Quantization and interpretation of the Dirac field

Solutions of the Dirac equation:

Since each solution of the Dirac equation has also to obey the Klein-Gordon equation, we know that the relativistic energy-momentum relation $m^2 = w_k^2 - \vec{k}^2$ has to be fulfilled and, thus, that there are again solutions with positive and negative frequency. Let's start with solutions with negative frequency and we make the following ansatz:

$$\Psi(x) = e^{-ikx} u(k)$$

where $u(k)$ is a four-component spinor. Using the Dirac equation we find

$$(\gamma^\mu k_\mu - m) u(k) = 0$$

There are two linear independent solutions which we choose to normalize as follows

$$u_s(k) = \sqrt{k^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{k}}{k^0+m} \chi_s \end{pmatrix} \; ; \; s = \pm\frac{1}{2}$$

The two-component spinors χ_s are defined as follows

$$\chi_{s=1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi_{s=-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Accordingly the solutions with positive frequency

$$\Psi(x) = e^{ikx} v(k)$$

with

$$(\gamma^\mu k_\mu + m) v(k) = 0$$

reads

$$v_s(k) = -\sqrt{k^0 + m} \begin{pmatrix} \frac{\vec{\sigma}\vec{k}}{k^0+m} \epsilon \chi_s \\ \epsilon \chi_s \end{pmatrix} \; ; \; s = \pm\frac{1}{2} \;\; \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Again a general solution of the Diarc equation is obtained as a superposition of plane-wave solutions with negative and positive frequencies

$$\Psi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{s=\pm 1/2} [u_s(k) a_s(\vec{k}) e^{-ikx} + v_s(k) b_s^\dagger(\vec{k}) e^{ikx}]$$

As a first step of quantizing the Dirac field we regard the Dirac spinor Ψ as a field operator. Consequently, the fourier coefficients are operator valued as well. We again require

$$a_s(\vec{k})|0>= 0 \ \text{ and } \ b_s(\vec{k})|0>= 0$$

To each four-momentum we can construct four one-particle states

$$a^\dagger_{s=+1/2}(\vec{k})|0> \quad \text{and} \quad a^\dagger_{s=-1/2}(\vec{k})|0>$$

$$b^\dagger_{s=+1/2}(\vec{k})|0> \quad \text{and} \quad b^\dagger_{s=-1/2}(\vec{k})|0>$$

The Lagrangian for a free Dirac field reads

$$\mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x)$$

so that the canonical conjugate momentum to Ψ is $\Pi(x) = i\bar{\Psi}(x)\gamma^0 = i\Psi^\dagger(x)$.

Thus, the Hamiltonian for the Dirac field reads

$$H = \int d^3x[\Pi \partial_0 \Psi - \mathcal{L}] = \int d^3x[\bar{\Psi} i\gamma^0 \partial_0 \Psi]$$

If we assume the same algebra for the annihilation and creation operators a_s, b_s as for the Klein-Gordon field, we find that the energy of the Hamiltonian can be negative

$$H = \int d^3k \omega_k \sum_s [a_s^\dagger(\vec{k}) a_s(\vec{k}) - b_s(\vec{k}) b_s^\dagger(\vec{k})]$$

and we get in trouble with the requirement of (micro)causality. With commutation relations we find

$$[\Psi(\vec{x}, t), \bar{\Psi}(\vec{y}, t)] \neq 0 \text{ for } \vec{x} \neq \vec{y}$$

Thus Dirac particles cannot be bosons. And we already know from experiment that electrons obey Fermi-statistics. If we assume *anticommutation relations*($\{A, B\} = AB + BA$)

$$\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\} = \delta_{rs} \delta^3(\vec{k} - \vec{k}')$$

$$\{b_r(\vec{k}), b_s^\dagger(\vec{k}')\} = \delta_{rs} \delta^3(\vec{k} - \vec{k}')$$

and zero for all other anticommutation relations, we find

$$\{\Psi(\vec{x},t),\bar{\Psi}(\vec{y},t)\} = \gamma^0\delta^3(\vec{x}-\vec{y})$$

where we have used the spin sums

$$\sum_s u_s(k)\bar{u}_s(k) = \gamma^\mu k_\mu + m$$

$$\sum_s v_s(k)\bar{v}_s(k) = \gamma^\mu k_\mu - m$$

For observable fields expressed in terms of the bilinear expressions $\bar{\Psi}M\Psi$ (e.g., $M = 1,\gamma^\mu,\gamma_5\gamma^\mu,\ldots$) we find commutation relations which are in agreement with the requirement of microcausality. For arbitrary 4×4 matrices $M_{1,2}$ we find (by using the operator identity $[AB,CD] = A\{B,C\}D - AC\{B,D\} - C\{A,D\}B + \{C,A\}DB$)

$$[\bar{\Psi}(x)M_1\Psi(x),\bar{\Psi}(y)M_2\Psi(y)] = 0 \text{ for } \vec{x}\neq\vec{y}$$

Moreover the use of anticommutation relations and normal ordering solves the problem of negative energy eigenvalues of the Hamiltonian

$$H = \int d^3k \omega_k \sum_s [a_s^\dagger(\vec{k}) a_s(\vec{k}) + b_s^\dagger(\vec{k}) b_s(\vec{k})]$$

Interpretation of the a, b operators:

While the $a_s^\dagger$ operator is the creation operator for matter (electron) with positive energy, the $b_s^\dagger$ operator can be interpreted as creation operator for antimatter (positron) with positive energy (or an annihilation operator for electrons with negative energy).

The Dirac Lagrangian is invariant under global phase transformations, so that there is a conserved current $\partial_\mu j^\mu = 0$ with $j^\mu = q\bar{\Psi}\gamma^\mu\Psi$. The quantized charge associated with this current reads:

$$Q = \int d^3x j^0 = q \int d^3k \sum_s [a_s^\dagger a_s - b_s^\dagger b_s]$$

We can interpret this as the current associated with the coupling to electromagnetism. Then $q(Q)$ corresponds to the electric charge (operator) and $b^\dagger$ is the creation operator for antimatter, i.e. a positron with opposite charge to the electron:

$$Qa_s^\dagger(\vec{k})|0>= qa_s^\dagger(\vec{k})|0> \quad \text{and} \quad Qb_s^\dagger(\vec{k})|0>= -qb_s^\dagger(\vec{k})|0>$$