{r(j)u}, k_{r(j)u}\rangle \ge 1/3$ *for all* $j \ge 2$ *and* $u \in S$;

(v) $\sum_{i=1}^{j} \|(G_{a(i)} - G_{b(i)})k_{r(j+1)u}\|_2 \le (j+1)^{-1}$ *for all* $j \in \mathbf{N}$ *and* $u \in S$;

(vi) $\sum_{i=1}^{j} \|(G_{a(j+1)} - G_{b(j+1)})k_{r(i)u}\|_2 \le 2^{-(j+1)}$ *for all* $j \in \mathbf{N}$ *and* $u \in S$.

Proof. Ler $r_0 \in (0,1)$ be such that $C_{4.6}(1-r_0)^{1/12} \le 1/12$. We will select $r(j)$, $b(j)$ and $a(j)$ inductively. We begin with arbitrary $0 < r(1) < 1$ and $0 < a(1) < b(1) < \infty$.

Suppose that $j \ge 1$ and that we have selected $r(i)$, $b(i)$ and $a(i)$ for $1 \le i \le j$. By Lemma 4.6, there is a $\rho \in (0,1)$ such that

$$\sum_{i=1}^{j} \|(G_{a(i)} - G_{b(i)})k_{ru}\|_2 \le \frac{1}{j+1} \quad \text{for all} \ \ \rho \le r < 1 \ \ \text{and} \ \ u \in S.$$

By Lemma 4.4(ii) and Lemma 4.5, there is a $\beta > 0$ such that

$$\sum_{i=1}^{j} \|(G_a - G_b)k_{r(i)u}\|_2 \le 2^{-(j+1)} \quad \text{for all} \ \ 0 < a < b \le \beta \ \ \text{and} \ \ u \in S.$$

We pick an $r(j+1)$ such that $\max\{1-2^{-j-1}, r_0, \rho\} \le r(j+1) < 1$ and $(1-r(j+1))^{1/3} \le \min\{a(j)/8, \beta\}$. Let $b(j+1) = (1-r(j+1))^{1/3}$. Then $b(j+1) \le a(j)/8$,

$$\sum_{i=1}^{j} \|(G_{a(i)} - G_{b(i)})k_{r(j+1)u}\|_2 \le \frac{1}{j+1}, \qquad u \in S,$$

and

$$\sum_{i=1}^{j} \|(G_a - G_{b(j+1)})k_{r(i)u}\|_2 \le 2^{-(j+1)} \quad \text{for all} \ \ 0 < a < b(j+1) \ \ \text{and} \ \ u \in S.$$

Since $r(j+1) \ge r_0$ and $C_{4.6}(1-r_0)^{1/12} \le 1/12$, by Lemma 4.6 we have $\|G_{b(j+1)}k_{r(j+1)u}\|_2 \le 1/12$, $u \in S$. By Lemma 4.4(iii) and Lemma 4.5, we can pick an $a(j+1) \in (0, b(j+1))$ such that $\langle G_{a(j+1)}k_{r(j+1)u}, k_{r(j+1)u}\rangle \ge (1/2) - (1/12)$ for all $u \in S$. Hence

$$\langle (G_{a(j+1)} - G_{b(j+1)})k_{r(j+1)u}, k_{r(j+1)u}\rangle \ge (1/2) - (1/12) - (1/12) = 1/3.$$

This completes the inductive selection of the sequences $\{r(j)\}$, $\{a(j)\}$ and $\{b(j)\}$. Since $r(j+1) \ge 1-2^{-(j+1)}$, we have $\lim_{k\to\infty} r(j) = 1$. Note that the inequalities $b(j+1) \le a(j)/8$ and $a(j) < b(j)$ imply $b(j+1) \le 8^{-j}b(1)$. Therefore $\lim_{j\to\infty} b(j) = 0$. This completes the proof. □

Let N_1 be an infinite subset of $\mathbf{N}$ such that the set $N_2 = \mathbf{N}\backslash N_1$ is also infinite.

Lemma 4.8. *There exists an infinite subset* N of N_1 *such that the limit*

$$\lim_{m\to\infty} \sum_{j\in N\cap\{1,2,\dots,m\}} (G_{a(j)} - G_{b(j)}) = F \tag{4.2}$$

exists in the strong operator topology.

Proof. By Lemma 4.4(i), $\sup_{j\geq 1} \|G_{a(j)} - G_{b(j)}\| < \infty$. By Lemma 4.7(ii) and Lemma 4.4(ii), we have the strong convergence $\lim_{j\to\infty}(G_{a(j)} - G_{b(j)}) = 0$. Each $G_{a(j)} - G_{b(j)}$ is compact and self-adjoint. Thus the desired conclusion follows from [14,Lemma 2.1]. □

Lemma 4.9. *If* $f \in \mathrm{VMO}$, *then* $[M_f, F]$ *is a compact operator on* $L^2(S, d\sigma)$.

Proof. We will show that F is in fact an example of the operator T defined at the beginning of Section 2. Then by Proposition 3.11, $[M_f, F]$ is compact for every $f \in \mathrm{VMO}$.

For each $a > 0$, again consider the function $\xi_a(t) = \xi(t/a)$, $t > 0$. Since $\xi_a'(t) = a^{-1}\xi'(t/a)$ and $\xi'(t/a) \neq 0$ only if $t \in (a/2, a)$, we have $0 \leq \xi_a'(t) \leq \|\xi'\|_\infty/t$ for all $t > 0$.

For each $j \in \mathbf{N}$, define the function $\psi_j(t) = \xi(a^{-1}(j)t) - \xi(b^{-1}(j)t)$, $t \in (0,\infty)$. Then, by the preceding paragraph, $|\psi_j'(t)| \leq \|\xi'\|_\infty/t$ for all $t > 0$. By the choice of ξ, we have $\psi_j \in C^\infty(0,\infty)$, $\psi_j = 0$ on $(0,\infty)\backslash(a(j)/2, b(j))$ and $0 \leq \psi_j \leq 1$ on $(0,\infty)$. Let

$$\psi(t) = \sum_{j\in N} \psi_j(t) = \sum_{j\in N}\{\xi(a^{-1}(j)t) - \xi(b^{-1}(j)t)\}.$$

By the condition $b(j+1) \leq a(j)/8$ (Lemma 4.7(iii)) and the above-mentioned properties of ψ_j, we have $\psi \in C^\infty(0,\infty)$, $0 \leq \psi(t) \leq 1$ and $|\psi'(t)| \leq \|\xi'\|_\infty/t$ for all $t > 0$. That is, ψ satisfies conditions (i) and (ii) required of the function ω in Section 2.

For each $\epsilon > 0$, define the operator F_ϵ by the formula

$$(F_\epsilon f)(u) = \int_{S\backslash B(u,\epsilon)} \frac{\psi(|1-\langle u,v\rangle|)}{(1-\langle u,v\rangle)^n} f(v)d\sigma(v).$$

For each $m \in N$, set $\epsilon_m = (a(m)/2)^{1/2}$. Then $\psi_j\chi_{[\epsilon_m^2,\infty)} = \psi_j$ if $j \leq m$ and $\psi_j\chi_{[\epsilon_m^2,\infty)} = 0$ if $j > m$. Thus by the definitions of ψ and G_a we have

$$F_{\epsilon_m} = \sum_{j\in N\cap\{1,2,\dots,m\}} (G_{a(j)} - G_{b(j)}).$$

Comparing this with (4.2), we see that F also satisfies (2.1). □

We now define $\tilde{F}$ to be the compression of F to the Hardy space $H^2(S)$. That is,

$$\tilde{F}g = PFg, \quad g \in H^2(S), \tag{4.3}$$

where P is the orthogonal projection from $L^2(S, d\sigma)$ onto $H^2(S)$.

Lemma 4.10. (a) *Let $i_1 < i_2 < ... < i_\nu < ...$ be any ascending sequence of the integers in the set N given by Lemma 4.8. Then for every $u \in S$ we have*

$$\liminf_{\nu\to\infty} \langle \tilde{F} k_{r(i_\nu)u}, k_{r(i_\nu)u} \rangle \geq 1/3.$$

(b) *Let $j_1 < j_2 < ... < j_\nu < ...$ be any ascending sequence of the integers in N_2. Then for every $u \in S$ we have*

$$\lim_{\nu\to\infty} \langle \tilde{F} k_{r(j_\nu)u}, k_{r(j_\nu)u} \rangle = 0.$$

Proof. For any given integer $j > 2$, it follows from (v) and (vi) in Lemma 4.7 that

$$\sum_{i\neq j} \|(G_{a(i)} - G_{b(i)})k_{r(j)u}\|_2 = \sum_{i=1}^{j-1} \|(G_{a(i)} - G_{b(i)})k_{r(j)u}\|_2 + \sum_{i=j+1}^{\infty} \|(G_{a(i)} - G_{b(i)})k_{r(j)u}\|_2$$

$$\leq \frac{1}{j} + \sum_{i=j+1}^{\infty} 2^{-i} = \frac{1}{j} + 2^{-j}. \tag{4.4}$$

Thus if $j \in N_2$ and $j > 2$, since $j \notin N$, we have

$$\|\tilde{F} k_{r(j)u}\|_2 \leq \sum_{i\in N} \|(G_{a(i)} - G_{b(i)})k_{r(j)u}\|_2 \leq \frac{1}{j} + 2^{-j},$$

which proves (b). To prove (a), we note that $\langle \tilde{F} g, g \rangle = \langle F g, g \rangle$ if $g \in H^2(S)$. Thus for $j \in N$, it follows from (4.2) that

$$\begin{aligned}\langle \tilde{F} k_{r(j)u}, k_{r(j)u} \rangle &= \langle F k_{r(j)u}, k_{r(j)u} \rangle \\ &\geq \langle (G_{a(j)} - G_{b(j)})k_{r(j)u}, k_{r(j)u} \rangle - \sum_{i\neq j} \|(G_{a(i)} - G_{b(i)})k_{r(j)u}\|_2.\end{aligned}$$

Combining this inequality with Lemma 4.7(iv) and (4.4), (a) follows. □

Lemma 4.11. *For each $f \in L^2(S, d\sigma)$, there is a Borel set Λ in S with $\sigma(\Lambda) = 0$ such that*

$$\lim_{r\uparrow 1} \|(f - f(u))k_{ru}\|_2 = 0$$

for every $u \in S\backslash\Lambda$.

Proof. For each $\varphi \in L^1(S, d\sigma)$, define the *Poisson integral*

$$\varphi(z) = \int P(z, \zeta)\varphi(\zeta)d\sigma(\zeta), \qquad |z| < 1,$$

where the Poisson kernel is given by the formula

$$P(z,\zeta) = |k_z(\zeta)|^2, \quad |\zeta| = 1, \quad |z| < 1.$$

See pages 40-41 in [16]. Let $f \in L^2(S, d\sigma)$ be given and define the function $h = |f|^2$. By [16,Theorem 5.3.1], there is a Borel set Λ in S with $\sigma(\Lambda) = 0$ such that each $u \in S\backslash\Lambda$ is a Lebesgue point for both f and h. By [16,Theorem 5.4.8], for each $u \in S\backslash\Lambda$ we have

$$\lim_{r\uparrow 1} f(ru) = f(u) \quad \text{and} \quad \lim_{r\uparrow 1} h(ru) = h(u) = |f(u)|^2. \tag{4.5}$$

But for every $u \in S$ and every $0 < r < 1$ we have

$$\begin{aligned}\|(f - f(u))k_{ru}\|_2^2 &= \|fk_{ru}\|_2^2 - 2\mathrm{Re}\langle fk_{ru}, f(u)k_{ru}\rangle + |f(u)|^2\|k_{ru}\|_2^2 \\ &= h(ru) - 2\mathrm{Re}\{f(ru)\overline{f(u)}\} + |f(u)|^2.\end{aligned}$$

Combining this with (4.5), the lemma follows. □

Lemma 4.12. *For any given $\varphi_1, ..., \varphi_m \in L^\infty(S, d\sigma)$, there exists a Borel set Ω in S with $\sigma(\Omega) = 0$ such that if $u \in S\backslash\Omega$, then the limit*

$$\lim_{r\uparrow 1}\langle T_{\varphi_1}...T_{\varphi_m}k_{ru}, k_{ru}\rangle$$

exists and equals $\varphi_1(u)...\varphi_m(u)$.

Proof. We use induction on m. The case $m = 1$ follows from Lemma 4.11. Suppose that $m \geq 2$ and that the desired assertion is true for $T_{\varphi_1}...T_{\varphi_{m-1}}$. Then

$$T_{\varphi_1}...T_{\varphi_m}k_{ru} = \varphi_m(u)T_{\varphi_1}...T_{\varphi_{m-1}}k_{ru} + T_{\varphi_1}...T_{\varphi_{m-1}}P(\varphi_m - \varphi_m(u))k_{ru}.$$

Thus the case for m follows from the induction hypothesis and another application of Lemma 4.11. □

Proposition 4.13. *If X is an operator belonging to the Toeplitz algebra $\mathcal{T}$, then there exists a Borel subset E of S with $\sigma(E) = 0$ such that the limit*

$$\lim_{r\uparrow 1}\langle Xk_{ru}, k_{ru}\rangle$$

exists for every $u \in S\backslash E$.

Proof. If $X \in \mathcal{T}$, then there exists a sequence $\{X_j\}$, where each X_j is the sum of a finite number of terms of the form $T_{\varphi_1}...T_{\varphi_m}$, $m \in \mathbf{N}$ and $\varphi_1, ..., \varphi_m \in L^\infty(S, d\sigma)$, such that

$$\lim_{j\to\infty}\|X - X_j\| = 0.$$

Thus this proposition is an immediate consequence of Lemma 4.12. □

Proof of Theorem 1.1. We want to show that the operator $\tilde{F}$ defined by (4.3) belongs to the essential commutant of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$ but does not belong to $\mathcal{T}$.

It is well known that if $f \in \mathrm{VMO}$, then $[M_f, P]$ is compact. Therefore it follows from Lemma 4.9 that $\tilde{F}$ belongs to the essential commutant of $\mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}})$.

To show that $\tilde{F} \notin \mathcal{T}$, recall from Lemma 4.7(i) that $\lim_{j\to\infty} r(j) = 1$. Thus Lemma 4.10 tells us that for no $u \in S$ does the limit

$$\lim_{r\uparrow 1}\langle \tilde{F}k_{ru}, k_{ru}\rangle$$

exist. By Proposition 4.13, this means $\tilde{F} \notin \mathcal{T}$. □

References

1. A. Calderón, *Inequalities for the maximal function relative to a metric*, Studia Math. **57** (1976), 297-306.
2. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241-250.
3. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611-635.
4. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, **242**, Springer-Verlag, Berlin-New York, 1971.
5. J. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Mathematics **11**, Springer-Verlag, New York, 1978.
6. K. Davidson, *On operators commuting with Toeplitz operators modulo the compact operators*, J. Funct. Anal. **24** (1977), 291-302.
7. X. Ding and S. Sun,*Essential commutant of analytic Toeplitz operators*, Chinese Sci. Bull. **42** (1997), 548-552.
8. M. Engliš, *Toeplitz operators and the Berezin transform on H^2*, Linear Algebra Appl. **223/224** (1995), 171-204.
9. J. Garnett, *Bounded analytic functions*, Academic Press, New York-London, 1981.
10. K. Guo and S. Sun, *The essential commutant of the analytic Toeplitz algebra and some problems related to it* (Chinese), Acta Math. Sinica (Chin. Ser.) **39** (1996), 300-313.
11. B. Johnson and S. Parrott, *Operators commuting with a von Neumann algebra modulo the set of compact operators*, J. Funct. Anal. **11** (1972), 39-61.
12. A. Korányi and S. Vági, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), 575-648.
13. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207-226.
14. P. Muhly and J. Xia, *On automorphisms of the Toeplitz algebra*, Amer. J. Math. **122** (2000), 1121-1138.
15. S. Popa, *The commutant modulo the set of compact operators of a von Neumann algebra*, J. Funct. Anal. **71** (1987), 393-408.
16. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York-Berlin, 1980.

17. E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, 1993.
18. D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), 97-113.
19. J. Xia, *Coincidence of essential commutant and the double commutant relation in the Calkin algebra*, J. Funct. Anal. **197** (2003), 140-150.
20. J. Xia, *Bounded functions of vanishing mean oscillation on compact metric spaces*, J. Funct. Anal. **209** (2004), 444-467.
21. J. Xia, *On the essential commutant of* $\mathcal{T}$(QC), Trans. Amer. Math. Soc. **360** (2008), 1089-1102.

Department of Mathematics
State University of New York at Buffalo
Buffalo, NY 14260
USA

E-mail: jxia@acsu.buffalo.edu