On certain Rankin-Selberg integrals on GE_6*

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Abstract

In this paper we begin the study of two Rankin-Selberg integrals defined on the exceptional group of type GE_6. We show that each factorizes and that the contribution from the unramified places is, in one case, the degree 54 Euler product $L^S(\pi \times \tau, E_6 \times GL_2, s)$ and in the other case the degree 30 Euler product $L^S(\pi \times \tau, \wedge^2 \times GL_2, s)$.

1 Introduction

In this paper, we begin the study of the tower of Rankin-Selberg integrals which was announced in [G-H3]. Specifically, we consider two integrals, which were labelled as (c3) and (c4) in [G-H3]. In more details, let π denote an irreducible cuspidal generic representation defined on the exceptional group $GE_6(\mathbf{A})$, and let τ denote an irreducible cuspidal representation of $GL_2(\mathbf{A})$. In the first integral we consider, we give a Rankin-Selberg construction for the partial L function $L^S(\pi \times \tau, E_6 \times GL_2, s)$. This is an L function of degree 54. For the second construction, let π denote an irreducible cuspidal representation of GL_6. The second L function we consider is the 30 degree L function $L^S(\pi \times \tau, \wedge^2 \times GL_2, s)$.

One of the main ingredients of these two constructions is the way that the cuspidal representation τ is built in it. Starting with τ, we build a residual representation defined on the group $GSpin_{10}(\mathbf{A})$, which we denote by θ_τ. This representation was constructed and studied in [G-H] where it was used in a slightly different way. In this paper, we build it inside an Eisenstein series defined on the group $GE_6(\mathbf{A})$. More precisely, let P denote one of the standard maximal parabolic subgroups of GE_6 whose Levi part is $GSpin_{10}$. Let $E_\tau(g, s)$ denote the Eisenstein series which is associated to the induced representation $Ind_{P(\mathbf{A})}^{GE_6(\mathbf{A})} \theta_\tau \delta_P^s$. In other words, our Eisenstein series is constructed using a residual representation which is associated with a cuspidal representation on

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$GL_2(\mathbf{A})$. As far as we know this is the first time that such a construction is used. With the above data, the first global integral we consider is given by

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi_\pi(g)\theta(g)E_\tau(g,s)dg$$

where $\theta(g)$ is a vector in the space of the minimal representation of the exceptional group E_6. The second integral is constructed using the same Eisenstein series, but it is integrated over a subgroup of GE_6.

In each of the two cases, we first unfold the global integrals, and show that they are Eulerian. This is now quite a standard procedure. Then, in each case, we carry out the unramified computations. A part of the unramified computations for the first integral involves an application of invariant theory to the Rankin-Selberg method in a way that seems to be new. Specifically, we use a theorem of D.I. Panyushev [P1] which takes as input an algebraic group G over an algebraically closed field of characteristic zero and a G-variety X, and gives as output a subgroup K and a subvariety Y such that Y has the structure of a K-variety and restriction of polynomial functions to a subvariety gives an isomorphism of the two algebras of U-invariants. (Here U is a maximal unipotent of G or K as appropriate.) This is applicable to our situation because checking that the summation that is obtained from our integral is equal to the desired L function amounts to the same thing as describing the decomposition of the symmetric algebra of the representation (ρ, V) used to form an L function, which, in turn, amounts to the same thing as describing the structure of the algebra $\mathbf{C}[V^*]^U$ of U invariant polynomial functions on the dual V^*.

In both of the cases considered in this paper $V = V_1 \otimes W$ where W is the standard two dimensional representation of $SL_2(\mathbf{C})$ and (ρ_1, V_1) is an irreducible representation of the other component in a product group. We pass from $\mathbf{C}[V^*]^U$ to $\mathbf{C}[V_1^* \times V_1^*]^U$. Then, using the same trick as in [G-H2] Section 4 of multiplying by a certain polynomial, we may simplify the summation. As described below this corresponds to passing to $\mathbf{C}[V_1^* \times C]^U$ where C is a certain cone. Each of these steps may be carried out for either of the two representations we consider. Now, suppose we take G to be $E_6(\mathbf{C})$ and X to be the variety $V_1^* \times C$ obtained in the first case. Then $K \simeq GL_6(\mathbf{C})$ and Y is isomorphic to the analogous variety obtained in the second case. The required identity then follows easily from the Littlewood-Richardson rule.

Interestingly enough, the treatment of the unramified computations corresponding to the second integral does not "factor through" this identity. This is because the summation obtained from the Rankin-Selberg integral in that case is not in a form which is amenable to being multiplied by the polynomial to pass to $\mathbf{C}[V_1^* \times C]^U$.

Finally, it should be mentioned that these L can be studied also using the Langlands Shahidi method of Whittaker coefficients of Eisenstein series. Indeed, $E_6 \times GL_2$ is a Levi subgroup of the exceptional group E_8, and E_7 has a parabolic subgroup whose Levi part is of type $A_5 \times A_1$. See [S] for details.

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with some of the algebro-geometric background for the work of Panyushev discussed in section 4.3.

2 The Global integral for $E_6 \times GL_2$

Let $G = GE_6$ denote the similitude group of the exceptional group E_6. For basic definitions and notations we refer the reader to [G]. We shall denote the six simple roots of G by α_i ordered by the Dynkin diagram as in [G]. For each root α there is a one dimensional unipotent subgroup U_α of G associated to α. We fix a family of isomorphisms $x_\alpha : \mathbf{G}_a \to U_\alpha$, where $\mathbf{G}_a$ is the additive group, as in [Gk-Se] so that the constants $N(\alpha, \beta)$ defined by

$$x_\alpha(r)x_\beta(s)x_\alpha(-r)x_\beta(-s) = x_{\alpha+\beta}(N(\alpha,\beta)rs)$$

are as in the table on p.416 of [Gk-Se]. We remark that they are all $0, 1$, or -1. (Here, we abuse notation: if $\alpha+\beta$ is not a root, there is no function $x_{\alpha+\beta}$, but $N(\alpha,\beta) = 0$ and $x_{\alpha+\beta}(0)$ is defined to be identity.) For $1 \le i \le 6$, let s_i denote the simple Weyl element of G corresponding to the simple root α_i. Let $w_i := x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1)$. Then w_i is a representative for s_i in G, and $w_i x_\beta(r) w_i^{-1} = x_{s_i\cdot\beta}(N(\alpha_i,\beta)r)$ with the same coefficients $N(\alpha,\beta)$ as above, except in the case $\beta = \pm\alpha_i$, in which case the appropriate coefficient is -1. We remark that these coefficients are all zero, 1 or -1, and that there will be no delicate points regarding these signs in the first of our two integrals. We shall denote by $w[i_1 i_2 \dots i_r]$ the product $w_{i_1} w_{i_2} \dots w_{i_r}$. As in [G] p. 104 we denote by $h(t_0, t_1, \dots, t_6)$ the maximal torus of the group G. The action of the Weyl group of G on this torus is described on that page.

Let π denote a generic cuspidal irreducible representation defined on the group $G(\mathbf{A})$, and for simplicity we shall assume that it has a trivial central character. Here $\mathbf{A}$ is the ring of adeles of some global field F. The precise definition of a generic representation is given in [G] section 1.2. For our construction we will need to work with the minimal representation of the group G. This representation was constructed and studied in [G-R-S]. The construction there is defined on the group E_6, however there are no problems to extend this definition to the similitude group. See [G-J] for a similar definition for the similitude exceptional group GE_7. In this paper we shall denote a function in the space of this representation by $\theta(g)$. Another representation we will need for our construction was defined and studied in [G-H], section 3. The representation constructed there was defined on the group $GSO_{10}(\mathbf{A})$. A similar definition holds for the group $GSpin_{10}(\mathbf{A})$. This representation depends on a cuspidal representation τ defined on $PGL_2(\mathbf{A})$, or, equivalently, defined on $GL_2(\mathbf{A})$ with trivial central character. We shall denote a vector in this space by $\theta_\tau(h)$ where $h \in GSpin_{10}(\mathbf{A})$.

To introduce the Eisenstein series we shall use, let P denote the maximal standard parabolic subgroup of G whose unipotent radical contains the one dimensional unipotent subgroup U_{α_6}. Hence, the Levi factor of P is isomorphic to $GSpin_{10} \times GL_1$. Let $E_\tau(g,s)$ denote the Eisenstein series defined on the group $G(\mathbf{A})$ from a vector $f_\tau(g,s)$ in the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})} \theta_\tau \delta_P^s$. Here s is a complex variable.

Consider the global integral

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi_\pi(g)\theta(g)E_\tau(g,s)dg \tag{1}$$

Here φ_π is a vector in the space of π and Z denotes the center of G. Fix a character ψ of the additive group $F\backslash\mathbf{A}$. We introduce a convenient shorthand for picking out particular characters of our unipotent groups. The unipotent radical $U(P)$ of P is generated by the subgroups U_α associated to those $\alpha = \sum_i n_i\alpha_i$ such that $n_6 > 0$. We put $\psi_{U(P)}(x_{\alpha_6}(r)u') = \psi(r)$, and what this indicates is the $\psi_{U(P)}$ is trivial on U_α for all α not listed. Similarly, if V denotes the maximal unipotent subgroup of the Levi of P associated to our choice of simple roots, we define a character ψ_V of V by $\psi_V(x_{\alpha_1}(r_1)x_{\alpha_2}(r_2)x_{\alpha_3}(r_3)x_{\alpha_5}(r_5)v') = \psi(r_1+r_2+r_3+r_4)$, and we define a character of the maximal unipotent subgroup $U = VU(P)$ of G by

$$\psi_U(x_{\alpha_1}(r_1)x_{\alpha_2}(r_2)x_{\alpha_3}(r_3)x_{\alpha_4}(r_4)x_{\alpha_5}(r_5)x_{\alpha_6}(r_6)u') = \psi(-r_1-r_2-r_3-r_4-r_5-r_6).$$

Then the main result of this section is
Theorem: *For $Re(s)$ large, the integral (1) is equal to*

$$\int_{Z(\mathbf{A})U_0(\mathbf{A})\backslash G(\mathbf{A})} \theta^{U(P),\psi}(g)\int_{\mathbf{A}^6} W_\pi(z_1(m_1,m_2,m_3,m_4)w[5645]g)f_\tau^{V,\psi}(z_2(l_1,l_2)w[45]g,s)dm_idl_idg,$$

where

$$z_1(m_1,m_2,m_3,m_4) = x_{-(000010)}(m_1)x_{-(000110)}(m_2)x_{-(000011)}(m_3)x_{-(000111)}(m_4)$$

and $z_2(l_1,l_2) = x_{-000110}(l_1)x_{000100}(-l_2)$, *and* U_0 *is the 34 dimensional unipotent group generated by* $\{x_{010000}(r)x_{001000}(-r)x_{010100}(s)x_{001100}(-s)\}$ *and the unipotent subgroups* U_α *corresponding to the other 32 positive roots.*
Remark: The above integral then factors as a product of local integrals. This follows from the fact that each of the functionals $\varphi_\pi \mapsto W_\pi(e), \theta \mapsto \theta^{U(P),\psi}(e), f_\tau(\cdot,s) \mapsto f_\tau^{V,\psi}(e,s)$ lies in a one-dimensional space spanned by a product of local functionals.
Proof: We unfold the global integral (1). Assume that $Re(s)$ is large. Unfolding the Eisenstein series, integral (1) equals

$$\int_{Z(\mathbf{A})P(F)\backslash G(\mathbf{A})} \varphi_\pi(g)\theta(g)f_\tau(g,s)dg. \tag{2}$$

Here $f_\tau(g,s)$ is a function in the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\theta_\tau\delta_P^s$. Next, we expand the function $\theta(g)$ along the unipotent group $U(P)$. We may sort the characters of $U(P)$ into orbits for the action of P by conjugation. By the smallness of the theta representation (see [G-R-S]) only two of them contribute to the expansion. One is the orbit consisting of the trivial character; the character $\psi_{U(P)}$ defined above is a representative for the other.

We have

$$\theta(g) = \theta^{U(P)}(g) + \sum_{\gamma\in L(F)\backslash P(F)} \theta^{U(P),\psi}(\gamma g), \tag{3}$$

where L is the stabilizer of $\psi_{U(P)}$ in P, $\theta^{U(P)}(g)$ is the constant term of $\theta(g)$ along $U(P)$, and

$$\theta^{U(P),\psi}(g)=\int\limits_{U(P)(F)\backslash U(P)(\mathbf{A})}\theta(ug)\psi_{U(P)}(u)du.$$

We plug the above expansion into (2). By the cuspidality of π, the first term contributes zero to the integral. We thus obtain

$$\int\limits_{Z(\mathbf{A})L(F)\backslash G(\mathbf{A})}\varphi_\pi(g)\theta^{U(P),\psi}(g)f_\tau(g,s)dg. \tag{4}$$

The group L may be described as follows. Let M denote the group generated by all unipotent elements $x_\alpha(r)$ where $\alpha=\sum_{i=1}^4 m_i\alpha_i$ and by all torus elements $h(t_0,t_1,\ldots,t_4,t_6^2,t_6)$. Here m_i are positive or negative. Thus $M/Z\cong GL_5$. Denote by V_1 the unipotent group generated by all $x_\alpha(r)$ where $\alpha=\sum_{i=1}^4 n_i\alpha_i+\alpha_5$. Thus $dimV_1=10$. With these notations we have $L=MV_1U(P)$.

Next we expand f_τ along V_1. The group M acts on the characters of $V_1(F\backslash\mathbf{A})$ via the exterior square representation. Thus there are three orbits. There are various ways to visualize this action. For example having noted that the representation of GL_5 is the exterior square, we may visualize its space as the space of 5×5 skew-symmetric matrices. Then the 3 orbits correspond to the three possibilities for the rank, which must be even. We also note that V_1 is contained in a Levi isomorphic to $GSpin_{10}$. For purposes of understanding unipotent subgroups there is no problem with passing to SO_{10}, and visualizing V_1 as the set of matrices $\left(\begin{smallmatrix} I & X\\ & I\end{smallmatrix}\right)$ in SO_{10}, defined as in [G-H2], in which case X is skew-symmetric about the non-standard diagonal. We may think of a character as given by a matrix A of coefficients with the same skew-symmetry property, by $\psi(\sum_{i,j}a_{ij}x_{ij})$. For this presentation it is important to remember that the action on characters is dual to the action on the matrices X. Alternatively, we may parameterize characters by the elements of the Lie algebra of the corresponding *lower* triangular parabolic of $\mathfrak{so}_{10}$ (cf. [G-R-S], p.93).

One can check that the contributions to (4) coming from the small orbits are both zero because of the cuspidality of π. Let us mention that in order to show this one has to use the invariance properties of the function $\theta^{U(P),\psi}(g)$, as described in [G-R-S] Theorem 5.4. We choose as representative for the big orbit the character ψ_1 given by $\psi_1(x_{010110}(r_1)x_{001110}(r_2)v_1')=\psi(r_1+r_2)$. Recall our convention from before: V_1 is the product of the groups U_α associated to roots $\alpha=\sum_{i=1}^6 n_i\alpha_i$ such that $n_6=0,n_5>0$, and what we mean is $\psi_1|_{U_\alpha}\equiv 1$ for all α other than the two named. Thus, integral (4) equals

$$\int\limits_{Z(\mathbf{A})M_1(F)V_1(F)U(P)(F)\backslash G(\mathbf{A})}\varphi_\pi(g)\theta^{U(P),\psi}(g)f_\tau^{V_1,\psi_1}(g,s)dg, \tag{5}$$

where $f_\tau^{V_1,\psi}(g,s)$ is defined in a similar way as $\theta^{U(P),\psi}(g)$, and M_1 is the stabilizer of the character ψ_1 inside M. It consists of a four-dimensional unipotent groups Y_1 which is the product of U_α, $\alpha\in\{(100000);(101000);(101100);(111100)\}$, and a reductive part isomorphic to GSp_4 contained in the standard Levi with simple roots $\alpha_2,\alpha_3,\alpha_4$. We expand $f_\tau^{V_1,\psi}$ along Y_1 and find that the nontrivial characters are permuted transitively by this copy of GSp_4. We choose the representative described with our convention by $\psi_2(y_1)=\psi_2(x_{100000}(r)y_1')=\psi(r)$. We denote the stabilizer of this character inside our GSp_4 by M_2. As above, the trivial orbit contributes zero by cuspidality, and we now factor the integration over the unipotent group $U_1=Y_1V_1U(P)$. Hence (5) equals

$$\int\limits_{Z(\mathbf{A})M_2(F)U_1(\mathbf{A})\backslash G(\mathbf{A})} \varphi_\pi^{U_1,\psi_3}(g)\theta^{U(P),\psi}(g)f_\tau^{V_2,\psi_2}(g,s)dg. \tag{6}$$

Here, we extended ψ_2 to a character of $V_2:=Y_1V_1$ by $\psi_2(y_1v_1)=\psi_2(y_1)\psi_1(v_1)$. Also the character ψ_3 of U_1 is given by $\psi_3(v_2u)=\psi_2^{-1}(v_2)\psi_{U(P)}^{-1}(u)$ for $v_2\in V_2$ and $u\in U(P)$.

The group M_2 consists of a reductive part generated by $U_{\pm(000100)}$, the center Z, the set of all $h(a,b)=h(ab^{-1},b^2,ab^2,ab^3,ab^4,b^2,b)$, and a three dimensional unipotent part

$$Y_2:=\{x_{010000}(r)x_{001000}(-r)x_{010100}(s)x_{001100}(-s)x_{011100}(t)\}.$$

Recall that $f_\tau(g,s)$ is a vector in the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\theta_\tau\delta_P^s$. Hence the unipotent integration that defines $f_\tau^{V_2,\psi_2}$ amounts to taking a certain Fourier coefficient of a function in the space of the representation θ_τ. In fact, it is essentially the *same* Fourier coefficient denoted by θ^{V,ψ_V} in equation (7) of [G-H2]. Repeating the arguments that appear in that paper, we first deduce that $f_\tau^{V_2,\psi_2}$ is invariant by $U_{011100}(\mathbf{A})$ on the left, and then obtain the identity

$$f_\tau^{V_2,\psi_2}(g,s)=\int_{\mathbf{A}^2} f_\tau^{V_4,\psi_4}(z_2(l_1,l_2)w[45]g,s)dl_i, \tag{7}$$

where $z_2(l_1,l_2)=x_{-(000110)}(l_1)x_{-(000100)}(l_2)$, and V_4 is the product of the subgroups U_α corresponding to all of the roots $\alpha=\sum_i n_i\alpha_i$ with $n_6=0$ and $n_5>0$ except for α_5 itself. The character ψ_4 of V_4 is given by

$$\psi_4(x_{100000}(r_1)x_{010000}(r_2)x_{001000}(r_3)v_4')=\psi(r_1+r_2+r_3).$$

We apply similar techniques to $\int_{F\backslash\mathbf{A}}\varphi_\pi^{U_1,\psi_3}(x_{011100}(r)g)dr$. Recall that U_1 is the product of the subgroups U_α corresponding to a certain set of roots. If, from this set, we delete the roots $(000010);(000110);(000011);(000111)$ and add $(001000);(001100);(011100);-(000010);-(000110)$, then the corresponding product of U_α's is again a group, which we denote U_2. By restricting ψ_3 to the common subgroup and then extending it trivially to U_2, we obtain a character of U_2 which we again denote by ψ_3. Next, let $U_3=w[5645]U_2w[5645]^{-1}$, and $\psi_5(u_5):=\psi_3(w[5645]^{-1}u_5w[5645])$. Then

$$\psi_5(x_{100000}(r_1)x_{010000}(r_2)x_{001000}(r_3)x_{000100}(r_4)u')=\psi(-r_1-r_2-r_3-r_4).$$

The identity

$$\int_{F\backslash\mathbf{A}}\varphi_\pi^{U_1,\psi_3}(x_{011100}(r)g)dr=\int_{\mathbf{A}^4}\varphi_\pi^{U_3,\psi_5}(z(m_1,m_2,m_3,m_4)w[5645]g)dm_i, \tag{8}$$

is an application of a trick, due to Jacquet-Shalika. The same trick appears on page 751 of [B-F-G] where it is explained in some detail. (The original instance, on page 218 of [J-S] is more complicated than our case here.) We now plug (7) and (8) into (6), and factor the integration over the unipotent part of M_2 to obtain

$$\int \varphi_\pi^{U_5,\psi_5}(z_1(m_1,m_2,m_3,m_4)w[5645]g)\theta^{U(P),\psi}(g)f_\tau^{V_4,\psi_4}(z_2(l_1,l_2)w[45]g,s)dl_i dm_j dg \tag{9}$$

Here the variable g is integrated over $Z(\mathbf{A})GL_2(F)U_4(\mathbf{A})\backslash G(\mathbf{A})$ and the variables l_i and m_j are integrated over $\mathbf{A}$. The group GL_2 in the integration domain is generated by the unipotent groups $x_{\pm(000100)}(r)$ and the torus $h(a,b)$ defined above. The group U_4 is the product of U_1 and the three dimensional unipotent part of M_2 described above. Finally, U_5 is the product of U_3 and this three dimensional unipotent part, which may also be described as the product of the subgroups U_α corresponding to all of the positive roots α except for α_5, α_6 and $\alpha_5+\alpha_6$.

Next we expand the function $\varphi_\pi^{U_5,\psi_5}$ along the unipotent group generated by $x_{000010}(r_1)$ and $x_{000011}(r_2)$ with points in $F\backslash\mathbf{A}$. Recall that M_2 contains a subgroup isomorphic to GL_2. After conjugation by $w[5645]$ this group acts with two orbits on this expansion. The trivial one contributes zero by cuspidality. For the other we choose the representative $x_{000010}(r_5)x_{000011}(r') \mapsto \psi(-r_5)$, and the stabilizer consists of U_{α_4} and the torus T_1 consisting of all $h(1,b)$ for $h(a,b)$ as above. Finally, we expand φ_π along $x_{000001}(r_6)$ with $r_6 \in F\backslash\mathbf{A}$. The nontrivial characters are permuted by $T_1(F)$ and we use $\psi(-r_6)$ as representative. Observe α_6 root corresponds under $w[5465]$ to α_4 and then under $w[45]$ to α_5. Hence when we factor the integration over $U_{\alpha_4}(F)Y_2(F)U_{011100}(\mathbf{A})\backslash U_{\alpha_4}(\mathbf{A})Y_2(\mathbf{A})$ we obtain $f_\tau^{V,\psi}(z_2(l_1,l_2)w[45]g)$, and the stated identity follows. ∎

3 Unramified computations for $E_6 \times GL_2$

In this section, F is a nonarchimedean local field at which all data is unramified. Denote by $I_1(W_\pi, \theta, f_{\tau,s})$ the integral

$$\int\limits_{ZU_0\backslash G}\int\limits_{F^6} W_\pi(z_1(m_1,m_2,m_3,m_4)w[5645]g)\theta^{U(P),\psi}(g)f_\tau^{V,\psi_V}(z_2(l_1,l_2)w[45]g,s)dl_i dm_j dg \tag{10}$$

Here $W_\pi, \theta^{U(P),\psi}$ and $f_\tau^{V,\psi}$ are the local functionals corresponding to the global objects of the same name appearing in the last section. Since we are at an unramified prime we may give formulae for them, as we will in due course. We shall prove

Proposition: *Assume all data is unramified. Then for $Re(s)$ large,*

$$I_1(W_\pi,\theta,f_{\tau,s}) = \frac{L(\pi\times\tau, E_6\times GL_2, 4s-3/2)}{L(\tau,12s-7/2)L(\tau,sym^3,12s-9/2)} \tag{11}$$

Proof: We have

$$z_1(m_1,m_2,m_3,m_4)w[5645] = w[5645]x_{000111}(m_1)x_{000110}(m_2)x_{000010}(m_3)x_{000011}(m_4).$$

We collapse the integrals over m_i and g to obtain one integral of g over $Z(\mathbf{A})U_0'\backslash G(\mathbf{A})$ where U_0' is the subgroup of U_0 which consists of all one dimensional unipotent root subgroups in U_0 not including the roots $000111; 000110; 000010; 000011$. Next we change variables $g \mapsto w[5645]g$. We obtain

$$\int\limits_{ZU_0''\backslash G}\int\limits_{F^2} W_\pi(g)\theta^{U(P),\psi}(w[5645]g)f_\tau^{V,\psi_V}(w[65]x_{-000111}(l_1)x_{-000110}(l_2)g,s)dl_i dg \quad (12)$$

Here $U_0'' = w[5645]U_0'w[5645]$. Let U denote the maximal unipotent radical of G. The quotient $U_0''\backslash U$ is 6 dimensional, and can be identified with the unipotent group $x_{000010}(m_1)x_{000011}(m_2)x_{000110}(m_3)x_{000111}(m_4)x_{010110}(m_5)x_{010111}(m_6)$. Observe that the functions, $W_\pi, \theta^{U(P),\psi}(w[5645]\cdot)$, and $f_\tau^{V,\psi}(w[65]\cdot, s)$ are all left-invariant by U_{010110}, U_{010111}. The only dependency of the integrand on m_5 is a $\psi(-m_5 l_2)$ that comes from the commutation relations and the equivariance of $f_\tau^{V,\psi}$ along U_{α_2}. We may now interpret the integration along l_2 as taking Fourier transform at m_5. Integrating m_5 returns the value of the original function at $m_5 = 0$. The situation with l_1 and m_6 is the same. Thus (12) equals

$$\int\limits_{ZU\backslash G}\int\limits_{F^4} W_\pi(g)\theta^{U(P),\psi}(w[5645]y(m_1,m_2,m_3,m_4)g)f_\tau^{V,\psi_V}(w[65]x_{000010}(m_1)x_{000011}(m_2)g,s)\psi(m_1)dm_i dg \quad (13)$$

Here $y(m_1,m_2,m_3,m_4) = x_{000010}(m_1)x_{000011}(m_2)x_{000110}(m_3)x_{000111}(m_4)$.
Lemma: *We may express $\theta^{U(P),\psi}$ as*

$$\theta^{U(P),\psi}(g) = \int\limits_F f_\theta(w[6]x_{000001}(r)g)\psi(r)dr$$

where f_θ is the unramified vector in the induced representation $Ind_P^G\delta_P^{1/4}$, normalized so that

$$\int\limits_F f_\theta(w[6]x_{000001}(r))\psi(r)dr = 1.$$

Proof: This follows from the construction of the minimal representation as a residue of an Eisenstein series [G-R-S], together with the fact, which we have already used in passing from the global integral to a product of local ones, that the functional $\theta \mapsto \theta^{U(P),\psi}(e)$, regarded as a $U(P)$-intertwining map from the minimal representation to the one dimensional representation of $U(P)$ by the character $\psi_{U(P)}$, is unique up to scalar and factors as a product of local functionals. Indeed, the proof of this uniqness statement is similar to the one stated in [G-J] page 41. See also [Gu] proposition 4.8.1. $\square$

Using this realization, we obtain the identity

$$\int_{F^2}\theta^{U(P),\psi}(w[5645]y(m_1,m_2,m_3,m_4)g)dm_3dm_4 =$$

$$\int\limits_{F^3} f_\theta(w[654]x_{000100}(r)x_{000110}(m_3)x_{000111}(m_4)g)\psi(r)drdm_3dm_4.$$

Next we write the Iwasawa decomposition for G in integral (13), replacing integration over $g \in ZU\backslash G$ by integration over $t \in Z\backslash T$. The appropriate measure is $\delta_{B(G)}^{-1}(t)dt$, where t is the Haar measure on T, and $\delta_{B(G)}$ is the modular quasicharacter of the Borel subgroup $B(G)$ of G. We recall the Casselman-Shalika formula, which may be formulated as follows. Let t_π be the semisimple conjugacy class in ${}^LG$ associated to the representation π. For $t \in Z\backslash T$ let $K_\pi(t) = W_\pi(t)\delta_{B(G)}^{-\frac{1}{2}}$. Let $n_i = v(\alpha_i(t))$, where v is the valuation in our local field. Consider $\sum_{i=1}^{6} n_i\varpi_i$ where $\{\varpi_i\}$ is the basis of fundamental weights for ${}^LG$ dual to the basis $\{\alpha_i\}$ of roots of G. If this weight is dominant, i.e., if all $n_i \geq 0$, then the value of $K_\pi(t)$ is the character (trace) of the irreducible finite dimensional representation of ${}^LG$ with this highest weight, evaluated at t_π. Otherwise, the value of $K_\pi(t)$ is zero. We denote this by $\chi_{E_6}(n_1, n_2, n_3, n_4, n_5, n_6)$ or simply $\chi_{E_6}(\underline{n})$, suppressing the dependence of t_π which is fixed throughout.

A similar description holds for $f_\tau^{V,\psi}(t)$. Let t_τ denote the semisimple conjugacy class in $SL_2(\mathbf{C})$ associated to τ and let $sym^2 t_\tau$ denote its image in $SL_3(\mathbf{C})$ under the symmetric square representation.

As in [G-H2], the construction of θ_τ is as a residue of an Eisenstein series [G-H2]. Let Q denote the parabolic of $GSpin_{10}$ used to define this Eisenstein series. Then, as in section 4 of [G-H2] we have

$$f_\tau^{V,\psi}(t,s) = \chi_{SL_3}(n_1,n_3)\chi_{SL_2}(n_2)\chi_{SL_2}(n_5)\delta_P^s(t)\delta_Q^{\frac{1}{3}}(t)\delta_{B(M_Q)}^{\frac{1}{2}}(t). \tag{14}$$

Here the SL_2 characters are evaluated at t_τ, and the SL_3 character is evaluated at $sym^2 t_\tau$.

We turn to the evaluation of

$$\int\limits_{F^3} f_\theta(w[654]x_{000100}(r)x_{000110}(m_3)x_{000111}(m_4)t)\psi(r)drdm_3dm_4.$$

Conjugating t to the left, making appropriate changes of variable, and using the fact that f_θ is an element of $Ind_P^G\delta_P^{\frac{1}{4}}$ we obtain a factor of $\delta_P^{\frac{1}{4}}(w[654]tw[654]^{-1})|\alpha_4^3\alpha_5^2\alpha_6(t))|$ times the integral

$$\int\limits_{F^3} f_\theta(w[654]x_{000100}(r)x_{000110}(m_3)x_{000111}(m_4))\psi(\alpha_4(t)r)drdm_3dm_4.$$

Direct computation shows that the value of this integral is the integer $n_4 + 1$.

We turn to

$$\int_{F^2} f_\tau^{V,\psi_V}(w[65]x_{000010}(m_1)x_{000011}(m_2)t,s)\psi(m_1)dm_i \tag{15}$$

We collect (14) into two pieces: let $\mu(t) = \delta_P^s(t)\delta_Q^{\frac{1}{3}}(t)\delta_{B(M_Q)}^{\frac{1}{2}}(t)$ and

$$\tilde{\chi}_{SL_2}(\underline{n}) = \chi_{SL_3}(n_1,n_3)\chi_{SL_2}(n_2)\chi_{SL_2}(n_5)$$

which we regard as a function on the weight lattice of $E_6(\mathbf{C})$. In the integral (15) we conjugate t to the left. It is convenient to introduce the notation $t' = w[65]tw[65]^{-1}$ and $f_\tau^{V,\psi}(t'; g, s) = \mu(t')^{-1} f_\tau^{V,\psi}(t'g, s)$. Then (15) equals

$$\mu(t')|\alpha_5^2\alpha_6(t)| \int_{F^2} f_\tau^{V,\psi_V}(t'; w[65]x_{000010}(m_1)x_{000011}(m_2), s)\psi(\alpha_5(t)m_1)dm_i$$

If we collect together all of the quasicharacters from all of the factors, the result is

$$\delta_P^{\frac{1}{4}}(w[654]tw[654]^{-1})|\alpha_4^3\alpha_5^4\alpha_6^2(t))|\delta_P^s(t')\delta_Q^{\frac{1}{3}}(t')\delta_{B(M_Q)}^{\frac{1}{2}}(t')\delta_{B(G)}^{-\frac{1}{2}}(t) = |\alpha_1^2\alpha_2^3\alpha_3^4\alpha_4^6\alpha_5^2\alpha_6(t)|^{4s-\frac{3}{2}}.$$

Let $x = q^{-4s-3/2}$, putting everything together, (13) equals

$$\sum_{n_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n})x^{2n_1+3n_2+4n_3+6n_4+2n_5+n_6}\int_{F^2} f_\tau^{V,\psi_V}(t'; w[65]x_{000010}(m_1)x_{000011}(m_2), s)\psi(p^{n_5}m_1)dm_i. \tag{16}$$

To compute the last integral we break the domain into four pieces depending on the Iwasawa decomposition of $w[65]x_{000010}(m_1)x_{000011}(m_2)$ We introduce a bit of notation which will help to keep the formulae short and focus attention where the action will be for the next few pages. Thus, let

$$\chi_{E_6}(\underline{n}'; a, b) = \chi_{E_6}(n_1, n_2, n_3, n_4, a, b),$$

$$\tilde{\chi}_{SL_2}(\underline{n}'; a) = \chi_{SL_3}(n_1, n_3)\chi_{SL_2}(n_2)\chi_{SL_2}(a),$$

$$\ell(\underline{n}) = 2n_1 + 3n_2 + 4n_3 + 6n_4 + 2n_5 + n_6.$$

Then the first contribution, corresponding to $|m_1|, |m_2| \le 1$ is

$$I_1 = \sum_{n_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}'; n_5, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6)x^{\ell(\underline{n})}.$$

Next we consider the case where $|m_1| \le 1$ and $|m_2| > 1$. In this case we get

$$\int\limits_{|m_2|>1} f_\tau^{V,\psi_V}(t'; \alpha_6^\vee(m_2^{-1}), s)dm_2$$

We have

$$\delta_P^s\delta_Q^{1/3}\delta_{B(GL_3)}^{1/2}\delta_{B(GSO_4)}^{1/2}(\alpha_6^\vee(m_2^{-1})) = |m_2|^{-12s+7/2}$$

The above integral is equal to

$$(1-q^{-1})\sum_{k_2=1}^{\infty}\tilde{\chi}_{SL_2}(\underline{n}'; n_6 - k_2)x^{3k_2}.$$

Since the volume of $|m_2| = q^{k_2}$ is $q^{k_2}(1-q^{-1})$, the contribution to (16) is

$$I_2 = (1-q^{-1})\sum_{n_i=0,\ n_6\ge k_2\ge 1}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}'; n_5, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6 - k_2)x^{\ell(\underline{n})+3k_2}.$$

$$= (1-q^{-1}) \sum_{n_i=0,\ k_2=0}^{\infty} (n_4+1)\chi_{E_6}(\underline{n}'; n_5, n_6+k_2+1)\tilde{\chi}_{SL_2}(\underline{n}'; n_6)x^{\ell(\underline{n})+4k_2+4}.$$

The remaining part is

$$\int\limits_F \int\limits_{|m_1|>1} f_\tau^{V,\psi_V}(t'; w[6]x_{000001}(m_2)x_{000010}(m_1^{-1})\alpha_5^\vee(m_1^{-1}), s)\psi(p^{n_5}m_1)dm_1dm_2.$$

Conjugating $x_{000010}(m_1^{-1})\alpha_5^\vee(m_1^{-1})$ to the right and changing variables, we obtain

$$\int\limits_F \int\limits_{|m_1|>1} f_\tau^{V,\psi_V}(t't_1(m_1^{-1}); w[6]x_{000001}(m_2), s)\psi(p^{n_5}m_1 + p^{n_6}m_2)|m_1|^{-12s+7/2}dm_1dm_2.$$

The character $\psi(p^{n_6}m_2)$ is obtained from the left invariant properties of the function f_τ^{V,ψ_V}. This is also equal to

$$\sum_{k_1=1}^{\infty} x^{3k_1} \int\limits_F f_\tau^{V,\psi_V}(t't_1(p^{k_1}); w_6x_{000001}(m_2), s)\psi(p^{n_6}m_2) \int\limits_{|\epsilon|=1} \psi(p^{n_5-k_1}\epsilon)d\epsilon dm_2.$$

If $|m_2| \le 1$ then we obtain

$$\sum_{k_1=1}^{n_5+1} \tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{3k_1} \int\limits_{|\epsilon|=1} \psi(p^{n_5-k_1}\epsilon)d\epsilon$$

$$= \sum_{k_1=1}^{n_5} \tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{3k_1} - q^{-1}\sum_{k_1=1}^{n_5+1} \tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{3k_1},$$

and the contribution to (16) is

$$I_3 = \sum_{n_i=0}^{\infty}\sum_{k_1=1}^{n_5} (n_4+1)\chi_{E_6}(\underline{n}'; n_5, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{\ell(\underline{n})+3k_1}$$

$$-q^{-1}\sum_{n_i=0}^{\infty}\sum_{k_1=1}^{n_5+1} (n_4+1)\chi_{E_6}(\underline{n}'; n_5, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{\ell(\underline{n})+3k_1}$$

$$= \sum_{n_i=0}^{\infty}\sum_{k_1=1}^{\infty} (n_4+1)\chi_{E_6}(\underline{n}'; n_5+k_1, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1)x^{\ell(\underline{n})+5k_1}$$

$$-q^{-1}\sum_{n_i=0}^{\infty}\sum_{k_1=0}^{\infty} (n_4+1)\chi_{E_6}(\underline{n}'; n_5+k_1, n_6)\tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1+1)x^{\ell(\underline{n})+5k_1+3}.$$

Similarly, when $|m_2| > 1$, we get

$$\sum_{k_1,k_2=1}^{\infty} x^{3k_1+3k_2}\tilde{\chi}_{SL_2}(\underline{n}'; n_6+k_1-k_2) \int\limits_{|\epsilon_i|=1} \psi(p^{n_5-k_1}\epsilon_1 + p^{n_6-k_2}\epsilon_2)d\epsilon_i,$$

and the contribution to (16) is

$$I_4=\sum_{n_i=0}^{\infty}\sum_{k_i=1}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2}$$

$$-q^{-1}\sum_{n_i,k_2=0}^{\infty}\sum_{k_1=1}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1-1)x^{\ell(\underline{n})+5k_1+4k_2+3}$$

$$-q^{-1}\sum_{n_i,k_1=0}^{\infty}\sum_{k_2=1}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1+1)x^{\ell(\underline{n})+5k_1+4k_2+3}$$

$$+q^{-2}\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2+6}.$$

Collecting all this together, (16) is equal to

$$\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2}$$

$$-q^{-1}x^3\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1+1)x^{\ell(\underline{n})+5k_1+4k_2}$$

$$-q^{-1}x^3\sum_{\substack{n_i,k_i=0\\(n_6,k_1)\neq(0,0)}}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1-1)x^{\ell(\underline{n})+5k_1+4k_2}$$

$$+q^{-2}x^6\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2}.$$

$$=(1-q^{-1}x^3\chi_{SL_2}(1)+q^{-2}x^6)\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2},$$

where $\chi_{SL_2}(1)$ denotes the character of the standard two-dimensional representation of SL_2, evaluated at the semisimple conjugacy class associated to τ, so that

$$(1-q^{-1}x^3\chi_{SL_2}(1)+q^{-2}x^6)=L(\tau,12s-7/2)^{-1}.$$

Thus the main equation (11) is reduced to

$$\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2} \tag{17}$$

$$=\frac{L(\pi\times\tau,E_6\times GL_2,4s-3/2)}{L(\tau,sym^3,12s-9/2)}$$

This identity is proved by the same method used in [G-H2]. We explain this method and state a number of lemmas from which (17) follows. These lemmas will be proved in the next section. Let $diag(\xi,\xi^{-1})$ be the conjugacy class in $SL_2(\mathbf{C})$, previously denoted t_τ, which is associated to τ. Let Γ_ν denote the irreducible finite dimensional $E_6(\mathbf{C})$-module of highest weight ν and $sym^k\Gamma_\nu$ its symmetric k-th power. Let t_π denote the semisimple conjugacy class in $E_6(\mathbf{C})$ associated to π as above. Then the right hand side of (17) is

$$(1-x^3\xi^3)(1-x^3\xi)(1-x^3\xi^{-1})(1-x^3\xi^{-3})\sum_{k=0}^{\infty}Tr(sym^k\Gamma_{\varpi_6}|t_\pi)\sum_{\ell=0}^{\infty}Tr(sym^\ell\Gamma_{\varpi_6}|t_\pi)x^{k+\ell}\xi^{k-\ell}.$$

Here $Tr(\Gamma|t)$ denotes the trace of t acting on Γ (which passes to a well-defined function on conjugacy classes).

To describe the next step we introduce the representation ring, $R[E_6]$ of $E_6(\mathbf{C})$. This is a formal ring generated by the irreducible finite dimensional representations. The trace maps $R[E_6]$ isomorphically to the ring $\mathbf{C}[T]^W$ of polynomial functions on the maximal torus which are invariant by the Weyl group. See [F-H] Section 23.2. Let $P(u)$ be the following element of $R[E_6][u]$ (i.e., a polynomial over the representation ring of E_6):

$$1-\Gamma_{\varpi_1}u^2+\Gamma_{\varpi_2}u^3-\Gamma_{\varpi_5}u^5+\Gamma_{\varpi_1+\varpi_6}u^6-\Gamma_{2\varpi_1}u^7-\Gamma_{2\varpi_6}u^8$$
$$+\Gamma_{\varpi_1+\varpi_6}u^9-\Gamma_{\varpi_3}u^{10}+\Gamma_{\varpi_2}u^{12}-\Gamma_{\varpi_6}u^{13}+u^{15}.$$

Then we have the following identity in $R[E_6][[u]]$:

Lemma:

$$P(u)\sum_{\ell=0}^{\infty}sym^n\Gamma_{\varpi_6}u^\ell=\sum_{\ell=0}^{\infty}\Gamma_{\ell\varpi_6}u^\ell.$$

Hence (17) follows from the following two assertions:

Lemma: *We have*

$$P(x\xi^{-1})\sum_{n_i,k_i=0}^{\infty}(n_4+1)\chi_{E_6}(\underline{n}';n_5+k_1,n_6+k_2)\tilde{\chi}_{SL_2}(\underline{n}';n_6+k_1)x^{\ell(\underline{n})+5k_1+4k_2}\qquad(18)$$

$$=(1-x^3\xi^{-1})(1-x^3\xi^{-1})\sum_{m_i=0}^{\infty}\chi_{E_6}(m_1,m_2,m_3,0,m_5,m_6)x^{\ell(\underline{m})}\xi^{m_2+2m_3-m_6}\frac{1-\xi^{2(m_1+1)}}{1-\xi^2}\frac{1-\xi^{2(m_6+1)}}{1-\xi^2}.$$

Lemma: *We have*

$$\sum_{k=0}Tr(sym^k\Gamma_{\varpi_6}|t_\pi)Tr(\Gamma_{\ell\varpi_6}|t_\pi)x^{k+\ell}\xi^{k-\ell}\qquad(19)$$

$$=(1-x^3\xi^3)^{-1}(1-x^3\xi)^{-1}\sum_{m_i=0}^{\infty}\chi_{E_6}(m_1,m_2,m_3,0,m_5,m_6)x^{\ell(\underline{m})}\xi^{m_2+2m_3-m_6}\frac{1-\xi^{2(m_1+1)}}{1-\xi^2}\frac{1-\xi^{2(m_6+1)}}{1-\xi^2}.$$

4 Lemmas for the Local Computations for $E_6 \times GL_2$.

4.1 On the polynomial P

To explain the existence of the polynomial P it is convenient to adopt a slightly different notation. Let t^μ denote the value of the weight μ at the element t of the torus. Let W denote the Weyl group of E_6 and l the length function defined on it. Let $A_\nu = \sum_{w \in W} t^{w\nu}$, so that the Weyl character formula expresses the character of the irreducible finite-dimensional representation with highest weight ν as $\frac{A_{\nu+\rho}}{A_\rho}$ where ρ is half the sum of the positive roots. Then

$$\sum_{n=0}^{\infty} u^n Tr(sym^n \Gamma_{\varpi_6}) = \prod_\nu (1 - t^\nu u)^{-1}$$

where the product is over all the weights of the representation Γ_{ϖ_6}. In our case, the set of weights is eqaul to the Weyl orbit of ϖ_6, so

$$A_\rho^{-1} \sum_{n=0} A_{n(\varpi_6+\rho)} u^n = A_\rho^{-1} \sum_\nu \frac{a(\nu)}{1 - t^\nu u},$$

where $a(\nu)$ is the sum of $(-1)^{l(w)} t^{w\rho}$ over only those elements of W such that $w\varpi_6 = \nu$. The polynomial P is obtained by putting this sum over a common denominator. It is clear that the degree is at most 26. What is not at once clear that the coefficient of u^n is in fact a virtual character of E_6. However, suppose we extend the map $t^\nu \mapsto A_\nu$ to an operator $\mathbf{C}[T][u] \to R[^L G][u]$ by $\mathbf{C}[u]$-linearity. Then

$$P(u) = A\left(\prod_{\nu \neq \varpi_6} (1 - t^\nu u)\right).$$

The product is over weights of the representation with highest weight ϖ_6 which are not the highest one.

Once we know that the polynomial P exists, we may find it via computer experimentation. In practice, it is better to work from both ends towards the middle, using the following insight. Computing the coefficient of u^k entails considering k-fold sums $\nu_1 + \cdots + \nu_k$ of weights that are not ϖ_6. But, the sum of all the weights in any representation is zero, so we may consider instead sums $-\nu_1 - \cdots - \nu_{n-k-1} - \varpi$. This gives an easy proof that the coefficient of u^{26} is zero (since $-\varpi_6 + \rho$ has a nontrivial stabilizer in the Weyl group) and extends to a more practical method of checking that the coefficients from 16 to 25 are also zero.

4.2 Proof of Identity (18):

We first collect the coefficient of $\chi_{E_6}(\underline{n})$ in the sum on the left hand side. An easy computation shows that

$$\sum_{k_1=0}^{n_5} \sum_{k_2=0}^{n_6} \chi_{SL_2}(n_6 + k_1 - k_2) x^{3k_1+3k_2} = x^{2n_5+n_6} \chi_{SL_3}(n_6, n_5) \begin{pmatrix} x^2 & & \\ & x^{-1}\xi^{-1} & \\ & & x^{-1}\xi^{-1} \end{pmatrix}, \quad (20)$$

That is, our sum of SL_2 characters, each of which is evaluated at $t_\tau = diag(\xi, \xi^{-1})$ as above, may be interpreted as an SL_3 character, now evaluated not at $sym^2 t_\tau$, but at the matrix specified. It follows that the coefficient of $\chi_{E_6}(\underline{n})$, which we denote by $c_{\underline{n}}(x,\xi)$, is given by

$$\chi_{SL_3}(n_1,n_3)\begin{pmatrix}\xi^2 & & \\ & 1 & \\ & & \xi^{-2}\end{pmatrix}\chi_{SL_2}(n_2)\begin{pmatrix}\xi & \\ & \xi^{-1}\end{pmatrix}\chi_{SL_3}(n_6,n_5)\begin{pmatrix}x^2 & & \\ & x^{-1}\xi^{-1} & \\ & & x^{-1}\xi^{-1}\end{pmatrix}(n_4+1)x^{\ell'(\underline{n})},$$

where now we reflect all of the semisimple conjugacy classes explicitly, and

$$\ell'(\underline{n}) = \ell(\underline{n}) + 2n_5 + n_6 = 2n_1 + 3n_2 + 4n_3 + 6n_4 + 4n_5 + 2n_6.$$

We recall a method of computing products of characters (and hence tensor products of finite dimensional representations) which is due to Brauer. Let A be as in the last section, so that the Weyl character formula is

$$\chi_{E_6}(\nu) = \frac{A_{\nu+\rho}}{A_\rho}, \tag{21}$$

for ν dominant. We may extend the definition of χ_{E_6} to all weights ν by setting it equal to the right hand side of (21). Then for $\chi_{E_6}(\lambda) = \sum_\nu m_\lambda(\nu)t^\nu$, we have

$$\chi_{E_6}(\lambda)\chi_{E_6}(\mu) = \sum_\nu m_\lambda(\nu)\chi_{E_6}(\mu+\nu).$$

Since, the weights $\mu+\nu$ appearing on the left hand side need not be dominant, we use the following facts: if $Stab_W(\eta+\rho)$ is nontrivial, then $\chi_{E_6}(\eta) = 0$, and if $w(\eta+\rho) = \eta'+\rho$, then $\chi_{E_6}(\eta) = (-1)^{l(w)}\chi_{E_6}(\eta')$.

We shall use this method to compute the products arising in the left hand side of (18), with the character from the polynomial P playing the role of $\chi_{E_6}(\lambda)$ and the weight $\underline{n}$ from the summation playing the role of μ. Thus, we obtain a sum over all weights ν which appear in any of the representations in P. There are 883 such weights, and some appear in more than one of the representations. It will be convenient to collect the terms corresponding to a specific weight, writing

$$P(u) = \sum_{\nu\in\Lambda} P_\nu(u)t^\nu,$$

where Λ is our set of 883 weights.

Let us fix a dominant weight $\underline{m}$. The coefficient of $\chi_{E_6}(\underline{m})$ on the left hand side is given by

$$\sum_{(w,\underline{n},\nu)} (-1)^{l(w)}c_{\underline{n}}(x,\xi)P_\nu(x\xi^{-1}), \tag{22}$$

where the expression $c_{\underline{n}}(x,\xi)$ was defined just after (20), and the sum is over triples $(w,\underline{n},\nu)$ with $w\in W, \nu\in\Lambda$, and $\underline{n}$ dominant satisfying

$$w(\underline{n}+\nu+\rho)-\rho = \underline{m}.$$

Thus, our claim is that this sum is described by the right hand side of (18). We may approximate (22) by

$$\sum_{\nu\in\Lambda} c_{\underline{m}-\nu}(x,\xi)P_\nu(x\xi^{-1}). \tag{23}$$

Indeed, it's easy to see that they are precisely equal when all m_i are sufficiently large. For general $\underline{m}$, they differ in two ways: (22) contains terms with $w \neq 1$, and (23) contains terms with $\underline{m}-\nu$ not dominant. Observe, however, that if $w(\underline{n}+\nu+\rho)-\rho = \underline{m}$, then $\underline{n} = w(\underline{m}-w\nu+\rho)-\rho$. We shall be able to use this fact to match up our two different sorts of discrepancies, once we make some observations about the properties satisfied by the weights ν appearing in our set Λ. Before we proceed with this, however, we record the following:

Lemma: *Let $\bar{c}_{\underline{n}}(x,\xi) = c_{\underline{n}}(x,\xi)/(n_4+1)$. Then,*

$$\bar{c}_{w[i](\underline{n}+\rho)-\rho}(x,\xi) = -\bar{c}_{\underline{n}}(x,\xi) \quad \textit{for } i \neq 4$$

$$c_{w[i](\underline{n}+\rho)-\rho}(x,\xi) = \begin{cases} -c_{\underline{n}}(x,\xi), \ i = 1,6 \\ -c_{\underline{n}}(x,\xi) - (n_i+1)\bar{c}_{\underline{n}}(x,\xi), \ i = 2,3,5. \end{cases}$$

Proof: This is immediate from the formula for c given above, and the fact that the jth entry of $w[i](\underline{n}+\rho)-\rho$ is given by

$$\begin{cases} -n_i - 2 \ \text{ if } j = i \\ n_i + n_j + 1 \ \text{ if the nodes corresponding to } \alpha_i, \alpha_j \text{ in the Dynkin diagram are connected,} \\ n_j \ \text{ otherwise.} \end{cases}$$

■

The next lemma rests on specific observations about the properties satisfied by all those weights ν that appear in the set Λ. The first is that for all such ν, we have $-2 \leq \nu_i \leq 2$ for all i.

Lemma: *Take $\underline{m}$ dominant and $\nu \in \Lambda$. Let w denote the product of all simple reflections $w[i]$ corresponding to indices i such that $\nu_i = 2$ and $m_i = 0$. (We shall see that this product may be taken in any order.) Then we have*

$$c_{\underline{m}-\nu} = (-1)^{l(w)} c_{w(\underline{m}-\nu+\rho)-\rho} - \sum_{i=2,3,5} \delta_{m_i,0}\delta_{\nu_i,2}\bar{c}_{\underline{m}-\nu} + \delta_{m_4,0}\delta_{\nu_4,2}\left(c_{\underline{m}-\nu} + c_{w[4](\underline{m}-\nu+\rho)-\rho}\right).$$

The δ's that appear here are Kronecker δ's. Furthermore, if $\underline{n} := w(\underline{m}-\nu+\rho)$ then either $\underline{n}$ is dominant, or $n_i = -1$ for some i.

Proof: We observe that if $\nu \in \Lambda$, then

- the set of indices i such that $\nu_i = 2$ has at most two elements,
- if $\nu_i = \nu_j = 2$, the nodes in the Dynkin diagram corresponding to i and j are not connected.
- with i, j as above, if nodes i and j are both connected to node k, then ν_k is strictly negative.

The assertion that the product of Weyl elements may be taken in any order follows from the second observation. The formula follows from the first and second observations and the previous lemma. For the assertion about $\underline{n}$ we require the third observation in addition to the first two. ■

Now, if $n_i = -1$ for some i, then $c_{\underline{n}}(x,\xi) = 0$, while if $\underline{n}$ is dominant, then the term $(-1)^{l(w)} c_{\underline{n}} P_\nu$ is precisely the contribution to the coefficient of $\chi_{E_6}(\underline{m})$ in (22) corresponding to the triple $(w, \underline{n}, w\nu)$. (Here we use that $w = w^{-1}$ and that $P_{w\nu} = P_\nu$.) Furthermore, all of the observations above remain true if 2 is replaced by -2, and from this it follows that every triple $(w, \underline{n}, \nu')$ which provides a nonzero contribution to (22) is accounted for. That is, (23) minus (22) equals

$$\sum_{i=2,3,5} \delta_{m_i,0} \sum_{\nu_i=2} P_\nu(x\xi^{-1}) \bar{c}_{\underline{m}-\nu}(x,\xi) - \delta_{m_4,0} \sum_{\nu:\nu_4=2} P_\nu(x\xi^{-1}) \left(\bar{c}_{\underline{m}-\nu}(x,\xi) - \bar{c}_{w[4](\underline{m}-\nu+\rho)-\rho}(x,\xi) \right)$$

(In the last sum we have used the fact that if $n_4 = -2$, then $c_{\underline{n}} = -\bar{c}_{\underline{n}}$ and $c_{w[4](\underline{n}} = \bar{c}_{w[4](\underline{n}}$.) At this point our main assertion follows from the following six identities:

$$\sum_{\nu\in\Lambda} \bar{c}_{\underline{m}-\nu}(x,\xi) P_\nu(x\xi^{-1}) = 0 \qquad \forall\, \underline{m}$$

$$\sum_{\nu\in\Lambda} \nu_4 \bar{c}_{\underline{m}-\nu}(x,\xi) P_\nu(x\xi^{-1}) = 0 \qquad \forall\, \underline{m}$$

$$\sum_{\nu\in\Lambda:\nu_i=2} \bar{c}_{\underline{m}-\nu}(x,\xi) P_\nu(x\xi^{-1}) = 0 \qquad \forall\, \underline{m} : m_i = 0, \qquad i = 2,3,5$$

$$\sum_{\nu\in\Lambda:\nu_4=2} P_\nu(x\xi^{-1}) \left(\bar{c}_{\underline{m}-\nu}(x,\xi) - \bar{c}_{w[4](\underline{m}-\nu+\rho)-\rho}(x,\xi) \right) = (1 - x^3\xi^{-3})(1 - x^3\xi^{-1}) \times$$
$$\times x^{\ell(\underline{m})} \xi^{m_2+2m_3-m_6} \frac{1-\xi^{2(m_1+1)}}{1-\xi^2} \frac{1-\xi^{2(m_6+1)}}{1-\xi^2} \qquad \forall\, \underline{m} :\ m_4 = 0.$$

Now, each of these identities may be rewritten as a single identity of polynomials by introducing auxiliary variables. Indeed, let $C_\nu(x, \xi, Y_1, Y_2, Y_3, Y_5, Y_6, X_5, X_6)$ equal

$$\begin{vmatrix} Y_1 Y_3 \xi^{-2\nu_1-2\nu_3} & 1 & Y_1^{-1} Y_3^{-1} \xi^{2\nu_1+2\nu_3} \\ Y_1 \xi^{2\nu_1} & 1 & Y_1^{-1} \xi^{-2\nu_1} \\ 1 & 1 & 1 \end{vmatrix} (Y_2 y^{-\nu_2} - Y_2^{-1} y^{\nu_2}) x^{-\ell'(\nu)} \times$$
$$\times \begin{vmatrix} X_5^2 X_6^2 x^{-2\nu_5-2\nu_6} & X_5^{-1} X_6^{-1} Y_5 Y_6 x^{\nu_5+\nu_6} \xi^{-\nu_5-\nu_6} & X_5^{-1} X_6^{-1} Y_5^{-1} Y_6^{-1} x^{\nu_5+\nu_6} \xi^{\nu_5+\nu_6} \\ X_5^2 x^{-2\nu_5} & X_5^{-1} Y_5 x^{\nu_5} \xi^{-\nu_5} & X_5^{-1} Y_5^{-1} x^{\nu_5} \xi^{\nu_5} \\ 1 & 1 & 1 \end{vmatrix},$$

where $|\cdot|$ denotes a determinant. Then

$$\bar{c}_{\underline{m}-\nu}(x,\xi) = x^{\ell'(\underline{m})} \frac{C_\nu(x, \xi, \xi^{2m_1+2}, \xi^{m_2+1}, \xi^{2m_3+2}, \xi^{m_5+1}, \xi^{m_6+1}, x^{m_5+1}, x^{m_6+1})}{C_{\underline{0}}(x, \xi, \xi^2, \xi, \xi^2, \xi, \xi, x, x)}.$$

Here, $\underline{0} = (0,0,0,0,0,0)$. Our first identity is equivalent to

$$\sum_{\nu\in\Lambda} P_\nu(x\xi^{-1}) C_\nu(x, \xi, \underline{Y}, \underline{X}) = 0.$$

With 883 terms, this is far too large to check by hand, but it is straightforward to verify by computer. The others are similar. We give some details for the last identity, as that is the only case in which the right hand side is nonzero. Let C_ν be as above and define C'_ν to be the expression obtained by replacing ν_i by $\nu_i + 1$ for $i = 2, 3, 5$ throughout, and multiplying by x^{12}. (So that $x^{-\ell'(\nu)}$ becomes $x^{-\ell'(\nu)-11+12} = x^{-\ell'(\nu)+1}$.) Then

$$\bar{c}_{w[4](\underline{m}-\nu+\rho)-\rho} = x^{\ell(\underline{m})} \frac{C'_\nu(x, \xi, \xi^{2m_1+2}, \xi^{m_2+1}, \xi^{2m_3+2}, \xi^{m_5+1}, \xi^{m_6+1}, x^{m_5+1}, x^{m_6+1})}{C_{\underline{0}}(x, \xi, \xi^2, \xi, \xi^2, \xi, \xi, x, x)}.$$

What is to be checked is

$$\sum_{\nu:\nu_4=2} P_\nu(x\xi^{-1})(C_\nu(x, \xi, \underline{Y}, \underline{X}) - C'_\nu(x, \xi, \underline{Y}, \underline{X}) = (\xi^2 - \xi^{-2})(\xi - \xi^{-1})(x^2 - x^{-1}\xi)(x^2 - x^{-1}\xi^{-1}) \times$$

$$\times (x^{-1}\xi - x^{-1}\xi^{-1})(1 - x^3\xi^{-1})(1 - x^3\xi^{-3})\xi^{-4} Y_2 Y_3 Y_6^{-1} X_5^{-2} X_6^{-1} X (Y_1 - 1)(Y_6^2 - 1).$$

4.3 Proof of Identity (19)

We first reduce (19) to the analogous statement corresponding to the next representation in our tower using work of D.I. Panyushev. To facilitate reference to the relevant papers, we adopt some of the notation of [P1]. Of note: in this section K is not the maximal compact, and superscript S means the points of a variety fixed by a certain subgroup S introduced below, rather than product over all places not in a finite set. We first reformulate the problem using an observation which is due to Littelmann [L]. It will be convenient to formulate things initially in some generality.

We begin with a reductive algebraic group G defined over $\mathbf{C}$, for which we have fixed a torus, T, and a $\mathbf{Z}$-basis of fundamental weights ϖ_i for the lattice of weights. We work in the category of G-varieties. Let V_ϖ denote affine space of the appropriate dimension equipped with an action of G by the irreducible representation with highest weight ϖ. Then the full symmetric algebra of V_ϖ may be identified with the algebra of polynomial functions on the G-module dual to V_ϖ, which we denote by V_ϖ^*. We also denote the highest weight of this G-module by ϖ^* so that $V_\varpi^* = V_{\varpi^*}$.

Now let ϖ be a fundamental weight. Under this interpretation, the subalgebra

$$\bigoplus_\ell \Gamma_{\ell\varpi} \subset \mathbf{C}[V_{\varpi^*}]$$

may be identified as the algebra of polynomial functions on the cone

$$C_{\varpi^*} := \{\lambda g \cdot v_H : \lambda \in \mathbf{C}, g \in G\},$$

where v_H is any highest weight vector in V_{ϖ^*}.

Consider the algebra $\mathbf{C}[V_{\varpi^*} \times C_{\varpi^*}]$. This algebra has a natural bi-grading corresponding to degree over V and over C individually. The (k, ℓ)-graded piece is precisely $Sym^k(\Gamma_\varpi) \otimes \Gamma_{\ell\varpi}$. The subalgebra $\mathbf{C}[V_{\varpi^*} \times C_{\varpi^*}]^U$ is preserved by the action of T and so it makes sense to speak of elements of this algebra having a weight. Indeed, the highest-weight vectors of irreducible components of $\mathbf{C}[V_{\varpi^*} \times C_{\varpi^*}]$ are all in the subalgebra of U-invariants, and describing its structure is equivalent to describing the decomposition of $Sym^k(\Gamma_\varpi) \otimes \Gamma_{\ell\varpi}$ into irreducibles for arbitrary k, ℓ.

In the case at hand, identity (19) amounts to the following description of the structure of $\mathbf{C}[V_{\varpi_1} \times C_{\varpi_1}]^U$: it is a polynomial algebra generated by 9 elements for which the triples (degree over V_{ϖ_1}, degree over C_{ϖ_1}; weight) are as follows:

$$(1,0;\varpi_6),(2,0;\varpi_1),(3,0;0),(0,1;\varpi_6),(1,1;\varpi_5),(1,1;\varpi_1),(2,1;0),(2,1;\varpi_2),(3,1;\varpi_3).$$

In this section, we relate this assertion to its analog for the next representation in our tower. That is, we prove

Lemma: *Let U and $\bar{U}$ denote the maximal unipotent subgroups of E_6 and SL_6 respectively. Let ϖ_1 (resp. ϖ_4') denote the first (resp. fourth) fundamental weight of E_6 (resp. SL_6) defined relative to U (resp. $\bar{U}$). Then we have*

$$\mathbf{C}[V_{\varpi_4'} \times C_{\varpi_4'}]^{\bar{U}} \simeq \mathbf{C}[V_{\varpi_1} \times C_{\varpi_1}]^U.$$

Remark: Clearly, the assertion remains true if we replace ϖ_1 by ϖ_6 and/or ϖ_4' by ϖ_2'.
Proof: This is proved by applying Theorem 1.8 of [P1] to $X = V_{\varpi_1} \times C_{\varpi_1}$. There are several intermediate steps. We sketch the general procedure and give the specifics of our situation. We consider the action of G on the product of X and a sort of "dual" G-variety X^*. In our case G is E_6 and X^* is simply $V_{\varpi_6} \times C_{\varpi_6}$. We need to compute a certain subgroup S and a closely related sub-semigroup $\mathcal{T}(X)$ of the semigroup of dominant weights. The group S is the stabilizer of a point in general position for the action of G on $X \times X^*$ which is "canonical," as defined on p.660 of [P1].

We first check that the stabilizer of a point in general position for the action of E_6 on $X \times X^* = V_{\varpi_1} \times V_{\varpi_6} \times C_{\varpi_1} \times C_{\varpi_6}$, is isomorphic to SL_2. By Lemmas 1 and 2 of [P2], this reduces to the same assertion about the stabilizer of a point in general position for the action of $Spin_{10}(\mathbf{C})$ on $V_{\varpi_1} \times V_{\varpi_6}$, which, may be computed by the procedure laid out explicitly in [P3]. Once we know that the group S is isomorphic to SL_2, it is immediate from the relations (3) and (4) between S and $\mathcal{T}(X)$ given on pp. 659-60 of [P1] that the unique root of S is α_4 and $\mathcal{T}(X)$ is the semigroup generated by $\{\varpi_i : i \neq 4\}$.

Next we need to find a subgroup K such that the identity component of the normalizer of S is K times the identity component of S. There is an element of the Weyl group that conjugates α_4 to the longest root, taking S to a conjugate S'. The identity component of the normalizer of S' is the product of S' and the standard Levi of E_6 isomorphic to GL_6. For K, we take the corresponding conjugate of this GL_6.

Observe that the S-fixed subspace $V_{\varpi_1}^S$ is a 15-dimensional K-module. We identify K with GL_6 in such a way that its highest weight is ϖ_4'. Then $C_{\varpi_4'}$ is identified with

$$\{\lambda k \cdot v_H : \lambda \in \mathbf{C}, k \in K\},$$

which is certainly contained in $C_{\varpi_1}^S = C_{\varpi_1'} \cap V_{\varpi_4'}$. Thus $V_{\varpi_4'} \times C_{\varpi_4'}$ is identified with a subvariety of $(V_{\varpi_1} \times C_{\varpi_1})^S$. To use Panyushev's result, we must check that $V_{\varpi_4'} \times C_{\varpi_4'}$ is a principal component of $(V_{\varpi_1} \times C_{\varpi_1})^S$, as defined on pp.658-9 of [P1]. The isomorphism is then given by restriction of functions. In fact, $V_{\varpi_4'} \times C_{\varpi_4'} = (V_{\varpi_1} \times C_{\varpi_1})^S$, as follows

from
Lemma: *We have*

$$Gv_H \cap V^S = Kv_H.$$

Proof: Let P_{ϖ_1} denote the maximal standard parabolic subgroup whose unipotent radical contains the root subgroup associated to the root α_1. The action of P_{ϖ_1} preserves the one dimensional subspace spanned by v_H.

We fix a set of representatives for the Weyl group of K in K, and expand it to a set of representatives for the Weyl group of G. We then fix a set $\dot{W}$ of representatives for $W/(W \cap P_{\varpi})$ such that every coset which contains an element of K is represented by one.

We may write an arbitrary element of g as uwp, with $p \in P_{\varpi_1}$, $w \in \dot{W}$, and $u \in U^w := U \cap w\bar{U}w^{-1}$. The action of p at most scales v_H, so we may assume that $p = 1$. Then wv_H is some vector v_λ on which T acts by the weight λ. This vector is in $V^S = V_{\varpi_4'}$ iff λ is one of the weights appearing in the irreducible representation of K on this space, in which case $w \in K$. But then the group U^w is contained in K as well, and so $gv_H \in Kv_H$. Now, suppose $v_\lambda \notin V^S$. The action of u is unipotent, so when $g \cdot v_H$ is written in terms of a basis of weight vectors including v_λ, the coefficient of v_λ is 1, and hence $g \cdot v_H$ is not in V^S. ■

In order to complete the proof of (19), we need to show that $\mathbf{C}[V_{\varpi_4} \times C_{\varpi_4}]^U$ is a polynomial algebra generated by nine elements for which the (degree over V, degree over C; weight) triples are:

$(1,0;\varpi_2)(0,1;\varpi_2)(2,0;\varpi_4)(1,1;\varpi_1+\varpi_3)(1,1;\varpi_4)(3,0;0),(2,1;0),(2,1;\varpi_1+\varpi_5),(3,1;\varpi_3+\varpi_5).$

This is equivalent to:

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Tr(sym^k\Gamma_{\varpi_2}) \otimes Tr(\Gamma_{l\varpi_2})x^k y^l \tag{24}$$

$$= \sum_{k_i=0}^{\infty} x^{k_1+2k_3+k_4+k_5+3k_6+2k_7+2k_8+3k_9} y^{k_2+k_4+k_5+k_7+k_8+k_9} \chi_{SL_6}(k_4+k_8, k_1+k_2, k_4+k_9, k_3+k_5, k_8+k_9),$$

where $\chi_{SL_6}(n_1, \ldots, n_5)$ denotes the character of the irreducible finite-dimensional representation of SL_6 with highest weight $n_1\varpi_1 + \cdots + n_5\varpi_5$. We omit the proof of (24). It is similar to, but much easier than, the Littlewood-Richardson computation that is done in 7.1.

5 The Global Integral for $\wedge^2 GL_6 \times GL_2$

We continue to use the notations of Section 1. Let Q denote the maximal parabolic subgroup of $G = GE_6$ with Levi part isomorphic to $GL_1 \times GL_6$. The unipotent radical of Q, denoted by $U(Q)$, is the product of the subgroups U_α associated to those positive roots $\alpha = \sum n_i\alpha_i$ such that $n_2 > 0$. We consider the subgroup H of the Levi of Q generated by $\{x_{\pm\alpha_i}(r) : i \neq 2\}$ and the subgroup of the maximal torus of G

consisting of elements of the form $h(t_2^{-1},t_1,t_2,t_3,t_4,t_5,t_6)$. This group is isomorphic to the subgroup of GL_6 consisting of elements with square determinant. The isomorphism may be described concretely as follows. We identify $x_{\alpha_1}(r)$ with $I+e_{1,2}r$. For each of the other roots $\alpha\in\{\pm\alpha_i : i\neq 2\}$ we identify $x_\alpha(r)$ with $I+e_{i,j}r$ for some i,j. The pair (i,j) is determined for all such α by the choice we made for α_1. This pins down a specific isomorphism between SL_6 and the subgroup generated by $\{x_{\pm\alpha_i}(r) : i\neq 2\}$. It follows from the discussion on p. 410 of [Gk-Se] that this isomorphism identifies $x_\alpha(r)$ with a matrix of the form $I+re_{i,j}$ for the remaining roots as well. We obtain a mapping of the torus of G to GL_6 by looking at the action on the root subgroups U_α. This mapping is

$$h(t_0,t_1,\ldots,t_6)\mapsto diag(t_1t_0,t_1^{-1}t_3,t_3^{-1}t_4,t_2t_4^{-1}t_5,t_2t_5^{-1}t_6,t_2t_6^{-1}). \tag{25}$$

In particular, the image of $h(t_2^{-1},t_1,t_2,t_3,t_4,t_5,t_6)$ is $diag(t_1t_2^{-1},t_1^{-1}t_3,t_3^{-1}t_4,t_2t_4^{-1}t_5,t_2t_5^{-1}t_6,t_2t_6^{-1})$.

An element of the center of the Levi of Q is of the form $h(t_2^3t_6^{-6},t_2^{-2}t_6^5,t_2,t_2^{-1}t_6^4,t_6^3,t_6^2,t_6)$. This torus contains the center of G, denoted by Z, given by the relations $t_2=a^3$ and $t_6=a^2$. The group H clearly contains Z. Using the action of the torus on the simple roots in G, and the commutation relations among the subgroups U_α, one can easily check that the group H commutes with the one dimensional unipotent subgroup U_{122321}. This root is the highest root in G.

Let φ_π denote a cuspform, in a generic cuspidal representation π defined on the group $GL_6(\mathbf{A})$. We shall assume that π has a trivial central character. The global integral we consider is given by

$$\int\limits_{Z(\mathbf{A})H(F)\backslash H(\mathbf{A})}\ \int\limits_{U(Q)(F)\backslash U(Q)(\mathbf{A})}\ \int\limits_{(F\backslash\mathbf{A})}\theta(ux_{122321}(r_1)h)\psi(r_1)dr_1\varphi_\pi(h)E_\tau(uh,s)dudh \tag{26}$$

The functions θ and E_τ were defined in Section 1. Since H commutes with $x_{122321}(r)$, the above integral is well defined.

In this section, we prove the following:

Theorem *Let W_π be the function in the Whittaker model of π corresponding to ϕ_π, and let N be the unipotent subgroup of GL_6 defined by*

$$N=\left\{\begin{pmatrix}1&x_1&x_2&y&*&*\\ &1&m&x_2&*&*\\ &&1&-x_1&*&*\\ &&&1&r_1&*\\ &&&&1&r_2\\ &&&&&1\end{pmatrix}\right\}$$

Then, the global integral (26) *is equal to*

$$\int\limits_{Z(\mathbf{A})N(\mathbf{A})\backslash H(\mathbf{A})}\ \int\limits_{U_1(Q)(\mathbf{A})} W_\pi(h)\theta^{U(P),\psi}(\widetilde{w}_0x_{111110}(1)x_{011210}(1)u_1h)\times \tag{27}$$

$$\int\limits_{\mathbf{A}^2} f_\tau^{V,\psi}(z_2(m_1,m_2)w[45]w_0u_1h,s)dm_idu_1dh.$$

Here, z_2 and $f_\tau^{V,\psi}$ are defined as in section 2.
Proof: To unfold this integral, we start by unfolding the Eisenstein series. We need to consider the space $P\backslash G/UH$. It is not hard to check that this space has three representatives given by $e, w[6542]$ and $w_0 = w[65423143542]$. The contribution to (26) from w_0 is given by

$$\int\limits_{Z(\mathbf{A})P_H(F)\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} \int\limits_{U_2(Q)(F)\backslash U_2(Q)(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \varphi_\pi(h)\theta(u_2 x_{122321}(r_1)u_1 h)\times \tag{28}$$
$$f_\tau(w_0 u_2 u_1 h, s)\psi(r_1)dr_1 du_2 du_1 dh,$$

where $P_H = H \cap w_0^{-1}Pw_0, U_2(Q) = U(Q) \cap w_0^{-1}Pw_0$, and $U_1(Q) = U_2(Q)\backslash U(Q)$. We may identify this quotient with the group $U(Q)\cap w_0^{-1}U(\overline{P})w_0$, where $\overline{P}$ is the parabolic subgroup opposite to P. The group $U_2(Q)$ is the product of U_α for the following roots:

$$010111, 011111, 111111, 011211, 111211, 011221, 112211, 111221, 112221, 112321. \tag{29}$$

Similar contributions corresponding to $w = e$ and $w[6542]$, vanish because $wU_{122321}w^{-1}$ is in the group $U(P)$ which leaves f_τ invariant. Thus (26) is equal to (28).
Lemma: *We have*

$$\int\limits_{(F\backslash \mathbf{A})} \theta(x_{122321}(r_1)g)\psi(r_1)dr_1 = \sum_{\delta\in F^{10}} \theta^{U(P),\psi}(\tilde{w}_0 z(\delta)g),$$

where $\tilde{w}_0 = w[5431243542]$ and

$$z(\delta) = x_{010000}(\delta_1)x_{010100}(\delta_2)x_{011100}(\delta_3)x_{010110}(\delta_4)x_{111100}(\delta_5)x_{011110}(\delta_6)\times$$
$$x_{111110}(\delta_7)x_{011210}(\delta_8)x_{111210}(\delta_9)x_{112210}(\delta_{10}).$$

Proof: We plug in the Fourier expansion (3) in the equivalent form

$$\theta(g) = \theta^{U(P)}(g) + \sum_{\epsilon\in F^*} \sum_{\gamma\in S(1,2,3,4)(F)\backslash M(P)} \theta^{U(P),\psi}(\alpha_5^\vee(\epsilon)\gamma g).$$

Here $M(P)$ is the Levi of P, and $S(1,2,3,4)$ is the maximal parabolic of this Levi whose unipotent radical contains U_{α_5}. For each coset in $S(1,2,3,4)(F)\backslash M(P)$ we choose a representative of the form $w\zeta$ where w is (the representative in G of) the element of minimal length in one of the cosets of $(W \cap S(1,2,3,4))\backslash(W \cap M(P))$ and ζ is an element of the maximal unipotent subgroup $V = U\cap M(P)$ corresponding to our choice of positive roots, with the property that $w\zeta w^{-1}$ is contained in the maximal unipotent $\overline{V}$ opposite to V. Thus we consider integrals of the form

$$\int_{F\backslash \mathbf{A}} \theta^{U(P),\psi}(\alpha_5^\vee(\epsilon)w\zeta x_{122321}(r_1)g), \tag{30}$$

with w and ζ as above. For all such w, the root $w \cdot \alpha_{122321}$ is positive. We conjugate $x_{122321}(r_1)$ to the left. If $w \cdot \alpha_{122321} \neq \alpha_6$ then $\theta^{U(P),\psi}$ is left-invariant by

$wx_{122321}(r_1)w^{-1}$ and we get zero. The unique element w with the property required above such that $w \cdot \alpha_{122321} = \alpha_6$ is $\tilde{w}_0$. Now, $\tilde{w}_0 x_{122321}(r_1)\tilde{w}_0^{-1} = x_{000001}(r_1)$, and $\theta^{U(P),\psi}(x_{000001}(r)g) = \psi(-r)\theta^{U(P),\psi}(g)$. Hence (30) is equal to $\theta^{U(P),\psi}(\alpha_5^\vee(\epsilon)\tilde{w}_0\zeta g)\int_{F\backslash\mathbf{A}}\psi(r_1(1-\epsilon))dr_1$. This integral is zero unless $\epsilon = 1$.

Finally, the function z is an explicit parameterization of $V \cap \tilde{w}_0^{-1}\overline{V}\tilde{w}_0$. □

The group P_H is a maximal unipotent subgroup of H. It's Levi M_H, contains the roots $\pm\alpha_i$ for $i = 1, 3, 4, 5$. It acts on $\{z(\delta) : \delta \in F^{10}\}$ with three orbits. (This action is essentially the same as the action of the group M on the characters of V_1 described after equation (4).) For δ in either of the two small orbits, $\theta^{U(P),\psi}(\tilde{w}_0 z(\delta)g)$ is invariant, as a function of g, by the unipotent radical of the group P_H. By the cuspidality of φ_π, these orbits contribute zero to our integral. We choose $z_0 := x_{111110}(1)x_{011210}(1)$ as a representative of the big orbit. The stabilizer in P_H consists of a reductive part

$$< x_{\pm\alpha_1}(r_1)x_{\pm\alpha_4}(-r_1), x_{\pm\alpha_3}(r_2), h(t_2^{-1}, t_1, t_2, t_3, t_4, t_1^{-2}t_4^2, t_1^{-1}t_4) > \simeq GSp_4 \times GL_1$$

and a 9 dimensional unipotent part L. This group L is the product of the unipotent radical L_1 of P_H, which corresponds to the five roots

$$\{000001, 000011, 000111, 001111, 101111\} \tag{31}$$

and another subgroup L_2 which corresponds to the four roots $\{000010, 000110, 001110, 101110\}$. The correspondence between a subgroup and a set of roots is that the subgroup is the product of the groups U_α for the roots listed. We shall continue to use this notion, keeping in mind that not all subsets correspond to groups and not all unipotent subgroups can be described in this way.

Since we have fixed an identification of H with a subgroup of GL_6, we can also describe this stabilizer in terms of matrices as:

$$\left\{ \begin{pmatrix} g & x_1 & x_2 \\ & d & y \\ & & d \end{pmatrix} : g \in GSp_4, d \in GL_1, x_1, x_2 \in Mat_{4\times 1}, y \in Mat_{1\times 1} \right\},$$

and L_1 and L_2 as

$$L_1 = \left\{ \begin{pmatrix} I_5 & l_1' \\ & 1 \end{pmatrix} : l_1' \in Mat_{5\times 1} \right\} \quad L_2 = \left\{ \begin{pmatrix} I_4 & l_2' & \\ & 1 & \\ & & 1 \end{pmatrix} : l_2' \in Mat_{4\times 1} \right\}$$

If we identify GSp_4 with its image above, we may now write (28) as

$$\int\limits_{Z(\mathbf{A})GSp_4(F)L(F)\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} \int\limits_{U_2(Q)(F)\backslash U_2(Q)(\mathbf{A})} \varphi_\pi(h)\theta^{U(P),\psi}(\tilde{w}_0 x_{111110}(1)x_{011210}(1)u_2u_1h)\times \tag{32}$$
$$f_\tau(w_0u_2u_1h, s)du_2du_1dh$$

Lemma: *The function $\theta^{U(P),\psi}(\tilde{w}_0 z_0 g)$ has the following left-equivariance properties:*

$$\theta^{U(P),\psi}(\tilde{w}_0 z_0 u_2 g) = \psi_{U2(Q)}(u_2)\theta^{U(P),\psi}(\tilde{w}_0 z_0 g),$$

$$\theta^{U(P),\psi}(\tilde{w}_0 z_0 l_1 g) = \psi_{L_1}(l_1)\theta^{U(P),\psi}(\tilde{w}_0 z_0 g),$$

where the characters $\psi_{U_2(Q)}, \psi_L$ are defined, using the shorthand introduced after (1), *by*

$$\psi_{U_2(Q)}(x_{011211}(r_1)x_{111111}(r_2)u_2') = \psi(-r_1 - r_2)$$

$$\psi_{L_1}(x_{000001}(r)l_1') = \psi(-r).$$

Proof: As noted in the proof of the last Lemma, $\theta^{U(P),\psi}(\tilde{w}_0 x_{122321}(r)g) = \psi(-r)\theta^{U(P),\psi}(\tilde{w}_0 g)$. On the other hand, if α is any positive root other than α_6, or any negative root in the span of $\{-\alpha_i : i = 1,2,3,4\}$, then $\theta^{U(P),\psi}$ is left-invariant by U_α. (See [G-R-S] Theorem 5.4.) ¿From this we deduce that the function $\theta^{U(P),\psi}(\tilde{w}_0 g)$ is left-invariant by U_α for all α listed in (29) and (31) above. Employing the notation $[a,b] = aba^{-1}b^{-1}$ for the commutator, we note that $\theta^{U(P),\psi}(\tilde{w}_0 g)$ is also left-invariant by $[z_0, x_\alpha(r)]$ for α as above, with only the following exceptions:

$$[z_0, x_{011211}(r_1)x_{111111}(r_2)] = x_{122321}(r_1 + r_2)$$

$$[z_0, x_{000001}(r)] = x_{011211}(r)x_{111111}(r)x_{122321}(r),$$

which account for $\psi_{U_2(Q)}$ and ψ_L respectively. □

Let $U_{1,2,3,4}$ denote the product of the groups U_α corresponding to the ten roots $\sum_{i=1}^4 n_i\alpha_i + \alpha_5$. It is the unipotent radical of the group $S(1,2,3,4)$ defined earlier. We recall that this group was a standard maximal parabolic not of G, but of the Levi $M(P)$ of P. It is not hard to check that $w_0 U_2(Q) w_0^{-1} = U_{1,2,3,4}$. If $\psi_{U_{1,2,3,4}}(u) := \psi_{U_2(Q)}(w_0^{-1} u w_0)$, then $\psi_{U_{1,2,3,4}}(x_{001110}(r_1)x_{010110}(r_2)u) = \psi(r_1 + r_2)$.

¿From all this we deduce that (32) equals

$$\int\limits_{Z(\mathbf{A})GSp_4(F)L_2(F)L_1(\mathbf{A})\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} \varphi_\pi^{L_1,\psi}(h)\theta^{U(P),\psi}(\tilde{w}_0 z_0 u_1 h) f_\tau^{U_{1,2,3,4},\psi}(w_0 u_1 h, s)\, du_1\, dh \tag{33}$$

Here

$$\varphi_\pi^{L_1,\psi}(h) = \int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \varphi_\pi(l_1 h)\psi_{L_1}(l_1)\, dl_1,$$

and

$$f_\tau^{U_{1,2,3,4},\psi}(g,s) = \int\limits_{U_{1,2,3,4}(F)\backslash U_{1,2,3,4}(\mathbf{A})} f_\tau(ug,s)\psi_{U_{1,2,3,4}}(u)\, du.$$

Next we consider the Fourier expansion of $\varphi_\pi^{L_1,\psi}(h)$ along $L_2(F)\backslash L_2(\mathbf{A})$. The group $GSp_4(F)$ acts on this expansion with two orbits. The trivial orbit contributes zero by cuspidality. Thus we have

$$\varphi_\pi^{L_1,\psi}(h) = \sum_{\gamma \in R_1(F)\backslash GSp_4(F)} \varphi_\pi^{L,\psi}(\gamma h)$$

Here

$$\varphi_\pi^{L,\psi}(h) = \int\limits_{L(F)\backslash L(\mathbf{A})} \varphi_\pi(lh)\psi_L(l)dl$$

is defined using the character $\psi_L(x_{000001}(r_1)x_{000010}(r_2)l') = \psi(-r_1 - r_2)$. This may also be described via the identification of H with a subgroup of GL_6 as $\psi_L(l) = \psi(-l_{4,5} - l_{5,6})$. We remark that one of the minus signs is dictated by ψ_{L_1} above and the other indicates our choice of a point in the open orbit here.

The subgroup R_1 of GSp_4 is the stabilizer of ψ_l inside GSp_4 and in matrices it is given by

$$R_1 = GL_2L_3 = \left\{ \begin{pmatrix} detg & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y \\ & 1 & & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} : g \in GL_2 \right\}$$

Returning to (33), we first plug in the expansion along L_2 and collapse summation with integration. Then we factor the integration over $L_2(F)\backslash L_2(\mathbf{A})$. We have

$$\theta^{U(P),\psi}(\widetilde{w}_0x_{111110}(1)x_{011210}(1)u_1l_2h) = \theta^{U(P),\psi}(\widetilde{w}_0x_{111110}(1)x_{011210}(1)u_1h).$$

Hence, (33) equals

$$\int\limits_{Z(\mathbf{A})R_1(F)L(\mathbf{A})\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} \varphi_\pi^{L,\psi}(h)\theta^{U(P),\psi}(\widetilde{w}_0x_{111110}(1)x_{011210}(1)u_1h)f_\tau^{V_1,\psi}(w_0u_1h,s)du_1dh, \tag{34}$$

where

$$f_\tau^{V_1,\psi}(w_0u_1h,s) = \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} f_\tau(vw_0u_1h,s)\psi_{V_1}(v)dv.$$

Here V_1 is the unipotent group of E_6 defined by $V_1 = U_{1,2,3,4}w_0L_2w_0^{-1}$, and

$$\psi_{V_1}(x_{010110}(r_1)x_{001110}(r_2)x_{100000}(r_3)v_1') = \psi(r_1 + r_2 + r_3).$$

This Fourier coefficient $f_\tau^{V_1,\psi}$ is the same, as the one denoted by $f_\tau^{V_2,\psi_2}$ in section 2. Applying again the arguments of [G-H2], we obtain

$$f_\tau^{V_1,\psi}(w_0u_1h,s) = \int\limits_{\mathbf{A}^2} f_\tau^{V_4,\psi}(z_2(m_1,m_2)w[45]w_0u_1h,s)dm_i \tag{35}$$

where V_4 is, as in section 2 the product of all the groups U_α lying in the Levi of the parabolic P, with the exception of U_{α_5}, and

$$f_\tau^{V_4,\psi}(z(m_1,m_2)w[45]w_0u_1h,s) = \int\limits_{V_4(F)\backslash V_4(\mathbf{A})} f_\tau(vz(m_1,m_2)w[45]w_0u_1h)\psi_{V_4}(v)dv.$$

The character ψ_{V_4} is given by $\psi_{V_4}(x_{100000}(r_1)x_{010000}(r_2)x_{001000}(r_3)v') = \psi(r_1+r_2+r_3)$. We now plug the expansion (35) into (34), and we factor the integration over the unipotent group L_3 appearing in the description of R_1 above. We obtain

$$\int\limits_{Z(\mathbf{A})GL_2(F)L_4(\mathbf{A})\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} \varphi_\pi^{L_4,\psi}(h)\theta^{U(P),\psi}(\widetilde{w}_0 x_{111110}(1)x_{011210}(1)u_1 h)\times \tag{36}$$
$$\int\limits_{\mathbf{A}^2} f_\tau^{V_2,\psi}(z(m_1,m_2)w[45]w_0u_1h,s)dm_i du_1 dh,$$

where $L_4 = LL_3$, and $\varphi_\pi^{L_4,\psi}$ can be written terms of matrices as

$$\varphi_\pi^{L_4,\psi}(h) = \int\limits_{L_4(F)\backslash L_4(\mathbf{A})} \varphi_\pi\left(\begin{pmatrix} 1 & x_1 & x_2 & y & * & * \\ & 1 & & x_2 & * & * \\ & & 1 & -x_1 & * & * \\ & & & 1 & r_1 & * \\ & & & & 1 & r_2 \\ & & & & & 1 \end{pmatrix} h\right)\psi(-r_1-r_2)dl_4.$$

(That is, the matrix appearing in the integrand gives an explicit parameterization of L_4.)

Expand the above integral along the unipotent group of matrices of the form $I_6 + n_1e_{1,2} + n_2e_{1,3}$ where $n_i \in F\backslash\mathbf{A}$. (The corresponding roots of E_6 are α_1 and $\alpha_1+\alpha_3$.) The group $GL_2(F)$, embedded as a subgroup of $R_1(F)$ defined above, acts on this expansion with two orbits. The contribution from the trivial one is zero by cuspidality. For the other we select the representative $I_6+n_1e_{1,2}+n_2e_{1,3} \mapsto \psi(-n_1)$. The stabilizer, P_0, consists of U_{α_3} and a one dimensional torus. Thus

$$\varphi_\pi^{L_4,\psi}(h) = \sum_{\gamma\in P_0(F)\backslash GL_2(F)} \int\limits_{L_5(F)\backslash L_5(\mathbf{A})} \varphi_\pi\left(\begin{pmatrix} 1 & x_1 & x_2 & y & * & * \\ & 1 & & x_3 & * & * \\ & & 1 & x_4 & * & * \\ & & & 1 & r_1 & * \\ & & & & 1 & r_2 \\ & & & & & 1 \end{pmatrix} \gamma h\right)\psi(-r_1-r_2-x_1-x_4)dl_5.$$

We plug this into (36) and factor the integration over U_{α_3}. We then perform another Fourier expansion along the group $I_6 + n_3e_{2,3}$, i.e., U_{α_3}. The zero term vanishes and the others are permuted by the torus contained in P_0. We choose $I_6+n_3e_{2,3} \mapsto \psi(-n_3)$ as a representative. Since $w[45]w_0x_{\alpha_3}(r)(w[45]w_0)^{-1} = x_{\alpha_5}(r)$, we finally obtain

$$\int\limits_{Z(\mathbf{A})N(\mathbf{A})\backslash H(\mathbf{A})} \int\limits_{U_1(Q)(\mathbf{A})} W_\pi(h)\theta^{U(P),\psi}(\widetilde{w}_0 x_{111110}(1)x_{011210}(1)u_1 h)\times \tag{37}$$
$$\int\limits_{\mathbf{A}^2} f_\tau^{V,\psi}(z(m_1,m_2)w[45]w_0u_1h,s)dm_i du_1 dh,$$

as desired. ■

6 Unramified computations for $\wedge^2 GL_6 \times GL_2$

Assume all data is unramified. We want to compute the corresponding local integral derived from (37). That is, we compute the integral

$$I(W_\pi, \theta, f_{\tau,s}) = \int\limits_{ZN\backslash H} \int\limits_{U_1(Q)} W_\pi(h)\theta^{U(P),\psi}(\widetilde{w}_0 x_{111110}(1)x_{011210}(1)u_1 h)\times \qquad (38)$$
$$\int\limits_{F^2} f_\tau^{V,\psi}(z_2(m_1,m_2)w_1u_1h,s)dm_idu_1dh$$

Here $\theta^{U(P),\psi}$ and $f_\tau^{V,\psi}$ are the defined as in section 3, and are the local functionals corresponding to the global objects of the same name encountered in the last section. Also z_2 and $\tilde{w}_0$ are as in the last section, i.e., $z_2(m_1,m_2) = x_{-000100}(m_1)x_{-000110}(m_2)$, and $\widetilde{w}_0 = w[5431243542]$ and we have introduced the notation $w_1 = w[45]w_0 = w[456]\tilde{w}_0$.

We shall prove

Proposition: *Assume all data is unramified. Then*

$$I(W_\pi, \theta, f_{\tau,s}) = \frac{L(\pi\times\tau, \wedge^2 GL_6 \times GL_2, 4s-3/2)}{L(\tau, 12s-7/2)L(\tau, sym^3, 12s-9/2)} \qquad (39)$$

Proof: Let U denote the maximal unipotent of H which contains the group N. The quotient $N\backslash U$ is two dimensional and inside G it can be identified with the group $x_{100000}(r_1)x_{101000}(r_2)$. Recall that the group $U_1(Q)$ is the unipotent subgroup of G generated by the one dimensional unipotent subgroups corresponding to the following 11 roots:

$$010000; 010100; 011100; 010110; 111100; 011110$$

$$111110; 011210; 111210; 112210; 122321$$

We make the change of variables $u_1 \mapsto x_{111110}(-1)x_{011210}(-1)u_1$, and then factor the integration over $N\backslash U$, which we identify with $x_{100000}(r_1)x_{101000}(r_2)$. The function W_π produces a factor of $\psi(r_1)$. Furthermore, $x_{100000}(r_1)x_{101000}(r_2)$ normalizes $U_1(Q)$ and is conjugated by $\tilde{w}_0$ to a unipotent element by which the function $\theta^{U(P),\psi}$ is invariant. We introduce the notation

$$y(r_1,r_2) = x_{100000}(r_1)x_{101000}(r_2)$$

$$z_0 = x_{111110}(1)x_{011210}(1),$$

Then, invoking the Iwasawa decomposition for H, we have

$$\int_{Z\backslash T} W_\pi(t)\delta_{B(H)}^{-1}(t)\int_{U_1(Q)}\int_{F^4}\psi(r_1)\theta^{U(P),\psi}(\tilde{w}_0u_1t)f_\tau^{V,\psi}(z_2(m_1,m_2)w_1z_0^{-1}y(r_1,r_2)u_1t,s)dr_idm_idt.$$

We conjugate t past u_1 and make a change of variables in u_1, obtaining a Jacobian $J(t)$. It will be convenient to hold off on writing $J(t)$ out explicitly.

We now record a trick which is useful for killing unipotent integration:
Lemma: *Suppose that Φ is a function with the property that, for any $\epsilon \in \mathfrak{o}, r \in F$ we have*

$$\Phi(x_\alpha(r)) = \Phi(x_\alpha(r)x_\beta(\epsilon)) = \Phi(x_\beta(\epsilon)x_{\alpha+\beta}(\pm\epsilon r)x_\alpha(r)) = \psi(\pm\epsilon r)\Phi(x_\alpha(r)).$$

(The two $\pm$'s need not be the same.) Then $\Phi(x_\alpha(r)) = 0$, unless $r \in \mathfrak{o}$.
The proof is self-evident. In applications, Φ is typically an inner integral, the first equality holds because we are in the unramified situation, and the third holds because we may conjugate $x_\alpha(r)$ and $x_{\alpha+\beta}(\pm\epsilon r)$ to the left and either absorb them into the integration or use left -invariance and -equivariance properties of W_π, $\theta^{U(P)}$ or $f_\tau^{V,\psi}$.

¿From this we obtain
Corollary: *Write u_1 as a product of elements $x_\alpha(r_\alpha)$ where α ranges over the roots listed above in any order. Then $\theta^{U(P),\psi}(\tilde{w}_0 u_1 t) = 0$ unless $r_\alpha \in \mathfrak{o}$ for all $\alpha \neq \alpha_{122321}$. If u_1 does satisfy this condition, then*

$$\theta^{U(P),\psi}(\tilde{w}_0 u_1 t) = \psi(-r_{\alpha_{122321}})\delta_P^{\frac{1}{4}}(\tilde{w}_0 t \tilde{w}_0^{-1}).$$

Proof: The left equivariance by x_{122321} comes from the fact that $\tilde{w}_0 x_{122321}(r)\tilde{w}_0^{-1} = x_{\alpha_6}(r)$. The relatively simple dependence on t stems from the fact that, as an element of the torus of H, it commutes with $x_{\alpha_{122321}}$, and the fact that the local minimal representation is the unramified constituent of an induced representation. To see that the rest of u_1 may simply be erased we inspect the list of roots α above, such that $U_\alpha \in U_1(Q)$. This is precisely the set of roots α such that $\alpha > 0$ and $\tilde{w}_0 \cdot \alpha < 0$. For each such α, let $\beta = 122321 - \alpha$. We observe that for each α on the list above, β is not on the list. It follows that the above lemma may be applied with this choice of β to restrict the integration in r_α to $\mathfrak{o}$. But then because we are in the unramified situation, this integration may be done away with entirely. ■
Motivated by this we put $\mu_1(t) = \delta_{B(H)}^{-1}(t)J(t)\delta_P^{\frac{1}{4}}(\tilde{w}_0 t \tilde{w}_0^{-1})$ and denote $x_{\alpha_{122321}}(r_{\alpha_{122321}})$ more simply by $z(r_3)$.

We have

$$\int_{Z\backslash T} W_\pi(t)\mu_1(t)\int_{F^5} \psi(r_1 - r_3) f_\tau^{V,\psi}(z_2(m_1, m_2)w_1 z_0^{-1} y(r_1, r_2)tz(r_3), s)dr_i dm_i dt. \quad (40)$$

Next we conjugate w_1 to the right, denoting the conjugates of z_0^{-1}, y, t, z by z_0', y', t', z'. Then $z_0' = x_{-010111}(-1)x_{-001111}(-1), y'(r_1, r_2) = x_{010100}(-r_1)x_{010110}(-r_2)$, and $z(r_3) = x_{-000111}(-r_3)$. Hence

$$z_2(m_1, m_2)z_0' y'(r_1, r_2) = y'(r_1, r_2)x_{\alpha_2}(r_1 m_1 + r_2 m_2)z_2(m_1, m_2)y''(r_1, r_2)z_0',$$

where $y''(r_1, r_2) = x_{-000011}(r_1)x_{-000001}(r_2)$, so that (40) equals

$$\int_{Z\backslash T} W_\pi(t)\mu_1(t)\int_{F^5} \psi(r_1 - r_3 - r_1 m_1 - r_2 m_2) f_\tau^{V,\psi}(z_2(m_1, m_2)y''(r_1, r_2)z_0' t' z'(r_3), s)dr_i dm_i dt.$$

Now we conjugate t to the left and make changes of variable in the unipotent integration. Because t was in the kernel of α_{122321}, t' is now in the kernel of α_{000111}, so the

Jacobian is 1. Let c, d, and e denote $\alpha_4(t'), \alpha_2(t')$ and $\alpha_3(t')$ respectively. Then we have

$$\int_{Z\backslash T} W_\pi(t)\mu_1(t) \int_{F^5} \psi(cr_1 - r_3 - r_1m_1 - r_2m_2)\times \tag{41}$$

$$\times f_\tau^{V,\psi}(t'z_2(m_1,m_2)y''(r_1,r_2)x_{-010111}(d)x_{-001111}(e)z'(r_3), s)dr_idm_idt.$$

Consider the inner integral over r_1, r_2, and r_3. By an argument similar to the one used to eliminate most of u_1 above, it is zero unless m_2 and $c - m_1$ are in $\mathfrak{o}$. Now conjugate $z_2(m_1, m_2)$ past $y''(r_1, r_2)$. This produces a factor of $x_{-000111}(-r_1m_1 - r_2m_2)$ which may be absorbed into r_3, simplifying the expression inside ψ. Now we may erase the integrals over m_1 and m_2, replacing $z_2(m_1, m_2)$ by $z_2(c, 0)$. We remark that this cancellation between our two factors of $r_1m_1 + r_2m_2$ may also be seen as the assertion that two threefold commutators are inverse to one another by tracing the genealogy of the equivariance property of $f^{V,\psi}$ along U_{α_2} back to the original character $\psi_{U(P)}$ of $U(P)$, which is also the origin of our $\psi(-r_3)$.

We now have

$$\int_{Z\backslash T} W_\pi(t)\mu_1(t) \int_{F^5} \psi(cr_1 + r_3)\times$$

$$\times f_\tau^{V,\psi}(t'x_{-000011}(r_1)x_{-000001}(r_2)x_{-000111}(-r_3)x_{-000100}(c)x_{-010111}(-d)x_{-001111}(-e), s)dr_idm_idt.$$

We now break the domain of integration into two pieces corresponding to $|e| \leq 1$ and $|e| > 1$. In the first piece, which we denote I_0, we may simply erase $x_{-001111}(-e)$. In the second, which we denote I_1, we may replace it by

$$\alpha_{001111}^\vee(e^{-1})x_{001111}(-e).$$

Here and throughout we use $\alpha^\vee$ for the coroot associated to the root α, which is a 1-parameter subgroup. We conjugate this expression to the left. Inside of $f_\tau^{V,\psi}$ we have

$$t'\alpha_{001111}^\vee(e^{-1})x_{001111}(-e)x_{001100}(r_1)x_{001110}(r_2)x_{001000}(-r_3)x_{-000011}(e^{-1}r_1)x_{-000001}(e^{-1}r_2)\times$$

$$\times x_{-000111}(-e^{-1}r_3)x_{-000100}(c)x_{-010111}(-d).$$

Now, $f_\tau^{V,\psi}$ is invariant by $U_{001111}U_{001100}U_{001110}$, but equivariant along U_{001000}. ¿From the definition of e, $\alpha_3(t'\alpha_{001111}^\vee(e^{-1})) = 1$, so we get a factor of $\psi(r_3)$. Making changes of variable in the r_i we obtain a Jacobian of $|e|^3$. Next, using the trick from above, we note that the inner integral vanishes whenever any of $|d|, |c|$, and $|r_3|$ exceeds one. Thus

$$I_1 = \int_{D_1} W_\pi(t)\mu_1(t)|e|^3 \int_{F^2} \psi(cer_1)f_\tau^{V,\psi}(t'\alpha_{001111}^\vee(e^{-1})x_{-000011}(r_1)x_{-000001}(r_2), s)dr_1dr_2dt,$$

where D_1 is the subset of $Z\backslash T$ defined by the conditions $|e| > 1, |c|, |d| \leq 1$. We return to I_0 and break it into two pieces I_{01} and I_{00} corresponding to $|d| > 1$ and $|d| \leq 1$. By arguments nearly identical to those just above, we get

$$I_{01} = \int_{D_{01}} W_\pi(t)\mu_1(t)|d|^3 \int_{F^2} \psi(dcr_1)f_\tau^{V,\psi}(t'\alpha_{010111}^\vee(d^{-1})x_{-000011}(r_1)x_{-000001}(r_2), s)dr_1dr_2dt,$$

where D_{01} is defined by $|d| > 1, |c|, |e| \leq 1$. Continuing, we break I_{00} into I_{000} and I_{001}. Corresponding to $|c| \leq 1$ and $|c| > 1$ respectively. This time, in I_{001}, when we conjugate $x_{000100}(c^{-1})\alpha_4^\vee(c^{-1})$ to the left we obtain inside

$$x_{-000011}(cr_1 - r_3)x_{-000001}(-r_2)x_{-000111}(-c^{-1}r_3),$$

so that when we make appropriate changes of variable in the r_i, the Jacobian is 1 and $\psi(cr_1 - r_3)$ becomes simply $\psi(r_1)$. Using the fact that $f_\tau^{V,\psi}$ is invariant by U_{α_4} on the left, we can once again eliminate the integration over r_3, obtaining

$$I_{001} = \int_{D_{001}} W_\pi(t)\mu_1(t) \int_{F^2} \psi(r_1) f_\tau^{V,\psi}(t\alpha_4^\vee(c^{-1})x_{-000011}(r_1)x_{-000001}(r_2), s)dr_1 dr_2 dt,$$

where D_{001} is the region defined by the conditions $|e|, |d| \leq 1, |c| > 1$. Finally, we break I_{000} into I_{0000} and I_{0001}. Observe that I_{0000} is the same basic shape as I_1, I_{01}, and I_{001}. We leave it alone for now, returning in a moment to do some manipulations valid for any integral of this shape. As for I_{0001}, we plug in $\alpha_{000111}^\vee(r_3^{-1})x_{000111}(-r_3)$ and conjugate them to the left, obtaining

$$I_{0001} = \int_{D_{000}} W_\pi(t)\mu_1(t) \int_{F-\mathfrak{o}} |r_3|^2 \psi(-r_3) \times$$

$$\times \int_{F^2} \psi(cr_1r_3) f_\tau^{V,\psi}(t'\alpha_{000111}^\vee(r_3^{-1})x_{-000011}(r_1)x_{-000001}(r_2), s dr_1 dr_2 dr_3 dt.$$

Here D_{000} is the subset of $Z\backslash T$ defined by $|c|, |d|, |e| \leq 1$. It shall emerge in a moment that the inner integral over r_1 and r_2 depends only on $|r_3|$. It follows that

$$I_{0001} = -\int_{D_{000}} W_\pi(t)\mu_1(t)q^2 \int_{F^2} \psi(cpr_1) f_\tau^{V,\psi}(t'\alpha_{000111}^\vee(p^{-1})x_{-000011}(r_1)x_{-000001}(r_2), s dr_1 dr_2 dr_3 dt$$

(p being a uniformizer and q^{-1} its absolute value).

We now turn to some manipulations for a general integral of the following shape

$$I'(\tilde{c}, t'') := \int_{F^2} \psi(\tilde{c}r_1) f_\tau^{V,\psi}(t''x_{-000011}(r_1)x_{-000001}(r_2), s)dr_1 dr_2.$$

We first introduce the notation to describe the answer. To avoid having two Q's we denote the maximal parabolic subgroup of $GSpin_{10}$ used to construct the Eisenstein series of which θ_τ is a residue (which was denoted by Q in section 3) by $Q^{(1)}$ here. Recall that if $n_i = v(\alpha_i(t))$, and $\mu_3(t) = \delta_P(t)^s \delta_{Q^{(1)}}(t)^{\frac{1}{3}}(t)\delta_{B(M_{Q^{(1)}})}^{\frac{1}{2}}(t)$, then, in the notation of section 3, we have

$$f_\tau^{V,\psi}(t) = \mu_3(t)\tilde{\chi}_{SL_2}(\underline{n}'; n_5).$$

We also reuse the notation $x = q^{-4s+3/2}$. Let $m_i = v(\alpha_i(t''))$, and let

$$S'(v(\tilde{c}), \underline{m}) = \sum_{k_1=0}^{v(\tilde{c})} \sum_{k_2=0}^{m_5} x^{3k_1+3k_2} \tilde{\chi}_{SL_2}(\underline{m}'; m_5 + k_1 - k_2).$$

Then we prove
Lemma: *We have*

$$I'(\tilde{c}, t'') = L(\tau, 12s - 7/2)\mu_3(t'')S'(v(\tilde{c}), \underline{m}).$$

Since the answer depends only on $v(\tilde{c})$, this allows for the simplification of I_{0001} noted above.
Proof: Using the same approach as above, we obtain $I' = I'_0 + I'_1$, where

$$I'_0 = \mathbf{1}_{\mathfrak{o}}(\tilde{c}) \int_F f_\tau^{V,\psi}(t'' x_{-000001}(r_2), s) dr_2,$$

($\mathbf{1}_{\mathfrak{o}}$ being the characteristic function of $\mathfrak{o}$) and

$$I'_1 = \int_{F-\mathfrak{o}} \psi(\tilde{c}r_1)|r_1| \int_F \psi(-\alpha_5(t'')r_2) f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}) x_{-000001}(r_2), s) dr_2 dr_1.$$

Let $II'_1(t'', r_1^{-1})$ denote the inner integral over r_2. It is equal to

$$\mathbf{1}_{\mathfrak{o}}(\alpha_5(t'')) f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}), s) + \int_{F-\mathfrak{o}} \psi(-\alpha_5(t'')r_2) f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}) \alpha_6^\vee(r_2^{-1}), s) dr_2.$$

Using the fact that

$$\int_{|r_2|=q^k} \psi(ar_2) dr_2 = q^k \times \begin{cases} (1-q^{-1}) \text{ if } k \le v(a), \\ -q^{-1} \text{ if } k = v(a)+1, \\ 0 \text{ if } k > v(a)+1, \end{cases}$$

we obtain

$$II'_1(t'', r_1^{-1}) = \mathbf{1}_{\mathfrak{o}}(\alpha_5(t'')) f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}), s) + \sum_{k_2=1}^{m_5} f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}) \alpha_6^\vee(p^{k_2}), s) q^{k_2}$$

$$-q^{-1} \sum_{k_2=1}^{m_5+1} f_\tau^{V,\psi}(t'' \alpha^\vee_{000011}(r_1^{-1}) \alpha_6^\vee(p^{k_2}), s) q^{k_2}.$$

We compute $q\mu_3 \circ \alpha_6^\vee(p)$ and find it equal to x^3. We get

$$\mu_3(t'' \alpha^\vee_{000011}(r_1^{-1})) \sum_{k_2=0}^{m_5} x^{3k_2} \left(\tilde{\chi}_{SL_2}(\underline{m}', m_5 - k_2) - q^{-1} x^3 \tilde{\chi}_{SL_2}(\underline{m}', m_5 - k_2 - 1) \right). \quad (42)$$

Now, using the formula for the integral of $\psi(\tilde{c}r_1)$ over the annulus $|r_1| = q^{k_1}$, we have

$$I'_1 = \sum_{k_1=1}^{v(\tilde{c})} q^{2k_1} II'_1(t'', p^{k_1}) - q^{-1} \sum_{k_1=1}^{v(\tilde{c})+1} q^{2k_1} II'_1(t'', p^{k_1}).$$

We compute $q^2\mu_3 \circ \alpha^\vee_{000011}(p)$, finding it equal to x^3 again. Plugging in (42) we obtain

$$\mu_3(t'') \left(\sum_{k_1=1}^{v(\tilde{c})} \sum_{k_2=0}^{m_5} x^{3k_1+3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5 + k_1 - k_2) - q^{-1} x^3 \sum_{k_1=1}^{v(\tilde{c})} \sum_{k_2=0}^{m_5} x^{3k_1+3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5 + k_1 - k_2 - 1) \right.$$

$$-q^{-1}x^3 \sum_{k_1=0}^{v(\tilde{c})} \sum_{k_2=0}^{m_5} x^{3k_1+3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5+k_1-k_2+1) + q^{-2}x^6 \sum_{k_1=0}^{v(\tilde{c})} \sum_{k_2=0}^{m_5} x^{3k_1+3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5+k_1-k_2) \Bigg).$$

A similar computation yields

$$I_0' = \mu_3(t'') \sum_{k_2=0}^{m_5} x^{3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5-k_2) - q^{-1}x^3 \sum_{k_2=0}^{m_5-1} x^{3k_2} \tilde{\chi}_{SL_2}(\underline{m}', m_5-k_2-1).$$

This time, the cut-off in the sum is provided by the support of $f_\tau^{V,\psi}$. The result follows. ■

Recalling the definition of $\tilde{\chi}_{SL_2}(\underline{m}', m_5+k_1-k_2)$ and the identity (20) mentioned earlier, we also have

$$S'(v,\underline{m}) = \chi_{SL_3}(m_1,m_3) \begin{pmatrix} \xi^2 & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix} \chi_{SL_2}(m_2) \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix} \chi_{SL_3}(v,m_5) \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix},$$

which we denote more briefly by $\chi^*_{SL_2}(v,\underline{m})$.

Returning to the main argument, we have broken our original integral into five pieces, each of which (in light of the observation that the inner integral in I_{0001} depends only on $|r_3|$) is of the form

$$\int_{D_\sigma} W_\pi(t)\mu_1(t)J_\sigma(t)I'(\tilde{c}_\sigma, t'T_\sigma)dt, \tag{43}$$

Here σ is simply standing in for one of our five labels $1, 01, \ldots, 0001$, and $J_\sigma, \tilde{c}_\sigma$, and T_σ are just the appropriate expressions from the corresponding integral, for example $J_1(t) = |e|^3$ and $\tilde{c}_{01} = cd$.

Now, recall that T is not the full torus of GE_6 but only the six-dimensional maximal torus of H. Because of this $\{\alpha_i(t') : 1 \le i \le 5\}$ provides a complete set of coordinates for $Z\backslash T$. Let $n_i = v(\alpha_i(t'))$. It is clear that each piece of the integrand above depends only on $n_1, \ldots, n_5$. We may therefore express each of our five pieces as a sum over $\underline{n}$ subject to constraints depending on the case. Let $\mu_2(t) = \delta_{B(H)}^{\frac{1}{2}}\mu_1(t)\mu_3(t')$.

Lemma: *We have $\mu_2(t) = x^{\ell(\underline{n})}$, where*

$$\ell(\underline{n}) = 2n_1 + 3n_2 + 4n_3 + 2n_4 + n_5.$$

Proof: We have $\mu_2(t) = \delta_{B(H)}^{\frac{1}{2}}(t)J(t)\delta_P^{\frac{1}{4}}(\tilde{t})\delta_P^{\frac{1}{4}}(t')\delta_{B(M_{Q^{(1)}})}^{\frac{1}{2}}(t')\delta_{Q^{(1)}}^{\frac{1}{3}}(t')$, where J is the Jacobian that emerged when t was conjugated past $u_1 \in U_1(Q)$. Each piece is naturally interpreted as an element of $\Lambda_R \otimes_{\mathbf{Z}} \mathbf{C}$, where Λ_R denotes the root lattice of GE_6. For the pieces where the argument is t or $\tilde{t}$ we apply the appropriate Weyl element to express them in terms of $\{\alpha_i(t')\}$. We find that

$$\begin{aligned}
J(t) &= (5,12,10,15,8,1) \qquad \text{in terms of } \{\alpha_i(t)\} \\
&= (-4,-6,-8,-15,-13,-11) \qquad \text{in terms of } \{\alpha_i(t')\} \\
\delta_{B(H)}^{-\frac{1}{2}}(t) &= (-\frac{5}{2},0,-4,-\frac{9}{2},-4,-\frac{5}{2}) \qquad \text{in terms of } \{\alpha_i(t)\} \\
&= (-4,-5,-7,-\frac{19}{2},-\frac{13}{2},-\frac{5}{2}) \qquad \text{in terms of } \{\alpha_i(t')\} \\
\delta_P^{\frac{1}{4}}(\tilde{t}) &= (2,3,4,6,5,4) \qquad \text{in terms of } \{\alpha_i(\tilde{t})\} \\
&= (2,3,4,3,2,1) \qquad \text{in terms of } \{\alpha_i(t')\} \\
\delta_P(t')^s &= (8s,12s,16s,24s,20s,16s) \qquad \text{in terms of } \{\alpha_i(t')\} \\
\delta_{B(M_{Q(1)})}^{\frac{1}{2}}(t') &= (1,\frac{1}{2},1,0,\frac{1}{2},0) \qquad \text{in terms of } \{\alpha_i(t')\} \\
\delta_{Q^{(1)}}^{\frac{1}{3}}(t') &= (2,3,4,6,3,0) \qquad \text{in terms of } \{\alpha_i(t')\}
\end{aligned}$$

Recall that for $t \in T$, t' is in the kernel of $(0,0,0,1,1,1)$. Reducing modulo the span of this element, and summing, we obtain

$$(8s-3, 12s-\frac{9}{2}, 16s-6, 8s-3, 4s-\frac{3}{2}, 0),$$

which gives the stated result. ■

Now, recall that $\delta_{B(H)}^{-\frac{1}{2}} W_\pi(t)$ is described by the Casselman-Shalika formula in terms of

$$(v(\alpha_1(t)), v(\alpha_3(t)), v(\alpha_4(t)), v(\alpha_5(t)), v(\alpha_6(t))) = (n_2+n_4, n_5, n_3+n_4, n_1, n_2+n_3).$$

Specifically, it is zero unless each of these integers is non-negative, in which case it is the trace of the irreducible representation of $SL_6(\mathbf{C})$ whose highest weight is given by the quintuple, evaluated at the conjugacy class in $SL_6(\mathbf{C})$ associated to the representation π. We denote this by $\chi_{SL_6}(\underline{n}'')$. Then $W_\pi(t)\mu_1(t)\mu_3(t') = \chi_{SL_6}(\underline{n}'')x^{\ell(n)}$.

Finally, in each case, $J_\sigma(t)\mu_3(T_\sigma)$ is some power of x. We have seen above that I_σ is evenly divisible by $L(12s-7/2,\tau)$ for all σ. Let $\hat{I}$ denote the quotient. Then each piece of our sum is of the following shape:

$$\sum \chi_{SL_6}(\underline{n}'')x^{\ell(n)+\Delta_\sigma}\chi^*(v(\tilde{c}_\sigma), m^\sigma).$$

Again, σ is simply one of our labels $1,\ldots,0001$.

We record the values of $\Delta, v(\tilde{c})$, and $\underline{m}$ in the following table, along with the constraints appropriate to each case:

case	constraints	x^{Δ_σ}	$v(\tilde{c}_\sigma)$	m_1	m_3	m_2	m_5
1	$n_3 < 0, n_i \geq 0, i \neq 3$	x^{-3n_3}	n_3+n_4	n_1+n_3	0	n_2+n_3	n_5
01	$n_2 < 0, n_i \geq 0, i \neq 3$	x^{-3n_2}	n_2+n_4	n_1	n_2+n_3	0	n_5
001	$n_4 < 0, n_i \geq 0, i \neq 3$	1	0	n_1	n_3+n_4	n_2+n_4	n_4+n_5
0000	$n_i \geq 0$ all i	1	n_4	n_1	n_3	n_2	n_5
0001	$n_i \geq 0$ all i	x^3	n_4-1	n_1	n_3-1	n_2-1	n_5

Note the order of the m_i. (It is chosen for convenience of plugging into $\tilde{\chi}_{SL_2}$.)

Our original claim is reduced to

$$\hat{I}_1+\hat{I}_{01}+\hat{I}_{001}+\hat{I}_{0000}-\hat{I}_{0001}\frac{L(\pi\times\tau,\wedge^2 GL_6\times GL_2,4s-3/2)}{L(\tau,sym^3,12s-9/2).} \tag{44}$$

This is essentially an identity of power series. To be precise, let R denote the representation ring of $SL_6(\mathbf{C})$. It may be identified with the ring of polynomial functions on the torus of $SL_6(\mathbf{C})$ which are symmetric with respect to the action of the Weyl group. The characters of irreducible representations form a basis for R as a $\mathbf{C}$-vector space. We consider the ring $R[Y_1,Y_2][[X]]$ (formal power series over a polynomial ring in two variables over R). Suppose that $diag(\xi,\xi^{-1})$ is a representative for the semisimple conjugacy class in $SL_2(\mathbf{C})$ associated to τ. Then for each σ there is an element $\tilde{I}_\sigma$ of $R[Y_1,Y_2][[X]]$ such that I_σ may be obtained from $\tilde{I}_\sigma$ by evaluating Y_1 at ξ, Y_2 at ξ^{-1}, X at $x=q^{-4s+3/2}$ and the characters in R at the semisimple conjugacy class in $SL_6(\mathbf{C})$ corresponding to π.

But $L(\pi\times\tau,\wedge^2 GL_6\times GL_2,4s-3/2)/L(\tau,sym^3,12s-9/2)$ is obtained by the same procedure from the power series Q, defined by

$$(1-X^3Y_1^3)(1-X^3Y_1)(1-X^3Y_2)(1-X^3Y_2^3)\sum_{n,m=0}^{\infty}Tr(sym^m\Gamma_{\varpi_2})Tr(sym^n\Gamma_{\varpi_2})Y_1^mY_2^nX^{m+n}.$$

Furthermore, other than the relation $\xi\xi^{-1}=1$, no specific information about the points we are evaluating at plays any role in the proof.

Thus (39) is reduced to the identity in $R[Y_1,Y_2][[X]]/\langle Y_1Y_2-1\rangle$:

$$Q=\tilde{I}_1+\tilde{I}_{01}+\tilde{I}_{001}+\tilde{I}_{0001}+\tilde{I}_{0000}, \tag{45}$$

which we prove in the appendix.

7 The proof of equation (45)

In this section we regard x as an indeterminate in a ring of formal power series, and ξ and ξ^{-1} and the images of Y_1 and Y_2, respectively, in $\mathbf{C}[Y_1,Y_2]/\langle Y_1Y_2-1\rangle$. It will be convenient to introduce $u:=x\xi$ and $v:=x\xi^{-1}$.

7.1 The Littlewood Richardson Rule

We first expand Q as a more explicit summation. Let (n_1,n_2,n_3,n_4,n_5) now denote the character of the irreducible representation of $SL_6(\mathbf{C})$ with highest weight $\sum_i n_i\varpi_i$, viewed as an element of the representation ring R. The decomposition of $sym^n\Gamma_{\varpi_2}$ is known:

$$Tr(sym^n\Gamma_{\varpi_2})=\sum_{a+2b+3c=n}(0,a,0,b,0).$$

(See [B].) Hence

$$Q=(1-u^3)(1-u^2v)(1-uv^2)(1-v^3)\sum_{m,n=0}^{\infty}Tr(sym^n\Gamma_{\varpi_2})Tr(sym^m\Gamma_{\varpi_2})u^nv^m$$

$$= (1-u^2v)(1-uv^2)\sum_{m_i,n_i=0}^{\infty}(0,m_1,0,m_2)(0,n_1,0,n_2,0)u^{m_1+2m_2}v^{n_1+2n_2}.$$

We now expand $(0,m_1,0,m_2)(0,n_1,0,n_2,0)$ using the Littlewood-Richardson rule. Thus we associate to $(0,n_1,0,n_2,0)$ the partition $(n_1+n_2)^2(n_2)^2$ and it's Young diagram, which consists of two rows of length n_1+n_2 and two of length n_2. To describe the multiplicities in $(0,m_1,0,m_2)(0,n_1,0,n_2,0)$ we consider all ways of adding m_1+m_2 boxes labeled a and an equal number labelled b, and then m_2 each labelled c and d, to the Young diagram of $(n_1+n_2)^2(n_2)^2$ subject to certain conditions, as described in [F-H], page 456. We let a_i denote the number of a's in row i and define b_i, c_i, d_i similarly. Then we have:

$$a_1 \geq b_2+b_3, \quad a_3 \geq b_4, \quad b_2 \geq c_3, \quad n_1 \geq a_3+b_3, \quad n_2 \geq a_5+b_5, \quad d_6 \geq c_5, \quad c_3 \geq d_4,$$
$$a_3+b_3 \geq b_4+c_4, \qquad b_2+b_3 \geq c_3+c_4, \qquad b_2+b_3+b_4 \geq c_3+c_4+c_5,$$
$$n_2+b_4 \geq a_5+b_5+c_5, \qquad n_2+b_4+c_4 \geq a_5+b_5+c_5+d_5, \qquad b_5+c_5 \geq d_6,$$
$$a_1+a_3 = b_2+b_3+b_4+b_5, \qquad b_6 = a_5, \qquad c_3+c_4+c_5 = d_4+d_5+d_6 = m_2,$$

and all variables not appearing in any of the above must be zero. Also, $a_1+a_3+a_5 = m_1+m_2$. We plug this into our sum and make appropriate changes of variable (e.g., $n_1 \mapsto n_1+a_3+b_3$) based on the inequalities in the first row. The first equality in the last row becomes $b_5 = a_1+a_3$. We eliminate m_1, m_2, b_5, b_6, and d_5, and obtain a sum in all remaining variables from 0 to ∞ subject to the following reduced set of constraints.

$$a_3+b_3 \geq c_4, \qquad b_2+b_3+b_4 \geq c_4+c_5, \qquad n_2+b_4+d_6 \geq c_3+c_5$$
$$a_1+a_3 \geq d_6, \qquad b_2+b_3 \geq c_4, \qquad n_2+b_4 \geq c_5, \qquad c_3+c_4 \geq d_6.$$

(The last constraint here results from the nonnegativity of the eliminated variable d_5.) The representation corresponding to a given value of these variables is obtained as follows: having added the boxes marked a, b, c, d to the original Young diagram, we now have the Young diagram of a new partition. To translate back to the quintuple notation, we simply subtract consecutive entries. Summarizing:

$$Q = (1-u^2v)(1-uv^2)\sum u^{n_1+2n_2+2a_1+3a_3+b_3+b_4}v^{a_1+a_3+a_5+b_2+b_3+b_4+2c_3+c_4+c_5+2d_4}$$

$$(a_1+b_3, \quad n_1+b_2, \quad a_3+b_3+c_3-c_4, \quad n_2+b_4-c_3-c_5+d_4+d_6, \quad a_1+a_3+c_3+c_4-2d_6),$$

where the summation is from 0 to ∞ in all variables subject to the constraints listed above. Observe that a_5 may be summed at once, canceling the factor of $(1-u^2v)$ in front.

7.2 Evaluation of the power series $\tilde{I}_\sigma$.

Now that both sides of (45) have been expressed as explicit summations, the claim is that the coefficient of the character $(m_1, m_2, m_3, m_4, m_5)$, is the same on both sides. This, in turn, is equivalent to the identity of power series in that we obtain by replacing $(m_1, m_2, m_3, m_4, m_5)$ with $t_1^{m_1} \dots t_5^{m_5}$ everywhere. By abuse of notation, we keep the same notation for the new power series. As we shall see, the power series $\tilde{I}_\sigma$ from section 6 are not difficult to evaluate in closed form. We first record a few lemmas. Each is proved by a straightforward computation.

Lemma: We have

$$\sum_{n,m=0}^{\infty} X_1^n X_2^m \chi_{SL_3}(n,m)| \begin{pmatrix} u_1 & & \\ & u_2 & \\ & & u_1 u_2^{-1} \end{pmatrix}$$

$$= \frac{1 - X_1 X_2}{(1-X_1u_1)(1-X_1u_2)(1-X_1u_1^{-1}u_2^{-1})(1-X_2u_1^{-1})(1-X_2u_2^{-1})(1-X_2u_1u_2)}$$

Lemma: We have

$$\sum_{n=0}^{\infty} X^n \chi_{SL_2}(n)| \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} = \frac{1}{(1-Xu)(1-Xu^{-1})}$$

Lemma: We have

$$\sum_{n=0}^{\infty} X^n \chi_{SL_3}(n,0)| \begin{pmatrix} u_1 & & \\ & u_2 & \\ & & u_1 u_2^{-1} \end{pmatrix} = \frac{1}{(1-Xu_1)(1-Xu_2)(1-Xu_1^{-1}u_2^{-1})}$$

Applying the symmetry of the Dynkin diagram to the last identity, we obtain

$$\sum_{n=0}^{\infty} X^n \chi_{SL_3}(0,n)| \begin{pmatrix} u_1 & & \\ & u_2 & \\ & & u_1 u_2^{-1} \end{pmatrix} = \frac{1}{(1-Xu_1^{-1})(1-Xu_2^{-1})(1-Xu_1u_2)}$$

Referring back to our table in section 6, we write out the formal power series $\tilde{I}_\sigma$:

$$\tilde{I}_{0000} = \sum_{n_i=0}^{\infty} t_1^{n_2+n_4} t_2^{n_5} t_3^{n_3+n_4} t_4^{n_1} t_5^{n_2+n_3} x^{2n_1+3n_2+4n_3+4n_4+2n_5} \times$$

$$\times\ \chi_{SL_3}(n_1,n_3)| \begin{pmatrix} \xi & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix} \chi_{SL_2}(n_2)\ \chi_{SL_3}(n_4,n_5)| \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix}.$$

$$\tilde{I}_{0001} = \sum_{n_i=0}^{\infty} t_1^{n_2+n_4} t_2^{n_5} t_3^{n_3+n_4} t_4^{n_1} t_5^{n_2+n_3} x^{2n_1+3n_2+4n_3+4n_4+2n_5+3} \times$$

$$\times\ \chi_{SL_3}(n_1,n_3-1)| \begin{pmatrix} \xi & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix} \chi_{SL_2}(n_2-1)\ \chi_{SL_3}(n_4-1,n_5)| \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix}.$$

$$\tilde{I}_{1} = \sum_{n_3<0} t_1^{n_2+n_4} t_2^{n_5} t_3^{n_3+n_4} t_4^{n_1} t_5^{n_2+n_3} x^{2n_1+3n_2+n_3+4n_4+2n_5} \times$$

$$\times\ \chi_{SL_3}(n_1+n_3,0)| \begin{pmatrix} \xi & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix} \chi_{SL_2}(n_2+n_3)\ \chi_{SL_3}(n_3+n_4,n_5)| \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix}.$$

$$\tilde{I}_{01} = \sum_{n_2<0} t_1^{n_2+n_4} t_2^{n_5} t_3^{n_3+n_4} t_4^{n_1} t_5^{n_2+n_3} x^{2n_1+4n_3+4n_4+2n_5} \times$$

$$\times\ \chi_{SL_3}(n_1,n_2+n_3)| \begin{pmatrix} \xi & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix}\ \chi_{SL_3}(n_2+n_4,n_5)| \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix}.$$

$$\tilde{I}_{001} = \sum_{n_4<0}^{\infty} t_1^{n_2+n_4} t_2^{n_5} t_3^{n_3+n_4} t_4^{n_1} t_5^{n_2+n_3} x^{2n_1+3n_2+4n_3+4n_4+2n_5} \times$$

$$\times\, \chi_{SL_3}(n_1, n_3+n_4)| \begin{pmatrix} \xi & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix} \chi_{SL_2}(n_2+n_4)\ \chi_{SL_3}(0, n_4+n_5)| \begin{pmatrix} \xi x & & \\ & x^{-2} & \\ & & \xi^{-1}x \end{pmatrix}.$$

In the last three sums, summation is from 0 to ∞ in the variables not indicated. Each of these is straightforward to sum using the lemmas above. For example, to compute $\tilde{I}_1$ we just have to make the change of variables $n_3 \mapsto -n_3 - 1, n_i \mapsto n_i + n_3 + 1, i = 1, 3, 4$ (which also has the effect $n_i + n_3 \mapsto n_i, i = 1, 3, 4$) to obtain summation from 0 to ∞ in all variables. We summarize the outcome. Let

$$A_{13} = (1 - u^3v^3t_3t_4t_5), \qquad B_{13} = (1 - u^2t_4)(1 - uvt_4)(1 - v^2t_4),$$
$$C_{13} = (1 - u^3vt_3t_5)(1 - u^2v^2t_3t_5)(1 - uv^3t_3t_5), \qquad A_{45} = (1 - u^3v^3t_1t_2t_3),$$
$$B_{45} = (1 - ut_2)(1 - vt_2)(1 - u^2v^2t_2), \qquad C_{45} = (1 - u^3v^2t_1t_3)(1 - u^2v^3t_1t_3)(1 - uvt_1t_3)$$
$$C_2 = (1u^2vt_1t_5)(1 - uv^2t_1t_5)$$

Then

$$\tilde{I}_{0000} = \frac{A_{13}A_{45}}{B_{13}C_{13}C_2B_{45}C_{45}}, \qquad \tilde{I}_{0001} = -u^6v^6x_1^2x_3^2x_5^2 \frac{A_{13}A_{45}}{B_{13}C_{13}C_2B_{45}C_{45}}$$

$$\tilde{I}_1 = \frac{u^3v^3t_1^2t_4A_{45}}{(1 - u^3v^3t_1^2t_4)B_{13}C_2B_{45}C_{45}}, \qquad \tilde{I}_{01} = \frac{u^3v^3t_3^2A_{13}A_{45}}{(1 - u^3v^3t_3^2)B_{13}C_{13}B_{45}C_{45}},$$

$$\tilde{I}_{001} = \frac{t_2t_5^2A_{13}}{(1 - t_2t_5^2)B_{13}C_{13}C_2B_{45}}.$$

Let $\tilde{I}_{000} = \tilde{I}_{0001} + \tilde{I}_{0000}$. It is indeed the power series corresponding to the integral I_{000} from section 6. Observe that a given quintuple $(m_1, m_2, m_3, m_4, m_5)$ will appear in only one of the power series $\tilde{I}_\sigma, \sigma \in \{1, 01, 001, 000\}$. For example, if $m_1 + m_3 - m_5$ is negative, it will only appear in $\tilde{I}_{001}$. This allows us to break Q into four parts Q_σ and compare like parts. This turns out to be more convenient than summing, since when we put everything over a common denominator, the numerator is irreducible of degree 42 in x.

7.3 Evaluation of the power series Q_σ

We recall the form of the quintuple that appears in our summation for Q:

$$(a_1{+}b_3, \quad n_1{+}b_2, \quad a_3{+}b_3{+}c_3{-}c_4, \quad n_2{+}b_4{-}c_3{-}c_5{+}d_4{+}d_6, \quad a_1{+}a_3{+}c_3{+}c_4{-}2d_6),$$

Comparing with $(n_2 + n_4, n_5, n_3 + n_4, n_1, n_2 + n_3)$ we find that the key quantities are

$$a_1 + c_4 - d_6, \qquad a_3 + c_3 - d_6, \qquad b_3 + d_6 - c_4$$

corresponding to the quantities n_2, n_3, and n_4 of subsection 7.2 respectively. To complete the proof of (39) from section 6 we must check

Proposition: For $\sigma = 1, 01, 001, 000$, we have $Q_\sigma = \tilde{I}_\sigma$.
Proof: We recall the form of the sum:

$$(1-uv^2)\sum u^{n_1+2n_2+2a_1+3a_3+b_3+b_4} v^{a_1+a_3+b_2+b_3+b_4+2c_3+c_4+c_5+2d_4}$$

$$(a_1+b_3, \quad n_1+b_2, \quad a_3+b_3+c_3-c_4, \quad n_2+b_4-c_3-c_5+d_4+d_6, \quad a_1+a_3+c_3+c_4-2d_6),$$

with summation from 0 to ∞ in all variables, subject to the constraints

$$\begin{array}{cccc} a_3+b_3 \geq c_4, & b_2+b_3+b_4 \geq c_4+c_5, & n_2+b_4+d_6 \geq c_3+c_5 & \\ a_1+a_3 \geq d_6, & b_2+b_3 \geq c_4, & n_2+b_4 \geq c_5, & c_3+c_4 \geq d_6. \end{array}$$

as well as the additional constraints which define the "piece" σ. We first observe that n_1 and d_4 do not appear in any constraints, and hence may be summed at once producing factors of $(1-ut_2)^{-1}$ and $(1-v^2t_4)^{-1}$ respectively. Also, it will be convenient to introduce $r_1 = n_2+b_4-c_3-c_5+d_6$ and eliminate the variable n_2. The resulting sum is

$$(1-uv^2)\sum u^{2r_1+2a_1+3a_3+b_3-b_4+2c_3+2c_5-2d_6} v^{a_1+a_3+b_2+b_3+b_4+2c_3+c_4+c_5}$$

$$(a_1+b_3, \quad b_2, \quad a_3+b_3+c_3-c_4, \quad r_1, \quad a_1+a_3+c_3+c_4-2d_6),$$

and the new constraints are:

$$\begin{aligned} b_2+b_3 &\geq c_4 && (46)\\ b_2+b_3+b_4 &\geq c_4+c_5 && (47)\\ a_3+b_3 &\geq c_4 && (48)\\ r_1+c_3 &\geq d_6 && (49)\\ a_1+a_3 &\geq d_6 && (50)\\ c_3+c_4 &\geq d_6 && (51)\\ r_1+c_3+c_5 &\geq b_4+d_6 && (52) \end{aligned}$$

The remainder of the computation is different in each case, but in all of the cases we make use of the following
Lemma: For $N \leq 0 \leq M$, we have

$$\sum_{\delta=N}^{M} \sum_{\substack{b_4,c_5=0\\ b_4-c_5=\delta}}^{\infty} (u^2v)^{c_5}(u^{-1}v)^{b_4} = \frac{1}{(1-u^2v)(1-u^{-1}v)} - \frac{(u^{-1}v)^{M+1}}{(1-u^{-1}v)(1-uv^2)} - \frac{(u^2v)^{-N+1}}{(1-u^2v)(1-uv^2)}.$$

Proof: We break up the sum as

$$\sum_{\delta=0}^{M}\sum_{c_5=0}^{\infty}(u^2v)^{c_5}(u^{-1}v)^{c_5+\delta} + \sum_{\delta=N}^{-1}\sum_{b_4=0}^{\infty}(u^2v)^{b_4-\delta}(u^{-1}v)^{b_4}. = \frac{1-(u^{-1}v)^{M+1}}{(1-u^{-1}v)(1-uv^2)} + \frac{u^2v-(u^2v)^{-N+1}}{(1-u^2v)(1-uv^2)}$$

and then simplify. It's worth noting that summing $(u^{-1}v)^\delta = \xi^{-2\delta}$ from 0 to ∞ would be invalid, but summing $u^2v = x^3\xi$ or $uv^2 = x^3\xi^{-1}$ is valid for $Re(s)$ sufficiently large.
■

7.3.1 The sum Q_1

For Q_1 we have the additional constraint $d_6 \geq a_3 + c_3 + 1$. When we make the change of variables $d_6 \mapsto d_6 + a_3 + c_3 + 1$, (49),(50), and (51) become

$$r_1 \geq d_6 + a_3 + 1, \qquad a_1 \geq d_6 + c_3 + 1, \qquad c_4 \geq d_6 + a_3 + 1$$

respectively. We make additional changes of variable

$$r_1 \mapsto r_1 + d_6 + a_3 + 1 \qquad a_1 \mapsto a_1 + d_6 + c_3 + 1, \qquad c_4 \mapsto c_4 + d_6 + a_3 + 1,$$

and (48) becomes $b_3 \geq c_4 + d_6 + 1$. Making the final change of variables $b_3 \mapsto b_3 + c_4 + d_6 + 1$, we now have a sum from 0 to ∞ in all variables subject only to:

$$b_2 + b_3 \geq a_3 \qquad b_2 + b_3 + b_4 \geq a_3 + c_5, \qquad r_1 + c_5 \geq b_4.$$

The summand is:

$$u^{2r_1+2a_1+3a_3+b_3+b_4+2c_3+c_4+2c_5+3d_6+3} v^{a_1+2a_3+b_2+b_3+b_4+3c_3+2c_4+c_5+3d_6+3}$$

$$t_1^{a_1+b_3+c_3+c_4+2d_6+2} t_2^{b_2} t_3^{b_3+c_3} t_4^{r_1+a_3+d_6+1} t_5^{a_1+c_4}.$$

The unconstrained variables a_1, c_3, c_4 and d_6 may be summed, yielding

$$\frac{u^3v^3t_1^2t_4}{(1-u^3v^3t_1^2t_4)(1-u^2vt_1t_5)(1-u^2v^3t_1t_3)(1-uv^2t_1t_5)}.$$

The remaining sum we may write as

$$\sum_{s=0}^{\infty}\sum_{b_2=0}^{s}\sum_{a_3=0}^{s}\sum_{r_1=0}^{\infty}(vt_2)^{b_2}(uvt_1t_3)^{s-b_2}(u^3v^2t_4)^{a_3}(u^2t_4)^{r_1}\sum_{\delta=a_3-s}^{r_1}\sum_{\substack{b_4,c_5=0\\ b_4-c_5=\delta}}^{\infty}(u^2v)^{c_5}(u^{-1}v)^{b_4}.$$

Now, let

$$G(\underline{X},Y,Z) := \sum_{s=0}^{\infty}\sum_{k_1=0}^{s}\sum_{k_2=0}^{s}\sum_{k_3=0}^{\infty}X_1^{k_1}X_2^{s-k_1}Z^{k_2}Y^{k_3} = (1-Y)^{-1}G_1(\underline{X},Z).$$

(The second equality defines G_1.) Then our sum is

$$\frac{G(\underline{X},Y,Z)}{(1-u^{-1}v)(1-u^2v)} - \frac{u^{-1}vG(\underline{X},u^{-1}vY,Z)}{(1-u^{-1}v)(1-uv^2)} - \frac{u^2vG(u^2v\underline{X},Y,u^{-2}v^{-1}Z)}{(1-u^2v)(1-uv^2)} \tag{53}$$

where

$$X_1 = vt_2, \qquad X_2 = uvt_1t_3, \qquad Y = u^2t_4, \qquad Z = u^3v^2t_4. \tag{54}$$

We now prove

Lemma: *We have the identity of power series*

$$G_1(\underline{X},Z) = \frac{1-X_1X_2Z}{(1-X_1)(1-X_2)(1-X_1Z)(1-X_2Z)}.$$

Proof: Performing the sums in k_1 and k_2 we obtain

$$(X_1 - X_2)^{-1}(1-Z)^{-1}\sum_{s=0}^{\infty}(1-Z^{s+1})(X_1^{s+1} - X_2^{s+1}),$$

which we break into four pieces and sum over s obtaining

$$(X_1 - X_2)^{-1}(1-Z)^{-1}\left(\frac{X_1}{1-X_1} - \frac{X_2}{1-X_2} - \frac{X_1 Z}{1-X_1 Z} + \frac{X_2 Z}{1-X_2 Z}\right).$$

When we place the sum in parentheses over a common denominator the numerator is precisely $(X_1 - X_2)(1-Z)(1-X_1X_2Z)$. ■

Returning to our specific situation, note that $(1 - X_iZ)$ is fixed when we replace X_i by u^2vX_i and Z by $u^{-2}v^{-1}Z$. Hence these factors are common to all three terms of (53). We easily combine the first two terms using the identity

$$\frac{1}{(1-u^2v)(1-Y)} - \frac{u^{-1}v}{(1-uv^2)(1-u^{-1}vY)} = \frac{(1-u^{-1}v)(1-uv^2Y)}{(1-u^2v)(1-uv^2)(1-Y)(1-u^{-1}vY)}.$$

Combining with the last term is more laborious and requires simplifying

$$(1-uv^2Y)(1-X_1X_2Z)(1-u^2vX_1)(1-u^2vX_2) - (u^2-uv^2Y)v(1-X_1X_2Zu^2v)(1-X_1)(1-X_2)$$

Noting that in (54) we have $uv^2Y = Z$, this simplifies to

$$(1-u^2v)(1-X_1Z)(1-X_2Z)(1-X_1X_2u^2v).$$

We cancel the $(1-u^2v)$ in the denominator and the $(1-X_1Z)(1-X_2Z)(1-X_1X_2u^2v)$ factored out earlier. The $(1-uv^2)$ in the denominator matches the one in front of the sum. Plugging in (54) we find that $1 - X_1X_2u^2v = 1 - u^3v^3t_1t_2t_3$, and that the terms which remain in the common denominator of (53) precisely match the part of the denominator of $\tilde{I}_1$ which has not already been accounted for, i.e.

$$(1-vt_2)(1-u^2v^2t_2)(1-uvt_1t_3)(1-u^3v^2t_1t_3)(1-u^2t_4)(1-uvt_4).$$

■

7.3.2 The sum Q_{001}

For Q_{001} we have the additional constraint $c_4 \geq b_3 + d_6 + 1$. Constraints (51) and (50) follow from this and the other constraints, and we eliminate them. When we make the change of variables $c_4 \mapsto c_4 + b_3 + d_6 + 1$, (46) and (48) become $b_2, a_3 \geq c_4 + d_6 + 1$ making the additional changes of variable $b_2 \mapsto b_2 + c4 + d_6 + 1$, $a_3 \mapsto a_3 + c_4 + d_6 + 1$ The new set of constraints is

$$b_2 + b_4 \geq c_5, \qquad r_1 + c_3 \geq d_6 \qquad r_1 + c_3 + c_5 \geq b_4 + d_6.$$

The remaining computation is entirely analogous to what was done for Q_1.

7.3.3 The sum Q_{01}

For Q_{01} we have the additional constraint $d_6 \geq a_1 + c_4 + 1$. Constraint (48) follows from the others. We make the change of variables $d_6 \mapsto d_6 + a_1 + c_4 + 1$, and then $a_3 \mapsto a_3 + d_6 + c_4 + 1$ and $c_3 \mapsto c_3 + a_1 + d_6 + 1$, leaving the sum

$$\sum u^{2r_1+2a_1+3a_3+b_3-b_4+2c_3+c_4+2c_5+3d_6+3} v^{3a_1+a_3+b_2+b_3+b_4+2c_3+2c_4+c_5+3d_6+3}$$

$$t_1^{a_1+b_3} t_2^{b_2} t_3^{a_1+a_3+b_3+c_3+2d_6+2} t_4^{r_1} t_5^{a_3+c_3},$$

subject to:

$$c_4 \leq b_2 + b_3, \qquad c_4 \leq r_1 + c_3, \qquad -(b_2 + b_3 - c_4) \leq b_4 - c_5 \leq r_1 + c_3 - c_4.$$

Summing the unconstrained variables a_1, a_3, d_6 we obtain a factor of

$$\frac{u^3 v^3 t_3^2}{(1 - u^3 v^3 t_3^2)(1 - u^2 v^3 t_1)(1 - u^3 v t_3 t_5)}$$

in front. Let

$$F(\underline{X}, \underline{Y}, Z) = \sum_{\ell=0}^{\infty} Z^\ell \sum_{s_1, s_2 = \ell}^{\infty} \sum_{k_1=0}^{s_1} X_1^{k_1} X_2^{s_1-k_1} \sum_{k_2=0}^{s_2} Y_1^{k_2} Y_2^{s_2-k_2} = \frac{\tilde{F}(\underline{X}, \underline{Y}, Z)}{(X_1 - X_2)(Y_1 - Y_2)}.$$

Then the remaining sum is

$$\frac{F(\underline{X}, \underline{Y}, uv^2)}{(1-u^2v)(1-u^{-1}v)} - \frac{u^{-1}vF(\underline{X}, u^{-1}v\underline{Y}, u^2v)}{(1-uv^2)(1-u^{-1}v)} - \frac{u^2vF(u^2v\underline{X}, \underline{Y}, u^{-1}v)}{(1-uv^2)(1-u^2v)} \tag{55}$$

$$= \frac{(1-uv^2)\tilde{F}(\underline{X}, \underline{Y}, uv^2) - (1-u^2v)\tilde{F}(\underline{X}, u^{-1}v\underline{Y}, u^2v) - (1-u^{-1}v)\tilde{F}(u^2v\underline{X}, \underline{Y}, u^{-1}v)}{(X_1 - X_2)(Y_1 - Y_2)(1-u^2v)(1-uv^2)(1-u^{-1}v)},$$

evaluated at

$$X_1 = vt_2, \qquad X_2 = uvt_1t_3, \qquad Y_1 = u^2t_4, \qquad Y_2 = u^2v^2t_3t_5. \tag{56}$$

Also,

$$\tilde{F}(\underline{X}, \underline{Y}, Z) = \sum_{i,j=1}^{2} \frac{X_i Y_j}{(1-X_i)(1-Y_j)(1-X_iY_jZ)}.$$

Observe that $(1 - X_iY_jZ)$ takes the same value, namely $(1 - X_iY_juv^2)$, in each of the three terms in (55), for all i, j. We check the identity

$$\frac{(1-uv^2)XY}{(1-X)(1-Y)} - \frac{u^{-1}v(1-u^2v)XY}{(1-X)(1-u^{-1}v)} - \frac{u^2(1-u^{-1}v)XY}{(1-u^2vX)(1-Y)}$$

$$= \frac{(1-u^{-1}v)(1-u^2v)(1-uv^2XY)XY}{(1-X)(1-Y)(1-u^{-1}vY)(1-u^2vX)},$$

and apply it to X_i, Y_j for each i, j. We then cancel $(1-u^{-1}v)(1-u^2v)(1-uv^2X_iY_j)$. The sum on i, j now factors into two separate sums, which are easy to compute. Plugging in (56), we check that the result matches $\tilde{I}_{001}$.

7.3.4 The sum Q_{000}

For Q_{000} we have the additional constraints

$$a_3 + c_3 \geq d_6 \tag{57}$$
$$a_1 + c_4 \geq d_6 \tag{58}$$
$$b_3 + d_6 \geq c_4 \tag{59}$$

Let

$$G(\underline{W}, \underline{X}, \underline{Y}, \underline{Z}) = \sum W_1^{a_1} W_2^{a_2} X_1^{b_2} X_2^{b_3} Y_1^{r_1} Y_2^{c_3} Z_1^{c_4} Z_2^{d_6}$$

with the sum subject to all of our constraints except (47) and (52). Then, viewing these two as defining a sum as considered in out lemma above, we find that our sum is

$$\frac{G(\underline{W}, \underline{X}, \underline{Y}, Z_1, Z_2)}{(1-u^{-1}v)(1-u^2v)} - \frac{u^{-1}vG(\underline{W}, \underline{X}, u^{-1}v\underline{Y}, Z_1, uv^{-1}Z_2)}{(1-u^{-1}v)(1-uv^2)} - \frac{u^2vG(\underline{W}, \underline{X}, \underline{Y}, u^2vZ_1, Z_2)}{(1-u^2v)(1-uv^2)} \tag{60}$$

evaluated at

$$W_1 = u^v t_1 t_5, \qquad W_2 = u^3 v t_3 t_5, \qquad X_1 = v t_2, \qquad X_2 = uvt_1t_3,$$
$$Y_1 = u^2 t_4, \qquad Y_2 = u^2 v^2 t_3 t_5, \qquad Z_1 = vt_3^{-1}t_5, \qquad Z_2 = u^{-2}t_5^{-2}.$$

Now, let

$$H(\underline{W}, \underline{X}, \underline{Y}, Z) = \sum W_1^{a_1} W_2^{a_3} X_1^{b_2} X_2^{b_3} Y_1^{r_1} Y_2^{c_3} Z^{\ell}$$

where the sum is subject to

$$\ell \leq \min(b_2 + b_3,\ a_3 + b_3,\ r_1 + c_3,\ a_1 + a_3,\ a_3 + c_3).$$

Lemma: *We have*

$$G(\underline{W}, \underline{X}, \underline{Y}, \underline{Z}) = \frac{1 - W_1X_2Y_2Z_1Z_2}{(1-X_2Z_1)(1-W_1Y_2Z_2)} H(\underline{W}, \underline{X}, \underline{Y}, Z_1Z_2),$$

and

$$H(\underline{W}, \underline{X}, \underline{Y}, Z) = \frac{1 - W_1W_2X_2Y_2Z}{(1-W_1)(1-W_2)(1-W_1X_2Y_2Z)} F(\underline{X}, \underline{Y}, ZW_2),$$

where F is defined as in the last section.

Proof: The proof in both cases is just to break into two pieces and make appropriate changes of variable in each piece. To prove the first identity we consider the subsum defined by the additional condition $c_4 \geq d_6$. Which renders (51) and (58) redundant. We then make the change of variable $c_4 \mapsto c_4 + d_6$, followed by $b_3 \mapsto b_3 + c_4$. The resulting constraint-set is that of H with the role of ℓ played by d_6. We obtain from this piece $(1-X_2Z_1)^{-1}H$. In the sum over $c_4 + 1 \leq d_6$ we find that (59) is redundant, and we make the change of variable $d_6 \mapsto d_6 + c_4 + 1$ followed by $c_3 \mapsto c_3 + d_6 + 1, a_1 \mapsto a_1 + d_6 + 1$. Then we again obtain the sum defining $H(\underline{W}, \underline{X}, \underline{Y}, Z_1Z_2)$, with the role of ℓ played by c_4 this time, and the sum over d_6 producing $(W_1Y_2Z_2)(1-W_1Y_2Z_2)^{-1}$. Simplifying the sum of the two terms in front, we obtain the first identity. The second identity is proved in the same manner, this time defining our two pieces by $a_3 \geq \ell$ and $\ell \geq a_3 + 1$.

$\square$

Corollary: *We have:*

$$G(\underline{W}, \underline{X}, \underline{Y}, \underline{Z}) = \frac{1 - W_1W_2X_2Y_2Z_1Z_2}{(1-W_1)(1-W_2)(1-X_2Z_1)(1-W_1Y_2Z_2)} F(\underline{X}, \underline{Y}, Z_1Z_2W_2).$$

Returning to the evaluation of (60), note that the expression in front of the F takes the same value in all three of the terms of (60), and that value is

$$\frac{1 - u^6v^6t_1^2t_3^2t_5^2}{(1-u^2vt_1t_5)(1-u^3vt_3t_5)(1-uv^2t_1t_5)(1-u^2v^3t_1t_3)}.$$

The remaining expression involving F is precisely (55), which has already been evaluated. Once again we check matching of every term.

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