MODEL-ERROR CONTROL SYNTHESIS USING APPROXIMATE RECEDING-HORIZON CONTROL LAWS

Jongrae Kim*
Department of Aerospace Engineering
Texas A&M University
College Station, TX 77843-3141

John L. Crassidis†
Department of Mechanical & Aerospace Engineering
University at Buffalo, The State University of New York
Amherst, NY 14260-4400

ABSTRACT

Model-Error Control Synthesis employs an optimal real-time nonlinear estimator to determine model error corrections to a nominal controller. Control compensation is achieved by using the estimated model error as a signal synthesis adaptive correction to the nominal control input so that maximum performance is achieved in the face of extreme model uncertainty and disturbance inputs. In this paper the capability of the model-error control synthesis is expanded by combining a nonlinear predictive filter using an approximate receding-horizon optimal solution with a Kalman filter in the overall control design. A robust stability analysis using the interlacing property from the Hermite-Biehler theorem is presented for the new approach. Simulation results for linear systems are shown to verify theoretical predictions.

INTRODUCTION

Robust control of dynamic systems is usually achieved using one of two schemes. The first scheme involves the design of a controller that is insensitive as possible to model uncertainty and/or disturbance inputs, e.g., H_∞ and μ-synthesis. The second scheme involves updating model parameters or control gains in real-time in order to achieve desired performance specifications. Adaptive control methods fall into this category. These control schemes can be used to provide robustness in a dynamic system with uncertainties, each with its own advantages and disadvantages.

Model-Error Control Synthesis (MECS) is a signal synthesis adaptive control method.[1] Robustness is achieved by applying a correction control to eliminate the effect of model error at the output. The error is estimated by a one step ahead predictive filter technique. The main advantage of this technique is that the model error is determined during the estimation process.[2] More details on the predictive filter can be found in Refs. [2] and [3]. In Ref. [1] MECS was first applied to suppress the wing rock motion of a slender delta wing, which is described by a highly nonlinear differential equation. In Ref. [4] a simple approach to test the stability of the closed loop system was presented using a Padé approximation. As shown by the benchmark problem example in Ref. [4], the one-step ahead prediction technique inherent in MECS could not stabilize the system, which has one pole at the origin and two poles on the imaginary axis. However, when using a different approach to determine the model error, given by an Approximate Receding-Horizon Control (ARHC) law, the closed loop system can be stabilized. The main topic of this paper is to expand the previous results so that the MECS can stabilize non-minimum phase unstable systems with uncertainty.

As shown in Ref. [4], since the future state is unknown in general, a trade-off between performance and stability exists. Hence, the weighting factor, W, for the model error has to be set to non-zero constant, i.e., perfect cancellation of the model error cannot be achieved. In addition, since the system measurement is given before calculating the model error, there is always the possibility that the predictive filter may have some bias error. To overcome this problem, we combine the predictive filter approach using an approximate receding-horizon optimal solution with a Kalman filter in the overall MECS approach. The closed form

*Graduate Student, Student Member AIAA.
†Assistant Professor, Senior Member AIAA.

solution of the approximate receding-horizon control using Quadratic Programming (QP) was first presented by Lu.[5] Also, in Ref. [5] the receding-horizon control concept was extended to the output tracking control problem. Although the problem is solved from a control standpoint, the algorithm can be reformulated as a filter and estimator problem.[2]

In this paper the model error is determined by the approximate receding-horizon optimal solution and the error is subtracted from the nominal control input. Finally, we obtain a new MECS scheme which has better characteristics than the one derived in Ref. [4]. To optimally design and analyze the new MECS scheme, we adopt the Hermite-Biehler theorem which gives the necessary and sufficient conditions for the closed loop system to be Hurwitz stable.[6] By the theorem, for the system to be Hurwitz stable, the closed loop characteristic polynomial has to satisfy an interlacing property, which can be used to define some useful graphical relations as stability checks. Specifically, we concentrate on sixth-order polynomials, which can be easily applied for any polynomial less than sixth-order.

The organization of this paper is as follows. First, we summarize the nonlinear predictive filter using an approximate receding-horizon optimal solution, and then combine the predictive filter with a Kalman filter. For robust design and analysis, we introduce the Hermite-Biehler theorem and derive some graphical relations for sixth-order characteristic equations to be Hurwitz stable. Using these relations we introduce a method to choose the optimal receding-horizon subinterval and weight matrices. Finally, we verify the results by some linear examples.

CONTROL DESIGN

In this section the predictive filter using an approximate receding-horizon scheme with a QP solution is summarized (see Ref. [4] for more details). In the nonlinear filter it is assumed that the state and output estimates are given by a preliminary model and a to-be-determined model error vector, given by

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] + \hat{G}\left[\hat{\mathbf{x}}(t)\right]\mathbf{u}(t) + \hat{G}\left[\hat{\mathbf{x}}(t)\right]\hat{\mathbf{u}}(t) \tag{1a}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{c}}\left[\hat{\mathbf{x}}(t)\right] \tag{1b}$$

where $\hat{\mathbf{f}} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is the assumed model vector, $\hat{G} : \mathcal{R}^n \rightarrow \mathcal{R}^{n\times q}$ is assumed control input and model error distribution matrix, $\hat{\mathbf{x}} \in \mathcal{R}^n$ is the state estimate vector, $\mathbf{u} \in \mathcal{R}^q$ is the control input, $\hat{\mathbf{u}} \in \mathcal{R}^q$ is the model error vector, $\hat{\mathbf{c}} : \mathcal{R}^n \rightarrow \mathcal{R}^m$ is the measurement vector, and $\hat{\mathbf{y}} \in \mathcal{R}^m$ is the estimated output vector. Both $\hat{\mathbf{f}}$ and $\hat{G}$ are C^2, i.e., a function itself, where the first and the second derivatives are continuous and $\hat{\mathbf{c}}\left(\hat{\mathbf{x}}\right)$ is sufficiently differentiable. In addition, $\hat{\mathbf{f}}\left(\mathbf{0}\right) = \mathbf{0}$ (if not, we can transform the states $\hat{\mathbf{x}}$ to some new states so that this condition holds). Also, we assume that a unique solution for $\hat{\mathbf{x}}$ exists.

The receding-horizon optimization problem is set up as follows:[5]

$$\min_{\hat{\mathbf{u}}} J\left[\hat{\mathbf{x}}(t),\, t,\, \hat{\mathbf{u}}\right] = \int_t^{t+T} \Big[\mathbf{e}^T(\xi)\, R^{-1}\, \mathbf{e}(\xi) + \hat{\mathbf{u}}^T(\xi)\, W\, \hat{\mathbf{u}}(\xi)\Big]\, d\xi \tag{2}$$

subject to the state equations (1a), (1b) and $\mathbf{e}(t+T) = 0$, where the residual error is defined by

$$\mathbf{e}(t) = \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}(t) \tag{3}$$

where $\tilde{\mathbf{y}}(t)$ is the measurement, and R and W are positive definite and positive semi-definite weighting matrices, respectively. Note that T is the receding-horizon subinterval, which in general is not the sampling interval.

At each time t the optimal model error solution, $\hat{\mathbf{u}}^*$, over a finite horizon $[t,\, t+T]$ is determined online. Then, the current model error $\hat{\mathbf{u}}(t)$ is set equal to $\hat{\mathbf{u}}^*$ and this process is repeated for every instant of time t, continuously. Define $h \equiv T/N$ for some integer $N \geq n/m$, where N is the number of subintervals on $[t,\, t+T]$. Now, $\hat{\mathbf{y}}(t+kh)$ for each $k = 1,\, 2,\, \cdots,\, Nh = T$ is approximated by an iterative first-order Taylor series. For simplicity and avoiding the cross-product terms of $\hat{\mathbf{u}}(t+ih)$ and $\hat{\mathbf{u}}(t+jh)$, $\hat{G}(\hat{\mathbf{x}}(t+kh)) \approx \hat{G}(\hat{\mathbf{x}}(t))$ and $\hat{F}(\mathbf{x}(t+kh)) \approx \hat{F}(\hat{\mathbf{x}}(t))$, where $\hat{F} \equiv \partial\hat{\mathbf{f}}/\partial\hat{\mathbf{x}}$. In addition, since future values of $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are not known, we assume that $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are constant over the finite horizon $[t,\, t+T]$. Then, we obtain the following expression for $1 \leq k \leq N$:[5]

$$\hat{\mathbf{y}}(t+kh) \approx \hat{\mathbf{y}}(t) + h\,\hat{C}\left[\left\{\sum_{i=0}^{k-1}\left(I + h\hat{F}\right)^i\right\}\hat{\mathbf{f}} + \sum_{i=0}^{k-1}\left(I + h\hat{F}\right)^i \hat{G}\,\left\{\mathbf{u}(t) + \hat{\mathbf{u}}(t+(k-1-i)h)\right\}\right] \tag{4}$$

where $\hat{C} \equiv \partial\hat{\mathbf{c}}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$ and $\hat{\mathbf{f}}$, $\hat{F}$, and $\hat{G}$ are evaluated at $\hat{\mathbf{x}}(t)$. Define the following:

$$L(k\,h) \equiv \mathbf{e}^T(t+k\,h)R^{-1}\mathbf{e}(t+k\,h) + \hat{\mathbf{u}}^T(t+k\,h)W\hat{\mathbf{u}}(t+k\,h) \tag{5}$$

The cost function to be minimized is approximated using a trapezoidal formula or Simpson's rule as follows:[5] when N is odd,

$$J \approx \bar{J} = \frac{T}{2N}\left[\frac{L(0)}{2} + L(h) + \cdots + \cdots + L((N-1)h) + \frac{L(Nh)}{2}\right] \tag{6}$$

when N is even,

$$J \approx \bar{J} = \frac{T}{6N}\left[L(0) + 4L(h) + 2L(2h) + +4L(3h) + 2L(4h) + \cdots + 4L((N-1)h) + L(Nh)\right] \tag{7}$$

With the following definition:

$$\boldsymbol{\nu}_0 \equiv \text{col}\left\{\hat{\mathbf{u}}(t), \hat{\mathbf{u}}(t+h), \cdots, \hat{\mathbf{u}}(t+(N-1)h)\right\} \quad (8)$$

where col represents a column vector, $\bar{J}$ can be rewritten in quadratic form as

$$\bar{J} = \frac{1}{2}\,\boldsymbol{\nu}_0^T H_0(\hat{\mathbf{x}})\,\boldsymbol{\nu}_0 + \mathbf{g}_0^T[\hat{\mathbf{x}}, \mathbf{u}, \tilde{\mathbf{y}}(t)]\,\boldsymbol{\nu}_0 + q_0[\hat{\mathbf{x}}, \mathbf{u}, \tilde{\mathbf{y}}(t)] \quad (9)$$

where H_0, $\mathbf{g}_0$ and q_0 are functions of $L(k\,h)$.[5] Also, the terminal constraint, $\mathbf{e}(t+T) = 0$, can be formulated as a constraint on $\boldsymbol{\nu}_0$ as follows:

$$\begin{aligned}
&\mathbf{e}(t+N\,h) \approx \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}(t+N\,h) \\
&\approx \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}(t) - \hat{C}\left[\left\{\sum_{i=0}^{N-1}\left(I+h\hat{F}\right)^i\right\}\hat{\mathbf{f}}(\hat{\mathbf{x}})\right. \\
&\left.+ \sum_{i=0}^{N-1}\left(I+h\hat{F}\right)^i \hat{G}\left\{\mathbf{u}(t) + \hat{\mathbf{u}}(t+(k-1-i)h)\right\}\right] \\
&= \mathbf{e}(t) - \hat{C}\left\{\sum_{i=0}^{N-1}\left(I+h\hat{F}\right)^i\right\}\left\{\hat{\mathbf{f}}(\hat{\mathbf{x}}) + \hat{G}\,\mathbf{u}(t)\right\} \\
&- M^T\,\boldsymbol{\nu} = 0
\end{aligned} \quad (10)$$

Hence,

$$M^T\boldsymbol{\nu} = \mathbf{d}(t) \quad (11)$$

where

$$M^T = C\left[(I+hF)^{N-1}G, \cdots, (I+hF)G, G\right] \quad (12)$$

and

$$\mathbf{d}(t) = \frac{1}{h}\,\mathbf{e}(t) - \hat{C}\sum_{i=0}^{N-1}(I+h\hat{F})^i\left\{\hat{\mathbf{f}} + \hat{G}\mathbf{u}(t)\right\} \quad (13)$$

Finally, the QP solution is given by

$$\begin{aligned}
\boldsymbol{\nu}_0 = &-\left[H_0^{-1} - H_0^{-1}M\left(M^TH_0^{-1}M\right)^{-1}M^TH_0^{-1}\right]\mathbf{g}_0(t) \\
&+ \left[H_0^{-1}M\left(M^TH_0^{-1}M\right)^{-1}\right]\mathbf{d}(t)
\end{aligned} \quad (14)$$

where the rank of M is m. The first q equations give a current model error minimizing the cost function, which leads to a predictive filter structure:

$$\hat{\mathbf{u}}(t;\,\hat{\mathbf{x}}(t),\,\mathbf{u}(t),\,\tilde{\mathbf{y}}(t),\,h) = I_{q\times N}\;\boldsymbol{\nu}_0 \quad (15)$$

In addition, as indicated in Fig. 1 for the $N = 3$ case, since the output is held constant during the given subinterval, the assumption becomes less accurate as the receding horizon time T increases and/or the speed of response increases. Therefore, the weights have to be adjusted accordingly. To accomplish this a simple exponential function is used, given by

$$r_i = e^{r_p}\,r_{i-1} \quad (16a)$$
$$w_i = e^{w_p}\,w_{i-1} \quad (16b)$$

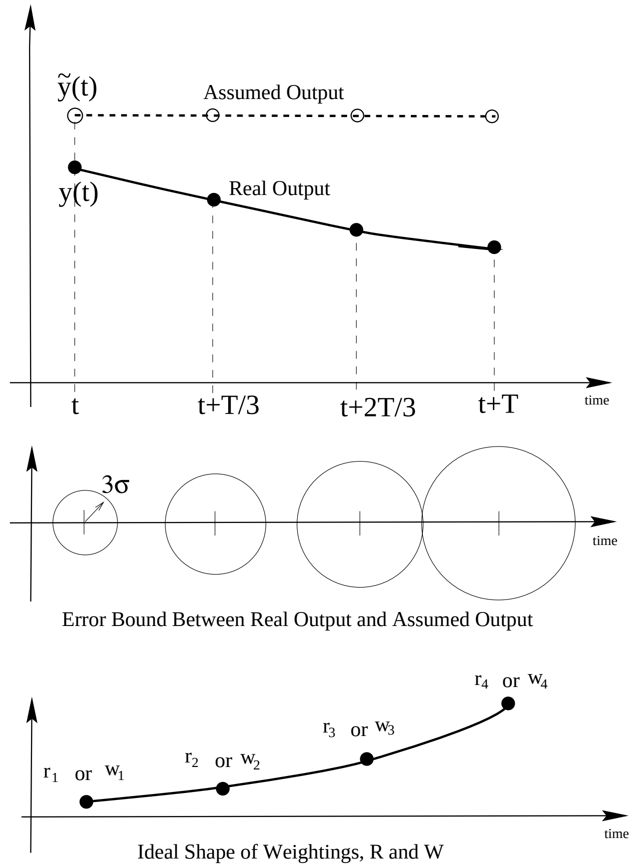

Fig. 1 Relation Between $\tilde{\mathbf{y}}(t)$ and R or W

where r_1 and w_1 are already given, r_p and w_p are nonnegative real values, $R = \text{diag}\,[r_1,\,r_2,\,r_3,\,r_4]$, and $W = \text{diag}\,[w_1,\,w_2,\,w_3,\,w_4]$. The weights are then reset at each measurement sample.

Since the system output (measurement) is given before subtracting the model error from the reference model, there is a possibility that the predictive filter may have a bias error. To remove this error, the predictive filter is combined with the Kalman filter. Since a response in the system must be given before compensation is applied, the following model error correction input is used:[1]

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) - \hat{\mathbf{u}}(t-\tau) \quad (17)$$

where $\bar{\mathbf{u}}(t)$ is the nominal control input, which can be any controller (e.g., Proportional-Integral-Derivative (PID), Linear Quadratic Regulator (LQR), Sliding Mode Control (SMC), etc.). The measurement time delay τ in $\hat{\mathbf{u}}$ is due to the fact that the model error is determined after the response was already given. The overall Kalman filter model can now be represented by

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x}) + \hat{G}(\mathbf{x})\left\{\mathbf{u} + \hat{\mathbf{u}}\right\} + \hat{G}\mathbf{w} \quad (18a)$$
$$\tilde{\mathbf{y}} = \mathbf{c}(\mathbf{x}) + \mathbf{v} \quad (18b)$$

where $\mathbf{w}$ and $\mathbf{v}$ are the process noise and the measurement noise with known covariances:

$$E\left\{\mathbf{w}\mathbf{w}^T\right\} = Q,\ E\left\{\mathbf{v}\mathbf{v}^T\right\} = R_m$$
$$E\left\{\mathbf{w}\right\} = \mathbf{0},\ E\left\{\mathbf{v}\right\} = \mathbf{0},\ E\left\{\mathbf{w}\mathbf{v}^T\right\} = 0$$

where R_m will be used as the first diagonal terms of R in the cost function corresponding to the current error, namely, r_1. After correcting the model error there is no deterministic error term in the system equation. We can design a Kalman filter for this system as follows:[7]

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{f}}(\hat{\mathbf{x}}) + \hat{G}(\hat{\mathbf{x}})\left\{\mathbf{u} + \hat{\mathbf{u}}\right\} \\ &\quad + K\left[\tilde{\mathbf{y}} - \hat{\mathbf{c}}(\hat{\mathbf{x}})\right] \end{aligned} \tag{19a}$$
$$\dot{P} = \hat{F}P + P\hat{F}^T - P\hat{C}^T R_m^{-1}\hat{C}P + Q \tag{19b}$$
$$K = PC^T R^{-1} \tag{19c}$$

where

$$\hat{F} = \frac{\partial \hat{\mathbf{f}}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}},\quad \hat{C} = \frac{\partial \hat{\mathbf{c}}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}$$

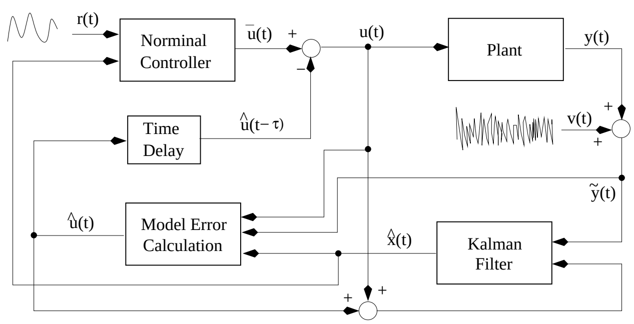

Fig. 2 System Block Diagram

The overall MECS system block diagram is shown in Fig. 2, where $r(t)$ is the reference command. The model error is determined using the estimated states, the control input, and the current measurement. The determined model error, $\hat{\mathbf{u}}$, corrects not only the nominal control input, $\bar{\mathbf{u}}$, but also the filter model. Also, the time delay τ is always applied because the response $\mathbf{y}(t)$ and the measurement $\tilde{\mathbf{y}}(t)$ are already given before the error is corrected.

ROBUST ANALYSIS

In this section a systematic approach for robust analysis using the Hermite-Biehler theorem is presented. This theorem gives the necessary and sufficient conditions for a system to be Hurwitz stable. A robust stability analysis method is derived for sixth-order characteristic equations, which can easily be applied for any characteristic equation of order less than six. The optimal subinterval size of the receding-horizon time, T, and the weighting matrices are determined to minimize an H_∞ or H_2 norm of the complementary sensitivity function or the sensitivity function.

HERMITE-BIEHLER THEOREM

The Hermite-Biehler theorem gives the necessary and sufficient conditions for a system to be Hurwitz stable.[6]

Theorem 1 *Hermite-Biehler Theorem*
Consider the following polynomial:

$$d_{cl}(s) = c_n\, s^n + c_{n-1}\, s^{n-1} + \cdots + c_2\, s^2 + c_1\, s + c_0 = 0$$

where $c_n \neq 0$ can be decomposed as

$$d_{cl}(s) = p(s) + sq(s)$$

where $p(s)$ contains even power terms and $q(s)$ contains odd power terms. Then $d_{cl}(s)$ is Hurwitz stable if and only if c_n and c_{n-1} are the same sign with all roots of $p(j\omega)$ and $q(j\omega)$ real, and the nonnegative roots satisfy the following interlacing property:

$$0 < \omega_{e1} < \omega_{o1} < \omega_{e2} < \omega_{o2} < \cdots$$

where ω_{ei} and ω_{oi} are the roots of $p(j\omega)$ and $q(j\omega)$, respectively. ■

We now concentrate on sixth-order polynomials. Let the characteristic equation be as follows:

$$d_{cl} = c_6\, s^6 + c_5\, s^5 + c_4\, s^4 + c_3\, s^3 + c_2\, s^2 + c_1\, s + c_0 \tag{20}$$

The characteristic equation can be decomposed as

$$p(s) = c_6 s^6 + c_4 s^4 + c_2 s^2 + c_0 \tag{21a}$$
$$q(s) = c_5 s^4 + c_3 s^2 + c_1 \tag{21b}$$

After substituting $s = j\omega$ we obtain

$$p(j\omega) = -c_6\omega^6 + c_4\omega^4 - c_2\omega^2 + c_0 \tag{22a}$$
$$q(j\omega) = c_5\omega^4 - c_3\omega^2 + c_1 \tag{22b}$$

Since the above equations have only even powers, let $\gamma \equiv \omega^2$, yielding

$$p(\gamma) = -c_6\gamma^3 + c_4\gamma^2 - c_2\gamma + c_0 \tag{23}$$
$$q(\gamma) = c_5\gamma^2 - c_3\gamma + c_1 \tag{24}$$

For the given characteristic equation in Eqn. (20) to be Hurwitz stable, c_6 and c_5 must have the same sign, and the cubic and the quadratic equations must have only positive real roots while satisfying the interlacing property. Since general solutions for cubic and quadratic equations exist, the roots can be directly calculated. Using the interlacing property a set of stability inequalities can be formulated, but these may not be easy to solve in general.

Let us consider the graphical properties of the given problem, shown by Fig. 3. First, without loss of generality, let c_6 be positive. So one of the necessary conditions for $p(\gamma)$ to have only real roots is

$$\text{I}:\ p(0) = c_0 > 0 \tag{25}$$

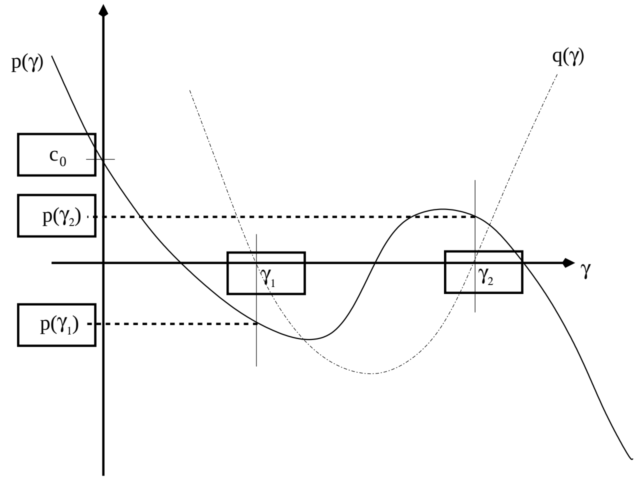

Fig. 3 Graphical Interpretation of I, II, and III

The following two roots of $q(\gamma)$ have to be positive:

$$\text{II}: \ \gamma_{1,2} = \frac{c_3 \pm \sqrt{c_3^2 - 4\,c_1\,c_5}}{2\,c_5} > 0 \tag{26a}$$

$$\Longleftrightarrow c_3^2 > 4\,c_1\,c_5 \text{ and } c_3 > 0, \ c_5 > 0 \tag{26b}$$

and finally, to satisfy the interlacing property, the values of $p(\gamma)$ at $\gamma_{1,2}$ must be

$$\text{III}: \ p(\gamma_1) < 0 \ \text{ and } \ p(\gamma_2) > 0 \tag{27}$$

Since $q(\gamma_1) = 0$ or $q(\gamma_2) = 0$ and $\gamma_1 < \gamma_2$, then

$$\begin{aligned} p(\gamma_1) &= -c_6\,\gamma_1^3 + c_4\,\gamma_1^2 - c_2\,\gamma_1 + c_0 \\ &= q(\gamma_1)\left\{-\frac{c_6}{c_5}\gamma_1 + \frac{1}{c_5^2}\,(c_4\,c_5 - c_3\,c_6)\right\} \\ &+ \frac{1}{c_5^2}\,\left(c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6\right)\gamma_1 \\ &+ \frac{1}{c_5^2}\,\left(c_0\,c_5^2 - c_1\,c_4\,c_5 + c_1\,c_3\,c_6\right) \\ &= \frac{1}{c_5^2}\,\left(c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6\right)\gamma_1 \\ &+ \frac{1}{c_5^2}\,\left(c_0\,c_5^2 - c_1\,c_4\,c_5 + c_1\,c_3\,c_6\right) \end{aligned} \tag{28}$$

Multiplying both sides of the above inequality by c_5^2 we obtain

$$\begin{aligned} &\left(c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6\right)\gamma_1 \\ &+ c_0\,c_5^2 - c_1\,c_4\,c_5 + c_1\,c_3\,c_6 < 0 \end{aligned} \tag{29}$$

Also, $\gamma_{1,\,2}$ is given by

$$\gamma_{1,\,2} = \frac{c_3 \pm \sqrt{c_3^2 - 4\,c_1\,c_5}}{2\,c_5} \tag{30}$$

and since c_5 has to be greater than zero by the second condition of II, then

$$\begin{aligned} &\left(c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6\right)\left(c_3 - \sqrt{c_3^2 - 4\,c_1\,c_5}\right) \\ &+ 2\,c_0\,c_5^3 - 2\,c_1\,c_4\,c_5^2 + 2\,c_1\,c_3\,c_5\,c_6 < 0 \end{aligned} \tag{31}$$

Define the following:

$$A \equiv c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6 \tag{32a}$$

$$B \equiv 2\,c_0\,c_5^3 - 2\,c_1\,c_4\,c_5^2 + 2\,c_1\,c_3\,c_5,\ c_6 \tag{32b}$$

then

$$c_3\,A + B < A\,\sqrt{c_3^2 - 4\,c_1\,c_5} \tag{33}$$

and for $\gamma = \gamma_2$

$$c_3\,A + B > -A\,\sqrt{c_3^2 - 4\,c_1\,c_5} \tag{34}$$

By Eqns. (33) and (34),

$$\left|c_3\,A + B\right| < \left|A\,\sqrt{c_3^2 - 4\,c_1\,c_5}\right| \tag{35}$$

Note that if the first condition of II is not satisfied, i.e., $c_3^2 < 4\,c_1\,c_5$, the above condition is not satisfied either. Squaring both sides of Eqn. (35) gives

$$4\,c_1\,c_5\,A^2 + 2\,c_3\,A\,B + B^2 < 0 \tag{36}$$

Finally, the inequalities to be satisfied are summarized as follows:

$$\text{I}: \ c_0 > 0 \tag{37a}$$

$$\text{II}: \ c_3 > 0 \text{ and } c_5 > 0 \tag{37b}$$

$$\text{III}: \ -\left(4, c_1\,c_5\,A^2 + 2\,c_3\,A\,B + B^2\right) > 0 \tag{37c}$$

with the assumption $c_6 > 0$. Now, bound each of the coefficients as follows:

$$\underline{c}_i \le c_i \le \bar{c}_i, \text{ for } i = 1,\, 2,\, \cdots,\, 6 \tag{38}$$

Then, the first two conditions are easy to check, i.e.,

$$\text{I}: \ \underline{c}_0 > 0 \tag{39a}$$

$$\text{II-1}: \ \min\ (c_3\,c_5) > 0 \tag{39b}$$

$$\text{II-2}: \ \min\ (c_5\,c_6) > 0 \tag{39c}$$

However, the third condition in Eqn. (37c) is difficult to solve analytically. We can formulate III as a minimization problem, but the problem is highly nonlinear and we cannot guarantee that the calculated value is a global minimum. To overcome this difficulty, extreme polynomials defined in Ref. [8] are used. If III is satisfied for the extreme polynomials, then it is satisfied for the rest of uncertainty space. The coefficients for the two extreme conditions are given by

$$\text{III-1}: \bar{c}_6,\ \underline{c}_5,\ \bar{c}_4,\ \bar{c}_3,\ \underline{c}_2,\ \underline{c}_1,\ \bar{c}_0 \tag{40a}$$

$$\text{III-2}: \underline{c}_6,\ \bar{c}_5,\ \underline{c}_4,\ \underline{c}_3,\ \bar{c}_2,\ \bar{c}_1,\ \underline{c}_0 \tag{40b}$$

In fact, we are not interested in the values of each condition but whether or not all the conditions are greater than zero. Therefore, we are only interested in the minimum value of all conditions, i.e.,

$$\kappa \equiv \min\left(\text{I},\ \text{II-1},\ \text{II-2},\ \text{III-1},\ \text{III-2}\right) \tag{41}$$

Define stability index, ϵ, as follows:

$$\epsilon \equiv \begin{cases} \text{sgn}(\kappa)|\log_{10}|\kappa||, & \text{for } \kappa \neq 0 \\ 0, & \text{for } \kappa = 0 \end{cases} \tag{42}$$

If $\epsilon > 0$ then the characteristic equation is Hurwitz stable, else it is not stable.

OPTIMAL T and R or W

The length of receding-horizon time, T, and the weights in the cost function, R and W, are critical factors for the stability and performance of MECS. To choose optimal values for T and R and/or W, transfer functions are derived for second order Single-Input-Single-Output (SISO) linear systems, represented by (we assume that $\hat{\mathbf{x}}(t) \approx \mathbf{x}(t)$)

$$y(t) = \frac{N_s(s)}{D_s(s)}\left[u(t)\right] + w(t) \tag{43}$$

where $y(t)$ is the system output, $u(t)$ is the control input, and $w(t)$ is output disturbance. In this paper the filter notation with zero initial conditions is used, i.e., $y(t) = G(s)[u(t)]$ can be treated as the signal $u(t)$ passing through the filter $G(s)$ and generating the output $y(t)$.[9] The nominal control input is given by

$$\bar{u}(t) = \frac{N_k(s)}{D_k(s)}\left[r(t) - y(t)\right] = \frac{N_k(s)}{D_k(s)}\left[r(t) - y(t)\right] \tag{44}$$

where $N_k(s)/D_k(s)$ is the transfer function of nominal controller and $r(t)$ is reference command. Also, the correction control input by MECS is given by

$$\begin{aligned} \hat{u}(t) = {} & a_1 y(t) + s a_2 y(t) - a_3 \hat{u}(t-\tau) + a_3 \bar{u}(t) \\ & + a_4\left[y(t) + v(t)\right] \end{aligned} \tag{45}$$

where the a_i's are constants determined by the predictive filer, and $v(t)$ is the measurement noise with known variance. Then

$$\begin{aligned} \hat{u}(t) = {} & \frac{d(s)}{d(s) + a_3 n(s)} \\ & \times \left\{ \frac{\left(a_1 + a_4 + s a_2\right) D_k(s) - a_3 N_k(s)}{D_k(s)} \left[y(t)\right] \right. \\ & \left. + \frac{a_3 N_k(s)}{D_k(s)}\left[r(t)\right] + a_4 v(t) \right\} \end{aligned} \tag{46}$$

where $n(s)/d(s)$ is a Padé approximation of $e^{-\tau s}$.[4] Substituting the control input into the system, the closed loop characteristic equation is given by

$$\begin{aligned} D_{cl}(s) = {} & \left\{d(s) + a_3 n(s)\right\} D_k(s) D_s(s) \\ & + \left\{\left(1 - a_3\right) d(s) + a_3 n(s)\right\} N_k(s) N_s(s) \\ & + d(s)\left(a_1 + a_4 + s a_2\right) D_k(s) N_s(s) \end{aligned} \tag{47}$$

Now the closed loop output system is

$$y(t) = \frac{N_r(s)}{D_{cl}(s)}\left[r(t)\right] + \frac{N_v(s)}{D_{cl}(s)}\left[v(t)\right] + \frac{N_w(s)}{D_{cl}(s)}\left[w(t)\right] \tag{48}$$

where

$$N_r(s) = \left\{\left(1 - a_3\right) d(s) + a_3 n(s)\right\} N_k(s) N_s(s) \tag{49a}$$

$$N_v(s) = -a_4 d(s) D_k(s) N_s(s) \tag{49b}$$

$$N_w(s) = \left\{d(s) + a_3 n(s)\right\} D_k(s) D_s(s) \tag{49c}$$

Then, the error dynamics is given by

$$\begin{aligned} e(t) &= r(t) - y(t) \\ &= \frac{D_{cl}(s) - N_r(s)}{D_{cl}(s)}\left[r(t)\right] - \frac{N_v(s)}{D_{cl}(s)}\left[v(t)\right] \\ &\quad - \frac{N_w(s)}{D_{cl}(s)}\left[w(t)\right] \\ &\equiv T(s)\left[r(t)\right] - S_v(s)\left[v(t)\right] + S(s)\left[w(t)\right] \end{aligned} \tag{50}$$

where $T(s)$ is the complementary sensitivity function, $S(s)$ is the sensitivity function, and $S_v(s)$ is the transfer function for the measurement noise (in this paper $T(\cdot)$ is used for the complementary sensitivity function and T is used for the length of receding-horizon time). Finally, we choose T, r_p and/or w_p by minimizing the following H_∞ norm or H_2 norm:

$$\begin{aligned} & \min \|W_S(j\omega) S(j\omega)\|_\infty \text{ or} \\ & \min \|W_T(j\omega) T(j\omega)\|_\infty \\ & \quad \text{subject to } \epsilon > 0 \end{aligned} \tag{51}$$

where $W_S(j\omega)$ and $W_T(j\omega)$ are design weighting functions, and each norm is minimized for the nominal values of c_i. A flowchart of the summarized design and analysis steps is given in Fig. 4.

EXAMPLES

In this section the previous results are applied to a second order marginally stable system and an unstable non-minimum phase system.

MARGINALLY STABLE SYSTEM

Here we will show the effect of the subinterval N for a fixed variation in each system parameter. The system is given by

$$\ddot{x} + \omega_n^2 x = b u(t) \tag{52a}$$

$$\tilde{y} = x + v \tag{52b}$$

Note that the given system has two poles on the pure imaginary axis (marginally stable system), and the measurement is displacement only with a zero-mean Gaussian white noise process having known covariance σ^2. The assumed model is given by:

$$\ddot{\hat{x}} + 2\hat{\zeta}\hat{\omega}_n\dot{\hat{x}} + \hat{\omega}_n^2\hat{x} = \hat{b}\bar{u} \tag{53a}$$

$$\hat{y} = \hat{x} \tag{53b}$$

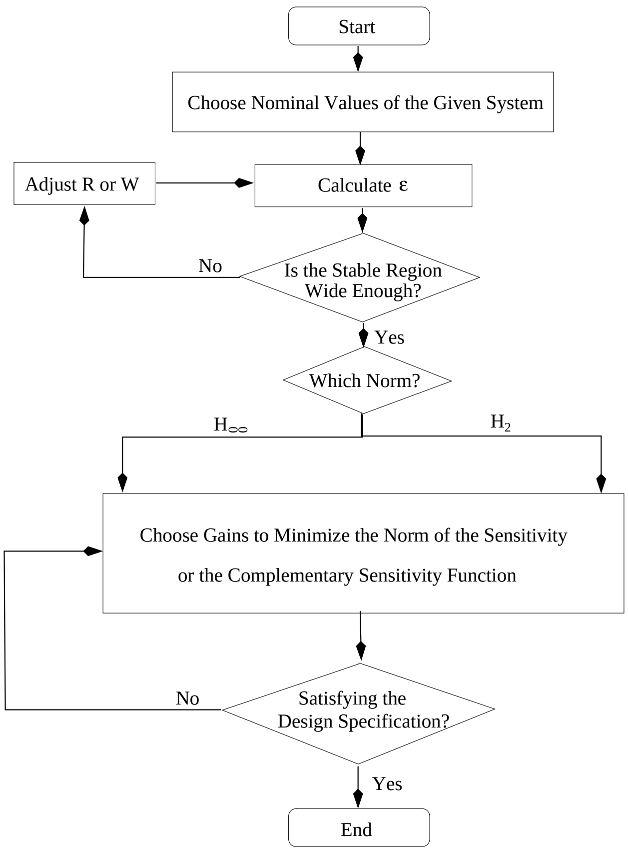

Fig. 4 Robust Design and Analysis Steps

where $\bar{u} = u + \hat{u}$. The state-space matrices are

$$\hat{F} = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_n^2 & -2\hat{\zeta}\hat{\omega}_n \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} 0 \\ \hat{b} \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{54}$$

To show the effect of the subinterval N, we first set $N = 2$. Then, the following matrices for the predictive filter are obtained:

$$H_0 = \frac{T}{6} \begin{bmatrix} \dfrac{h^4 \hat{b}^2}{r_2} + w_1 & 0 \\ 0 & 4w_2 \end{bmatrix}, \quad M = \begin{bmatrix} h\hat{b} \\ 0 \end{bmatrix} \tag{55}$$

where $R = \text{diag}[\, r_1,\, r_2 \,]$ and $W = \text{diag}[\, w_1,\, w_2 \,]$. The QP solution of Eqn. (14) is

$$\boldsymbol{\nu}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{3}{2Tw_2} \end{bmatrix} \mathbf{g}_0(t) + \begin{bmatrix} \dfrac{1}{h\hat{b}} \\ 0 \end{bmatrix} d(t) \tag{56}$$

where

$$\mathbf{g}_0(t) = \begin{bmatrix} g_1 \neq 0 \\ 0 \end{bmatrix} \tag{57}$$

$$d(t) = \frac{\tilde{y} - \hat{y}}{h} - 2\hat{x}_2 - h\left\{ -\hat{\omega_n}^2 \hat{x}_1 - \hat{\zeta}\hat{\omega}_n \hat{x}_2 + \hat{b}u(t) \right\} \tag{58}$$

Therefore, for $N = 2$ the model error $\hat{u}(t)$ is given by

$$\begin{aligned} \hat{u}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{\nu}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ 0 \end{bmatrix} + \frac{1}{h\hat{b}} d(t) \\ &= 0 + \frac{1}{h\hat{b}} d(t) = \frac{1}{h\hat{b}} d(t) \end{aligned} \tag{59}$$

Since the first row elements of the matrix in front of $\mathbf{g}_0(t)$ in Eqn. (56) are zero when $N = 2$, $\hat{u}(t)$ is not a function of weights R and W, but a function of N and T (i.e. h) only.

On the other hand, for $N = 3$, since the first row elements of the matrix in front of $\mathbf{g}_0$ are not all zeros, the determined model error, $\hat{u}(t)$, is a function of the weights as follows:

$$\begin{aligned} &\hat{u}(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ &\times \Big[-\left\{ H_0^{-1} - H_0^{-1} M \left(M^T H_0^{-1} M \right)^{-1} M^T H_0^{-1} \right\} \mathbf{g}_0(t) \\ &+ \left\{ H_0^{-1} M \left(M^T H_0^{-1} M \right)^{-1} \right\} d(t) \Big] \end{aligned} \tag{60}$$

where

$$H_0 = \frac{T}{6} \times \tag{61a}$$

$$\begin{bmatrix} \dfrac{(2\,r_4 + r_3) h^4 + w_1 r_3 r_4 \hat{m}^2}{r_3\, r_4\, \hat{m}^2} & \dfrac{h^4}{r_4\, \hat{m}^2} & 0 \\ \dfrac{h^4}{r_4\, \hat{m}^2} & \dfrac{h^4 + 2w_2 r_4 \hat{m}^2}{r_4\, \hat{m}^2} & 0 \\ 0 & 0 & 2\,w_3 \end{bmatrix} \tag{61b}$$

$$M^T = \hat{C} \left[\left(I + h\hat{F} \right)^2 \hat{G},\ \left(I + h\hat{F} \right) \hat{G},\ \hat{G} \right], \tag{61c}$$

$R = \text{diag}[\, r_1,\, r_2,\, r_3,\, r_4 \,]$, $W = \text{diag}[\, w_1,\, w_2,\, w_3,\, w_4 \,]$, $\mathbf{g}_0(t)$ is the first order terms of $\boldsymbol{\nu}_0$ in the cost function, and $d(t)$ is given by

$$\begin{aligned} d(t) = &\frac{1}{h} \left\{ \tilde{y}(t) - \hat{y}(t) \right\} \\ &- \sum_{i=0}^{2} \left\{ \hat{C} \left(I + h\hat{F} \right)^i \left(\hat{F}\hat{\mathbf{x}}(t) + \hat{G}u(t) \right) \right\} \end{aligned} \tag{62}$$

As a nominal controller we simply use the following Proportional-Derivative (PD) controller:

$$u(t) = -2\hat{x}_1(t) - 10\hat{x}_2(t) \tag{63}$$

The simulation parameters are as follows: the weights, R and W, are set to $1.0 \times 10^{-6} I$ and identity matrix with appropriate dimension for $N = 2$ and $N = 3$ cases, respectively, and the variance of the measurement noise, σ^2, is 1.0×10^{-6}. The length of horizon, T, is 1 sec, and the initial conditions of the system and the estimator are set to $\mathbf{x}_0 = [\, -1.5267\ 5.5540 \,]^T$ and $\hat{\mathbf{x}}_0 = [\, -1.5286\ 5.5749 \,]^T$, respectively.

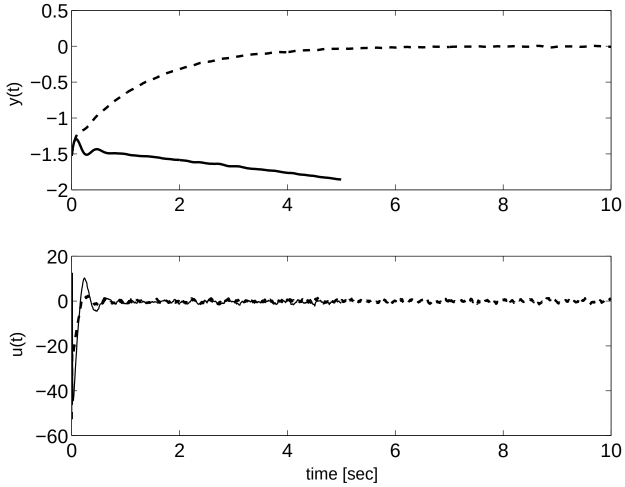

Fig. 5 System Output and Control Input for $N = 2$ (Solid) and $N = 3$ (Dotted) Cases

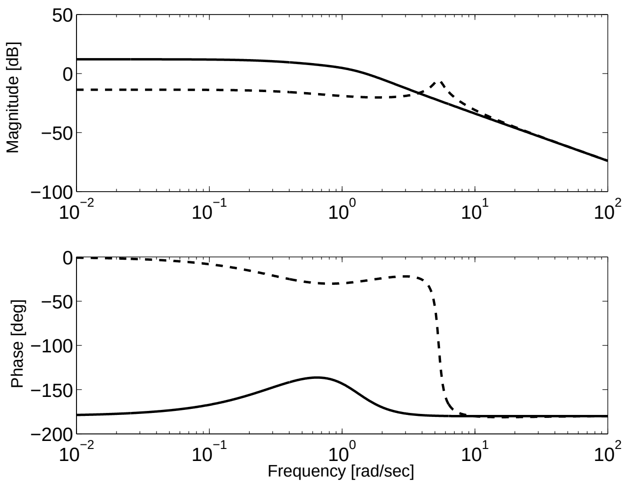

Fig. 6 PD Only(Solid) and PD+MECS(Dotted)

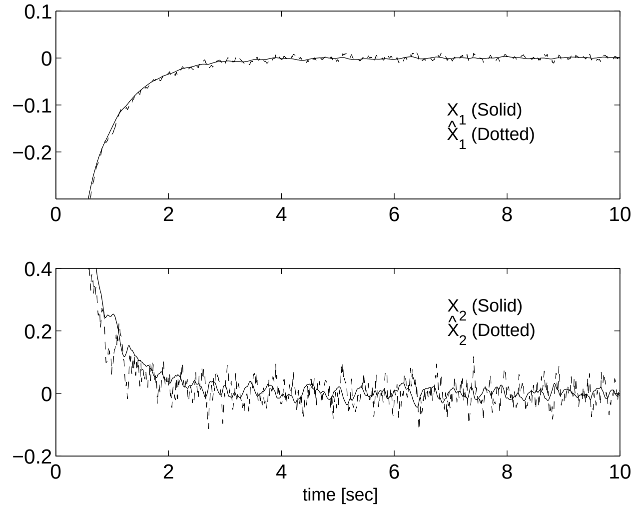

Fig. 7 $N = 3$ Case: Time History of the States

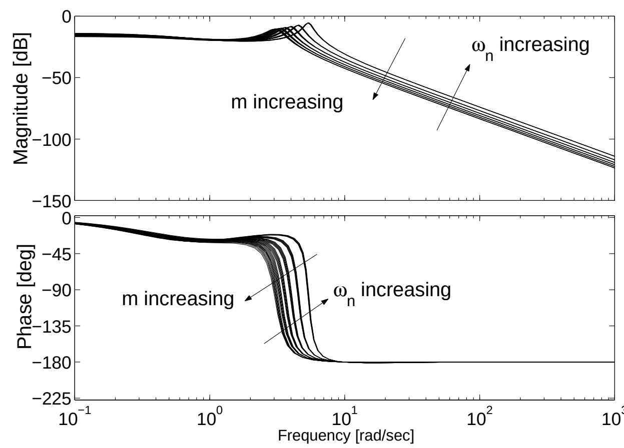

Fig. 8 PD+MECS with Parameters Uncertainty Variations

As shown in Fig. 5, for $N = 2$ with parameter variations given by $\omega_n = 0.5$, $b = 2$, $\hat{\zeta} = 0.707$, $\hat{\omega}_n = 1$, and $\hat{b} = 1$, the output diverges. However, for $N = 3$ the response is stable with almost the same control input as the $N = 2$ case. Since we have more design parameters for the $N = 3$ case than $N = 2$, this result coincides with our intuition. However, this result does not imply that for $N = 2$ the algorithm fails to stabilize the system. This example merely shows that dividing T into more intervals, a wider range of parameters variations can be tolerated.

Fig. 6 compares the Bode plots of PD control only with PD control combined with MECS for $N = 3$. The same parameter variations from the previous simulation are used in this comparison. For the PD only case the gain margin is -12.126 dB and the phase margin is 20.102°, so the given closed loop system for this case is unstable. But for the PD with MECS case the gain margin is 30.283 dB and the phase margin is infinity, yielding a stable closed loop system. Fig. 7 shows the time histories of the states and estimation errors. At the beginning there is transient response in the estimated state trajectories. However, even though the measurement is displacement only with high noise, the Kalman filter combined with predictive filter estimates both states (displacement and velocity) well.

Fig. 8 shows Bode plots of PD control with MECS for various parameter variations. The nominal values are the same as in the above case. The actual values of the system vary as follows: ω_n varies from 0.5 to 1.5 and b varies from 0.67 to 2.0. With the given parameter variations, which represent more than ±30% variations from the nominal values, the gain margins for all cases are at least 20 dB and the phase margins are infinity. In the magnitude plot for high frequencies the slopes are about -40 dB/dec, hence, the overall system has good high frequency rejection.

NONMINIMUM PHASE UNSTABLE SYSTEM

In this example the advantages of the MECS design are shown for non-minimum phase unstable systems. The system is given by

$$y(t) = \frac{(\delta_3+1)\,s - (\delta_4+1)}{s^2 - (\delta_1 - 0.8)\;s - (\delta_2+0.2)}\,[u(t)] \qquad (64)$$

For the nominal values, i.e., all $\delta_i = 0$, the system poles are -1 and 0.2, and the zero is 1. Let each δ_i be given by

$$-0.1 \le \delta_i \le 0.1 \quad \text{for } i = 1,\,2,\,3,\,4 \qquad (65)$$

The nominal controller is PI control, as follows:

$$\bar{u}(t) = \frac{k_p s + k_i}{s}\,[\hat{y}(t)] \qquad (66)$$

where $k_p = -0.43$ and $k_i = -0.02$.[10] Using a (3,3) Padé approximation for the time delay τ in the MECS design we have[11]

$$e^{-\tau s} \approx \frac{-\tau^3 s^3 + 12\tau^2 s^2 - 60\tau s + 120}{\tau^3 s^3 + 12\tau^2 s^2 + 60\tau s + 120} \qquad (67)$$

From Eqn. (47) a sixth-order characteristic equation is obtained. Before proceeding further, note that for each calculated c_i the uncertainty factors δ_i's appear linearly, so each c_i can be written as

$$c_i = \mathbf{v}_i^T \mathbf{x} \;\; \text{for} \;\; i = 0,\,1,\,2,\,\cdots,\,6 \qquad (68)$$

where $\mathbf{v}_i$ is a vector whose elements are coefficients of δ_i, and $\mathbf{x} = [\;\delta_1\;\delta_2\;\delta_3\;\delta_4\;1]^T$. Then, the upper and the lower bounds of each coefficient are given by

$$\bar{c}_i = \mathbf{v}_i^T \begin{bmatrix} \text{if sgn}(v_{ij}) > 0 \text{ then } \max(\delta_j) \\ \text{else } \min(\delta_j) \text{ for } j = 1,\,2,\,3,\,4 \\ 1 \end{bmatrix} \qquad (69)$$

where v_{ij} is the j^{th} element of $\mathbf{v}_i$. Similarly, the lower bound is given by

$$\underline{c}_i = \mathbf{v}_i^T \begin{bmatrix} \text{if sgn}(v_{ij}) > 0 \text{ then } \min(\delta_j) \\ \text{else } \max(\delta_j) \text{ for } j = 1,\,2,\,3,\,4 \\ 1 \end{bmatrix} \qquad (70)$$

Then II-1 and II-2 in Eqn. (39) can be formulated as follows:

$$\min\;(c_i c_j) > 0 \qquad (71a)$$

$$\Longleftrightarrow \min\;\left(\mathbf{x}^T \mathbf{v}_i \mathbf{v}_j^T \mathbf{x}\right) > 0 \qquad (71b)$$

Hence, if T is chosen for a given τ so that $\mathbf{v}_3\mathbf{v}_5^T$ and $\mathbf{v}_5\mathbf{v}_6^T$ are positive definite, then the conditions II-1 and II-2 are completely decoupled from the uncertainty factors. However, since $\mathbf{v}_i\mathbf{v}_j^T$ is always rank 1, the decoupling is impossible. The equation can be formulated as a minimization problem as follows:

$$\min\;\left\{\boldsymbol{\delta}^T V_{11}\boldsymbol{\delta} + (\mathbf{v}_{12}+\mathbf{v}_{21})^T\boldsymbol{\delta} + v_{44}\right\} > 0 \qquad (72)$$

subject to $-\boldsymbol{\delta}_l \le \boldsymbol{\delta} \le \boldsymbol{\delta}_u$, where V_{11} is the left upper 4× 4 matrix, $\mathbf{v}_{12}$ is the right upper 4×1 vector, $\mathbf{v}_{21}^T$ is the left lower 1×4 vector, and v_{44} is the scalar of the lower right element of the following:

$$\mathbf{v}_i\mathbf{v}_j^T = \begin{bmatrix} V_{11} & \mathbf{v}_{12} \\ \mathbf{v}_{21}^T & v_{44} \end{bmatrix} \qquad (73)$$

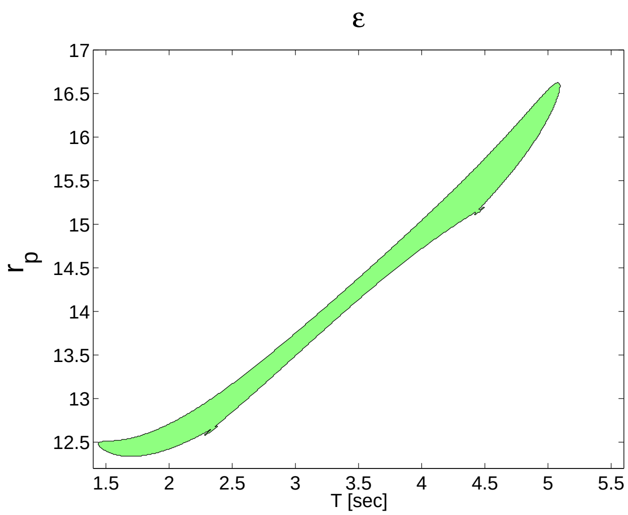

Fig. 9 Stability Index ϵ with Respect to T and r_p

For $N = 3$, $\mathbf{g}_0 = [g_i]$, $H_0 = [h_{ij}]$, $\hat{F}$, $\hat{G}$, and $\hat{C}$ are given in appendix. With the time delay $\tau = 0.01$ sec, $w_1 = 1$, $w_p = 0$, and $r_1 = 1\times 10^{-6}$, the stability index in Eqn. (42) with respect to variations in T and r_p is shown in Fig. 9. Inside the contour line, the closed loop system is guaranteed to be Hurwitz stable within the given uncertainty bounds.

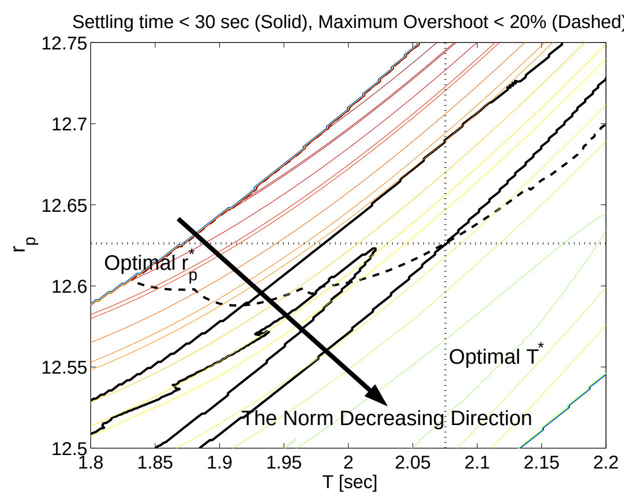

Fig. 10 $||W_T(j\omega)T(j\omega)||_\infty$

Now, in the stable region we search for the optimal T and r_p to minimize the H_∞ or H_2 norm of the weighted complementary sensitivity function or the

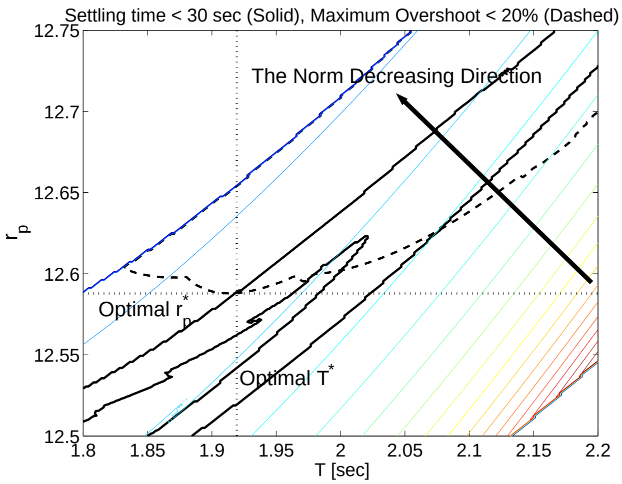

Fig. 11 $||W_S(j\omega)S(j\omega)||_\infty$

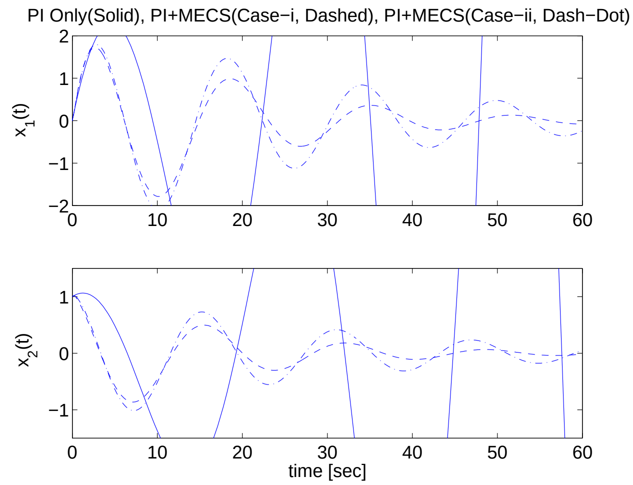

Fig. 12 Time History of the State

weighted sensitivity function using the nominal values of the system. The weighting functions are given by

$$W_T(s) = W_S(s) = \frac{s + 0.1}{s + 1} \tag{74}$$

Figs. 10 and 11 show contour lines of the weighted complementary sensitivity function and the weighted sensitivity function, respectively. For each figure inside the solid line the settling time of the weighted $T(j\omega)$ is less than 30 sec and inside the dashed line the overshoot of the weighted $T(j\omega)$ is less than 20%. For each case the optimal receding-horizon subinterval T^* and weighting factor r_p^* are as follows:

$$\begin{aligned} \text{Case-i:} \quad & \min||W_T(j\omega)T(j\omega)||_\infty \\ & T^* = 2.077 \text{ sec}, \; r_p^* = 12.627 \end{aligned} \tag{75a}$$

$$\begin{aligned} \text{Case-ii:} \quad & \min||W_S(j\omega)S(j\omega)||_\infty \\ & T^* = 1.919 \text{ sec}, \; r_p^* = 12.588 \end{aligned} \tag{75b}$$

The choice of Case-i or Case-ii depends on which norm we want to minimize. For each case the determined $\hat{u}(t)$ are as follows:

$$\begin{aligned} \text{Case-i:} \quad \hat{u}(t) \approx & -0.296\,\hat{x}_1 - 0.163\,\hat{x}_2 \\ & - 0.606\,u(t) + 0.515\,\tilde{y}(t) \end{aligned} \tag{76a}$$

$$\begin{aligned} \text{Case-ii:} \quad \hat{u}(t) \approx & -0.349\,\hat{x}_1 - 0.122\,\hat{x}_2 \\ & - 0.552\,u(t) + 0.533\,\tilde{y}(t) \end{aligned} \tag{76b}$$

The time histories of each state for the above cases are shown in Fig. 12, where $\mathbf{x}(0) = [\,0\;1\,]^T$ and $\hat{\mathbf{x}}(0) = [\,0\;0\,]^T$. Note that for the PI control only case the true state is used, and for the other cases the estimated state and the measurement are used. The uncertainty factors are $\delta_i = 0.1$ for $i = 1, 2, 3, 4$. As shown in the time history of the states, Fig. 12, and the control input, Fig. 13, PI control with MECS is vastly more robust than PI control only.

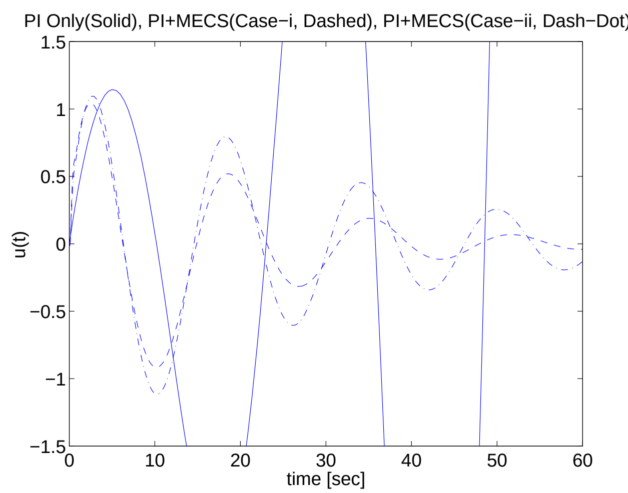

Fig. 13 Time History of the Control

CONCLUSION

An improved MECS controller is derived by combining a predictive filter with a Kalman filter. The model error in the plant is determined by an approximate receding-horizon optimal solution, which is used to update the nominal control input. Also, systematic robust design and analysis processes for sixth-order characteristic equations are presented. A robust controller was developed for given uncertainty bounds to satisfy certain design specifications. As shown in the examples the MECS design system can be applied for both minimum phase stable systems and non-minimum phase unstable systems. Furthermore, the closed loop system can tolerate relatively large uncertainties, which leads to an extremely robust control methodology.

REFERENCES

[1]Crassidis, J. L., “Robust Control of Nonlinear Systems Using Model-Error Control Synthesis,” *Journal of Guidance,*

Control, and Dynamics, Vol. 22, No. 4, July-Aug. 1999, pp. 595–601.

[2]Crassidis, J. L. and Markley, F. L., "Predictive Filtering for Nonlinear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May-June 1997, pp. 566–572.

[3]Crassidis, J. L., Markley, F. L., Anthony, T. C., and Andrews, S. F., "Nonlinear Predictive Control of Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 6, Nov.-Dec. 1997, pp. 1096–1103.

[4]Kim, J. and Crassidis, J. L., "Linear Stability Analysis of Model-Error Control Synthesis," *AIAA Guidance, Navigation, and Control Conference*, Denver, CO, Aug. 2000, AIAA-2000-3963.

[5]Lu, P., "Approximate Nonlinear Receding-Horizon Control Laws in Closed Form," *International Journal of Control*, Vol. 71, No. 1, 1998, pp. 19–34.

[6]Gantmacher, F. R., *The Theory of Matrices*, Vol. II, Chelsea Publishing Company, 1959.

[7]Gelb, A., *Applied Optimal Estimation*, MIT Press, 1974.

[8]Argoun, M. B., "On the Stability of Low-Order Perturbed Polynomials," *IEEE Transactions on Automatic Control*, Vol. 35, No. 2, Feb. 1999, pp. 180–182.

[9]Ioannou, P. A. and Sun, J., *Robust Adaptive Control*, Prentice-Hall, 1996.

[10]Ho, M. T., Datta, A., and Bhattacharyya, S. P., "A Generalization of the Hermite Biehler Theorem," *Proceedings of the 34th Conference on Decision & Control*, New Orleans, LA, Dec. 1995, pp. 130–131.

[11]Wang, Z. and Hu, H., "Robust Stability Test for Dynamic Systems with Short Delay by Using Padé Approximation," *Nonlinear Dynamics*, Vol. 18, No. 6, 1999, pp. 275–287.

APPENDIX

The specific variables for the second example are given by

$$
\begin{aligned}
g_1 = &-\frac{e^{-r_p}\,h^2}{6250\,r_1}\left\{4264\,e^{-2\,r_p}\,h^5 - 22275\,e^{-2\,r_p}\,h^4\right.\\
&+\left(43250\,h^3\,e^{-2\,r_p} + 9450\,h^3\,e^{-r_p}\right)h^3\\
&-\left(40000\,e^{-2\,r_p} + 23250\,e^{-r_p}\right)h^2\\
&+\left(18750\,e^{-2\,r_p} + 21250\,e^{-r_p} + 5000\right)h\\
&\left.-3125\,e^{-2\,r_p} - 6250\,e^{-r_p} - 6250\right\}\,\hat{x}_1(t)\\
&+\frac{e^{-r_p}\,h^3}{1250\,r_1}\left\{861\,e^{-2\,r_p}\,h^4 - 4350\,e^{-2\,r_p}\,h^3\right.\\
&+\left(9000\,e^{-2\,r_p} + 1800\,e^{-r_p}\right)\,h^2\\
&-\left(8250\,e^{-2\,r_p} + 5500\,e^{-r_p}\right)\,h\\
&\left.+1875\,e^{-2\,r_p} + 2500\,e^{-r_p} + 1250\right\}\,\hat{x}_2(t)\\
&+\frac{e^{-r_p}\,h^3}{1250\,r_1}\left\{1681\,e^{-2\,r_p}\,h^4 - 9225\,e^{-2\,r_p}\,h^3\right.\\
&+\left(16250\,e^{-2\,r_p} + 4050\,e^{-r_p}\right)\,h^2\\
&-\left(10125\,e^{-2\,r_p} + 6750\,e^{-r_p}\right)\\
&\left.+1875\,e^{-2\,r_p} + 2500 * e^{-r_p} + 1250\right\}\,u(t)\\
&+\frac{e^{-r_p}\,h^2}{50\,r_1}\left\{-41\,e^{-2\,r_p}\,h^2 + \left(90\,e^{-2\,r_p} + 90\,e^{-r_p}\right)\,h\right.\\
&\left.-25\,e^{-2\,r_p} - 50\,e^{-r_p} - 50\right\}\,\tilde{y}(t)
\end{aligned}
$$

$$
\begin{aligned}
g_2 = &\frac{h^2}{1250\,r_1}\left\{936\,e^{-3\,r_p}\,h^4 - 3355\,e^{-3\,r_p}\,h^3\right.\\
&+\left(4275\,e^{-3\,r_p} + 1050\,e^{-2\,r_p}\right)\,h^2\\
&-\left(2625\,e^{-3\,r_p} + 2000\,e^{-2\,r_p}\right)\,h\\
&\left.+625\,e^{-3\,r_p} + 1250\,e^{-2\,r_p}\right\}\,\hat{x}_1(t)\\
&-\frac{h^3}{250\,r_1}\left\{189\,e^{-3\,r_p}\,h^3 - 645\,e^{-3\,r_p}\,h^2\right.\\
&+\left(975\,e^{-3\,r_p} + 200\,e^{-2\,r_p}\right)\,h\\
&\left.-\left(375\,e^{-3\,r_p} + 500\,e^{-2\,r_p}\right)\right\}\,\hat{x}_2(t)\\
&-\frac{h^3}{250\,r_1}\left\{369\,e^{-3\,r_p}\,h^3 - 1420\,e^{-3\,r_p}\,h^2\right.\\
&+\left(1350\,e^{-3\,r_p} + 450\,e^{-2\,r_p}\right)\,h\\
&\left.-375\,e^{-3\,r_p} - 500\,e^{-2\,r_p}\right\}\,u(t)\\
&+\frac{h^2}{10\,r_1}\left(9\,e^{-3\,r_p}\,h - 5\,e^{-3\,r_p} - 10\,e^{-2\,r_p}\right)\,\tilde{y}(t)
\end{aligned}
$$

$$
\begin{aligned}
g_3 = &-\frac{h^2\left(104\,h^3 - 315\,h^2 + 300\,h\right)\,e^{-3\,r_p}}{250\,r_1}\,\hat{x}_1(t)\\
&+\frac{3\,h^3\left(7\,h^2 - 20\,h + 25\right)\,e^{-3\,r_p}}{50\,r_1}\,\hat{x}_2(t)\\
&+\frac{h^3\left(41\,h^2 - 135\,h + 75\right)\,e^{-3\,r_p}}{50\,r_1}\,u(t) - \frac{h^2\,e^{-3\,r_p}}{2\,r_1}\,\tilde{y}(t)
\end{aligned}
$$

$$
\begin{aligned}
h_{11} = &\frac{h}{1250\,r_1}\left\{1681\,e^{-3\,r_p}\,h^6 - 7380\,e^{-3\,r_p}\,h^5\right.\\
&+\left(10150\,e^{-3\,r_p} + 4050\,e^{-2\,r_p}\right)\,h^4\\
&-4500\,\left(e^{-3\,r_p} + e^{-2\,r_p}\right)\,h^3\\
&\left.+\left(625\,e^{-3\,r_p} + 1250\,e^{-2\,r_p} + 1250\,e^{-r_p}\right)\,h^2 + 625\,w_1\,r_1\right\}
\end{aligned}
$$

$$h_{12} = h_{21} = \frac{h^3\,e^{-3\,r_p}\,\left(5 - 9\,h\right)\,\left(41\,h^2 - 90\,h + 25 + 50\,e^{r_p}\right)}{250\,r_1}$$

$$h_{13} = h_{31} = \frac{h^3\,\left(41\,h^2 - 90\,h + 25\right)}{e^{3\,r_p}\,r_1}$$

$$
\begin{aligned}
h_{22} = &\frac{h}{50\,r_1}\left\{81\,e^{-3\,r_p}\,h^4 - 90\,e^{-3\,r_p}\,h^3\right.\\
&\left.+\left(25\,e^{-3\,r_p} + 50\,e^{-2\,r_p}\right)\,h^2 + 50\,w_1\,r_1\,e^{w_p}\right\}
\end{aligned}
$$

$$h_{23} = h_{32} = \frac{h^3\,\left(5 - 9\,h\right)}{e^{3\,r_p} r_1}$$

$$h_{33} = \frac{h\left(h^2 + 2\,e^{3\,w_p}\,e^{3\,r_p}\,w_1\,r_1\right)}{2\,e^{3\,r_p}\,r_1}$$

where $h = T/3$.

$$\hat{F} = \begin{bmatrix} -0.8 & 1 \\ 0.2 & 0 \end{bmatrix},\ \hat{G} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$