ONE HUNDRED YEARS OF DIAGONALIZATION

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Three basic conventions for these talks:

(1) All Hilbert spaces are **separable**.

(2) The word "operator" means **bounded** operator, although some of the results also hold in unbounded situations. The point is that in situations where the results can be generalized from bounded case to unbounded case, the generalization itself is **NOT** very interesting.

(3) In all the citations below the year refers to the year when the result **appeared in print**.

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First of all, what is a diagonal operator?

An operator D on a Hilbert space $\mathcal{H}$ is said to be **diagonal** if it is a diagonal matrix with respect to an orthonormal basis in $\mathcal{H}$. This can be depicted more graphically, as follows:

Let ℓ^2_+ be the Hilbert space of complex sequences $\{a_1,..., a_j, ...\}$ with $\sum_{j=1}^{\infty} |a_j|^2 < \infty$.

Given a bounded sequence c_1, ..., c_j, ... in $\mathbf{C}$, define the operator $\mathrm{diag}(c_j)_{j=1}^{\infty}$ on ℓ^2_+ by the formula

$$\mathrm{diag}(c_j)_{j=1}^{\infty}\{a_1, ..., a_j, ...\} = \{c_1 a_1, ..., c_j a_j, ...\}.$$

An operator D on an infinite dimensional Hilbert space $\mathcal{H}$ is **diagonal** if and only if it is unitarily equivalent to a $\mathrm{diag}(c_j)_{j=1}^{\infty}$ on ℓ^2_+.

1. Diagonalization of a Single Operator

Our story starts with a well-known result of Hermann Weyl published in 1909:

Theorem. If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then A can be diagonalized modulo a compact operator. In other words, there exists a self-adjoint compact operator K such that the operator

$$D = A + K$$

is diagonal.

Historical note. It is commonly accepted that "Functional Analysis" did not become a subject until Riesz introduced L^p spaces in the 1920s. "Operator Theory" came even later. So Weyl's theorem was way ahead of its time. Particularly noteworthy is the fact that Weyl's theorem appeared only five years after the publication of Lebesgue's

Lecons sur l'intégration,

where the notion of "measure" made its first appearance.

As we will see, **measures** are very relevant to the problem of diagonalization.

In 1935, von Neumann gave the following improvement of Weyl's theorem:

Theorem. If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then A can be diagonalized modulo the Hilbert-Schmidt class. In other words, there exists a self-adjoint operator K in the Hilbert-Schmidt class such that the operator

$$D = A + K$$

is diagonal. Furthermore, $\|K\|_2$ can be required to be less than any given $\epsilon > 0$.

To describe the next improvement, let us introduce **norm ideals**, of which the Hilbert-Schmidt class is one of the best known examples.

As we will see, in addition to measures, norm ideals are the other essential ingredient in these talks.

Let $\mathcal{H}$ be a Hilbert space. A **norm ideal** is a two-sided ideal $\mathcal{C}$ in $\mathcal{B}(\mathcal{H})$ equipped with a norm $\|.\|_{\mathcal{C}}$ which has the following properties:

(a) For any $S, T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}$, $\|SAT\|_{\mathcal{C}} \le \|S\|\|A\|_{\mathcal{C}}\|T\|$.

(b) If $A \in \mathcal{C}$, then $A^* \in \mathcal{C}$ and $\|A^*\|_{\mathcal{C}} = \|A\|_{\mathcal{C}}$.

(c) For any $A \in \mathcal{C}$, $\|A\| \le \|A\|_{\mathcal{C}}$, and the equality holds when $\mathrm{rank}(A) = 1$.

(d) $\mathcal{C}$ is complete with respect to $\|.\|_{\mathcal{C}}$.

(e) $\mathcal{C} \neq \{0\}$.

In some books and papers, such a $\mathcal{C}$ is called a *symmetrically normed ideal*. See, for example, the famous book by Gohberg and Krein. But I have always used the term "norm ideal", which is due to Schatten.

Example. For each $1 \le p < \infty$, let $\mathcal{C}_p = \{K : \|K\|_p < \infty\}$, where

$$\|K\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}.$$

Such $\mathcal{C}_p$'s are called **Schatten classes**.

$\mathcal{C}_1$ is also called the **trace class**, which is the smallest of all norm ideals. The **Hilbert-Schmidt class** is just $\mathcal{C}_2$.

Another well-known example is $\mathcal{K}$, the collection of compact operators.

Although $\mathcal{B}(\mathcal{H})$ itself is a norm ideal by this definition, for the rest of the talks our norm ideals are always assumed to be contained in $\mathcal{K}$.

The next improvement of Weyl's theorem came in 1958:

Theorem. (Kuroda.) If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then A can be diagonalized modulo any norm ideal $\mathcal{C}$ which is **not the trace class**. That is, if $\mathcal{C}$ is not the trace class, there exists a self-adjoint $K \in \mathcal{C}$ such that the operator

$$D = A + K$$

is diagonal. Furthermore, $\|K\|_{\mathcal{C}}$ can be required to be less than any given $\epsilon > 0$.

This leads to the obvious question: what happens in the case of trace-class perturbation?

The first "no-go" result in diagonalization follows from a theorem proved **independently** by Kato and Rosenblum in 1957. To discuss the Kato-Rosenblum theorem, let us first review the spectral decomposition of self-adjoint operators.

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(A) \oplus \mathcal{H}_{\mathrm{s}}(A)$$

with the following properties:

(1) Both subspaces $\mathcal{H}_{\mathrm{ac}}(A)$ and $\mathcal{H}_{\mathrm{s}}(A)$ are invariant under A.

(2) The spectral measure of $A_{\mathrm{ac}} = A|\mathcal{H}_{\mathrm{ac}}(A)$ is absolutely continuous with respect to the Lebesgue on $\mathbf{R}$.

(3) The spectral measure of $A_{\mathrm{s}} = A|\mathcal{H}_{\mathrm{s}}(A)$ is singular to the Lebesgue on $\mathbf{R}$.

A_{ac} is called the **absolutely continuous part** of A.

A_{s} is called the **singular part** of A.

A is said to be **purely absolutely continuous** if $\mathcal{H}_{\mathrm{s}}(A) = \{0\}$.

A is said to be **purely singular** if $\mathcal{H}_{\mathrm{ac}}(A) = \{0\}$.

Kato-Rosenblum Theorem. Let A be a self-adjoint operator. Let K be a self-adjoint operator **in the trace class** and write

$$B = A + K.$$

Then A_{ac} and B_{ac} are unitarily equivalent.

In other words, the absolutely continuous part of a self-adjoint operator cannot be changed by trace-class perturbation.

In particular, if A is not purely singular, then A cannot be diagonalized modulo the trace class.

On the other hand, in 1976 Carey and Pincus proved the following:

Theorem. Let A be a self-adjoint operator. If A is purely singular, then there exists a self-adjoint operator K in the trace class such that the operator

$$D = A + K.$$

is diagonal. Furthermore, $\|K\|_1$ can be required to be arbitrarily small.

Summarizing the story of diagonalization of a single self-adjoint operator A, we have

(1) **Kuroda:** A can be diagonalized modulo any norm idea $\mathcal{C}$ which is not the trace class.

(2) **Carey-Pincus and Kato-Rosenblum:** A can be diagonalized modulo the trace class if and only if it is purely singular.

Before concluding the story about a single self-adjoint operator A, let us see a proof of Weyl's original theorem, which provides some idea of what is involved.

To prove Weyl's original theorem, it suffices to consider the simplified scenario where we assume:

(1) $\mathcal{H} = L^2([0,1), d\mu)$, where $d\mu$ is a Borel measure concentrated on a subset of $[0,1)$.

(2) A is the natural multiplication operator on $L^2([0,1), d\mu)$, i.e., $(Af)(x) = xf(x)$ for $f \in L^2([0,1), d\mu)$.

Now, for each pair of integers $k \geq 1$ and $1 \leq j \leq 2^k$, define

$$I_j^k = \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right) \qquad \text{and}$$

$$e_j^k = \begin{cases} \frac{1}{\sqrt{\mu(I_j^k)}}\chi_{I_j^k} & \text{if} \quad \mu(I_j^k) \neq 0 \\ \\ 0 & \text{if} \quad \mu(I_j^k) = 0 \end{cases}.$$

Let $E_k = \text{span}\{e_j^k : 1 \leq j \leq 2^k\}$ and let

$$P_k = \sum_{j=1}^{2^k} e_j^k \otimes e_j^k.$$

Then P_k is the orthogonal projection from $L^2([0,1), d\mu)$ onto E_k. We have

$$\begin{aligned} [A, P_k] &= \sum_{j=1}^{2^k} [A, e_j^k \otimes e_j^k] \\ &= \sum_{j=1}^{2^k} (\{(A - x(k,j)e_j^k\} \otimes e_j^k - e_j^k \otimes \{(A - x(k,j)e_j^k\}), \end{aligned}$$

where $x(k,j)$ is any chosen point in I_j^k. Therefore

$$\|[A, P_k]\| \le 2 \max_{1\le j\le 2^k} \max_{x\in I_j^k} |x - x(k,j)| \le 2^{-k+1}.$$

If we define $Q_k = P_k - P_{k-1}$ for $k \ge 2$, then

$$\|[A, Q_k]\| \le 2^{-k+1} + 2^{-k+1+1} \le 2^{-k+3}.$$

Note that $E_k \subset E_{k+1}$. Hence

$$Q_k Q_{k'} = 0 \quad \text{for} \quad k \ne k', \qquad \text{and} \qquad P_m Q_k = 0 \quad \text{for} \quad k > m.$$

For any $m \ge 1$, we have

$$1 = P_m + \sum_{k=m+1}^{\infty} Q_k.$$

Therefore

$$\begin{aligned} A = 1A1 = P_m A P_m &+ \sum_{k=m+1}^{\infty} Q_k A Q_k \\ &+ \sum_{k=m+1}^{\infty} (P_m A Q_k + Q_k A P_m) \\ &+ \sum_{m+1\le j<k<\infty} (Q_j A Q_k + Q_k A Q_j). \end{aligned}$$

Note that the operator

$$D = P_m A P_m + \sum_{k=m+1}^{\infty} Q_k A Q_k$$

is diagonal, because each term has a finite rank and the ranges of the terms are orthogonal to each other.

To complete the proof, it suffices to show that the operators

$$K_1 = \sum_{k=m+1}^{\infty} (P_m A Q_k + Q_k A P_m),$$

$$K_2 = \sum_{m+1 \leq j < k < \infty} (Q_j A Q_k + Q_k A Q_j)$$

are compact. But for $k > j \geq m+1$, since $Q_j Q_k = 0$, we have

$$\|Q_j A Q_k\| = \|Q_j [A, Q_k]\| \leq \|[A, Q_k]\| \leq 2^{-k+3}.$$

Therefore

$$\sum_{m+1 \leq j < k < \infty} \|Q_j A Q_k\| \leq \sum_{j=m+1}^{\infty} \sum_{k=j+1}^{\infty} 2^{-k+3} = 2^{-m+3}$$

Since each $Q_j A Q_k$ has a finite rank, it follows that K_2 is compact with $\|K_2\| \leq 2^{-m+4}$. A similar argument shows that K_1 is also compact with $\|K_1\| \leq 2^{-m+4}$. If we write $K = K_1 + K_2$, then

$$A = D + K$$

where K is compact with $\|K\| \leq 2^{-m+5}$.

The proof of Weyl's theorem in the general case follows from this special case and the fact that a general A can be decomposed into the orthogonal sum of **countably** many operators of multiplication by the coordinate function.

The same line of argument can also be used to prove Kuroda's improvement of Weyl's theorem. Recall that we have shown

$$\|[A, P_k]\| \le 2^{-k+1}$$

in the above. Obviously,

$$\mathrm{rank}([A, P_k]) \le 2^{k+1}.$$

Kuroda's observation was that if $\mathcal{C}$ is not the trace class, then the above two inequalities imply

$$\lim_{k\to\infty} \|[A, P_k]\|_{\mathcal{C}} = 0.$$

(This follows, for example, from a duality argument using the fact that $\|.\|_{\mathcal{C}}$ must be strictly weaker than the trace norm.) Thus there is a subsequence $\{P_{k(i)}\}_{i=1}^{\infty}$ such that

$$\|[A, P_{k(i)}]\|_{\mathcal{C}} \le 2^{-i+1}$$

For every $i \ge 1$. Using the subsequence $\{P_{k(i)}\}_{i=1}^{\infty}$ in place of the original sequence of projections $\{P_k\}_{k=1}^{\infty}$, the argument on the previous pages yield

$$A = D + K \quad \text{with} \quad K \in \mathcal{C}.$$

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Next we move on to the problem of diagonalization for commuting tuples of self-adjoint operators, which is where things really become interesting.

2. Diagonalization of Commuting Tuples

The first result for diagonalization for commuting tuples was for commuting **pairs**:

Theorem. (Berg, 1971.) If N is a normal operator on a Hilbert space $\mathcal{H}$, then N can be diagonalized modulo a compact operator. In other words, there exists a compact operator K such that the operator

$$D = N + K \tag{1}$$

is both normal and diagonal. Furthermore, $\|K\|$ can be required to be less than any given $\epsilon > 0$.

In the same paper, Berg asked whether the K in (1) can be required to be in the Hilbert-Schmidt class, but he did not have an answer.

The next leap in the subject was due to the work of Voiculescu, who answered this question and much, much more.

Definition. Let $T = (T_1, ..., T_n)$ be a **commuting** tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{C}$ be a norm ideal. We say that T is simultaneously diagonalizable modulo $\mathcal{C}$ if there exists a **commuting** tuple of self-adjoint diagonal operators $(D_1, ..., D_n)$ such that

$$T_j - D_j \in \mathcal{C}$$

for every $1 \leq j \leq n$.

If $\mathcal{C}$ is a norm ideal, then we denote by $\mathcal{C}^{(0)}$ the $\|.\|_{\mathcal{C}}$-closure of the collection of finite-rank operators in $\mathcal{C}$.

Equipped with the same norm $\|.\|_{\mathcal{C}}$, $\mathcal{C}^{(0)}$ is itself a norm ideal.

It is well-known that $\mathcal{C}^{(0)}$ can be strictly smaller than $\mathcal{C}$. We will see such examples later. (Recall that we assume $\mathcal{C} \subset \mathcal{K}$, so such examples are not trivial.)

Theorem. (Voiculescu, 1979.) Let $(T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators and let $\mathcal{C}$ be a norm ideal. Then the following are equivalent:

(a) $(T_1, ..., T_n)$ can be simultaneously diagonalized modulo $\mathcal{C}^{(0)}$.

(b) $(T_1, ..., T_n)$ can be simultaneously diagonalized modulo $\mathcal{C}^{(0)}$ with arbitrarily small perturbation. That is, for any $\epsilon > 0$, there is a commuting tuple of self-adjoint diagonal operators $(D_1, ..., D_n)$ such that $T_j - D_j \in \mathcal{C}^{(0)}$ with $\|T_j - D_j\|_{\mathcal{C}} \leq \epsilon$.

(c) There exists a sequence $\{P_k\}$ of finite-rank orthogonal projections such that

$$\text{s-}\lim_{k\to\infty} P_k = 1 \quad \text{and} \quad \lim_{k\to\infty} \sum_{j=1}^{n} \|[T_j, P_k]\|_{\mathcal{C}} = 0.$$

(d) There is a sequence $\{G_k\}$ of compact operators such that

$$\text{s-}\lim_{k\to\infty} G_k = 1 \quad \text{and} \quad \lim_{k\to\infty} \sum_{j=1}^{n} \|[T_j, G_k]\|_{\mathcal{C}} = 0.$$

As it turned out, there is one more equivalent condition.

Theorem. (Voiculescu, 1990.) Let $(T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators and let $\mathcal{C}$ be a norm ideal. If $(T_1, ..., T_n)$ is simultaneously diagonalizable modulo $\mathcal{C}$, then it is also simultaneously diagonalizable modulo the (possibly smaller) ideal $\mathcal{C}^{(0)}$.

Berg's question was answered affirmatively:

Theorem. (Voiculescu, 1979.) Suppose that $n \geq 2$. Then any n-tuple of commuting self-adjoint operators $(T_1, ..., T_n)$ is simultaneously diagonalizable modulo the Schatten class $\mathcal{C}_n$.

This should be compared with the fact that a single self-adjoint operator A is not necessarily diagonalizable modulo $\mathcal{C}_1$.

Perhaps more important than the above are Voiculescu's obstruction results (or "no-go" results) for diagonalization.

3. Obstruction to Diagonalization

To discuss these obstruction results, we need to talk a little more about norm ideals in general.

Suppose that $\mathcal{C}$ is any norm ideal. Then there exists a norm ideal which is the Banach-space dual of $\mathcal{C}^{(0)}$. That is, there exists a norm ideal $\mathcal{C}'$ with the following two properties:

(a) If $X \in \mathcal{C}^{(0)}$ and $Y \in \mathcal{C}'$, then $XY \in \mathcal{C}_1$ with

$$|\mathrm{tr}(XY)| \leq \|X\|_{\mathcal{C}} \|Y\|_{\mathcal{C}'}.$$

(b) If φ is a bounded linear functional on $\mathcal{C}^{(0)}$, then there is a unique $Y_\varphi \in \mathcal{C}'$ with $\|Y_\varphi\|_{\mathcal{C}'} = \|\varphi\|$ such that

$$\varphi(X) = \mathrm{tr}(XY_\varphi)$$

for every $X \in \mathcal{C}^{(0)}$.

For example, if $1 < p < \infty$, then $(\mathcal{C}_p)' = \mathcal{C}_{p/(p-1)}$.

$(\mathcal{C}_1)' = \mathcal{B}(\mathcal{H})$.

$\mathcal{K}' = \mathcal{C}_1$.

Let us introduce two other classes of norm ideals.

Suppose that $1 \leq p < \infty$. For each compact operator K, define

$$\|K\|_p^+ = \sup_{m \geq 1} \frac{\sum_{j=1}^m s_j(K)}{\sum_{j=1}^m j^{-1/p}}$$

and

$$\|K\|_p^- = \sum_{j=1}^{\infty} \frac{s_j(K)}{j^{(p-1)/p}}.$$

(Here, $s_1(K)$, ..., $s_j(K)$, ... are the so-called s-numbers of K.) With these norms, we can define the norm ideals

$$\mathcal{C}_p^+ = \{K : \|K\|_p^+ < \infty\}$$

and

$$\mathcal{C}_p^- = \{K : \|K\|_p^- < \infty\}.$$

For $1 < p < p' < \infty$ we have **proper** inclusions

$$\mathcal{C}_p^- \subset \mathcal{C}_p \subset \mathcal{C}_p^+ \subset \mathcal{C}_{p'}^-.$$

But note that $\mathcal{C}_1^- = \mathcal{C}_1$ and that $\mathcal{C}_1^+ \neq \mathcal{C}_1$.

We have $(\mathcal{C}_p^-)^{(0)} = \mathcal{C}_p^-$. But,

$$(\mathcal{C}_p^+)^{(0)} \neq \mathcal{C}_p^+.$$

This is the most commonly given example to demonstrate the fact that $\mathcal{C}^{(0)}$ need not coincide with $\mathcal{C}$.

For $1 < p < \infty$, $\mathcal{C}^+_{p/(p-1)}$ is the dual of $\mathcal{C}^-_p$, i.e.,

$$(\mathcal{C}^-_p)' = \mathcal{C}^+_{p/(p-1)}.$$

Theorem. (Voiculescu, 1979.) Suppose that $n \geq 2$ and let m_n denote the n-dimensional Lebesgue measure on $\mathbf{R}^n$. Then the tuple $(M_1, ..., M_n)$ on $L^2([0,1)^n, dm_n)$ defined by the formula

$$(M_j f)(x_1, ..., x_n) = x_j f(x_1, ..., x_n), \qquad f \in L^2([0,1)^n, dm_n),$$

is **not** simultaneously diagonalizable modulo $\mathcal{C}^-_n$.

This was proved by considering the singular integral operators

$$(S_j f)(x) = \int_{[0,1)^n} \frac{x_j - y_j}{|x-y|^2} f(y) dm_n(y)$$

$1 \leq j \leq n$, on $L^2([0,1)^n, dm_n)$. Voiculescu showed that, when $n \geq 2$, $S_1, ..., S_n \in \mathcal{C}^+_{n/(n-1)}$, the dual of $\mathcal{C}^-_n$. Note that

$$\sum_{j=1}^n [M_j, S_j] = 1 \otimes 1,$$

where 1 is the constant function of value 1 in $L^2([0,1)^n, dm_n)$. The fact $\mathrm{tr}(1 \otimes 1) \neq 0$ plus the fact that $S_1, ..., S_n$ belong to the dual of $\mathcal{C}^-_n$ implies that the tuple $(M_1, ..., M_n)$ is not simultaneously diagonalizable modulo $\mathcal{C}^-_n$.

This approach to obstruction to diagonalization was later generalized further.

Proposition. (Voiculescu, 1990.) Let $(T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators and let $\mathcal{C}$ be a norm ideal. Suppose that there exist $S_1, ..., S_n \in \mathcal{C}'$ such that the operator

$$\Gamma = \sum_{j=1}^{n} [T_j, S_j]$$

satisfies **either one** of the following two conditions:
(a) $\Gamma \in \mathcal{C}_1$ and $\mathrm{tr}(\Gamma) \neq 0$;
(b) Γ is a positive operator with $0 < \mathrm{tr}(\Gamma) \leq \infty$.
Then the tuple $(T_1, ..., T_n)$ is **NOT** simultaneously diagonalizable modulo $\mathcal{C}^{(0)}$.

(Recall that Voiculescu also showed that if $(T_1, ..., T_n)$ is not simultaneously diagonalizable modulo $\mathcal{C}^{(0)}$ then $(T_1, ..., T_n)$ is not simultaneously diagonalizable modulo $\mathcal{C}$ either.)

This proposition is now the basis for proving all obstruction results. In other words, nowadays the common approach to showing that $(T_1, ..., T_n)$ is not simultaneously diagonalizable modulo $\mathcal{C}$ is to look for $S_1, ..., S_n$ satisfying the conditions in the proposition.

By spectral decomposition, $(T_1, ..., T_n)$ is the orthogonal sum of countably many tuples of multiplication operators on L^2 of different measures μ on $\mathbf{R}^n$. Diagonalization problems always boil down to this situation. Therefore, when one tries to establish obstruction to diagonalization, the $S_1, ..., S_n$ are **usually** singular integral operators. We will have more to say about this "usual" approach at the end of these talks.

A few more conventions:

Convention. For the rest of the talks, the symbol μ will denote a compactly supported regular Borel measure on $\mathbf{R}^n$ with a nonzero total mass.

Notation. Let $(M_1^\mu, ..., M_n^\mu)$ denote the tuple on $L^2(\mathbf{R}^n, d\mu)$ defined by the formula

$$(M_j^\mu f)(x_1, ..., x_n) = x_j f(x_1, ..., x_n),$$

$f \in L^2(\mathbf{R}^n, d\mu)$.

Notation. For each $1 \leq j \leq n$, let S_j^μ denote the singular integral operator on $L^2(\mathbf{R}^n, d\mu)$ defined by the formula

$$(S_j^\mu f)(x) = \int \frac{x_j - y_j}{|x-y|^2} f(y) d\mu(y),$$

$f \in L^2(\mathbf{R}^n, d\mu)$.

(Keep in mind that $\sum_{j=1}^n [M_j^\mu, S_j^\mu] = 1 \otimes 1$, and that $\mathrm{tr}(1 \otimes 1) = \mu(\mathbf{R}^n)$, the total mass of μ.)

Notation. For $x \in \mathbf{R}^n$ and $r > 0$, we write

$$B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\},$$

the usual Euclidian ball.

Theorem. (David and Voiculescu, 1990.) Suppose that $1 < p < \infty$. Suppose that there is a constant $0 < C < \infty$ such that

$$\mu(B(x,r)) \leq Cr^p \tag{2}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Then $S_1^\mu, ..., S_n^\mu \in \mathcal{C}^+_{p/(p-1)}$. Thus the tuple $(M_1^\mu, ..., M_n^\mu)$ is not simultaneously diagonalizable modulo $\mathcal{C}_p^-$.

Because $m_n(B(x,r)) \leq Cr^n$, this generalizes the earlier obstruction result for the tuple $(M_1, ..., M_n)$ on $L^2([0,1)^n, dm_n)$. But the result of David and Voiculescu can be improved even further.

Theorem. (J.X., 2003.) Suppose that $1 < p < \infty$. Suppose that there is a $t > 1/(p-1)$ such that

$$\sup_{0<r\leq 1} \int \left(\frac{\mu(B(x,r))}{r^p}\right)^t d\mu(x) < \infty. \tag{3}$$

Then $S_1^\mu, ..., S_n^\mu \in \mathcal{C}^+_{p/(p-1)}$. Thus the tuple $(M_1^\mu, ..., M_n^\mu)$ is not simultaneously diagonalizable modulo $\mathcal{C}_p^-$.

Obviously, (2) implies (3). But there exists μ for which (3) holds and yet

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^p} = \infty \tag{4}$$

for every x in the support $\mathcal{X}$ of μ. (4) implies that the p-dimensional Hausdorff measure of $\mathcal{X}$ is zero, which means that $\mathcal{X}$ is a "small" set. Thus there exists a μ which is singular to the p-dimensional Hausdorff measure and for which $(M_1^\mu, ..., M_n^\mu)$ is not simultaneously diagonalizable modulo $\mathcal{C}_p^-$.

4. Perturbation Invariance

Recall that for a single self-adjoint A on a Hilbert space $\mathcal{H}$, we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(A) \oplus \mathcal{H}_{\mathrm{s}}(A). \tag{5}$$

Moreover, $A_{\mathrm{s}} = A|\mathcal{H}_{\mathrm{s}}(A)$ is diagonalizable modulo the trace class whereas $A_{\mathrm{ac}} = A|\mathcal{H}_{\mathrm{ac}}(A)$ is not diagonalizable modulo the trace class. What happens with tuples?

Theorem. (Voiculescu, 1981.) Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{C}$ be a norm ideal. Then there is an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}) \oplus \mathcal{H}_{\mathrm{d}}(T;\mathcal{C}) \tag{6}$$

withe the following properties:

(a) Both subspaces $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C})$ and $\mathcal{H}_{\mathrm{d}}(T;\mathcal{C})$ are invariant under the tuple T.

(b) $T|\mathcal{H}_{\mathrm{d}}(T;\mathcal{C})$ is simultaneously diagonalizable modulo $\mathcal{C}^{(0)}$.

(c) $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C})$ contains no nonzero invariant subspace of T on which T is simultaneously diagonalizable modulo $\mathcal{C}^{(0)}$.

But there is a huge difference between decompositions (5) and (6). Note that (5) is given in terms of the spectral measure of A, whereas (6) makes no mention of the spectral property of the tuple T. In fact, the general diagonalization problem can be simply stated as

determine the subspaces $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C})$ and $\mathcal{H}_{\mathrm{d}}(T;\mathcal{C})$ in terms of the spectral measure of T.

On the other hand, there is a perturbation invariance associated with (6). It is “a kind of” Kato-Rosenblum theorem for tuples.

Theorem. (Voiculescu, 1981.) Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{C}$ be a norm ideal **which is not the trace class**. Suppose that $T' = (T'_1, ..., T'_n)$ is another commuting tuple of self-adjoint operators on $\mathcal{H}$. If

$$T_j - T'_j \in \mathcal{C}^{(0)} \quad \text{for every } \ 1 \le j \le n,$$

then $T|\mathcal{H}_{\text{nd}}(T;\mathcal{C})$ and $T'|\mathcal{H}_{\text{nd}}(T';\mathcal{C})$ are unitarily equivalent.

Note the assumption that $\mathcal{C}$ is not the trace class above. One would naturally ask, what happens in the case of trace class $\mathcal{C}_1$?

Theorem. (J.X., 1998.) Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$. If $T' = (T'_1, ..., T'_n)$ is another commuting tuple of self-adjoint operators on $\mathcal{H}$ such that

$$T_j - T'_j \in \mathcal{C}_1 \quad \text{for every } \ 1 \le j \le n,$$

then $T|\mathcal{H}_{\text{nd}}(T;\mathcal{C}_1)$ and $T'|\mathcal{H}_{\text{nd}}(T';\mathcal{C}_1)$ are unitarily equivalent.

Usually, to obtain the desired unitary equivalence, one needs to construct **wave operators** from T and T', which involves a strong convergence. This is the case in the original Kato-Rosenblum theorem and in the case where $\mathcal{C} \neq \mathcal{C}_1$. But in the case of trace class perturbation for tuples, such a strong convergence is not available. The 1998 result was proved using properties of trace class operators and weak convergence alone.

We call these two theorems "a kind of" Kato-Rosenblum theorem because a spectral description of $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C})$ for a general $\mathcal{C}$ is not yet available. In fact, even a spectral description of $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1)$ is not yet available, and that is one of the big open problems, as we will discuss at the end of these talks.

On the other hand, for any norm ideal $\mathcal{C}$ for which a spectral description of $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C})$ **is** available, the above is the analogue of the Kato-Rosenblum theorem.

We will see that in the case of Schatten classes $\mathcal{C}_p$ with $1 < p < \infty$, the spectral description of $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_p)$ is available. The same is true for certain Orlicz ideals $\mathcal{C}_G$.

5. Diagonalization in the Twenty-first Century

To discuss the results that appeared (and those which are unpublished) in the new century we need more definitions.

Definition. Let $1 < p < \infty$. A compactly supported regular Borel measure μ on $\mathbf{R}^n$ is said to be ***p*-singular** if

$$(7) \quad \int_0^1 \left(\frac{\mu(B(x,r))}{r^p} \right)^{1/(p-1)} \frac{dr}{r} = \infty \quad \text{for } \ \mu\text{-a.e. } \ x \in \mathbf{R}^n.$$

Roughly speaking (very roughly speaking), if μ is p-singular, then the rate at which $\mu(B(x,r))$ tends to 0 as $r \downarrow 0$ cannot be much faster than r^p. And this very rough statement is made precise by condition (7).

Definition. Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{E}$ denote the spectral measure of T. For each $\xi \in \mathcal{H}$, define the measure μ_ξ be the formula

$$\mu_\xi(\Delta) = \langle \mathcal{E}(\Delta)\xi, \xi \rangle.$$

For any given $1 < p < \infty$, the tuple T is said to be **purely *p*-singular** if the measure μ_ξ is p-singular for every $\xi \in \mathcal{H}$.

Theorem 2000. Let $1 < p < \infty$. A commuting tuple of self-adjoint operators $(T_1, ..., T_n)$ is simultaneously diagonalizable modulo the Schatten class $\mathcal{C}_p$ if and only if it is purely p-singular.

As it turns out, the special case of 2-singularity admits a particularly simple description.

Theorem. Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{E}$ denote the spectral measure of T. For each $\xi \in \mathcal{H}$, define the measure μ_ξ be the formula

$$\mu_\xi(\Delta) = \langle \mathcal{E}(\Delta)\xi, \xi \rangle.$$

Then the tuple T is simultaneously diagonalizable modulo the Hilbert-Schmidt class $\mathcal{C}_2$ if and only if

$$\iint \frac{1}{|x-y|^2} d\mu_\xi(x) d\mu_\xi(y) = \infty \tag{8}$$

for every **nonzero** vector ξ in $\mathcal{H}$.

This leads to a particularly curious result about diagonalization modulo $\mathcal{C}_2$. To state this curious result, let us introduce

Definition. Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators. For each vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$, denote

$$\alpha \cdot T = \alpha_1 T_1 + ... + \alpha_n T_n.$$

To motivate the curious result, let us recall that Voiculescu showed that every commuting pair (T_1, T_2) of self-adjoint operators is simultaneously diagonalizable modulo $\mathcal{C}_2$, but a commuting triple (T_1, T_2, T_3) of self-adjoint operators need not be simultaneously diagonalizable modulo $\mathcal{C}_3^-$ and, consequently, need not be simultaneously diagonalizable modulo $\mathcal{C}_2$. In other words, triples are the first place where obstructions to diagonalization modulo the Hilbert-Schmidt class **can and do** occur. In some sense, the curious result below says any such obstruction can always be traced back to triples.

Theorem. (Unpublished.) Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators. Suppose that for **every** triple of vectors $\alpha, \beta, \gamma \in \mathbf{R}^n$, the operator triple

$$(\alpha \cdot T, \beta \cdot T, \gamma \cdot T)$$

is simultaneously diagonalizable modulo the Hilbert-Schmidt class. Then the tuple T itself is simultaneously diagonalizable modulo the Hilbert-Schmidt class.

The proof of this result uses the diagonalization criterion (8), Fourier transform of measures, and some algebra involving matrices.

One redeeming value of this curious theorem is that it illustrates (or at least **its proof** illustrates) that explicit diagonalization criteria can be useful for unexpected reasons. This goes some way towards justifying the study of diagonalization problems.

The next progress came in the form of a general obstruction result. To state it, we need more preliminaries.

Let $\hat{c}$ be the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for each sequence $a_j \neq 0$ only for a finite number of j's.

A map $\Phi : \hat{c} \to [0, \infty)$ is said to be a **symmetric gauge function** if it has the following properties:

(a) Φ is a norm on $\hat{c}$.
(b) $\Phi(\{1, 0, ..., 0, ...\}) = 1$.
(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

For a finite index set $F = \{s_1, ..., s_m\}$, we define

$$\Phi(\{c_s\}_{s\in F}) = \Phi(\{c_{s_1}, ..., c_{s_m}, 0, ..., 0, ...\}).$$

For an arbitrary index set E, we further define

$$\Phi(\{c_s\}_{s\in E}) = \sup\{\Phi(\{c_s\}_{s\in F}) : F \subset E,\ \mathrm{card}(F) < \infty\}.$$

Given a symmetric gauge function Φ, the set

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi = \Phi(\{s_j(A)\}_{j\in\mathbf{N}}) < \infty\}$$

is a norm ideal. Conversely, if we begin with a norm ideal $\mathcal{C}$ on a Hilbert space $\mathcal{H}$ and if $\{\xi_j\}_{j\in\mathbf{N}}$ is an orthonormal set in $\mathcal{H}$, then the formula

$$\Phi_{\mathcal{C}}(\{a_j\}_{j\in\mathbf{N}}) = \left\| \sum_{j=1}^{\infty} a_j \xi_j \otimes \xi_j \right\|_{\mathcal{C}}, \qquad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

defines a symmetric gauge function and we have $\|A\|_{\mathcal{C}} = \|A\|_{\Phi_{\mathcal{C}}}$ for every $A \in \mathcal{C}^{(0)}$. We will call $\Phi_{\mathcal{C}}$ the **symmetric gauge function associated with $\mathcal{C}$**.

Definition. A norm ideal $\mathcal{C}$ is said to satisfy condition (QK) if there exist constants $0 < t < 1$ and $0 < B < \infty$ such that

$$\|\overbrace{X \oplus ... \oplus X}^{k \text{ times}}\|_{\mathcal{C}} \leq Bk^t \|X\|_{\mathcal{C}}$$

for every finite-rank operator X and every $k \in \mathbf{N}$.

I call this condition (QK) because it is a **Q**uantitative variant of the a condition of **K**uroda and because I cannot think of any better term.

(QK) is a very mild condition. For $1 < p < \infty$, the ideals

$$\mathcal{C}_p, \quad \mathcal{C}_p^+, \quad \mathcal{C}_p^-$$

satisfy condition (QK). We will give more examples of ideals satisfying condition (QK) later.

One obvious example of a norm ideal **NOT** satisfying condition (QK) is the trace class $\mathcal{C}_1$. In fact, the statement that $\mathcal{C}$ satisfies (QK) means that $\mathcal{C}$ differs from $\mathcal{C}_1$ by some quantifiable measure.

We now need to discuss dyadic decomposition, which is somewhat technical.

Let Q denote the unit cube $[0,1)^n$ in $\mathbf{R}^n$.

For each $\ell \in \mathbf{N}$, let W_ℓ denote the collection of words of length ℓ with {1, 2, 3, ..., 2^n} being the alphabet. That is,

$$W_\ell = \{w_1...w_\ell : w_j \in \{1, 2, 3, ..., 2^n\}, j = 1, ..., \ell\}.$$

We denote the length of each word w by $|w|$, i.e., $|w| = \ell$ for $w \in W_\ell$.

Let

$$\mathcal{W} = \bigcup_{\ell=1}^{\infty} W_\ell.$$

Let γ_1, ..., γ_{2^n} be an enumeration of the vectors $\{(\epsilon_1, ..., \epsilon_n) : \epsilon_i \in \{0, 1\}, i = 1, ..., n\}$.

For each $w = w_1...w_\ell \in W_\ell$, defined the cube

$$Q_w = Q_{w_1...w_\ell} = [0, 2^{-\ell})^n + 2^{-1}\gamma_{w_1} + 2^{-2}\gamma_{w_2} + ... + 2^{-\ell}\gamma_{w_\ell}.$$

For arbitrary w, $w' \in \mathcal{W}$, we have either $Q_w \cap Q_{w'} = \emptyset$, or $Q_w \supset Q_{w'}$, or $Q_{w'} \supset Q_w$.

Although this labelling system for cubes is quite cumbersome, it solves problems. The main reason for this cumbersomeness is that in proofs we need to consider composite words

$$wu = w_1...w_\ell u_1...u_k.$$

But in these talks we need not get to that level of technicality.

As we have previously explained, diagonalization problems always boil down to the consideration of a measure concentrated on a cube.

Recall that for each $1 \leq j \leq n$, S_j^μ denotes the singular integral operator on $L^2(\mathbf{R}^n, d\mu)$ defined by the formula

$$(S_j^\mu f)(x) = \int \frac{x_j - y_j}{|x-y|^2} f(y) d\mu(y),$$

whereas M_j^μ denotes the multiplication operator

$$(M_j^\mu f)(x) = x_j f(x), \qquad x = (x_1, ..., x_n),$$

on $L^2(\mathbf{R}^n, d\mu)$.

Theorem 2004. Suppose that μ is concentrated on the unit cube Q. Let $\mathcal{C}$ be a norm ideal satisfying condition (QK). If

$$\Phi_{\mathcal{C}'}(\{2^{|w|}\mu(Q_w)\}_{w \in \mathcal{W}}) < \infty, \tag{9}$$

then $S_1^\mu, ..., S_n^\mu \in \mathcal{C}'$ and, consequently, the tuple $(M_1^\mu, ..., M_n^\mu)$ is not simultaneously diagonalizable modulo $\mathcal{C}$.

Even though (9) looks technical, in applications this condition is readily verifiable. In fact, this theorem subsumes **all** existing obstruction results because (9) can be easily verified in the case of each and every existing obstruction result.

Let us give more examples of norm ideals satisfying condition (QK).

For each pair of $1 \leq p < \infty$ and $0 \leq s < 1$, the formula

$$\|K\|_{p,s} = \left(\sum_{j=1}^{\infty} j^{-s} (s_j(K))^p \right)^{1/p}$$

defines a norm, and the collection

$$\mathcal{L}_{p,s} = \{K : \|K\|_{p,s} < \infty\}$$

is a norm ideal. These are called **Lorentz ideals**. Obviously, $\mathcal{L}_{p,0} = \mathcal{C}_p$, and $\mathcal{L}_{1,s} = \mathcal{C}^-_{1/(1-s)}$.

Proposition. The Lorentz ideal $\mathcal{L}_{p,s}$ and its dual $\mathcal{L}'_{p,s}$ satisfy condition (QK) if either $p \neq 1$ or $s \neq 0$.

Consider a non-increasing sequence π of positive numbers:

$$1 = \pi_1 \geq \pi_2 \geq ... \geq \pi_k \geq ... \quad .$$

Furthermore, assume that π is **binormalizing** in the sense of Gohberg and Krein:

$$\sum_{k=1}^{\infty} \pi_k = \infty \qquad \text{and} \qquad \lim_{k \to \infty} \pi_k = 0.$$

Such a sequence gives rise to the norm

$$\|K\|_\pi = \sup_{k \geq 1} \frac{\sum_{j=1}^{k} s_j(K)}{\sum_{j=1}^{k} \pi_j}.$$

Then $\mathcal{C}_\pi = \{K : \|K\|_\pi < \infty\}$ is a norm ideal. The ideals $\mathcal{C}_p^+$ are examples of this class.

In the study of $\mathcal{C}_\pi$, a sort of minimal assumption that one imposes on the sequence π is that it be **regular**, meaning that there is a constant $0 < M < \infty$ such that

$$M\pi_k \geq \frac{\pi_1 + ... + \pi_k}{k} \qquad \text{for every} \quad k \geq 1.$$

Those $\mathcal{C}_\pi$ which fail to be regular are generally considered to be difficult to deal with.

But there exists a sequence π which is **NOT** regular and yet the corresponding the norm ideal $\mathcal{C}_\pi$ satisfies condition (QK)! I think that this fact is a significant indication of the inclusiveness of condition (QK).

The next class of norm ideals satisfying (QK) are Orlicz ideals with certain growth conditions.

Consider a continuous, strictly increasing function $g : [0, \infty) \to [0, \infty)$ which has the following properties:

(a) $g(0) = 0$ and $g(s) \to \infty$ as $s \to \infty$.

(b) There are $1 \le M < \infty$ and $0 < \epsilon \le 1$ such that $Mg(Cs) \ge C^\epsilon g(s)$ for all $1 \le C < \infty$ and $0 \le s < \infty$.

(c) There are $1 \le N < \infty$ and $0 < \delta \le 1$ such that $NCg(s) \ge g(C^\delta s)$ for all $1 \le C < \infty$ and $0 \le s < \infty$.

We let

(1) $h = g^{-1}$, the **inverse** of g.

(2) $G(t) = \int_0^t g(s)ds$, $t \in [0, 1)$.

(3) $H(t) = \int_0^t h(s)ds$, $t \in [0, 1)$.

(4) α be the unique number in $[0, \infty)$ such that $G(\alpha) = 1$.

Then the formula

$$R^G(\{a_j\}_{j\in\mathbf{N}}) = \alpha \inf \left\{ \lambda > 0 : \sum_{j=1}^{\infty} G\left(\frac{|a_j|}{\lambda}\right) \le 1 \right\}$$

defines a symmetric gauge function on $\hat{c}$. Thus

$$\mathcal{C}_G = \{K : R^G(\{s_j(K)\}_{j\in\mathbf{N}}) < \infty\}$$

is a norm ideal, called **Orlicz ideal**. We can similarly define R^H and $\mathcal{C}_H$. As it turns out, the dual of $\mathcal{C}_G$ is the set $\mathcal{C}_H$ with an equivalent norm.

Both $\mathcal{C}_G$ and $\mathcal{C}_H$ satisfy condition (QK).

The class of such $\mathcal{C}_G$'s is quite large. Most noticeable examples are functions of the form

$$G(t) = \int_0^t \frac{s^b}{\{\log(1+\frac{1}{s})\}^d} ds,$$

where $0 < b < \infty$ and $0 \leq d < \infty$. If we set $d = 0$ in this particular case, we recover all $\mathcal{C}_p$, $1 < p < \infty$. But there are much more exotic G's. For example, we can take a g which behaves like a power of s times a "super logarithm". We can also take a g which mimics **TWO** different powers of s at different scales.

For this class of $\mathcal{C}_G$, not only is the obstruction result of 2004 applicable, but we have the complete story about diagonalization, similar to the Schatten class result of 2000.

Definition. A compactly supported regular Borel measure μ on $\mathbf{R}^n$ is said to be G**-singular** if

$$\int_0^1 \frac{H(r^{-1}\mu(B(x,r)))}{\mu(B(x,r))} \frac{dr}{r} = \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbf{R}^n.$$

Note that the G-singularity is actually stated using H, the **Young's function complementary** to G.

Definition. Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{E}$ denote the spectral measure of T. For each $\xi \in \mathcal{H}$, define the measure μ_ξ be the formula

$$\mu_\xi(\Delta) = \langle \mathcal{E}(\Delta)\xi, \xi \rangle.$$

The tuple T is said to be **purely** G**-singular** if the measure μ_ξ is G-singular for every $\xi \in \mathcal{H}$.

Theorem 2006. A commuting tuple of self-adjoint operators $(T_1, ..., T_n)$ is simultaneously diagonalizable modulo the Orlicz ideal $\mathcal{C}_G$ if and only if it is purely G-singular.

The latest result on diagonalization is somewhat surprising, because it requires absolutely **NO** assumption on the norm ideal $\mathcal{C}$.

Recall the dyadic decomposition Q_w, $w \in \mathcal{W}$, of the cube Q we discussed earlier. Also recall that diagonalization problems always boil down to the setting of some μ concentrated on Q.

Theorem 2008. Suppose that μ is concentrated on Q. Let $\mathcal{C}$ be any norm ideal and let $\Phi_{\mathcal{C}}$ be the symmetric gauge function associated with $\mathcal{C}$. If there exists a set of non-negative numbers $\{\lambda_w\}_{w\in\mathcal{W}}$ such that

$$\Phi_{\mathcal{C}}(\{\lambda_w\}_{w\in\mathcal{W}}) < \infty$$

and

$$\sum_{w\in\mathcal{W}} 2^{|w|}\lambda_w \chi_{Q_w}(x) = \infty \quad \text{for } \mu\text{-a.e. } x \in Q, \tag{10}$$

then the tuple of multiplication operators $(M_1^\mu, ..., M_n^\mu)$ is simultaneously diagonalizable modulo $\mathcal{C}$.

This brings up the obvious question, is the condition in this theorem **necessary** for diagonalization modulo $\mathcal{C}$?

At present we have no examples where this condition fails to be necessary. Moreover, we have

Proposition. In the case of the Orlicz ideals discussed above, the 2008 condition is necessary for diagonalization modulo $\mathcal{C}_G$.

Notwithstanding the issue of necessity in the general case, this latest theorem is a satisfying result.

In fact, one can show that this theorem subsumes all previous diagonalization results, again by explicitly verifying that the condition in this theorem is satisfied in the case of each and every existing diagonalization result.

What initially lead me to the general diagonalization result of 2008 was an attempt to prove the following result about diagonalization modulo the norm ideal $\mathcal{C}_p^+$:

Theorem. (Unpublished.) If

$$\int_0^1 \left(\frac{\mu(B(x,r))}{r^p}\right)^{1/p} \frac{dr}{r} = \infty \quad \text{for } \ \mu\text{-a.e. } \ x \in \mathbf{R}^n, \tag{11}$$

then the tuple $(M_1^\mu, ..., M_n^\mu)$ is simultaneously diagonalizable modulo $\mathcal{C}_p^+$.

It is easy to verify that (11) implies that there exists a set of non-negative numbers $\{\lambda_w\}_{w\in\mathcal{W}}$ such that

$$\Phi_p^+(\{\lambda_w\}_{w\in\mathcal{W}}) < \infty,$$

where Φ_p^+ is the symmetric gauge function for $\mathcal{C}_p^+$, and

$$\sum_{w\in\mathcal{W}} 2^{|w|}\lambda_w \chi_{Q_w}(x) = \infty \quad \text{for } \ \mu\text{-a.e. } \ x \in Q.$$

6. Open Problems

The first problem is the obvious problem that we already mentioned:

Problem 1. Is the condition in Theorem 2008 necessary for diagonalization modulo $\mathcal{C}$?

As we have mentioned, the answer is affirmative in the case of those Orlicz ideals $\mathcal{C}_G$ we discussed earlier. So the question is, what about other ideals?

In the case where $\mathcal{C}$ satisfies condition (QK), the condition in Theorem 2008 is at least **close to being necessary**. This can be seen in the following way. Note that (10) is equivalent to

$$\sum_{w \in \mathcal{W}} 2^{|w|} \lambda_w \mu(E \cap Q_w) = \infty$$

for **every** Borel set $E \subset Q$ with $\mu(E) > 0$. The emphasis here is on the word "every", because we have

Proposition. Let $\mathcal{C}$ be a norm ideal satisfying condition (QK). Suppose that the commuting tuple $(M_1^\mu, ..., M_n^\mu)$ is simultaneously diagonalizable modulo $\mathcal{C}$. Then for any Borel set $E \subset Q$ with $\mu(E) > 0$, there exists a set of non-negative numbers $\{a_w\}_{w \in \mathcal{W}}$, **which may depend on** E, such that

$$\Phi_{\mathcal{C}}(\{a_w\}_{w \in \mathcal{W}}) < \infty$$

and

$$\sum_{w \in \mathcal{W}} 2^{|w|} a_w \mu(E \cap Q_w) = \infty.$$

Definition. Suppose that μ is concentrated on Q. Let $\mathcal{C}$ be a norm ideal. We say that μ belongs to the class $\Omega(\mathcal{C})$ if it has the following two properties:

(a) If $\{\lambda_w\}_{w\in\mathcal{W}}$ is a set of non-negative numbers such that $\Phi_{\mathcal{C}}(\{\lambda_w\}_{w\in\mathcal{W}}) < \infty$, then $\mu(\{x \in Q : \sum_{w\in\mathcal{W}} 2^{|w|}\lambda_w\chi_{Q_w}(x) < \infty\}) > 0$.

(b) For any Borel set $E \subset Q$ with $\mu(E) > 0$, there exists a set of non-negative numbers $\{a_w\}_{w\in\mathcal{W}}$ such that $\Phi_{\mathcal{C}}(\{a_w\}_{w\in\mathcal{W}}) < \infty$ and $\sum_{w\in\mathcal{W}} 2^{|w|}a_w\mu(E \cap Q_w) = \infty$.

It is easy to see that property (a) is equivalent to

(a′) There is a Bore set $B \subset Q$ with $\mu(B) > 0$ such that if $\{\lambda_w\}_{w\in\mathcal{W}}$ is a set of non-negative numbers satisfying the condition $\Phi_{\mathcal{C}}(\{\lambda_w\}_{w\in\mathcal{W}}) < \infty$, then $\sum_{w\in\mathcal{W}} 2^{|w|}\lambda_w\chi_{Q_w}(x) < \infty$ for μ-a.e. $x \in B$.

Suppose that $\mathcal{C}$ is a norm ideal satisfiying condition (QK). If μ is concentrated on Q but does **not** belong to the measure class $\Omega(\mathcal{C})$, then the question of whether the tuple $(M_1^\mu, ..., M_n^\mu)$ is simultaneously diagonalizable modulo $\mathcal{C}$ is completely settled:

In fact, if $\mu \notin \Omega(\mathcal{C})$, then **either** μ fails to have property (a), in which case $(M_1^\mu, ..., M_n^\mu)$ is simultaneously diagonalizable modulo $\mathcal{C}$; **or** μ fails to have property (b), in which case $(M_1^\mu, ..., M_n^\mu)$ is not simultaneously diagonalizable modulo $\mathcal{C}$

The fact that the condition in Theorem 2008 is necessary for diagonalization modulo $\mathcal{C}_G$ is a consequence of the fact that

$$\Omega(\mathcal{C}_G) = \emptyset.$$

For the class of norm ideals satisfying condition (QK), the diagonalization problem is reduced to

Problem 2. (1) Does there exist a norm ideal $\mathcal{C}$ for which $\Omega(\mathcal{C})$ is **not** empty?
(2) If there exists a norm ideal $\mathcal{C}$ such that $\Omega(\mathcal{C}) \neq \emptyset$ and if $\mu \in \Omega(\mathcal{C})$, then is $(M_1^\mu, ..., M_n^\mu)$ simultaneously diagonalizable modulo $\mathcal{C}$? Or for such a μ is there obstruction to such diagonalization?

Any $\mu \in \Omega(\mathcal{C})$, if it exists, must be a very odd measure.

We are particular interested in Problem 2 in the case of $\mathcal{C}_p^-$ and $\mathcal{C}_p^+$, $1 < p < \infty$.

Another big question is, what happens in the case of the trace class $\mathcal{C}_1$? Recall that $\mathcal{C}_1$ does not satisfy condition (QK). So it is a different kind of problem, and it will be discussed separately.

Switching gears, our second problem is the simultaneous diagonalization of a **non-separable** family of operators. This very specific diagonalization problem has connections to functions space, VMO, etc.

Recall that the C^*-algebra of **quasi-continuous** functions on the unit circle $\mathbf{T}$ is defined to be

$$\mathrm{QC} = (H^\infty + C(\mathbf{T})) \cap \overline{(H^\infty + C(\mathbf{T}))}.$$

Alternately, we have

$$\mathrm{QC} = L^\infty(\mathbf{T}) \cap \mathrm{VMO}.$$

The norm on QC is the usual $\|.\|_\infty$. Two interesting facts:

(1) QC is not separable with respect to the norm $\|.\|_\infty$.

(2) QC contains no projections other than 0 and 1. In other words, the maximal ideal space of QC is connected. (This is an easy exercise in measure theory.)

Given the background of QC (connections with Teoplitz operators and Hankel operators), it will be interesting to investigate

Problem 3. Is the family of multiplication operators

$$\{M_f : f \in \mathrm{QC}\}$$

on the Hilbert space $L^2(\mathbf{T})$ simultaneously diagonalizable modulo $\mathcal{K}$? In other words, does there exist a commuting family of diagonal operators

$$\{D_f : f \in \mathrm{QC}\}$$

such that $M_f - D_f \in \mathcal{K}$ for every $f \in \mathrm{QC}$?

Definition. Suppose that $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. A **sequence** of finite-rank positive contractions $\{F_k\}$ on $\mathcal{H}$ is said to be **quasi-central approximate units** for $\mathcal{A}$ if

$$\lim_{k\to\infty} F_k = 1$$

in the strong operator topology and

$$\lim_{k\to\infty} \|[F_k, A]\| = 0 \quad \text{for every} \quad A \in \mathcal{A}.$$

The following was known in the 1970's (Voiculescu, Arveson):

Proposition. If $\mathcal{A}$ is separable, then $\mathcal{A}$ possesses **a sequence of** quasi-central approximate units.

Although QC is not separable, we have

Proposition. (Unpublished, easy to prove.) There exists a sequence $\{P_k\}$ of finite-rank orthogonal projections on $L^2(\mathbf{T})$ such that

$$\lim_{k\to\infty} P_k = 1$$

in the strong operator topology and

$$\lim_{k\to\infty} \|[P_k, M_f]\| = 0 \quad \text{for every} \quad f \in \mathrm{QC}.$$

The existence of **a sequence of** quasi-central approximate units seems to suggest that $\{M_f : f \in \mathrm{QC}\}$ should be be simultaneously diagonalizable modulo $\mathcal{K}$. But because of the non-separability of QC, the usual diagonalization procedure does not apply. So this is still a big open problem.

Finally, we consider the problem of diagonalization modulo the trace class. Let $T = (T_1, ..., T_n)$ be a commuting tuple of self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Recall that we have the rather abstract orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1) \oplus \mathcal{H}_{\mathrm{d}}(T;\mathcal{C}_1)$$

such that $T|\mathcal{H}_{\mathrm{d}}(T;\mathcal{C}_1)$ is simultaneously diagonalizable modulo $\mathcal{C}_1$ whereas $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1)$ contains no nonzero invariant subspace on which T is simultaneously diagonalizable modulo $\mathcal{C}_1$.

The problem of diagonalization modulo the trace class is just to give a spectral description of the subspaces $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1)$ and $\mathcal{H}_{\mathrm{d}}(T;\mathcal{C}_1)$.

Let $\mathcal{E}$ be the spectral measure for $T = (T_1, ..., T_n)$ and define μ_ξ, $\xi \in \mathcal{H}$, by the formula $\mu_\xi(\Delta) = \langle \mathcal{E}(\Delta)\xi, \xi\rangle$ as before. It is easy to show that

$$\mathcal{H} = \mathcal{H}^1_{\mathrm{inv}}(T) \oplus \mathcal{H}^1_{\mathrm{sin}}(T),$$

where $\mathcal{H}^1_{\mathrm{sin}}(T)$ is the collection of $\xi \in \mathcal{H}$ such that

$$\limsup_{r\downarrow 0} \frac{\mu_\xi(B(x,r))}{r} = \infty \quad \text{for } \mu_\xi\text{-a.e. } x \in \mathbf{R}^n$$

and $\mathcal{H}^1_{\mathrm{inv}}(T)$ is the collection of $\xi \in \mathcal{H}$ such that

$$\limsup_{r\downarrow 0} \frac{\mu_\xi(B(x,r))}{r} < \infty \quad \text{for } \mu_\xi\text{-a.e. } x \in \mathbf{R}^n.$$

In the special case $n = 1$, this is just the usual decomposition in terms of absolute continuity with respect to the Lebesgue measure on $\mathbf{R}$. It is known that $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1) \subset \mathcal{H}^1_{\mathrm{inv}}(T)$ for any $n \in \mathbf{N}$. The natural guess is

Conjecture 4. $\mathcal{H}_{\mathrm{nd}}(T;\mathcal{C}_1) = \mathcal{H}^1_{\mathrm{inv}}(T)$ and, therefore, $\mathcal{H}_{\mathrm{d}}(T;\mathcal{C}_1) = \mathcal{H}^1_{\sin}(T)$.

This conjecture is equivalent to its "working form":

Conjecture 5. Suppose that there is a constant $0 < C < \infty$ such that

$$\mu(B(x,r)) \le Cr \tag{12}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Then the tuple $(M^\mu_1, ..., M^\mu_n)$ is not simultaneously diagonalizable modulo the trace class $\mathcal{C}_1$.

Since $(\mathcal{C}_1)' = \mathcal{B}(\mathcal{H})$, to prove this conjecture, we only need to find **bounded** operators $B_1, ..., B_n$ such that

$$\Gamma = \sum_{j=1}^n [M^\mu_j, B_j]$$

is a trace-class operator of nonzero trace. The "natural" candidates are $B_j = S^\mu_j$, where

$$(S^\mu_j f)(x) = \int \frac{x_j - y_j}{|x-y|^2} f(y) d\mu(y).$$

Unfortunately, there are μ's for which (12) holds and yet the S^μ_j's fail to be bounded. In other words, the "natural" candidates do not work in this case.

But that does not mean that some other, perhaps "unnatural", $B_1, ..., B_n$ will not give us the desired Γ.

So the search goes on ...