Abstract
In this paper, we study the Empirical Risk Minimization problem in the non-interactive local model of differential privacy. We first show that if the ERM loss function is $(\infty, T)$-smooth, then we can avoid a dependence of the sample complexity, to achieve error $\alpha$, on the exponential of the dimensionality $p$ with base $1/\alpha$ (i.e., $\alpha^{-p}$), which answers a question in (Smith et al., 2017). Our approach is based on Bernstein polynomial approximation. Then, we propose player-efficient algorithms with 1-bit communication complexity and $O(1)$ computation cost for each player. The error bound is asymptotically the same as the original one. Also with additional assumptions we show a server efficient algorithm with polynomial running time. At last, we propose (efficient) non-interactive locally differential private algorithms, based on different types of polynomial approximations, for learning the set of k-way marginal queries and the set of smooth queries.

1. Introduction
In the big data era, a tremendous amount of individual data are generated every day. Such data, if properly used, could greatly improve many aspects of our daily lives. However, due to the sensitive nature of such data, a great deal of care needs to be taken while analyzing them. Private data analysis seeks to enable the benefits of learning from data with the guarantee of privacy-preservation. Differential privacy (Dwork et al., 2006) has emerged as a rigorous notion for privacy which allows accurate data analysis with a guaranteed bound on the increase in harm for each individual to contribute her data. Methods to guarantee differential privacy have been widely studied, and recently adopted in industry (Near, 2018; Erlingsson et al., 2014).

Two main user models have emerged for differential privacy: the central model and the local one. In the central model, data are managed by a trusted central entity which is responsible for collecting them and for deciding which differentially private data analysis to perform and to release. A classical use case for this model is the one of census data (Haney et al., 2017). In the local model instead, each individual manages his/her proper data and discloses them to a server through some differentially private mechanisms. The server collects the (now private) data of each individual and combines them into a resulting data analysis. A classical use case for this model is the one aiming at collecting statistics from user devices like in the case of Google’s Chrome browser (Erlingsson et al., 2014), and Apple’s iOS-10 (Near, 2018; Tang et al., 2017).

In the local model, there are two basic kinds of protocols: interactive and non-interactive. Smith et al. (2017) have recently investigated the power of non-interactive differentially private protocols. These protocols are more natural for the classical use cases of the local model: both the projects from Google and Apple use the non-interactive model. Moreover, implementing efficient interactive protocols in such applications is more difficult due to the latency of the network. Despite being used in industry, the local model has been much less studied than the central one. Part of the reason for this is that there are intrinsic limitations in what one can do in the local model. As a consequence, many basic questions, that are well studied in the central model, have not been completely understood in the local model, yet.

In this paper, we study differentially private Empirical Risk Minimization in the non-interactive local model. Before showing our contributions and discussing comparisons with previous work, we firstly discuss our motivations.

Problem setting (Smith et al., 2017) Given a convex, closed and bounded constraint set $C \subseteq \mathbb{R}^p$, a data universe $D$, and a loss function $\ell : C \times D \mapsto \mathbb{R}$. A dataset $D = \{x_1, x_2, \cdots, x_n\} \in \mathbb{D}^n$ defines an empirical risk function: $L(\theta; D) = \frac{1}{n} \sum_{i=1}^n \ell(\theta, x_i)$. When the inputs are drawn i.i.d from an unknown underlying distribution.
\[ P \] on \( D \), we can also define the population risk function:
\[ L_P(\theta) = \mathbb{E}_{D \sim P}[\ell(\theta; D)]. \]

Now we have the following two kinds of excess risk:

**Empirical:** \( \text{Err}_D(\theta_{\text{priv}}) = \hat{L}(\theta_{\text{priv}}; D) - \min_{\theta \in C} \hat{L}(\theta; D), \)

**Population:** \( \text{Err}_P(\theta_{\text{priv}}) = L_P(\theta_{\text{priv}}) - \min_{\theta \in C} L_P(\theta). \)

The problem that we study in this paper is for finding \( \theta_{\text{priv}} \in C \) under non-interactive local differential privacy (see Definition 1) which makes the empirical and population excess risk as small as possible. Alternatively, we can express this problem in terms of sample complexity as finding as small of \( n \) as possible for achieving \( \text{Err}_D \leq \alpha \) and \( \text{Err}_P \leq \alpha \), where \( \alpha \) is the user specified error tolerance (or simply called error). We denote this problem as NILDP-ERM.

**Motivation** Smith et al. (2017) prove the following result concerning NILDP-ERM for general convex 1-Lipschitz loss functions over a bounded constraint set.

**Theorem.** (Smith et al., 2017) Under the assumptions above, there is an \( \epsilon \) non-interactive LDP algorithm such that for all distribution \( P \) on \( D \), with probability \( 1 - \beta \), we have \( \text{Err}_P \leq \hat{O}(\sqrt{n \log^2 (1/\beta) / \epsilon^2 n}) \). A similar result holds for \( \text{Err}_D \). Alternatively, to achieve error \( \alpha \), the sample complexity must satisfies \( n = \Omega(\sqrt{p} \epsilon^{-2\alpha - (p+1)}) \), where \( c \) is some constant.

More importantly, they also show that the dependence of the sample size over the dimensionality \( p \), in the terms \( \alpha^{-(p+1)} \) and \( c^p \), is unavoidable.

This situation is somehow undesirable: when the dimensionality is high and the target error is low, the dependency on \( \alpha^{-(p+1)} \) could make the sample size quite large. However, several results have already shown that for some specific loss functions, the exponential dependency on the dimensionality can be avoided. For example, Smith et al. (2017) show that, in the case of linear regression, there is a non-interactive \((\epsilon, \delta)\)-LDP algorithm\(^1\) whose sample complexity for achieving error \( \alpha \) for the empirical risk is \( n = \Omega(p \log(1/\delta) \epsilon^{-2} \alpha^{-2}) \). Similarly, Zheng et al. (2017) showed that for logistic regression, if the sample complexity satisfies \( n > O((\frac{r}{c})^{4r \log \log(8 \epsilon/\alpha)} (\frac{8}{c})^{2r \log \log(8 \epsilon/\alpha)} + 2(\frac{1}{\alpha^2})), \) where \( c \) and \( r \) are independent on \( p \), then there is an non-interactive \((\epsilon, \delta)\)-LDP such that \( \text{Err}_P(\theta_{\text{priv}}) \leq \alpha \).

So, we have a gap between the general case and the case of specific loss functions. For this reason, in this paper we will study the following natural question: can we give natural conditions on the loss function that guarantee non-interactive \( \epsilon \)-LDP with sample complexity that is not exponential in the dimensionality \( p \)?

**Our contributions** We first show that if the loss function is \((\infty, T)\)-smooth (Definition 5), then, there is a non-interactive \( \epsilon \)-LDP algorithm, such that when \( n \geq \tilde{O}(\epsilon^2 D_p^2 p^2 e^{-2 \alpha^{-4}}) \), we have empirical excess risk \( \text{Err}_D \leq \alpha \), where \( D_p \) depends only on \( p \). Interestingly, to obtain this result we do not need the loss function to be convex. However, if the loss function is convex and 1-Lipschitz, results of population excess risk can also be achieved. For example, with \( n \geq \tilde{O}(D_p^2 p^3 e^{2} \epsilon^{-\frac{4}{3}} \alpha^{-\frac{3}{2}}) \), we can have population excess risk \( \text{Err}_P \leq \alpha \). Note that in these results the dependence on \( \alpha \) is \( \alpha^{-4} \) and \( \alpha^{-\frac{3}{2}} \), respectively, rather than \( \alpha^{-\frac{p+1}{2}} \). To prove these results, we use multivariate Bernstein polynomials to approximate the loss function and an LDP algorithm to estimate the coefficients (Section 4).

Next, we address the efficiency issue, which has not been well studied before (Smith et al., 2017). Following an approach similar to (Bassily & Smith, 2015), we propose an algorithm which has only 1-bit communication cost and \( O(1) \) computation cost for each client, and which achieves asymptotically the same error bound as the original one. Additionally, we show also a novel analysis for the server. This shows that if the loss function is convex and Lipschitz and the convex set satisfies some natural conditions, then we have an algorithm which achieves the error bound of \( O(p \alpha) \) and runs in polynomial time when \( n \) is the same as in the previous part.

At last, we show the generality of our technique by applying polynomial approximation to other problems. We give a non-interactive LDP algorithm for answering the class of k-way marginals queries, by using Chebyshev polynomials approximation, and a non-interactive LDP algorithm for answering the class of smooth queries, by using trigonometric polynomials approximation.

In this paper, we focus on eliminating the dependency on the term \( \alpha^{-1}(p) \). The methods we propose still have a dependency on a term \( c^p \) which comes from the perturbation of the coefficients. We leave the development of methods without this dependency for future works.

Due to the space limit, all the proofs and some details of algorithms can be found in the supplemental material.

**2. Related Work**

ERM in the local model of differential privacy has been studied in (Kasiviswanathan et al., 2011; Beimel et al., 2008; Duchi et al., 2017; 2013; Zheng et al., 2017; Smith et al., 2017). Kasiviswanathan et al. (2011) showed a general equivalence between learning in the local model and learn-
ing in the statistical query model. Beimel et al. (2008) showed the lower bound of the squared error of distributed protocols for mean estimation. Duchi et al. (2017; 2013) gave the lower bound $O\left(\frac{d}{\epsilon^4 n}\right)$ and optimal algorithms for general convex optimization; however, their optimal procedure needs many rounds of interactions. The works that are most related to ours are (Zheng et al., 2017; Smith et al., 2017). Zheng et al. (2017) considered some specific loss functions in high dimensions, such as sparse linear regression and kernel ridge regression. Note that although it also studied a class of loss functions (i.e., Smooth Generalized Linear Loss functions) and used the polynomial approximation approach, the functions investigated in our paper are more general, which include linear regression and logistic regression, and the approximation techniques are quite different. Smith et al. (2017) studied general convex loss functions for population excess risk and showed that the dependence on the exponential of the dimensionality is unavoidable. In this paper, we show that such a dependence in the term of $\alpha$ is actually avoidable for a class of loss functions, and this even holds for non-convex loss functions, which is a big difference from all existing works. In addition, our algorithms are simpler and more efficient. Kulkarni et al. (2017) recently studied the problem of releasing $k$-way marginal queries in LDP. They compare different LDP methods to release marginal statistics but they do not consider methods based on polynomial approximation.

For other problems under LDP model, (Bun et al., 2017; Bassily & Smith, 2015; Bassily et al., 2017; Hsu et al., 2012) studied heavy hitter problem, (Ye & Barg, 2017; Kairouz et al., 2016; Wang et al., 2017b; Ye & Barg, 2017) considered local private distribution estimation. ERM in central differentially private has been studied well, such as (Bassily et al., 2014; Talwar et al., 2015; Chaudhuri et al., 2011; Wang et al., 2017a). The polynomial approximation approach has been used under central model in (Aldà & Rubinstein, 2017; Wang et al., 2016; Thaler et al., 2012; Zheng et al., 2017).

3. Preliminaries

Differential privacy in the local model. In LDP, we have a data universe $D$, $n$ players with each holding a private data record $x_i \in D$, and a server that is in charge of coordinating the protocol. An LDP protocol proceeds in $T$ rounds. In each round, the server sends a message, which we sometime call a query, to a subset of the players, requesting them to run a particular algorithm. Based on the queries, each player $i$ in the subset selects an algorithm $Q_i$, run it on her data, and sends the output back to the server.

Definition 1. (Kasiviswanathan et al., 2011; Smith et al., 2017) An algorithm $Q$ is $\epsilon$-locally differentially private (LDP) if for all pairs $x, x' \in D$, and for all events $E$ in the output space of $Q$, we have

$$\Pr[Q(x) \in E] \leq e^\epsilon \Pr[Q(x') \in E].$$

A multi-player protocol is $\epsilon$-LDP if for all possible inputs and runs of the protocol, the transcript of player i’s interaction with the server is $\epsilon$-LDP. If $T = 1$, we say that the protocol is $\epsilon$ non-interactive LDP.

Since we only consider non-interactive LDP through the paper, we will use LDP as non-interactive LDP below.

As an example that will be useful in the sequel, the next lemma shows an $\epsilon$-LDP algorithm for computing 1-dimensional average.

Algorithm 1 1-dim LDP-AVG

1: Input: Player $i \in [n]$ holding data $v_i \in [0, b]$, privacy parameter $\epsilon$.
2: for Each Player $i$ do
3: send $z_i = v_i + \text{Lap}\left(\frac{\epsilon}{b}\right)$
4: end for
5: for The Server do
6: Output $a = \frac{1}{n} \sum_{i=1}^{n} z_i$
7: end for

Lemma 1. Algorithm 1 is $\epsilon$-LDP. Moreover, if player $i \in [n]$ holds value $v_i \in [0, b]$ and $n > \log \frac{2}{\beta} \geq 0 < \beta < 1$, then, with probability at least $1 - \beta$, the output $a \in \mathbb{R}$ satisfies:

$$|a - \frac{1}{n} \sum_{i=1}^{n} v_i| \leq \frac{2b\sqrt{\log \frac{2}{\beta}}}{\sqrt{n\epsilon}}. \quad (1)$$

Bernstein polynomials and approximation. We give here some basic definitions that will be used in the sequel; more details can be found in (Aldà & Rubinstein, 2017; Lorentz, 1986; Micchelli, 1973).

Definition 2. Let $k$ be a positive integer. The Bernstein basis polynomials of degree $k$ are defined as $b_{v,k}(x) = \binom{k}{v} x^v (1-x)^{k-v}$ for $v = 0, \cdots, k$.

Definition 3. Let $f : [0, 1] \rightarrow \mathbb{R}$ and $k$ be a positive integer. Then, the Bernstein polynomial of $f$ of degree $k$ is defined as $B_k(f)(x) = \sum_{v=0}^{k} f(v/k) b_{v,k}(x)$. We denote by $B_k$ the Bernstein operator $B_k(f)(x) = B_k(f, x)$.

Bernstein polynomials can be used to approximate some smooth functions over $[0, 1]$.

Definition 4. (Micchelli, 1973) Let $h$ be a positive integer. The iterate Bernstein operator of order $h$ is defined as the sequence of linear operators $B_h^{(h)} = I - (I - B_h)^{h} = \sum_{i=1}^{h} \binom{h}{i} (-1)^{i-1} B_k^{i}$, where $I = B_0^1$ denotes the identity operator and $B_k^i$ is defined as $B_k^i = B_k \circ B_k^{i-1}$. The iterated
Bernstein polynomial of order \( h \) can be computed as
\[
B_k^{(h)}(f; x) = \sum_{v=0}^{k} f(x) b_v^{(h)}(x),
\]
where \( b_v^{(h)}(x) = \sum_{i=1}^{h} (-1)^{i-1} b_{v,i}(h; x) \).

Iterate Bernstein operator can well approximate univariate \((h, T)\)-smooth functions.

**Definition 5.** (Micchelli, 1973) Let \( h \) be a positive integer and \( T > 0 \) be a constant. A function \( f : [0, 1]^p \mapsto \mathbb{R} \) is \((h, T)\)-smooth if it is in class \( C^h([0, 1]^p) \) and its partial derivatives up to order \( h \) are all bounded by \( T \). We say it is \((\infty, T)\)-smooth if, for every \( h \in \mathbb{N} \) it is \((h, T)\)-smooth.

**Theorem 1.** (Micchelli, 1973) If \( f : [0, 1]^p \mapsto \mathbb{R} \) is a \((2h, T)\)-smooth function, then for all positive integers \( k \) and \( y \in [0, 1] \), we have \( |f(y) - B_k^{(h)}(f; y)| \leq TD_h k^{-h} \), where \( D_h \) is a constant independent of \( k, f \) and \( y \).

The theorem above is for univariate functions, Aldà & Rubinstein (2017) extend it to multivariate functions.

**Definition 6.** Assume \( f : [0, 1]^p \mapsto \mathbb{R} \) and let \( k_1, \ldots, k_p, h \) be positive integers. The multivariate iterated Bernstein polynomial of order \( h \) at \( y = (y_1, \ldots, y_d) \) is defined as:
\[
B_{k_1,\ldots,k_p}^{(h)}(f; y) = \sum_{j=1}^{p} k_j \sum_{v_j=0}^{k_j} \frac{f(y)}{k_j!} \frac{v_j!}{k_j^p} \prod_{j=1}^{p} b_v^{(h)}(y_j).
\]

We denote \( B_k^{(h)} = B_{k_1,\ldots,k_p}^{(h)}(f; y) \) if \( k = k_1 = \cdots = k_p \).

**Theorem 2.** (Aldà & Rubinstein, 2017) If \( f : [0, 1]^p \mapsto \mathbb{R} \) is a \((2h, T)\)-smooth function, then for all positive integers \( k \) and \( y \in [0, 1] \), we have \( |f(y) - B_k^{(h)}(f; y)| \leq O((pTD_h k^{-h})^p) \).

**Our settings** We conclude this section by making explicitly the settings that we will consider throughout the paper. We assume that there is a constraint set \( \mathcal{C} \subseteq [0, 1]^p \) and for every \( x \in \mathcal{D} \) and \( \theta \in \mathcal{C} \), \( \ell(\cdot, x) \) is well defined on \([0, 1]^p \) and \( \ell(\theta, x) \in [0, 1] \). These closed intervals can be extended to arbitrarily bounded closed intervals. Our settings are similar to the ‘Typical Settings’ in (Smith et al., 2017), where \( \mathcal{C} \subseteq [0, 1]^p \) appears in their Theorem 10, and \( \ell(\theta, x) \in [0, 1] \) from their 1-Lipschitz requirement and \( ||\mathcal{C}||_2 \leq 1 \).

**4. Main Result**

Definition 6 and Theorem 2 tell us that if we know the value of the empirical risk function, i.e. the average of the sum of loss functions, on each of the grid points \((v_1, \ldots, v_p)\), where \( (v_1, \ldots, v_p) \in \mathcal{T} = \{0, 1, \ldots, k\}^p \) for some large \( k \), then we can approximate it well. Our main observation is that this can be done in the local model by estimating the average of the sum of loss functions on each of the grid points using Algorithm 1. This is the idea of Algorithm 2.

**Algorithm 2 Local Bernstein Mechanism**

1. **Input:** Player \( i \in [n] \) holding data \( x_i \in \mathcal{D} \), public loss function \( \ell : [0, 1]^p \times \mathcal{D} \mapsto [0, 1] \), privacy parameter \( \epsilon > 0 \), and parameter \( k \).
2. Construct the grid \( \mathcal{T} = \{v_1, \ldots, v_p\} \), where \( \{v_1, \ldots, v_p\} \in \{0, 1, \ldots, k\}^p \).
3. **for** Each grid point \( v = (v_1, \ldots, v_p) \in \mathcal{T} \) **do**
4. **for** Each Player \( i \in [n] \) **do**
5. Calculate \( \ell(v, x_i) \).
6. **end for**
7. Run Algorithm 1 with \( \epsilon = \frac{\epsilon}{(d+1)p} \) and \( b = 1 \) and denote the output as \( \tilde{L}(v; D) \).
8. **end for**
9. **for** The Server **do**
10. Construct Bernstein polynomial, as in (2), for the perturbed empirical loss \( \tilde{L}(v; D) \). Denote \( \tilde{L}(\cdot, D) \) the corresponding function.
11. **Compute** \( \theta_{\text{priv}} = \arg\min_{\theta \in \mathcal{C}} \tilde{L}(\theta; D) \).
12. **end for**

**Theorem 3.** For \( \epsilon > 0, 0 < \beta < 1 \), Algorithm 2 is \( \epsilon \)-LDP. Assume that the loss function \( \ell(\cdot, x) \) is \((2h, T)\)-smooth for all \( x \in \mathcal{D} \) for some positive integer \( h \) and constant \( T \). If \( n, \epsilon \) and \( \beta \) satisfy \( n = \Omega\left(\frac{\log \frac{1}{\beta} e^{\epsilon^2 p}}{\epsilon^2 D_h^2 p} \right) \) for some constant \( c \), then setting \( k = \mathcal{O}\left(\frac{D_h^2 p^2 \beta^2 e^{(h+1)p} \epsilon}{\epsilon^2 \log \frac{1}{\beta}} \right) \) we have with probability at least \( 1 - \beta \):
\[
\text{Err}_D(\theta_{\text{priv}}) \leq \tilde{O}\left(\frac{\log \frac{1}{\beta} D_h^2 p^2 \beta^2 e^{(h+1)p} \epsilon}{n \log \frac{1}{\beta} e^{\epsilon^2 D_h^2}} \right),
\]
where \( \tilde{O} \) hides the log and \( T \) terms.

From (3) we can see that in order to achieve error \( \alpha \), the sample complexity needs to be \( n = \tilde{O}(\log \frac{1}{\beta} D_h^2 p^2 \beta^2 e^{(h+1)p} \epsilon) \). As a particular case, we have the following.

**Corollary 1.** If the loss function \( \ell(\cdot, x) \) is \((\infty, T)\)-smooth for all \( x \in \mathcal{D} \) for some constant \( T \), and if \( n, \epsilon, \beta, k \) satisfy the condition in Theorem 3 with \( h = p \), then with probability at least \( 1 - \beta \), the output \( \theta_{\text{priv}} \) of Algorithm 2 satisfies:
\[
\text{Err}_D(\theta_{\text{priv}}) \leq \tilde{O}\left(\frac{\sqrt{\log \frac{1}{\beta} D_h^2 p^2 \beta^2 e^{(h+1)p}}}{n^{1/4}} \right),
\]
where \( \tilde{O} \) hides the log and \( T \) terms. So, to achieve error \( \alpha \), with probability at least \( 1 - \beta \), we have sample complexity:
\[
n = \tilde{O}\left(\max\{c_{\epsilon^2 p}^2 \log \frac{1}{\beta} D_h^2 p^2 \epsilon^2, \frac{1}{\epsilon^4 D_h^2}\} \right),
\]
for some constants $c, c_1$.

It is worth noticing that from (3), when the term $\frac{b}{p}$ grows, the term $\alpha$ decreases. Thus, for loss functions that are $(\infty, T)$-smooth, we can get a smaller dependency than the term $\alpha^{-4}$ in (5). For example, if we take $h = 2p$, then the sample complexity is $n = \Omega(\max\{\beta^2 p^{1.4} D_p^2 \sqrt{p} \epsilon^{-3} - \log \frac{p}{\epsilon^2 D_p^2}\})$. When $h \to \infty$, the dependency on the error becomes $\alpha^{-2}$, which is the optimal bound, even for convex functions. However, the dependency on $c^b$ makes it still impractical for $h \to \infty$.

Our analysis of the empirical excess risk does not use the convexity assumption. While this gives a bound which is not optimal, even for $p = 1$, it also says that our result holds for non-convex loss functions and constrained domain set, as long as they are smooth enough.

Using the convexity assumption of the loss function, and a lemma in (Shalev-Shwartz et al., 2009), we can also give a bound on the population excess risk.

**Theorem 4.** Under the conditions in Corollary 1, if we further assume the loss function $\ell(x, y)$ to be convex and 1-Lipschitz for all $x \in D$ and the convex set $C$ satisfying $\|C\|_2 \leq 1$, then with probability at least $1 - 2\beta$, we have:

$$\text{Err}_p(\theta_{\text{prev}}) \leq \tilde{O}\left(\frac{\beta \sqrt{\log(1/\beta)}}{n \epsilon^2} + \frac{1}{\sqrt{\beta d}}\right).$$

That is, if we have sample complexity $n = \tilde{O}(\max\{\beta \sqrt{\log(1/\beta)} D_p^2 \sqrt{p} \epsilon^{-3} - \log \frac{p}{\epsilon^2 D_p^2}\})$, then we have $\text{Err}_p(\theta_{\text{prev}}) \leq \alpha$. Here $c, c_1, c_2$ are some constants.

Corollary 1 and Theorem 4 provide an answer to our motivating question. That is, for loss functions which are $(\infty, T)$-smooth, there is an $\epsilon$-LDP algorithm for empirical and population excess risks achieving error $\alpha$ with sample complexity which is independent from the dimensionality $p$ in the term $\alpha$. This result does not contradict the results by Smith et al. (2017). Indeed, the example they provide whose sample complexity must depend on $\alpha^{-\Omega(p)}$, to achieve the $\alpha$ error, is actually non-smooth.

In our result, like in the one by Smith et al. (2017), there is still a dependency of the sample complexity in the term $c^b$, for some constant $c$. Furthermore ours has also a dependency in the term $D_p$. There is still the question about what condition would allow a sample complexity independent from this term. We leave this question for future works and we focus instead on the efficiency and further applications of our method.

5. More Efficient Algorithms

5.1. Player-Efficient Algorithms

Algorithm 2 has computational time and communication complexity for each player which is exponential in the dimensionality. This is clearly problematic for every realistic practical application. For this reason, in this section, we study more efficient algorithms.

Consider the following lemma, showing an $\epsilon$-LDP algorithm for computing $p$-dimensional average (notice the extra conditions on $n$ and $p$ compared with Lemma A).

**Lemma 2.** (Nissim & Stemmer, 2017) Consider player $i \in [n]$ holding data $v_i \in \mathbb{R}^p$ with coordinate between 0 and $b$. Then for $0 < \beta < 1$, $0 < \epsilon$ such that $n \geq 8p \log(\frac{2b}{\beta \epsilon})$ and $\sqrt{n} \geq \frac{\epsilon \sqrt{p}}{\beta}$, there is an $\epsilon$-LDP algorithm, LDP-AVG, that with probability at least $1 - \beta$, returns a vector $a \in \mathbb{R}^p$ satisfying:

$$\max_{j \in [d]} |a_j - \frac{1}{n} \sum_{i=1}^{n} v_{ij}| \leq O\left(\frac{bp \sqrt{p \log(p)}}{\sqrt{n} \epsilon \sqrt{\log(p)}}\right).$$

Moreover, the computation cost for each user is $O(1)$.

By using this lemma and by discretizing the grid with interval steps of $O(\frac{3}{n \epsilon \sqrt{\log(\frac{2}{\beta \epsilon})}})$ (this procedure does not affect the error bound of the average), we can design an algorithm which requires $O(1)$ computation time and $O(\log n)$-bits communication per player (we report this algorithm in the supplemental material). However, we would like to do even better and obtain constant communication complexity.

Instead of discretizing the grid, we apply a technique, firstly proposed by Bassily & Smith (2015), which permits to transform any ‘sampling resilient’ $\epsilon$-LDP protocol into a protocol with 1-bit communication complexity. Roughly speaking, a protocol is sampling resilient if its output on any dataset $S$ can be approximated well by its output on a random subset of half of the players.

Since our algorithm only uses the LDP-AVG protocol, we can show that it is indeed sampling resilient. Inspired by this result, we propose Algorithm 6 and obtain the following theorem.

**Theorem 5.** For $\epsilon \leq \ln 2$ and $0 < \beta < 1$, Algorithm 6 is $\epsilon$-LDP. If the loss function $\ell(x, y)$ is $(\infty, T)$-smooth for all $x \in D$ and $n = \Omega(\max\{\log \frac{p}{\epsilon^2 D_p^2}, p(k+1)p \log(k+1), \frac{1}{p \log(\frac{2}{\beta \epsilon})}\})$ for some constant $c$, then by setting $k = O\left(\frac{bp \sqrt{p \log(p)}}{\sqrt{n} \epsilon \sqrt{\log(p)}}\right)$, (4) holds with probability at least $1 - 4\beta$. Moreover, for each player the time complexity is $O(1)$, and the communication complexity is 1-bit.
Algorithm 3 Player-Efficient Local Bernstein Mechanism with 1-bit communication per player

1: **Input:** Player \( i \in [n] \) holding data \( x_i \in \mathcal{D} \), public loss function \( \ell : [0, 1]^p \times \mathcal{D} \rightarrow [0, 1] \), privacy parameter \( \epsilon \leq \ln 2 \), and parameter \( k \).

2: **Preprocessing:**
3: Generate \( n \) independent public strings \( y_1 = \text{Lap}(\frac{1}{2}), \cdots, y_n = \text{Lap}(\frac{1}{2}) \).
4: Construct the grid \( \mathcal{T} = \{ \frac{y_1}{\epsilon}, \cdots, \frac{y_n}{\epsilon} \} \{ v_1, \cdots, v_p \} \), where \( \{ v_1, \cdots, v_p \} \in \{ 0, 1, \cdots, k \}^p \).
5: Partition randomly \( [n] \) into \( d = (k+1)^p \) subsets \( I_1, I_2, \cdots, I_d \), and associate each \( I_j \) to a grid point \( \mathcal{T}(j) \in \mathcal{T} \).
6: for Each Player \( i \in [n] \) do
7: Find \( I_i \) such that \( i \in I_i \). Calculate \( v_i = \ell(\mathcal{T}(l); x_i) \).
8: Compute \( p_i = \frac{1}{2} \Pr[v_i + \text{Lap}(\frac{1}{2}) = y_i] \) - \( \Pr[\text{Lap}(\frac{1}{2}) = y_i] \).
9: Sample a bit \( b_i \) from Bernoulli \( (p_i) \) and send it to the server.
end for
10: for The Server do
11: for \( i = 1 \cdots n \) do
12: Check if \( b_i = 1 \), set \( \tilde{e}_i = y_i \), otherwise \( \tilde{e}_i = 0 \).
13: end for
14: for each \( l \in [d] \) do
15: Compute \( v_l = \frac{1}{n} \sum_{i \in I_i} \tilde{e}_i \).
16: Denote the corresponding grid point \( (\frac{v_1}{\epsilon}, \cdots, \frac{v_n}{\epsilon}) \in \mathcal{T} \) of \( I_i \), then denote \( \tilde{L}(\frac{v_1}{\epsilon}, \cdots, \frac{v_n}{\epsilon}); D) = v_l \).
17: end for
18: Construct Bernstein polynomial for the perturbed empirical loss \( \tilde{L} \) as in Algorithm 2. Denote \( \tilde{L}(\theta; D) \) the corresponding function.
19: Compute \( \theta_{\text{priv}} = \arg \min_{\theta \in \mathcal{C}} \tilde{L}(\theta; D) \).
21: end for

5.2. Server-Efficient Algorithm

Now we study the algorithm from the server’s complexity perspective. The construction time complexity is \( O(n) \), where the most inefficient part is finding \( \theta_{\text{priv}} = \arg \min_{\theta \in \mathcal{C}} \tilde{L}(\theta; D) \). In fact, this function may be non-convex; but unlike general non-convex functions, it can be \( \alpha \)-uniformly approximated by a convex function \( \tilde{L}(\cdot; D) \) if the loss function is convex (by the proof of Theorem 3), although we do not have access to it. Thus, we can see this problem as an instance of Approximately-Convex Optimization, which has been studied recently by Risteski & Li (2016).

**Definition 7.** (Risteski & Li, 2016) We say that a convex set \( \mathcal{C} \) is \( \mu \)-well-conditioned for \( \mu \geq 1 \), if there exists a function \( F : \mathbb{R}^p \rightarrow \mathbb{R} \) such that \( \mathcal{C} = \{ x | F(x) \leq 0 \} \) and for every \( x \in \partial \mathcal{C} : \frac{\| \nabla^2 F(x) \|_2}{\| \nabla F(x) \|_2} \leq \mu \).

**Lemma 3** (Theorem 3.2 in (Risteski & Li, 2016)). Let \( \epsilon, \Delta \) be two real numbers such that
\[
\Delta \leq \max \left\{ \frac{c^2 - 2\epsilon}{\mu\sqrt{p}}, \frac{\epsilon}{p} \right\} \times \frac{1}{16348}.
\]
Then, there exists an algorithm \( \mathcal{A} \) such that for any given \( \Delta \)-approximate convex function \( f \) over a \( \mu \)-well-conditioned convex set \( \mathcal{C} \subseteq \mathbb{R}^p \) of diameter 1 (that is, there exists a 1-Lipschitz convex function \( f : \mathcal{C} \rightarrow \mathbb{R} \) such that for every \( x \in \mathcal{C} \), \( |f(x) - f(x')| \leq \Delta \)), \( \mathcal{A} \) returns a point \( \tilde{x} \in \mathcal{C} \) with probability at least \( 1 - \delta \) in time \( \text{poly}(p, \frac{1}{\epsilon}, \log \frac{1}{\Delta}) \) and with the following guarantee \( \tilde{f}(\tilde{x}) \leq \min_{x \in \mathcal{C}} f(x) + \epsilon \).

Based on Lemma 3 (for \( \tilde{L}(\theta; D) \)) and Corollary 1, and taking \( \epsilon = O(\alpha) \), we have the following result.

**Theorem 6.** Under the conditions in Corollary 1, and assuming that \( n = \tilde{O}(\epsilon^2 \log(1/\beta)D^2p \epsilon^{-2}\alpha^{-4}) \), that the loss function \( \ell(\cdot, x) \) is 1-Lipschitz and convex for every \( x \in \mathcal{D} \), that the constraint set \( \mathcal{C} \) is convex and \( \| \mathcal{C} \|_2 \leq 1 \), and satisfies \( \mu \)-well-conditioned property (see Definition 7), if the error \( \alpha \) satisfies \( \alpha \leq C \frac{\mu}{p\sqrt{p}} \) for some universal constant \( C \), then there is an algorithm \( \mathcal{A} \) which runs in Poly\( n, p, \frac{1}{\alpha}, \log \frac{1}{\Delta} \) time for the server, and with probability \( 1 - 2\beta \) the output \( \theta_{\text{priv}} \) of \( \mathcal{A} \) satisfies
\[
\tilde{L}(\theta_{\text{priv}}; D) \leq \min_{\theta \in \mathcal{C}} \tilde{L}(\theta; D) + O(\alpha),
\]
which means that \( \text{Err}_D(\theta_{\text{priv}}) \leq O(\alpha) \).

Combining with Theorem 5, 6 and Corollary 1, and taking \( \alpha = \frac{\Delta}{p} \), we have our final result:

**Theorem 7.** Under the conditions of Corollary 1, Theorem 5 and 6, and for any \( C \frac{\mu}{p\sqrt{p}} > \alpha > 0 \), if we further set \( n = \tilde{O}(\epsilon^2 \log(1/\beta)D^2p \epsilon^{-2}\alpha^{-4}) \), then there is an \( \epsilon \)-LDP algorithm, with \( O(1) \) running time and 1-bit communication per player, and Poly\( n, p, \frac{1}{\alpha}, \log \frac{1}{\Delta} \) running time for the server. Furthermore, with probability at least \( 1 - 5\beta \), the output \( \theta_{\text{priv}} \) satisfies \( \text{Err}_D(\theta_{\text{priv}}) \leq O(\alpha) \).

Note that compared with the sample complexity in Theorem 7 and Corollary 1, we have an additional factor of \( p^4 \); however, the \( \alpha \) terms are the same.

6. LDP Algorithms for Learning K-way Marginals Queries and Smooth Queries

In this section, we will show further applications of our idea by giving \( \epsilon \)-LDP algorithms for answering sets of queries. All the queries we consider in this section are linear, that is, of the form \( q_f(D) = \frac{1}{|D|} \sum_{x \in D} f(x) \) for some function \( f \). It will be convenient to have a notion of accuracy for the algorithm we will present with respect to a set of queries. This is defined as follow:
We define a where \( x \in D \) has each of the attributes set to \( x \). This definition guarantees that \( \forall q \in Q, |\mathcal{A}(D, q) - q(D)| \geq \alpha \leq \beta \).

### 6.1. K-way Marginals Queries

Now we consider a database \( D = (\{0, 1\}^p)^n \), where each row corresponds to an individuals record. A marginal query is specified by a set \( S \subseteq [p] \) and a pattern \( t \in \{0, 1\}^{|S|} \). Each such query asks: “What fraction of the individuals in \( D \) has each of the attributes set to \( t \)?” We will consider here k-way marginals which are the subset of marginal queries specified by a set \( S \subseteq [p] \) with \( |S| \leq k \). K-way marginals permit to represent several statistics over datasets, including contingency tables, and the problem to release them under differential privacy has been studied extensively in the literature (Hardt et al., 2012; Gupta et al., 2013; Thaler et al., 2012; Gaboardi et al., 2014). All these previous works have considered the central model of differential privacy, and only the recent work (Kulkarni et al., 2017) studies this problem in the local model, while their methods are based Fourier Transform. We now use the LDP version of Chebyshev polynomial approximation to give an efficient way of constructing a sanitizer for releasing k-way marginals.

Since learning the class of k-way marginals is equivalent to learning the class of monotone k-way disjunctions (Hardt et al., 2012), we will only focus on the latter. The reason why we locally privately learning them is that they form a Q-Function Family.

**Definition 9** (Q-Function Family). Let \( Q = \{q_y\}_{y \in Y_Q \subseteq \{0, 1\}^m} \) be a set of counting queries on a data universe \( D \), where each query is indexed by an \( m \)-bit string. We define the index set of \( Q \) to be the set \( Y_Q = \{y \in \{0, 1\}^m | q_y \in Q \} \).

We define a Q-Function Family \( \mathcal{F}_Q = \{f_{Q,x} : \{0, 1\}^m \rightarrow \{0, 1\}\}_{x \in D} \) as follows: for every data record \( x \in D \), the function \( f_{Q,x} : \{0, 1\}^m \rightarrow \{0, 1\} \) is defined as \( f_{Q,x}(y) = q_y(x) \). Given a database \( D \in \mathbb{D}^n \), we define \( f_{Q,D}(y) = \frac{1}{n} \sum_{i=1}^{n} f_{Q,x}(y) = \frac{1}{n} \sum_{i=1}^{n} q_y(x_i) = g_y(D) \), where \( x_i \) is the \( i \)-th row of \( D \).

This definition guarantees that Q-function queries can be computed from their values on the individual’s data \( x \). We can now formally define the class of monotone k-way disjunctions.

**Definition 10**. Let \( D = \{0, 1\}^p \). The query set \( Q_{\text{disj},k} = \{q_y\}_{y \in Y_k \subseteq \{0, 1\}^p} \) of monotone k-way disjunctions over \( \{0, 1\}^p \) contains a query \( q_y \) for every \( y \in Y_k = \{y \in \{0, 1\}^p | |y| \leq k \} \). Each query is defined as \( q_y(x) = \vee_{j \in Y_k} y_j x_j \). The Q-disj,k-function family \( \mathcal{F}_{Q_{\text{disj},k}} = \{f_{x} : x \in \{0, 1\}^p \} \) contains a function \( f_x(y_1, y_2, \ldots, y_p) = \vee_{j=1}^{p} y_j x_j \) for each \( x \in \{0, 1\}^p \). This definition guarantees that if we can uniformly approximated the function \( f_{Q,x} \) by polynomials \( p_x \), then we can also have an approximation of \( f_{Q,D} \), i.e. we can approximate \( q_y(D) \) for every \( y \) or all the queries in the class \( Q \).

### Theorem 8

For \( \epsilon > 0 \) Algorithm 4 is \( \epsilon \)-LDP. Also, for \( 0 < \beta < 1 \), there are constants \( C, C_1 \) such that \( \mathcal{F}_{Q_{\text{disj},k}} \) is a set of counting queries on a data universe \( D \), where each query is indexed by an \( m \)-bit string. We define the index set of \( Q \) to be the set \( Y_Q = \{y \in \{0, 1\}^m | q_y \in Q \} \).

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that for every \( k, p, n \in \mathbb{N} \) with \( k \leq p \), if \( n \geq \Omega(\max\{p^{\mathcal{O}(\sqrt{\log \frac{1}{\epsilon}})}, n^{\mathcal{O}(\sqrt{\log \frac{1}{\epsilon}})}\}) \), this algorithm is \((\alpha, \beta)\)-accuracy with respect to \( Q_{\text{disj}, k} \). The running time for player is \( \mathcal{O}(p^{O(\sqrt{\log \frac{1}{\epsilon}})}) \), and the running time for server is at most \( \mathcal{O}(n) \) and the time for answering a query is \( \mathcal{O}(p^{\mathcal{O}(\sqrt{\log \frac{1}{\epsilon}})}) \) for some constant \( C_2 \). Moreover, as in Section 5, the communication complexity can be improved to 1-bit per player.

### 6.2. Smooth Queries

We now consider the case where each player \( i \in [n] \) holds a data \( x_i \in \mathbb{R}^p \) and we want to estimate the kernel density for a given point \( x_0 \in \mathbb{R}^p \). A natural question is: If we want to estimate Gaussian kernel density of a given point \( x_0 \) with many different bandwidths, can we do it simultaneously under \( \epsilon \) local differential privacy?

We can see this kind of queries as a subclass of the smooth queries. So, like in the case of k-way marginals queries, we will give an \( \epsilon \)-LDP sanitizer for smooth queries. Now we consider the data universe \( D = [-1, 1]^p \) and dataset \( D \in \mathcal{D}^n \). For a positive integer \( h \) and constant \( T > 0 \), we denote the set of all \( p \)-dimensional \((h, T)\)-smooth function (Definition 5) as \( C^h_T \), and \( Q_{C^h_T} = \{ qf(D) = \frac{1}{n} \sum_{x \in D} f(D), f \in C^h_T \} \) the corresponding set of queries. The idea of the algorithm is similar to the one used for the k-way marginals; but instead of using Chebyshev polynomials, we will use trigonometric polynomials. We now assume that the dimensionality \( p \), \( h \), and \( T \) are constants so all the result in big \( O \) notation will be omitted. The idea of Algorithm 5 is actually based on the following Lemma.

**Lemma 5.** (Wang et al., 2016) Assume \( \gamma > 0 \). For every \( f \in C^h_T \), defined on \([-1, 1]^p \), let \( g_{\theta_1}(\theta_2, \ldots, \theta_p) = f(\cos(\theta_1), \ldots, \cos(\theta_p)) \), for \( \theta_i \in [-\pi, \pi] \). Then there is an even trigonometric polynomial \( p \) whose degree for each variable is \( t(\gamma) = (\frac{1}{\gamma})^\frac{p+p}{p} \):

\[
p(\theta_1, \ldots, \theta_p) = \sum_{0 \leq r_1, \ldots, r_p < t(\gamma)} c_{r_1, \ldots, r_p} \prod_{i=1}^p \cos(r_i \theta_i),
\]

such that 1) \( p \) \( \gamma \)-uniformly approximates \( g_f \), i.e. \( \sup_{x \in [-1, 1]^p} |p(x) - g_f(x)| \leq \gamma \). The coefficients are uniformly bounded by a constant \( M \) which only depends on \( h, T \) and \( p \). 2) Moreover, the whole set of the coefficients can be computed in time \( \mathcal{O}(\frac{h^2 + 2p}{\gamma} \log \frac{1}{\epsilon}) \).

By (9), we can see that all the \( p(x) \) which corresponds to \( g_f(x) \), representing functions \( f \in C^h_T \), have the same basis \( \prod_{i=1}^p \cos(r_i \theta_i) \). So, we can use Lemma 2 to estimate the average of the basis. Then, for each query \( f \) the server can only compute the corresponding coefficients \( \{c_{r_1, r_2, \ldots, r_p}\} \).

This idea is implemented in Algorithm 5 for which we have the following result.

**Theorem 9.** For \( \epsilon > 0 \), Algorithm 5 is \( \epsilon \)-LDP. Also for \( \alpha > 0, 0 < \beta < 1 \), if \( n \geq \Omega(\max\{\log \frac{n^{p+2}}{\epsilon^2}, \frac{1}{\epsilon^2} \log \frac{1}{\epsilon^2}\}) \) and \( t = \mathcal{O}(\sqrt{n}) \), then Algorithm 5 is \((\alpha, \beta)\)-accurate with respect to \( Q_{C^h_T} \). The time for answering each query is \( \tilde{O}(\sqrt{n}) \), where \( O \) omits \( h, T, p \) and some log terms. For each player, the computation and communication cost could be improved to \( O(1) \) and 1 bit, respectively, as in Section 5.

**Algorithm 5 Local Trigonometry Mechanism for \( Q_{C^h_T} \)**

1. **Input:** Player \( i \in [n] \) holding data \( x_i \in [-1, 1]^p \), privacy parameter \( \epsilon > 0 \), error bound \( \alpha \), and \( t \in \mathbb{N} \).
2. **For Each Player \( i \in [n] \) do**
   a. Compute \( p_i = \{p_{i,v} \in T_i^p \} \).
3. **For Each \( v = (v_1, v_2, \ldots, v_p) \in T_i^p \) do**
   a. Compute \( p_{1,v} = \cos(v_1 \theta_1(x_i)), \ldots, \cos(v_p \theta_p(x_i)) \).
4. **end for**
5. **end for**
6. Let \( \bar{p}_i = \{p_{i,v} \in T_i^p \} \).
7. **end for**
8. **For The Server do**
9. Run LDP-AVG from Lemma 2 on \( \{p_i\}_{i=1}^n \) with parameter \( \epsilon, b = 1 \), denote the output as \( \tilde{p}_D \).
10. For each query \( qf \in Q_{C^h_T} \).
    a. Let \( g_f(\theta) = f(\cos(\theta_1), \cos(\theta_2), \ldots, \cos(\theta_p)) \).
    b. Compute the trigonometric polynomial approximation \( p_f(\theta) \) of \( g_f(\theta) \), where \( p_f(\theta) = \sum_{r=0}^p \cos(r \theta_1) \cdots \cos(r \theta_p) \).
11. **end for**
12. Compute \( \tilde{p}D \cdot c \).
13. **end for**

### 7. Conclusion and Discussion

In this paper, we studied ERM under non-interactive LDP and proposed an algorithm which is based on Bernstein polynomial approximation. We showed that if the loss function is smooth enough, then the sample complexity to achieve \( \alpha \) error is \( \alpha^{-c} \) for some positive constant \( c \), which improves significantly on the previous result of \( \alpha^{-\frac{1}{p+1}} \). Moreover, we proposed efficient algorithms for both player and server views. We also showed how a similar idea based on other polynomial approximations can be used to answering k-way marginals and smooth queries in the local model.

In our algorithms the sample complexity still depends on the dimension \( p \), in the term of \( \epsilon^p \) for constant \( c \). We will focus on removing this dependency in future work. Additionally, we will study the difference between strongly convex and convex loss functions in the non-interactive LDP setting.
References


A. Details in Section 3

**Lemma.** (Nissim & Stemmer, 2017) Suppose that \( x_1, \ldots, x_n \) are i.i.d sampled from \( \text{Lap}(\frac{\beta}{\epsilon}) \). Then for every \( 0 \leq t < \frac{2n}{\epsilon} \), we have

\[
\Pr\left( \left| \sum_{i=1}^{n} x_i \right| \geq t \right) \leq 2 \exp\left( -\frac{t^2}{4n} \right).
\]

**Proof of Lemma 1.** Consider Algorithm 1. We have \( |a - \frac{1}{n} \sum_{i=1}^{n} v_i| = \left| \sum_{i=1}^{n} x_i \right| \), where \( x_i \sim \text{Lap}(\frac{\beta}{\epsilon}) \). Taking \( t = \frac{2\sqrt{n\log \frac{2}{\epsilon}}}{\epsilon} \) and applying the above lemma, we prove the lemma.

B. Details in Section 4

**Proof of Theorem 3.** The proof of the \( \epsilon \)-LDP comes from Lemma 1 and composition theorem. W.l.o.g., we assume \( T = 1 \). To prove the theorem, it is sufficient to estimate \( \sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - \tilde{L}(\theta; D)| \leq \alpha \) for some \( \alpha \), since if it is true, denote \( \theta^* = \arg \min_{\theta \in \mathcal{C}} \bar{L}(\theta; D) \), we have \( \bar{L}(\theta_{\text{priv}}; D) - \bar{L}(\theta^*; D) \leq \bar{L}(\theta_{\text{priv}}; D) - \bar{L}(\theta_{\text{priv}}; D) + \bar{L}(\theta_{\text{priv}}; D) - \bar{L}(\theta^*; D) + \bar{L}(\theta^*; D) - \bar{L}(\theta^*; D) \leq 2\alpha \).

Since we have \( \sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - \tilde{L}(\theta; D)| \leq \sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - B_k^{(h)}(\tilde{L}, \theta)| + \sup_{\theta \in \mathcal{C}} |B_k^{(h)}(\tilde{L}, \theta) - \bar{L}(\theta; D)| \). The second term is bounded by \( O(D_{h,p} \frac{1}{k^p}) \) by Theorem 2.

For the first term, by (2) and the algorithm, we have

\[
\sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - B_k^{(h)}(\tilde{L}, \theta)| \leq \max_{v \in T} |\bar{L}(v; D) - \tilde{L}(v; D)| \sup_{\theta \in \mathcal{C}} \sum_{j=1}^{p} \sum_{i=0}^{k} |\prod_{i=1}^{p} b_{v_i,k}(\theta_i)|. \tag{10}
\]

By Proposition 4 in (Aldà & Rubinstein, 2017), we have \( \sum_{j=1}^{p} \sum_{v_j=0}^{k} |\prod_{i=1}^{p} b_{v_i,k}(\theta_i)| \leq (2^h - 1)^p \). Next lemma bounds the term \( \max_{v \in T} |\bar{L}(v; D) - \tilde{L}(v; D)| \), which is obtained by Lemma A.

**Lemma.** If \( 0 < \beta < 1 \), \( k \) and \( n \) satisfy that \( n \geq p \log(2/\beta) \log(k+1) \), then with probability at least \( 1 - \beta \), for each \( v \in T \),

\[
|\bar{L}(v; D) - \tilde{L}(v; D)| \leq O\left( \frac{\sqrt\log \frac{1}{\beta} \sqrt{p} \sqrt{\log(k+1)^p}}{\sqrt{n \epsilon}} \right). \tag{11}
\]

**Proof.** By Lemma 1, for a fixed \( v \in T \), if \( n \geq \frac{2}{\beta} \), we have with probability \( 1 - \beta \), \( |\bar{L}(v; D) - \tilde{L}(v; D)| \leq \frac{2\sqrt{\log \frac{2}{\epsilon}}}{\sqrt{n \epsilon}} \). Taking the union of all \( v \in T \) and then taking \( \beta = \frac{\beta}{(k+1)^p} \) (since there are \( (k+1)^p \) elements in \( T \)) and \( \epsilon = \frac{\epsilon}{(k+1)^p} \), we get the proof.

By \( (k+1) < 2k \), we have

\[
\sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - \tilde{L}(\theta; D)| \leq O\left( \frac{D_{h,p} \frac{1}{k^p}}{\sqrt{\log \frac{1}{\beta} \sqrt{p} \sqrt{\log(k+1)^p}}} \right). \tag{12}
\]

Now we take \( k = O\left( \frac{D_{h,p} \frac{1}{k^p}}{\sqrt{\log \frac{1}{\beta} \sqrt{p} \sqrt{\log(k+1)^p}}} \right) \). Since \( n = \Omega\left( \frac{\epsilon^{ph}}{\sqrt{p} \beta D_{h}} \right) \), we have \( \log k > 1 \). Plugging it into (12), we get

\[
\sup_{\theta \in \mathcal{C}} |\bar{L}(\theta; D) - \tilde{L}(\theta; D)| \leq O\left( \frac{\log \frac{k}{h} \left( \frac{1}{2} D_{h}^{\frac{1}{p}} \frac{1}{2} \sqrt{\log \frac{1}{\beta} \sqrt{p} \sqrt{\log(k+1)^p}}} {\sqrt{h + p n \frac{\epsilon^{ph}}{\sqrt{p} \beta D_{h}} \epsilon^{ph}}} \right) = O\left( \frac{\log \frac{k}{h} \left( \frac{1}{2} D_{h}^{\frac{1}{p}} \frac{1}{2} \sqrt{\log \frac{1}{\beta} \sqrt{p} \sqrt{\log(k+1)^p}}} {n \frac{\epsilon^{ph}}{\sqrt{p} \beta D_{h}} \epsilon^{ph}} \right). \tag{13}
\]

Also we can see that \( n \geq p \log(2/\beta) \log(k+1) \) is true for \( n = \Omega\left( \frac{\epsilon^{ph}}{\sqrt{p} \beta D_{h}} \right) \). Thus, the theorem follows.
Proof of Corollary 1. Since the loss function is \((\infty, T)\)-smooth, it is \((2p, T)\)-smooth for all \(p\). Thus, taking \(h = p\) in Theorem 3, we get the proof.

Lemma. (Shalev-Shwartz et al., 2009) If the loss function \(\ell\) is \(L\)-Lipschitz and \(\mu\)-strongly convex, then with probability at least \(1 - \beta\) over the randomness of sampling the data set \(\mathcal{D}\), the following is true,

\[
\text{Err}_{\mathcal{P}}(\theta) \leq \sqrt{\frac{2L^2}{n\mu}} \sqrt{\text{Err}_{\mathcal{D}}(\theta)} + \frac{4L^2}{\beta \mu n}.
\]

Proof of Theorem 4. For the general convex loss function \(\ell\), we let \(\hat{\ell}(\theta; x) = \ell(\theta; x) + \frac{\mu}{2} \|\theta\|^2\). By the above lemma for Err\(_{\mathcal{D}}\),

\[
\text{Err}_{\mathcal{P}, \ell}(\theta) \leq \text{Err}_{\mathcal{P}, \hat{\ell}}(\theta) + \frac{\mu}{2}.
\]

By the above lemma for Err\(_{\mathcal{P}, \ell}(\theta)\), where \(\hat{\ell}(\theta; x) = 1 + ||\theta|| = O(1)\)-Lipschitz, thus we have the following,

\[
\text{Err}_{\mathcal{P}, \ell}(\theta) \leq \hat{\Theta} \left( \frac{2}{n\mu} \log \frac{1}{\beta} \frac{D^\frac{1}{\beta} D^\frac{1}{p} \log(\frac{1}{\epsilon})}{n \sqrt{\epsilon}} + \frac{4}{\beta \mu n} + \frac{\mu}{2} \right).
\]

Taking \(\mu = O\left(\frac{1}{\sqrt{\beta n}}\right)\), we get

\[
\text{Err}_{\mathcal{P}, \ell}(\theta) \leq \hat{\Theta} \left( \frac{1}{\sqrt{\beta n}} \right).
\]

Thus, we have the theorem.

C. Details in Section 5

Algorithm 6 Player-Efficient Local Bernstein Mechanism with \(O(\log n)\)-bits communication per player

1. **Input:** Each user \(i \in [n]\) has data \(x_i \in \mathcal{D}\), privacy parameter \(\epsilon\), public loss function \(\ell: \{0, 1\}^p \times \mathcal{D} \mapsto [0, 1]\), and parameter \(k\) (we will specify it later).
2. **Preprocessing:**
3. Construct the grid \(\mathcal{T} = \{\frac{v_1}{k}, \frac{v_2}{k}, \cdots, \frac{v_p}{k}\}\) \(v_1, v_2, \cdots, v_p\), where \(\{v_1, v_2, \cdots, v_p\} = \{0, 1, \cdots, k\}\).
4. Discretize the interval \([0, 1]\) with grid steps \(O\left(\frac{1}{\sqrt{n}} \sqrt{\frac{1}{\epsilon}} \log \left(\frac{1}{\epsilon}\right)\right)\). Denote the set of grids by \(\mathcal{G}\).
5. Randomly partition \([n]\) in to \(d = (k + 1)^p\) subsets \(I_1, I_2, \cdots, I_d\), with each subset \(I_j\) corresponding to a grid in \(\mathcal{T}\) denoted as \(T(j)\).
6. **For** Each Player \(i \in [n]\) **do**
7. Find the subset \(I_j\) such that \(i \in I_j\). Calculate \(v_i = \ell(T(j); x_i)\).
8. Denote \(z_i = v_i + \text{Lap}(\frac{1}{\epsilon})\), round \(z_i\) into the grid set \(\mathcal{G}\), and let the resulting one be \(\tilde{z}_i\).
9. Send \((\tilde{z}_i, \ell)\).
10. **End for**
11. **For** The Server **do**
12. **For** Each \(\ell \in [d]\) **do**
13. Compute \(v_\ell = \frac{n}{d} \sum_{i \in I_\ell} \tilde{z}_i\).
14. Denote the corresponding grid point \((\frac{v_1}{k}, \frac{v_2}{k}, \cdots, \frac{v_p}{k})\) \(\in \mathcal{T}\) as \(\ell\); then let \(\check{L}((\frac{v_1}{k}, \frac{v_2}{k}, \cdots, \frac{v_p}{k}); D) = v_\ell\).
15. **End for**
16. Construct perturbed Bernstein polynomial of the empirical loss \(\check{L}\) as in Algorithm 2, where each \(\check{L}((\frac{v_1}{k}, \frac{v_2}{k}, \cdots, \frac{v_p}{k}); D)\) is replaced by \(\hat{L}((\frac{v_1}{k}, \frac{v_2}{k}, \cdots, \frac{v_p}{k}); D)\). Denote the function as \(\check{L}(\cdot, D)\).
17. Compute \(\theta_{\text{priv}} = \arg\min_{\theta \in C} \hat{L}(\theta; D)\).
18. **End for**
Proof of Theorem 5. By (Bassily & Smith, 2015) it is $\epsilon$-LDP. The time complexity and communication complexity is obvious. As in (Bassily & Smith, 2015), it is sufficient to show that the LDP-AVG is sampling resilient. Here the STAT is the average, and $\phi(x, y) = \max_{y \in \mathbb{R}} |x - y|$. By Lemma 2, we can see that with probability at least $1 - \beta$, $\phi(\text{Avg}(v_1, v_2, \ldots, v_n); a) = O\left(\frac{\sqrt{np}}{\sqrt{\log \frac{p}{\beta}}}\right)$. Now let $S$ be the set obtained by sampling each point $v_i, i \in [n]$ independently with probability $\frac{1}{2}$. Note that by Lemma 2, we have on the subset $S$. If $|S| \geq \Omega(\max\{p \log(\frac{\beta}{p}), \frac{1}{\epsilon} \log(\frac{\beta}{p})\})$ with probability at least $1 - \beta$, $\phi(\text{Avg}(S); \text{LDP-AVG}(S)) = O\left(\frac{\sqrt{p}N}{\sqrt{|S|^\epsilon}} \sqrt{\log \frac{p}{\beta}}\right)$. Now by Hoeffdings Inequality, we can get $|n/2 - |S|| \leq \sqrt{n \log \frac{4}{\beta}}$ with probability $1 - \beta$. Also since $n = \Omega(\log \frac{1}{\beta})$, we know that $|S| \geq O(n) = O(\Omega(p \log(\frac{\beta}{p}))$ is true. Thus, with probability at least $1 - 2\beta$, $\phi(\text{Avg}(S); \text{LDP-AVG}(S)) = O\left(\frac{\sqrt{p}N}{\sqrt{|S|^\epsilon}} \sqrt{\log \frac{p}{\beta}}\right)$.

Actually, we can also get $\phi(\text{Avg}(S); \text{Avg}(v_1, v_2, \ldots, v_n)) \leq O\left(\frac{4d}{\sqrt{\log \frac{d}{\beta}}}\right)$. We now first assume that $v_i \in \mathbb{R}$. Note that $\text{Avg}(S) = \frac{1}{n} \sum_{i=1}^{n} x_i$, where $x_i \sim \text{Bernoulli}(\frac{1}{2})$. Denote $M = x_1 + x_2 + \cdots + x_n$, by Hoeffdings Inequality, we have with probability at least $1 - \frac{\beta}{2}, |M - \frac{n}{2}| \leq \sqrt{n \log \frac{4}{\beta}}$. Denote $N = v_1x_1 + \cdots + v_nx_n$. Also, by Hoeffdings inequality, with probability at least $1 - \beta$, we get $|N - \frac{v_1 + \cdots + v_n}{2}| \leq b\sqrt{n \log \frac{2}{\beta}}$. Thus, with probability at least $1 - \beta$, we have:

$$\frac{|N - \frac{v_1 + \cdots + v_n}{n}|}{n} \leq \frac{|N - \sum_{i=1}^{n} v_i/2|}{M} + \left|\sum_{i=1}^{n} v_i/2\right| - \frac{2}{n} \leq \frac{|N - \sum_{i=1}^{n} v_i/2|}{M} + \frac{nb}{2} - \frac{2}{n}.$$  

The second term $\frac{n/\sqrt{M}}{\sqrt{\log \frac{p}{\beta}}}$, we know from the above $|n/2 - M| \leq \sqrt{n \log \frac{4}{\beta}}$. Also since $n = \Omega(\log \frac{1}{\beta})$, we get $M \geq O(n)$. Thus, $\frac{1}{M} - \frac{2}{n} \leq O\left(\frac{\log \frac{4}{\beta}}{\sqrt{\log \frac{4}{\beta}}}\right)$. The upper bound of the second term is $O\left(\frac{\log \frac{4}{\beta}}{\sqrt{\log \frac{4}{\beta}}}\right)$. The same for the first term. For $p$ dimensions, we just choose $\beta = \frac{\beta}{p}$ and take the union. Thus, we have $\phi(\text{Avg}(S); \text{Avg}(v_1, v_2, \ldots, v_n)) \leq O\left(\frac{\sqrt{p}N}{\sqrt{|S|^\epsilon}} \sqrt{\log \frac{p}{\beta}}\right)$.

In summary, we have shown that $\phi(\text{AVG-LDP}(S); \text{AVG}(v_1, v_2, \ldots, v_n)) \leq O\left(\frac{\sqrt{p}N}{\sqrt{|S|^\epsilon}} \sqrt{\log \frac{p}{\beta}}\right)$ with probability at least $1 - 4\beta$.

Recently, (Bun et al., 2017) proposed a generic transformation, GenProt, which could transform any $(\epsilon, \delta)$-LDP protocol to an $O(\epsilon)$-LDP protocol with the communication complexity for each layer being $O(\log \log n)$, which removes the condition of ‘sample resilient’ in (Bassily & Smith, 2015). The detail is in Algorithm 2. The transformation uses $O(n \log \frac{1}{\beta})$ independent public string. The reader is referred to (Bun et al., 2017) for details. Actually, by Algorithm 2, we can easily get an $O(\epsilon)$-LDP algorithm with the same error bound.

Theorem 10. With $\epsilon \leq \frac{1}{4}$, under the condition of Corollary 1, Algorithm 7 is 10c-LDP. If $T = O(\log \frac{n}{\beta})$, then with probability at least $1 - 2\beta$, Corollary 1 holds. Moreover, the communication complexity of each layer is $O(\log \log n)$ bits, and the computational complexity for each layer is $O(\log \log n)$.

Proof of Theorem 6. Let $\theta^* = \arg \min_{\theta \in C} \hat{L}(\theta; D)$, $\theta_{\text{priv}} = \arg \min_{\theta \in C} \hat{L}(\theta; D)$. Under the assumptions of $\alpha, n, k, \epsilon, \beta$, we know from the proof of Theorem 3 and Corollary 1 that $\sup_{\theta \in C} |\hat{L}(\theta; D) - \hat{L}(\theta; D)| \leq \alpha$. Also by setting $\epsilon = 16348p\alpha$ and $\alpha \leq \frac{1}{10^{48} p^{\frac{1}{4}}}$, we can see that the condition in Lemma 3 holds for $\Delta = \alpha$. So there is an algorithm returns

$$\hat{L}(\theta_{\text{priv}}; D) \leq \min_{\theta \in C} \hat{L}(\theta; D) + O(p\alpha).$$  

Thus, we have

$$\hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta^*; D) \leq \hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta_{\text{priv}}; D) + \hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta^*; D),$$  

where

$$\hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta_{\text{priv}}; D) \leq \hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta_{\text{priv}}; D) + \hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta_{\text{priv}}; D) \leq \alpha + O(p\alpha) = O(p\alpha).$$
Algorithm 7 Player-Efficient Local Bernstein Mechanism with $O(\log \log n)$ bits communication complexity.

1: **Input:** Each user $i \in [n]$ has data $x_i \in D$, privacy parameter $\epsilon$, public loss function $\ell : [0, 1]^p \times D \mapsto [0, 1]$, and parameter $k, T$.

2: **Preprocessing:**
3: For every $(i, T) \in [n] \times [T]$, generate independent public string $y_{i,t} = \text{Lap}(\perp)$.
4: Construct the grid $\mathcal{T} = \left\{ \frac{v_1}{K}, \frac{v_2}{K}, \ldots, \frac{v_p}{K} \right\} v_1, v_2, \ldots, v_p$, where $\{v_1, v_2, \ldots, v_p\} = \{0, 1, \ldots, k\}^p$.
5: Randomly partition $[n]$ into $d = (k + 1)^p$ subsets $I_1, I_2, \ldots, I_d$, with each subset $I_j$ corresponding to an grid in $\mathcal{T}$ denoted as $\mathcal{T}(j)$.

6: **for** Each Player $i \in [n]$ **do**
7: Find the subset $I_\ell$ such that $i \in I_\ell$. Calculate $v_i = \ell(\mathcal{T}(\ell) ; x_i)$.
8: For each $t \in [T]$, compute $p_{i,t} = \frac{1}{2} \Pr[|v_i + \text{Lap}(\frac{1}{2})| = y_{i,t}]$.
9: **for** every $t \in [T]$, if $p_{i,t} \notin \left[ \frac{e^{-2\epsilon}}{2}, \frac{2e^{\epsilon}}{2} \right]$, then set $p_{i,t} = \frac{1}{2}$.
10: **for** every $t \in [T]$, sample a bit $b_{i,t}$ from Bernoulli($p_{i,t}$).
11: Denote $H_i = \{ t \in [T] : b_{i,t} = 1 \}$
12: If $H_i = \emptyset$, set $H_i = [T]$
13: Sample $g_i \in H_i$ uniformly, and send $g_i$ to the server.
14: **end for**
15: **for** The Server **do**
16: Compute $v_\ell = \frac{1}{|I_\ell|} \sum_{i \in I_\ell} g_i$.
17: Denote the corresponding grid point $(\frac{v_1}{K}, \frac{v_2}{K}, \ldots, \frac{v_p}{K}) \in \mathcal{T}$ as $\ell$; then let $L((\frac{v_1}{K}, \frac{v_2}{K}, \ldots, \frac{v_p}{K}); D) = v_\ell$.
18: **end for**
19: Construct perturbed Bernstein polynomial of the empirical loss $\hat{L}$ as in Algorithm 2. Denote the function as $L(\cdot, D)$.
20: Compute $\theta_{\text{priv}} = \arg \min_{\theta \in \mathbb{C}} L(\theta; D)$.
21: **end for**

Also $\hat{L}(\theta_{\text{priv}}; D) - \hat{L}(\theta^*; D) \leq \hat{L}(\theta^*; D) - \hat{L}(\theta^*; D) \leq \alpha$. The theorem follows. The running time is determined by $n$. This is because when we use the algorithm in Lemma 3, we have to use the first order optimization. That is, we have to evaluate some points at $L(\theta; D)$, which will cost at most $O(\text{poly}(n))$ time (note that $\hat{L}$ is a polynomial with $(k + 1)^p \leq n$ coefficients).

D. Details in Section 6

**Proof of Theorem 7.** It is sufficient to prove that

$$\sup_{y \in Y_k} |\hat{p}_D(y) - q_y(D)| \leq \gamma + \frac{T(\frac{p+tk}{tk})^2 \sqrt{\log (\frac{\epsilon}{\beta})}}{\sqrt{n}} \cdot (19),$$

where $T = n^{O(\sqrt{T \log (\frac{1}{\epsilon})})}$. Now we denote $p_D \in \mathbb{R}^{(\frac{p+tk}{tk})}$ as the average of $\hat{q}_t$. That is, it is the unperturbed version of $\hat{p}_D$. By Lemma 4, we have $\sup_{y \in Y_k} |p_D(y) - q_y(D)| \leq \gamma$. Thus it is sufficient to prove that

$$\sup_{y \in Y_k} |\hat{p}_D(y) - p_D(y)| \leq \frac{T(\frac{p+tk}{tk})^2 \sqrt{\log (\frac{\epsilon}{\beta})}}{\sqrt{n}} \cdot (20).$$

Since $\hat{p}_D, p_D$ can be viewed as a vector, we have

$$\sup_{y \in Y_k} |\hat{p}_D(y) - p_D(y)| \leq \|\hat{p}_D - p_D\|_1 \cdot (21).$$

Also, since each coordinate of $p_D(y)$ is bounded by $T$ by Lemma 4, by Lemma 2, we can see that if $n = \Omega(\max\{\frac{1}{\gamma}, (\frac{p+tk}{tk}) \log (\frac{p+tk}{tk} \log 1/\beta)\})$, then with probability at least $1 - \beta$, the following is true $\|\hat{p}_D - p_D\|_1 \leq$
\[
\frac{T^{(p+tk)^2} \sqrt{\log \left( \frac{p+tk}{\beta} \right)}}{\sqrt{mc}}, \text{ thus take } \gamma = \frac{\alpha}{2} \text{ and } (p+tk) = p^{O(tk)}. \text{ This gives us the theorem.} \]

Proof of Theorem 9. Let \( t = (\frac{1}{\gamma})^k \). It is sufficient to prove that \( \sup_{q_f \in \mathcal{Q}_{c_f}} |\bar{p}_D \cdot c_f - q_f(D)| \leq \alpha \). Let \( p_D \) denote the average of \( \{p_i\}_{i=1}^n \), i.e. the unperturbed version of \( \bar{p}_D \). Then by Lemma 5, we have \( \sup_{q_f \in \mathcal{Q}_{c_f}} |p_D \cdot c_f - q_f(D)| \leq \gamma \). Also since \( \|c_f\|_\infty \leq M \), we have \( \sup_{q_f \in \mathcal{Q}_{c_f}} |\bar{p}_D \cdot c_f - p_D \cdot c_f| \leq O(\|\bar{p}_D - p_D\|_1) \). By Lemma 2, we know that if \( n = \Omega(\max\{\frac{1}{\beta} \log \frac{1}{\beta}, t_{2p} \log \frac{1}{\beta} \}) \), then \( \|\bar{p}_D - p_D\|_1 \leq O(\frac{\|p_D\|_1}{\sqrt{nmc}}) \) with probability at least \( 1 - \beta \). Thus, we have \( \sup_{q_f \in \mathcal{Q}_{c_f}} |\bar{p}_D \cdot c_f - q_f(D)| \leq O(\gamma + \frac{(1/\gamma) \sqrt{\log(1/\beta)}}{\sqrt{nmc}}) \). Taking \( \gamma = O((1/\sqrt{nmc})^{2b/\sqrt{mc}}) \), we get \( \sup_{q_f \in \mathcal{Q}_{c_f}} |\bar{p}_D \cdot c_f - q_f(D)| \leq O(\sqrt{\log(1/\beta)}(\frac{1}{\sqrt{nmc}}) \frac{2b}{\sqrt{mc}}) \leq \alpha \). The computational cost for answering a query follows from Lemma 5 and \( b \cdot c = O(t_p) \).