Note: Where appropriate, the “final answer” for each problem is given in bold italics for those not interested in the discussion of the solution.

I. Formulas
This section contains the formulas you will need for this homework set. It builds off of the notation used in the previous problem set solutions.

1. Expected Return on a Two-Asset Portfolio:

\[ E(r_p) = w_1E(r_1) + w_2E(r_2) \]  

where \( r_p \) is the return on the portfolio, \( r_1 \) and \( r_2 \) are the returns on assets 1 and 2, \( w_1 \) and \( w_2 = 1 - w_1 \) are the weights on the first and second assets (i.e. the fraction of the total dollar investment in each asset). This is a special case of the more general result that the expected return of a portfolio of N-assets is the allocation weighted average of the individual asset expected returns.

2. Return Variance of a Two-Asset Portfolio:

\[ Var(r_p) = \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}, \]  

where \( w_1 \) and \( w_2 = 1 - w_1 \) are the weights on the first and second assets (i.e. the fraction of the total dollar investment in each asset).

3. CAPM:

\[ E(r_i) = r_f + \beta[E(r_m) - r_f] \]
where \( r_i \) is the return on an arbitrary asset \( i \), \( r_f \) is the return on a risk-free asset, \( r_m \) is the return on the market and

\[
\beta = \frac{\text{Cov}(r_i, r_m)}{\text{Var}(r_m)} = \frac{\sigma_{im}}{\sigma_m^2} \quad (4)
\]

4. Beta on a Two-Asset portfolio:

\[
\beta_p = w_1 \beta_1 + w_2 \beta_2 \quad (5)
\]

where \( w_1 \) and \( w_2 = 1 - w_1 \) are the weights on the first and second assets and \( \beta_1 \) and \( \beta_2 \) are the betas on the first and second asset, with beta defined above. This is a special case of the more general result that the beta of a portfolio of \( N \)-assets is the allocation weighted average of the individual asset betas.

5. Constant Dividend Growth Formula:

\[
P_t = \frac{D_{t+1}}{r - g} \quad (6)
\]

where \( P_t \) is the price (or value) at time \( t \), \( D_{t+1} \) is the dividend received next period, \( r \) is the discount rate (e.g. cost of equity capital) and \( g \) is the constant growth rate of dividends. Note: This formula applies more generally to any stream of cash flows that grow at a constant rate. In other words, we can use this formula to value securities other than dividend paying equities.

Since dividends are equal to the payout ratio \( (d_t) \) times earnings \( (E_t) \), I sometimes write

\[
P_t = \frac{d_{t+1}E_{t+1}}{r - g}.
\]

II. Problems

1. We begin with the following information:

\[
\mathbb{E}(r_m) = 0.15 \\
r_f = 0.06.
\]
1.a
From the definition of beta in equation (4), the beta for Powergas is

\[ \beta_P = \frac{Cov(r_P, r_m)}{Var(r_m)} = \frac{0.0135}{(0.15)^2} = 0.6. \]

where \( r_P \) is the return to Powergas and \( r_m \) is the return to the market. Again, from the definition of beta we have

\[ \beta_S = \frac{Cov(r_S, r_m)}{Var(r_m)} \]

\[ 1.6 = \frac{Cov(r_S, r_m)}{(0.15)^2} \]

where \( r_S \) is the return to Supertech. **Solving for the covariance yields** \( Cov(r_S, r_m) = 0.036 \).

1.b
The correlation between Powertech and the market is:

\[ Corr(r_P, r_m) = \frac{Cov(r_P, r_m)}{SD(r_P)SD(r_m)} = \frac{0.0135}{0.45 \times 0.15} = 0.2, \]

where \( SD(r_i) \) is the standard deviation of the return to asset \( i \).

The correlation between Supertech and the market is:

\[ Corr(r_S, r_m) = \frac{Cov(r_S, r_m)}{SD(r_S)SD(r_m)} = \frac{0.036}{0.40 \times 0.15} = 0.6. \]

1.c
Begin by using the CAPM (equation (3)) to obtain the expected returns for each asset. First, Powergas:

\[ \mathbb{E}(r_P) = 0.06 + 0.6 (0.15 - 0.06) = 0.114 \]

Second, Supertech:

\[ \mathbb{E}(r_S) = 0.06 + 1.6 (0.15 - 0.06) = 0.204 \]
The expected rate of return on the market is 15%. We need to find a weight \( w \) such that

\[ 0.15 = w \times 0.114 + (1 - w) \times 0.204 \]

(This is the equation for the expected return on a portfolio of two assets, equation (1).) Solving for \( w \) yields 0.6 implying \textit{60% of our wealth should go into Powergas and 40% into Supertech.}

An alternative, but equivalent, approach recognizes that a portfolio with identical expected return to the market portfolio means that portfolio has the same beta as the market portfolio, \( \beta = 1 \). The beta of a portfolio is a weighted average of the individual betas (equation (5)) so,

\[ 1 = w \times 0.6 + (1 - w) \times 1.6. \]

The solution is again \( w = 0.6 \).

\textbf{1.d}

Using equation (1), we have

\[ 0.24 = w \times 0.114 + (1 - w) \times 0.204. \]

The solution is \( w = -0.4 \), which implies \textit{we should be short 40% in Powergas, and long 140% in Supertech.} (Note: an alternative approach is to use the betas to solve for the portfolio weights as in part 1.c)

To compute the risk (i.e. standard deviation), we begin by computing the variance:

\[
\sigma^2 = w^2 \sigma_P^2 + (1 - w)^2 \sigma_S^2 + 2w(1 - w) \sigma_P \sigma_S \rho_{PS} \\
= (-0.4)^2 (0.45)^2 + (1.4)^2 (0.4)^2 + 2(-0.4) (1.4) (0.45) (0.4) (0.5) \\
= 0.2452
\]

The standard deviation is the square root of the variance so that \textit{the risk is:}

\[ \sigma = \sqrt{0.2452} = 0.4952. \]
1.e
The portfolio in part 1.d has an expected return of 24%. We should be able to replicate this return with a lower standard deviation by forming a portfolio consisting of the market portfolio and the risk-free asset.

\[ 0.24 = w(0.06) + (1 - w)(0.15) \]

The solution is -1.0, implying we should short 100% of our investment in the risk-free asset (i.e. borrow money at the riskless rate) and go long 200% in the market.

The variance of this portfolio is

\[ \sigma^2 = (2)^2 (0.15)^2 = 0.09 \]

Thus, the risk, or standard deviation, is \( \sigma = \sqrt{0.09} = 0.3 \), which is less than the standard deviation of the portfolio above (0.4952).

2.
A brief summary of the information we are given is:

- Market Cap of Small Cap = $100 million
- \( \beta_{\text{Small Cap}} = 1.5 \)
- Market Cap of Low Cost = $400 million
- \( \beta_{\text{Low Cost}} = 1.2 \)
- \( r_f = 0.05 \)
- \( r_m - r_f = 0.08 \Rightarrow r_m = 0.13 \)

where “market cap” is short for the market capitalization of the firm (i.e. the price per share times the total number of shares).

2.a
The beta of the merged company is the weighted average of the betas of the individual companies (equation (5)). The weights are:

\[ w_{\text{Small Cap}} = \frac{100}{500} = 0.2 \]

which implies the weight on LowCost is \((1 - 0.2) = 0.8\). The beta of the merged company is thus

\[ \beta_{\text{Merge}} = 0.20 \times 1.5 + 0.80 \times 1.2 = 1.26 \]
2.b
The expected return follows from the CAPM model:

\[
\mathbb{E}(r_P) = r_f + \beta (\mathbb{E}(r_m) - r_f)
\]

\[
= 0.05 + 1.26 (0.08)
\]

\[
= 0.1508
\]

2.c
The required rate of return also follows from the CAPM:

\[
\mathbb{E}(r) = 0.05 + 1.2 (0.08)
\]

\[
= 0.1460
\]

The key is to recognize that we should still use the beta for LowCost since the two lines of business are unrelated. Thus, the merger has not reduced SmallCap’s cost of capital.

2.d
The exposure of the fund is unchanged. The fund holds both companies in proportion to their value weights. The market values are unaffected by the merger (there are no synergies and no capital structure issues) so the fund’s holdings are unaffected.

3.
3.a
There are three ways to solve this problem.

Method 1. Standardize the expected return of each fund with the risk-free asset to be equal to the market return. The expected returns are:

\[
\begin{align*}
\text{Fund 1 : } \mathbb{E}(r_1) &= 0.05 + 0.8 (0.08) = 0.114 \\
\text{Fund 2 : } \mathbb{E}(r_2) &= 0.05 + 1.2 (0.08) = 0.146.
\end{align*}
\]

Now find the weights for a portfolio of the fund and the risk-free asset that match the expected return for the market (0.13). First, fund 1:

\[
0.13 = w0.114 + (1 - w) 0.05,
\]
implies \( w = 1.25 \). The variance of this portfolio is:

\[
(1.25)^2 (0.20)^2 = 0.0625.
\]

For fund 2,

\[
0.13 = w0.146 + (1 - w) 0.05
\]

which implies \( w = 0.833 \). The variance of this portfolio is

\[
(0.83)^2 (0.32)^2 = 0.0705
\]

Therefore, **we prefer fund 1 because it has a lower variance for the same rate of return.**

**Method 2.** Now let’s compare the riskiness of two funds with the same beta. To do this, we form a portfolio of the fund and the risk-free asset. The beta for our portfolio is a weighted average of the individual asset betas. Since the risk-free asset has a beta of zero, we only need to concern ourselves with the relevant fund in the portfolio. We normalize the beta of the portfolio’s to equal 1 (this number is arbitrary. The portfolio betas just need to be the same). This implies our fund 1 portfolio will contain

\[
1.0 = w\beta_1 = w0.8 \Rightarrow w = 1.25,
\]

or 125% of our wealth invested in fund 1 to obtain a portfolio beta of 1, exactly as before. Our fund 2 portfolio will contain

\[
1.0 = w\beta_2 = w1.2 \Rightarrow \beta_2 = 0.83,
\]

or 83% of our wealth invested in fund 2. Calculating the variance of each of these portfolios was done above and shows that a portfolio of fund 1 and the risk-free asset is preferred.

**Method 3.** A final method examines the Sharpe-ratios for these two funds. This allows us to compare the expected return normalized by the standard deviation, that is, the expected reward per unit of risk. For fund 1,

\[
Sharpe_1 = \frac{\mathbb{E}(r_1) - r_f}{\sigma_1} = \frac{0.114 - 0.05}{0.20} = 0.32
\]

where the expected return for the fund was computed using the CAPM. For fund 2,

\[
Sharpe_2 = \frac{\mathbb{E}(r_2) - r_f}{\sigma_2} = \frac{0.146 - 0.05}{0.32} = 0.3.
\]

Again, fund 1 is preferred because it offers a greater reward per unit of risk.

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1. Technical note: It is does not matter if we compare variances or standard deviations since they are monotonic transformations of one another on the positive half-line.
3. b
We know from part 3.a that we prefer fund 1 to fund 2. The portfolio weights to generate an expected return of 14.6% are

\[ 0.146 = w0.114 + (1 - w)0.05 \]

implying \( w = 1.5 \). Thus we buy fund one with 150\% of our wealth (i.e. 50\% on margin) and short 50\% of the risk-free asset. The standard deviation of this portfolio is

\[ \sigma_p = (1.5)(0.20) = 0.3, \]

which is less than fund 2’s standard deviation.

3. c
If \((1 - w)\) is negative, than our investment in the risk-free asset is a short position, which means that we have borrowed money. But now we can only borrow at 7\%. This changes the calculation to

\[ 0.146 = w0.114 + (1 - w)0.07 \]

implying \( w = 1.7273 \). The standard deviation of this portfolio is

\[ \sigma_p = 1.7273(0.20) = 0.3456, \]

which is greater than the standard deviation of fund 2. Thus, we should put all of our money into fund 2.

4.
We want to find the abnormal return for each fund and choose the highest. Let \( \alpha \) be the abnormal return, equal to the actual return minus the expected return from the CAPM. Then,

\[ \alpha = r - [r_f + \beta (E(r_m) - r_f)] \]

where \( r \) is the ex-post realized return.

\[ \alpha_A = 0.30 - (0.06 + 0.8(0.30 - 0.06)) = 0.048 \]
\[ \alpha_B = 0.34 - (0.06 + 1.2(0.30 - 0.06)) = -0.008 \]
\[ \alpha_C = 0.33 - (0.06 + 1.0(0.30 - 0.06)) = 0.030, \]

implying that fund A offered the best risk-adjusted returns.
5.  
5.a  
The CAPM predicts an expected return of 
\[ \mathbb{E}(r_A) = 0.07 + 1.5(0.15 - 0.07) = 0.19. \]
A single share sells at a discount of 19% implying 
\[ \text{Price} = \frac{100}{1.19} = \$84.03. \]

5.b  
The problem gives us the following information: 
\[ \rho_{KM} = 0.7, \quad \sigma_K = 0.20, \quad \sigma_M = 0.15. \]
The implied beta of KnowLode is 
\[ \beta_K = \frac{0.7(0.2)(0.15)}{(0.15)^2} = 0.933, \]
which follows from equations (4) and the definition of the correlation coefficient. Since the riskless asset has a beta of zero, we need only concern ourselves with Knowlode. 
\[ 1.5 = 0.933w \Rightarrow w = 1.608 \]
Therefore, 160.8% of our wealth should be invested in KnowLode and -60.8% into the risk-free asset (i.e. borrow money).

6.  
6.a  
Using the CAPM 
\[ Er_A = 0.05 + 0.9(0.08) = 0.122 \]
\[ Er_B = 0.05 + 1.2(0.08) = 0.146 \]
6.b

The merged company beta is

\[ \beta_{\text{Merge}} = \frac{10}{30} 0.9 + \frac{20}{30} 1.2 = 1.1, \]

implying the required rate of return is

\[ E r_{\text{Merge}} = 0.05 + 1.1 (0.08) = 0.138. \]

The variance of the merged company is

\[ \sigma^2_{\text{Merge}} = \left( \frac{10}{30} \right)^2 (0.25)^2 + \left( \frac{20}{30} \right)^2 (0.30)^2 + 2 \left( \frac{10}{30} \right) \left( \frac{20}{30} \right) (0.25) (0.30) (0.2) \]

\[ = 0.0536 \]

implying the standard deviation is \( \sqrt{0.0536} = 0.2315. \)

Since the two lines of business are unrelated, the discount factor has not changed. Any project undertaken by one of the division’s needs to be discounted at that divisions specific discount rate. However, the merger did reduce the risk of the company’s cash flow by taking advantage of diversification. Bottom line: the merger did not create value (in the absence of synergies and leverage considerations), but it did reduce risk. However, investors can take care of this by diversifying their own portfolio directly.

6.c

The constant dividend growth model (equation (6)) implies

\[ P_0 = \frac{D_1}{r - g} = \frac{(1 + g) D_0}{r - g} \]

assuming the growth rate over the first year is \( g. \) Letting \( y = D_0/P_0 \) represent the dividend yield,

\[ g = \frac{r - y}{1 + y} \]

Substituting for the unknowns implies

\[ g = \frac{0.122 - 0.05}{1 + 0.05} = 0.0686. \]
Similarly, for firm B

\[ g = \frac{0.146 - 0.01}{1 + 0.01} = 0.1347 \]

The dividend for each company may be obtained by manipulating the constant dividend growth equation. For company A

\[ D_0 = \frac{P_0 (r - g)}{1 + g} = \frac{10 (0.122 - 0.0686)}{1 + 0.0686} = 0.4997 \text{ billion} \]

For company B

\[ D_0 = \frac{20 (0.146 - 0.1347)}{1 + 0.1347} = 0.1992 \text{ billion} \]

The dividend of the merged company is the sum of these two figures, $0.700 billion, and the dividend yield of the merged company is

\[ \frac{D_0}{P_0} = \frac{0.70}{30} = 0.023. \]

The growth rate for the merged company is thus

\[ g = \frac{0.138 - 0.023}{1 + 0.023} = 0.1124 \]

7.
You are given the following information about possible investments:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean Return</th>
<th>Market Value</th>
<th>Standard Deviation</th>
<th>Correlation with Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Estate</td>
<td>8%</td>
<td>$5 trillion</td>
<td>15%</td>
<td>0.5</td>
</tr>
<tr>
<td>Growth Stocks</td>
<td>12%</td>
<td>$2 trillion</td>
<td>22.5%</td>
<td>1.0</td>
</tr>
<tr>
<td>Antiques</td>
<td>15%</td>
<td>$2.5 trillion</td>
<td>15%</td>
<td>0.25</td>
</tr>
</tbody>
</table>

7.a
The equal- and value-weighted returns are 11.67% and 10.68%, respectively.
7.b
Use the fact that:

\[ \beta = \frac{Cov(r_j, r_m)}{\sigma_m^2} = \frac{\rho_{j,m} \sigma_j \sigma_m}{\sigma_m^2} = \frac{\rho_{j,m} \sigma_j}{\sigma_m} \]

Plugging in the given information we get that the real estate beta = 0.5, the growth stock beta is 1.5, and the Antiques beta is 0.25.

7.c
We need to solve two equations in two unknowns. The first equation is the expected return for real estate:

\[ 0.08 = r_f + 0.5[E(r_m) - r_f] \]

The second equation is the expected return for growth stocks:

\[ 0.12 = r_f + 1.5[E(r_m) - r_f] \]

Solving these two equations implies that the market expected return is equal to 10% and the risk-free rate is equal to 6%.

7.d
The expected return to Antiques is too high relative to the prediction of the CAPM (or SML). To take advantage of this mispricing, we need to form a portfolio using real estate and growth stocks that has the same beta as Antiques. This can be done by finding weights \( w \) and \((1 - w)\) such that the weighted sum of the constituent securities equals 0.25:

\[ 0.25 = w\beta_{\text{Real Estate}} + (1 - w)\beta_{\text{Growth Stocks}} \]

Solving implies that \( w = 1.25 \).  

† These solutions are produced by Michael R. Roberts. Thanks go to Jen Rother for her excellent assistance, an anonymous TA, and Alon Brav. Any remaining errors are mine.
8.

8.a
Use the fact that:

\[ \beta = \frac{\text{Cov}(r_j, r_m)}{\sigma_m^2} = \frac{\rho_{j,m} \sigma_j \sigma_m}{\sigma_m^2} = \frac{\rho_{j,m} \sigma_j}{\sigma_m} \]

The value stock beta is 1.5, growth stock beta is 0.5, gold beta is -0.5 and the T-bill beta is 0.

8.b
For value stocks the SML implies:

\[ 0.18 = 0.06 + 1.5[E(r_m) - 0.06] \]

which yields an expected return on the market equal to 14%.

For Growth stocks the SML implies:

\[ 0.06 + 1.5[0.14 - 0.06] = 0.10 \]

For Gold the SML implies:

\[ 0.06 - 0.5[0.14 - 0.06] = 0.02 \]

8.c
BadCo's expected return is -14%, while the CAPM predicts that the expected return should be

\[ 0.06 - 2.0[0.14 - 0.06] = 0.10. \]

Thus, the expected return is too low implying that BadCo is overpriced.

To take advantage of this mispricing, we should sell BadCo's shares and buy a portfolio with the same beta. There are many ways to create such a
portfolio. For example, we can use a portfolio of the market and T-bills by matching the portfolio beta to that of BadCo.

\[-2 = (1 - w) \beta_{\text{riskless}} + w \beta_{\text{Market}} = (1 - w) \times 0 + w \times 1 = w\]

Thus our portfolio should contain -200 percent of our wealth in the market and 300% in T-bills. The expected gain on this strategy is 4%.

8.d

No. We eliminated systematic risk but there is still residual risk. On average we stand to make positive expected return but we are still exposed to variability due to the idiosyncratic variability of the firm’s return.