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Would a risk-averse newsvendor order less at a higher selling price? *

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1. Introduction

Consider a newsvendor who must decide how many newspapers to order from the publisher in the morning for sale during the day. If he orders too few, he will lose potential sales and may also face additional shortage costs such as a loss of goodwill. If he orders too many, he must salvage all unsold newspapers to the publisher at a lower value. The newsvendor’s objective is to choose an optimal order quantity to balance his cost (or disutility) of ordering too many against his cost (or disutility) of ordering too few. Because of its simple but elegant structure, the single-period newsvendor model has contributed insight to a variety of settings such as inventory control, capacity planning, yield management, insurance, and supply chain contracts.

The standard newsvendor problem is based upon risk neutrality so that managers will place orders to maximize expected profits. However, in practice, there are many examples that imply managers’ decisions do not always correspond to the expected profit-maximization order quantity (e.g., Kahn, 1992; Fisher and Raman, 1996; Patsuris, 2001). Therefore, developing alternative choice models rather than risk neutrality to describe manager’s newsvendor decision-making behavior is becoming more important. Within this research stream, some researchers have studied risk-averse newsvendor decisions within the expected utility theory (EUT) framework (e.g., Eeckhoudt et al., 1995; Agrawal and Seshadri, 2000a,b).

Although those risk-averse newsvendor models provide useful guidance to managers on their optimal inventory decisions, none of them pay enough attention to a limitation of EUT in the economics field, i.e., risk aversion within the EUT framework implies that people are approximately risk-neutral when economic stakes are small (Arrow, 1971). For example, Rabin (2000) exposed some of the problematic consequences of this limitation for the question of whether or not to accept a gamble. His risk aversion calibration theorem shows that within EUT, even very little risk aversion over modest stakes implies an absurd degree of risk aversion over large stakes, e.g., if a person turns down gambles where she loses $100 or gains $110, each with 50% probability, at any wealth level, then she will turn down a 50–50 bet of losing $1000 or gaining an infinite sum of money. That is an absurd rate for the utility of money to deteriorate, thus showing a limitation of risk aversion.

This paper is especially motivated by the following result we observed in a numerical study in Section 4:

Suppose a manager chooses a commonly used exponential utility function within EUT to describe his risk aversion newsvendor decision-making. If demand is uniformly distributed between 0 and 100, then a slightly risk-averse manager will order 49 at unit cost $100 if his selling price is $400, but will order 47 if his selling price is $600. In other words, the quantity that maximizes his...
expected utility is 49 when the opportunity cost of a lost sale is only $300, but when his opportunity cost is higher at $500, his optimal order quantity decreases, and in fact, approaches zero as selling price continues to increase.

The above numerical example shows that as selling price becomes higher and higher, i.e., the prospect of selling a perishable product (e.g., newspaper) becomes better and better, a risk-averse manager prefers ordering less and less according to EUT. This anomalous result illustrates that the limitation of EUT also exists in the risk-averse newsvendor problem.

The main purpose of our research is to show when, how, and why the limitation of EUT noted by Arrow and Rabin is manifested in newsvendor decision-making. We use a series of theorems to characterize the relationship between a risk-averse newsvendor’s optimal order quantity and selling price. For most commonly used classes of risk aversion utility functions within EUT, e.g., CARA, IARA, and bounded DARA utility functions\(^2\), we find that a risk-averse newsvendor will order less than an arbitrarily small quantity as selling price gets larger if price is higher than a threshold value. Our results suggest that: (1) some care is in order when interpreting results based on newsvendor models and EUT, and (2) investigation of types of models suitable for risk-averse newsvendor behavior has merit.

This paper is organized as follows. In Section 2, we briefly review the relevant literature. In Section 3, we analyze the risk-averse newsvendor problem and derive some theorems and insights. In Section 4, we use a numerical example to illustrate our results. Finally, in Section 5, we offer concluding remarks and suggest opportunities for future research.

2. Related literature

The literature related to this research can be divided into two general categories: papers on the newsvendor problem with alternative utility functions to risk neutrality, and papers addressing limitations of EUT.

The traditional newsvendor model is based upon risk neutrality. We refer interested readers to Porteus (1990) and Khouja (1999) for reviews of this part of literature. Some researchers have attempted to use risk aversion within EUT to describe the decision-making behavior in the newsvendor problem. Eeckhoudt et al. (1995) study a risk-averse newsvendor who is allowed to obtain additional orders if demand is higher than his initial order. They find that a risk-averse newsvendor will order strictly less than a risk-neutral newsvendor. Agrawal and Seshadri (2000a) investigate a risk-averse and price-setting newsvendor problem. They find that a risk-averse newsvendor will charge a higher price and order less than the risk-neutral newsvendor if the demand distribution has the multiplicative form of relationship with price. Also, the risk-averse newsvendor will charge a lower price if the demand distribution has the additive form of relationship with price, but the effect on the quantity ordered depends on the demand sensitivity to selling price. Agrawal and Seshadri (2000b) consider an important role of intermediaries in supply chains to reduce the financial risk faced by risk-averse retailers. They show that a risk-neutral distributor can offer a menu of mutually beneficial contracts to the retailers so that the supply chain inefficiency due to risk-averse retailers can be avoided. Keren and Pliskin (2006) study a risk-averse newsvendor model under uniform demand. They derive the closed form solution and discuss its properties and application for assessing the newsvendor utility function parameters.

In addition to risk aversion, some researchers have used loss aversion within Prospect Theory (Kahneman and Tversky, 1979) to describe the decision-making behavior in the newsvendor problem. Wang and Webster (forthcoming) study a loss-averse newsvendor problem. They find that if shortage cost is negligible, then a loss-averse newsvendor may order more than a risk-neutral newsvendor. Wang and Webster (2007) consider a decentralized supply chain in which a single risk-neutral manufacturer is selling a perishable product to a single loss-averse retailer facing uncertain demand. They investigate the role of a gain/loss sharing provision for mitigating the loss aversion effect, which drives down the retailer order quantity and total supply chain profit.

We next briefly review research on EUT and its limitations. EUT may be traced back to Bernoulli (1954) in response to the famous St. Petersburg paradox.\(^3\) Later, the development of EUT with a set of appealing axioms on preference by von Neumann and Morgenstern (1944) provided the basis for most subsequent analysis of economic behavior under uncertainty. In particular, EUT allows for a bounded utility function, thus avoiding the St. Petersburg paradox. We refer to Schoemaker (1982) for a comprehensive review of EUT.

Although EUT is well accepted, empirical studies dating from the early 1950s (e.g., Allais, 1953) have shown some patterns in choice behavior inconsistent with EUT. The most recent paper reinforcing this theme is Rabin (2000), who characterizes a relationship between risk attitudes over small and large economic stakes. His calibration theorem applies to lotteries with two possible outcomes. He shows that within EUT, anything but virtual risk neutrality over modest stakes implies an absurd degree of risk aversion over large stakes. The results raise questions into validity of conclusions from experiments and analyses that rely on same utility function over large and small stakes. What is not clear is whether EUT presents any difficulties in a more complex newsvendor setting, a setting that can be viewed as a lottery with not two, but many possible outcomes contingent upon an order quantity. In addition, the choice is not whether or not to accept a gamble but how much to buy among a range of alternatives.

3. The risk-averse newsvendor model under EUT

We consider a risk-averse newsvendor with initial wealth \(W_0\) selling short-life-cycle products with uncertain demand. At the beginning of the selling season, the newsvendor initially orders \(Q\) products at a unit cost \(w\) from a supplier and sells at a retail price \(p > w\) during the selling season. Demand \(X\) is stochastic with PDF \(f(x)\) and CDF \(F(x)\) defined over the continuous interval \([a, b]\). To simplify notation, we

\(^2\) Utility functions within EUT are commonly classified into three categories of absolute risk aversion: (1) decreasing absolute risk aversion (DARA), which states that as an individual becomes wealthier, he will be less risk-averse; (2) increasing absolute risk aversion (IARA), which states that as an individual becomes wealthier, he will be more risk-averse; and (3) constant absolute risk aversion (CARA), which states that an individual's degree of risk aversion is independent of his wealth level.

\(^3\) Suppose a utility function \(U(.)\) is unbounded (e.g., as is the case for a risk neutral utility function), so that for every integer \(n\) there is an amount of money \(x_n\) with \(U(x_n) > 2^n\). Consider the following lottery: we toss a coin repeatedly until tails comes up. If this happens in the \(n\)th toss, then the monetary payoff from the lottery is \(x_n\). Since the probability of this outcome is \(2^{-n}\), it is clear that the expected utility of this lottery is infinity. But this means that an individual should be willing to give up all his wealth for the opportunity to play this lottery, an absurd conclusion.
assume without loss of generality that \( a = 0 \) (e.g., if \( a > 0 \), then \( Q \) represents the amount to increase the order beyond the minimum possible demand). If realized demand \( x \) is higher than \( Q \), then unit shortage cost penalty \( s > w - p \) is incurred on \( x - Q \) units. The case of \( s \in (w - p, 0) \) corresponds to situations where \( x - Q \) units can be purchased and sold after demand is realized at the lower unit margin of \(-s\) instead of \( p - w \). For example, \( Eckhoudt \text{ et al.} (1995) \) consider a newsvendor who is allowed to obtain additional newspapers at a cost \( w' \) satisfying \( w < w' < p \). Thus, the newsvendor is still able to make money on shortfalls units by placing a second order to satisfy unmet demand (i.e., the shortage cost penalty is \( s = w' - p \in (w - p, 0) \)). The possibility of \( s \leq w - p \) is excluded because such a case would imply that the newsvendor can make at least as much profit from a stock out as from selling a unit, i.e., there would be no reason to order prior to observing customer demand. If realized demand \( x \) is lower than \( Q \), then the newsvendor salvages \( Q - x \) unsold products at a unit value \( v < w \). As with most of the newsvendor models, we assume \( F(x) \) is continuous, differentiable, invertible, and strictly increasing over \( I \).

The newsvendor is risk-averse in the sense that he is unwilling to take a bet that is actuarially fair when facing uncertainty (Arrow, 1971). For example, a newsvendor who turns down a gamble of losing $100 or gaining $110, each with 50% probability, is risk-averse.

The function \( U(w) \) defines the newsvendor’s utility over his final wealth \( W \) where \( U(W) \) is twice differentiable. We have the following assumptions for the utility function \( U(W) \) of the risk-averse newsvendor.

A1. \( U'(W) > 0 \) for all \( W \)
A2. \( U''(W) < 0 \) for all \( W \)

A1 implies that \( U(W) \) is a strictly increasing function of \( W \), which simply says that more wealth is desirable. A2 implies the \textit{diminishing-marginal-utility-of-wealth theory} of risk aversion, i.e., a dollar that helps us avoid poverty is more valuable than a dollar that helps us become very rich. The assumptions A1 and A2 are common in the economics literature (see, e.g., Arrow, 1971; Pratt, 1964 and Rabin, 2000).

We let \( r(W) = -U''(W)/U'(W) \) denote the well-known Arrow–Pratt measure of absolute risk aversion, which measures the insistence of an individual for more-than-fair bets (Pratt, 1964; Arrow, 1971). We introduce the following additional assumption on the nature of \( r(W) \).

A3. \( \lim_{W \to \infty} r(W) > 0 \)

Assumptions A1 and A2 imply that \( r(W) > 0 \) for any finite \( W \). A3 says that the newsvendor is strictly risk-averse even as wealth goes to infinity. \( Rabin (2000) \) counterintuitive result (i.e., Corollary, p. 1291) also relies on A3. All CARA and IARA utility functions satisfy A3. A3 eliminates some unbounded DARA utility functions such as logarithmic and power functions with \( \lim_{W \to \infty} r(W) = 0 \). However, as pointed out by Arrow (1971), within EU1, the utility function must be bounded to avoid the St. Petersburg Paradox (Bernoulli, 1954). In addition, most of the commonly used DARA utility functions in economics and finance have the property of being characterized by the mixture of exponential functions, i.e., \( U(W) = \int_{0}^{W} \exp(-g(t)) \; dt \), where \( g(t) \) is a nondecreasing and bounded distribution function on \([0, \infty)\) (Caballé and Pomansky, 1996). If \( g(t) \) is a nondecreasing and bounded distribution function on \([0, \infty)\), then from Proposition 6.2 of Caballé and Pomansky (1996), \( \lim_{W \to \infty} r(W) = 0 \). For example, if \( g(t) = 0 \) for \( 0 < t < 1 \), \( g(t) = 1 \) for \( 1 < t < 2 \), and \( g(t) = 2 \) for \( 2 < t < \infty \), then \( U(W) = 2 - e^{-W} - e^{-2W} \); if \( g(t) = 0 \) for \( 0 < t < 1 \) and \( g(t) = t^{-4} \) for \( 1 < t < \infty \), then \( U(W) = 1 - e^{-W} + 1/e^{(W+1)}/(W+1) \). Both utility functions are DARA with \( \lim_{W \to \infty} r(W) = 1 \). In summary, there is a large class of DARA utility functions that satisfy A3 and have desirable properties for preference relationships, modeling aggregate economic behavior, and utility measurement assessment (Hammond, 1974; Brockett and Golden, 1987).\(^4\)

The newsvendor has the following payoff function:

\[
\pi(x, Q) = \begin{cases} 
\pi_-.(x, Q) = W_0 + px + v(Q - x) - wQ & x \leq Q, \\
\pi_.(x, Q) = W_0 + pQ - wQ - (x - Q) & x > Q. 
\end{cases}
\]

The newsvendor’s problem is to find an optimal order quantity \( Q^* \) to maximize his expected utility \( E[U(\pi(X, Q))] \), which can be expressed as follows:

\[
E[U(\pi(X, Q))] = \int_0^Q U(\pi_-.(x, Q))f(x)dx + \int_Q^\infty U(\pi_.(x, Q))f(x)dx.
\]

After taking the first and second derivatives of expression (2) with respect to \( Q \), we get:

\[
dE[U(\pi(X, Q))]/dQ = (p + s - w) \int_0^Q U'(\pi_.(x, Q))f(x)dx - (w - v) \int_0^Q U'(\pi_.(x, Q))f(x)dx
\]

(3)

and

\[
d^2E[U(\pi(X, Q))]/dQ^2 = (p + s - w)^2 \int_0^Q U''(\pi_.(x, Q))f(x)dx + (w - v)^2 \int_0^Q U''(\pi_.(x, Q))f(x)dx - (p + s - w)U'(\pi_.(Q, Q))F(Q)
\]

\[-(w - v)U'(\pi_.(Q, Q))F(Q) < 0.
\]

(4)

From (3) and (4) we see that \( dE[U(\pi(X, 0))]/dQ > 0, dE[U(\pi(X, b))]/dQ < 0, \) and \( d^2E[U(\pi(X, Q))]/dQ^2 < 0 \) for all \( Q \in I \). Therefore, there exists a unique optimal order quantity \( Q^* \) that satisfies the first-order condition

\[
(p + s - w) \int_0^Q U'(\pi_.(x, Q^*))f(x)dx - (w - v) \int_0^Q U'(\pi_.(x, Q^*))f(x)dx = 0
\]

(5)

and is an interior point, i.e., \( 0 < Q^* < b \).

\(^4\) For example, the sum-of-exponential utility function, \( U(W) = \sum_{i=1}^{k} \left( \frac{1}{1+e^{-x}} \right) a_i \), is a mixture of exponential functions over a discrete measure studied by Hammond (1974). It is easy to verify that it is DARA and satisfies A3. Brockett and Golden (1987) show that the sum-of-exponential utility function can also be used in practice to construct the decision maker’s utility function which is not known exactly at all wealth levels but can be determined at a finite number of points through testing.
If the newsvendor is risk-neutral, then the first-order condition (5) reduces to
\[(p + s - w)F(Q_s^*) - (w - v)F(Q_s^*) = 0\]  
and we get the risk-neutral newsvendor’s optimal order quantity
\[Q_s^* = F^{-1}\left(\frac{p + s - W}{p + s - v}\right).\]

From (5) and (6), we can see some differences between the risk-averse and risk-neutral newsvendor’s decision-making behavior. For a risk-neutral newsvendor, the first term in (6) is his marginal benefit in expected profit due to an increase in the initial order quantity whereas the second term is his marginal loss in expected profit. Given that there is an underage with a probability of \(F(Q_s^*)\), if one more unit is ordered, then a unit underage cost \(c_o = p + s - w\) will be saved. Similarly, given that there is an overage with a probability of \(F(Q_s^*)\), if one more unit is ordered, then a unit overage cost \(c_o = w - v\) will be incurred.

For the risk-averse newsvendor, the first term in (5) is his marginal benefit in expected utility due to an increase in the initial order quantity whereas the second term is his marginal loss in expected utility. Given that there is an underage with a probability of \(F(Q_s^*)\), if one more unit is ordered, then a unit underage cost \(c_o\) will be saved. However, in contrast with the risk-neutral newsvendor, there is also a decrease in the marginal utility of underage \(U'(\pi_s)\) through a wealth effect, i.e., the diminishing marginal utility if the newsvendor is richer (by \(U'(W) < 0\)). Similarly, given that there is an overage with a probability of \(F(Q_s^*)\), if one more unit is ordered, then a unit overage cost \(c_o\) will be incurred, and the decrease in wealth will be accompanied by an increase in the marginal utility of overage \(U'(\pi_s)\) through a wealth effect.

**Theorem 1.** For utility function \(U(W)\) satisfying A1–A3, let \(Q^*(p)\) denote the optimal order quantity at selling price \(p.\) If \(s \leq 0,\) then for any \(q > 0,\) there exists a finite threshold selling price \(p_T(q)\) such that \(Q^*(p) < q\) for all \(p > p_T(q).\)

**Theorem 1** suggests that if shortage cost penalty is negligible (i.e., \(s = 0\)), or the newsvendor is still able to make money on shortfall units by placing a second order to satisfy unmet demand (i.e., \(s < 0\)), then a risk-averse newsvendor with any risk aversion utility function satisfying A1–A3 will order less than an arbitrarily small quantity as selling price gets larger once price is beyond a threshold value. The result in **Theorem 1** is anomalous in the sense that while the traditional newsvendor models predict that a risk-neutral newsvendor will order more products as selling price increases, a risk-averse newsvendor within EUT will order less products as selling price increases beyond a threshold price. The result in **Theorem 1** holds for various types of risk aversion utility functions within EUT, including CARA, IARA, bounded DARA, and some unbounded DARA known as mixed utility functions (see **Definition 1**). It also holds for any continuous risk aversion utility functions exhibiting IARA, CARA, or DARA in certain range of wealth levels.

We next prove that as selling price increases, for any shortage cost penalty \(s,\) the newsvendor order quantity will eventually become arbitrarily small if the utility function exhibits mixed risk aversion, which is a class of utility functions comprised of a mixture of exponential functions that have positive odd derivatives and negative even derivatives.

**Definition 1 (Theorem 2.2, Caballé and Pomansky, 1996).** A utility function \(U_m(W)\) defined on \([0, \infty)\) displays risk aversion if and only if it admits the functional representation
\[U_m(W) = \int_0^\infty \frac{1 - e^{-w}}{t} \, dG(t),\]  
for some nondecreasing and right-continuous distribution function \(G\) on \([0, \infty)\) satisfying
\[\int_1^\infty \frac{dG(t)}{t} < \infty,\]  
where (8) is a necessary and sufficient condition for the convergence of the integral defining \(U_m(W)\) for \(W \in [0, \infty)\).

As pointed out by Pratt and Zeckhauser (1987), most of the commonly used utility functions in economics and finance such as power, exponential, logarithmic, and HARA (Hyperbolic Absolute Risk Aversion) have the property of being characterized by the mixture of exponential functions (see also Brockett and Golden, 1987; Caballé and Pomansky, 1996; Dachraoui et al., 2004).

Since \(\lim_{W \to \infty} r(W) > 0\) (by A3), from Proposition 6.2 of Caballé and Pomansky (1996), \(\lim_{W \to \infty} r(W) = \inf \{t | G(t) > 0\} > 0.\) Thus, we can rewrite (7) as follows:
\[U_m(W) = \int_{t_0}^\infty \frac{1 - e^{-w}}{t} \, dG(t)\]  
with
\[U'_m(W) = \int_{t_0}^\infty e^{-w} dG(t).\]

**Theorem 2.** For any mixed utility function \(U_m(W)\) satisfying (7) and (8) and A1–A3, let \(Q_m(p)\) denote the optimal order quantity at selling price \(p.\) Then for any \(q > 0,\) there exists a finite threshold selling price \(p_T(q)\) such that \(Q_m(p) < q\) for all \(p > p_T(q).\)

The conclusion in **Theorem 2** holds for any shortage cost penalty \(s\) and requires that the functional form of the utility function is a mixture of exponential functions. The results in **Theorem 1** and **Theorem 2** are largely consequences of the fact that, within EUT, a large increase in wealth becomes close to meaningless as an individual becomes wealthier. While it is true that a large decrease in wealth also becomes close to meaningless, the decrease in wealth from ordering one unit too many remains fixed at \(w - v\) as selling price increases. The details underlying this effect can be seen by first noting that (5) can be rewritten as
where \( c_0 = p + s - w \) and \( c_0 = w - v \). Leave \( Q^* \) fixed and consider the impact of increasing \( p \). If the newsvendor is risk neutral, then \( U'(W) = 1 \) for all \( W \). The selling price \( p \) only affects the risk-neutral newsvendor's decision-making through \( c_0 \). If \( p \) increases, then \( c_0 \) increases and \( c_0 \) remains unchanged so that the marginal benefit due to an increase in order quantity is larger than the marginal loss. Thus, the optimal quantity increases.

If the newsvendor is risk-averse, then again, as \( p \) increases, \( c_0 \) increases and \( c_0 \) remains unchanged. However, \( p \) also enters the decision-making process by increasing the value of the payoff function for any positive realization of demand. And due to the effect of increasing wealth, the expected change in utility per unit increase in wealth gets smaller. More specifically, an increase in \( p \) causes \( c_0U'(\pi(x,X)) \) to decrease at a rate proportional to \( c_0U'(\pi(x,X)) \). Consequently, the newsvendor perceives a higher cost rate of an underage (higher \( c_0 \)) and lower disutility from an overage (lower \( c_0U'(\pi(x,X)) \)). Both of these effects suggest that the optimal newsvendor quantity should increase. However, the increase in \( p \) also causes \( E(U'(\pi(x,X))) \) to get smaller, and when \( p \) is large, the value of \( c_0U'(\pi(x,X)) \) decreases at a rate approximately proportional to \( c_0U'(\pi(x,X)) \). The decrease is amplified by the large value of \( c_0 \) relative to \( c_0 \), and eventually the decrease in \( c_0U'(\pi(x,X)) \) dominates the decrease in \( c_0U'(\pi(x,X)) \). Thus, the optimal quantity decreases.

**Corollary 1.** For CARA utility function \( U_0(W) = w_0 - e^{-\gamma W} \) with \( \gamma > 0 \) and \( w_0 \) being a constant, let \( Q_0(p) \) denote the optimal order quantity at selling price \( p \). For any \( q > 0 \), there exists a finite threshold selling price \( p_T(q) \) such that \( Q_0(p) < q \) for all \( p > p_T(q) \).

Corollary 1 investigates the CARA utility function, which is a special form of mixed utility functions (Caballé and Pomansky, 1996, p. 490) and commonly used in economics and applied fields. The result in Corollary 1 suggests that for any shortage cost penalty \( s \), a risk-averse newsvendor with a CARA utility function will order less than an arbitrarily small quantity as selling price gets larger once price is beyond a certain threshold value. More specifically, Corollary 1 shows that as the prospect of an investment on selling newspapers (or other perishable goods) becomes better and better, a risk-averse newsvendor with a CARA utility function prefers ordering less and less.

For any CARA utility function \( U \) under CARA, the marginal utility gain of a unit below \( Q^* \) is everywhere, which is essentially the same as \( \gamma > 0 \). Thus, the newsvendor will not lose money when fulfilling emergency orders, i.e., the newsvendor is not worse-off in an increase in risk aversion (Pratt, 1964). The following proposition highlights the relationship between the optimal order quantities under utility functions \( U(W) \) and \( U(W) \) for the case of \( s \leq 0 \).

**Proposition 1.** If \( s \leq 0 \), then a newsvendor with utility function \( U(W) \) will always order less than a newsvendor with CARA utility function \( U_0(W) = w_0 - e^{-\gamma W} \).
To see if Watt (2002) different result still exists in the risk-averse newsvendor problem, next consider two commonly used unbounded DARA utility functions – logarithmic and power function – to investigate the newsvendor problem. Both utility functions satisfy \( \lim_{W \to -\infty} r(W) = 0 \) and thus violate A3. Since it is possible that the newsvendor will face negative payoff and both utility functions require positive outcomes as an input value, we need the following assumption on the initial wealth level \( W_0 \) to ensure the newsvendor’s final wealth level is positive at any realized demand \( x \leq l \).

**A4.** \( W_0 > \max\{(w - v)b, sb\}. \)

Note the term \((w - v)b\) in A4 is the newsvendor’s maximum overage loss, i.e., he orders \( b \), the highest potential demand, but the realized demand is zero. Also, the term \( sb \) is the newsvendor’s maximum underage loss, i.e., he orders nothing, but demand is realized at its highest potential, \( b \). A4 is not restrictive in the sense that we must use this assumption if we use those two utility functions to study the risk-averse newsvendor problem.

**Proposition 2.** For logarithmic function \( U_{\ell}(W) = \ln(W) \), let \( Q_{\ell} \) denote the optimal order quantity. If A4 holds, then \( dQ_{\ell}/dp > 0 \) for all \( p > w \).

**Proposition 3.** For power function \( U_p(W) = W^z(0 < z < 1) \), let \( Q_p \) denote the optimal order quantity. If A4 holds, then \( dQ_p/dp > 0 \) for all \( p > w \).

As shown by Proposition 2 and Proposition 3, results based upon unbounded DARA functions such as the logarithmic and power functions that violate A3 differ sharply from our previous results. Under both unbounded DARA utility functions, the newsvendor will order more as selling price gets higher, which is consistent with results based upon risk neutrality. From Propositions 2 and 3, it is natural to ask whether unbounded DARA utility functions that violate A3 are better choice models for describing the newsvendor decision bias than risk aversion utility functions satisfying A3 because of their consistent and intuitive results when we vary the selling price from low to high. Unbounded DARA utility functions have been suggested by Arrow (1971) and are applied in economics. However, while these unbounded DARA utility functions avoid the anomalous property of Theorem 1, they also fall prey to the counterintuitive St. Petersburg paradox, e.g., these utility functions imply that an individual would be willing to give up all his wealth for a gamble with a 50% probability that the utility of the pay-off will only be $2.01.

### 4. Numerical illustration

To illustrate our results, we assume the newsvendor’s utility function is CARA and satisfies \( U(W) = 1 - e^{-W} \). We also assume demand is uniformly distributed on \([0, D]\). We let \( s = v - W_0 = 0 \) (i.e., the newsvendor’s salvage value and initial wealth are zero). After mapping the demand and utility function into (2) and plugging in the parameters, we can rewrite (2) as

\[
E[U(\pi(X, Q))] = \int_0^D [1 - e^{-(p-w)\xi}]f(x)dx + \int_0^Q [1 - e^{-(p-w)\xi}]f(x)dx = 1 - \frac{D - Q}{D}e^{-(p-w)Q} + \frac{e^{-(p-w)Q} - e^{wQ}}{rpD} \tag{12}
\]

and at selling price \( p \), there exists a unique optimal order quantity \( Q^*(p) \in (0, D) \) which satisfies the following first-order condition:

\[
\frac{dE[U(\pi(X, Q^*(p)))]}{dQ} = \frac{e^{-(p-w)Q^*(p)}}{pD} - \frac{rp(p-w)(D - Q^*(p)) - w(e^{pQ^*(p)} - 1)}{e^{pQ^*(p)} - 1} = 0. \tag{13}
\]

**Fig. 1** illustrates the newsvendor’s optimal order quantity \( Q^*(p) \) with respect to his selling price \( p \) under various degrees of risk aversion \( r \). As shown in **Fig. 1**, if the newsvendor is a little risk-averse (e.g., \( r = 0.00001 \)) and selling price is relatively small (e.g., \( p < \$2000 \)), then his optimal order quantity is increasing in selling price. However, if selling price is higher than a threshold value (e.g., \( p \approx \$2000 \)), then the optimal order quantity begins to decrease in selling price. If the newsvendor is more risk-averse (e.g., \( r = 0.001 \)), then the threshold selling price becomes very small and the optimal order quantity is deceasing in selling price. The result in **Fig. 1** illustrates that EUT predicts a consistent relationship between selling price and order quantity (i.e., optimal order quantity is increasing in selling price) over a reasonable price range when risk aversion is relatively low (e.g., \( r = 0.00001 \) and \( p \in [\$200, \$2000] \)). However, the threshold price can be quite low at higher levels of risk aversion, which raises questions into the appropriateness of EUT in these settings.

![Fig. 1. Newsvendor’s optimal order quantity under alternative selling prices (p) and degrees of risk aversion (r) at D = 100 and w = $100.](image-url)
Proposition 4. For any \( q \in (0, D] \), define \( H(p) = p(p - w)(D - q) - \frac{w}{r}(e^{rpq} - 1) \). Then \( H(p) \) has at most two roots satisfying \( H(p) = 0 \).

(i) If \( H(p) \) has two roots, then the threshold price \( p_1(q) \) is the larger of two values of \( p \) that satisfies \( H(p) = 0 \);
(ii) If \( H(p) \) has only one root, then the threshold price \( p_1(q) \) is the unique value of \( p \) that satisfies \( H(p) = 0 \);
(iii) If \( H(p) \) has no root, then the threshold price satisfies \( p_1(q) = w \).

Proposition 4 characterizes the unique threshold price \( p_1(q) \) such that for any given \( q \in (0, D] \), the optimal order quantity \( Q^*(p) < q \) for all \( p > p_1(q) \). More specifically, if the function \( H(p) \) has one or two roots, then the unique threshold price \( p_1(q) \) can be characterized by the following equation:

\[
H(p_1(q)|q|) = p_1(q)(p_1(q) - w)(D - q) - \frac{w}{r}(e^{rp_1(q)q} - 1) = 0.
\] (14)

Proposition 5. If the unique threshold price \( p_1(q) \) satisfies (14), then

(i) \( p_1(q) \) is decreasing in \( q \);
(ii) if \( p_1(q) > \frac{1}{b} \), then \( p_1(q) \) is decreasing in \( r \).

Proposition 5(i) implies that as \( q \) increases, its corresponding threshold selling price \( p_1(q) \) becomes smaller. For example, our numerical results show that when \( D = 100 \) and \( r = 0.00001 \), if \( q \) is relatively small (e.g., \( q = 5 \)), then the threshold selling price is relatively high (e.g., \( p_1(5) = $268,774 \)). However, if \( q \) is relatively large (e.g., \( q = 85 \)), then the threshold selling price becomes much smaller (e.g., \( p_1(85) = $3368 \)). Proposition 5(ii) identifies a condition under which the threshold price \( p_1(q) \) is decreasing in the newsvendor’s degree of risk aversion \( r \). For example, our numerical results show that when \( D = 100 \) and \( q = 5 \), if the newsvendor is a little risk-averse, then the threshold price is relatively high (e.g., \( p_1(5) = $268,774 \) when \( r = 0.00001 \)). However, if the newsvendor becomes more risk-averse, then the threshold price becomes much smaller (e.g., \( p_1(5) = $1527 \) when \( r = 0.001 \)).

5. Discussion and conclusion

This paper investigates the application of EUT to the newsvendor problem. We find that within EUT, a risk-averse newsvendor will order less than an arbitrarily small quantity as selling price gets larger if price is higher than a threshold value. Our results provide the following implications.

First, the results support Rabin (2000) criticism on the limitation of EUT, i.e., within EUT, risk aversion at small stakes implies unrealistic risk aversion over large stakes. While this limitation has been recognized for lottery procedures in experimental economics (see Rabin, 2000 for a list of references), there is a lack of critical evaluation of EUT for more complex settings such as the newsvendor problem – a fundamental decision problem in the operations management literature. We build on Rabin (2000) risk aversion calibration theorem for the two-payoff distribution and go/no-go decision by considering the newsvendor problem with a continuous payoff distribution and a quantity decision. Our results show that the weakness in EUT exposed by Rabin’s calibration theorem is also present in the newsvendor problem. Given the wide applications of the newsvendor model in inventory control, capacity planning, yield management, insurance, finance, and supply chain contracts, our findings suggest that some care is in order when using EUT to derive policy guidelines, especially when there is significant risk aversion.

Second, for managers facing the risk-averse newsvendor problem, our study sheds light on when anomalous results that may arise and the implications regarding the choice of utility functions. For example, one alternative is an unbounded DARA utility function such as the logarithmic and power functions. Although such functions are also subject to the St. Petersburg paradox, our results show that they can avoid the anomalous effect of order quantity decreasing in selling price.

Appendix

Lemma 1. For any \( t_0 > 0 \), define \( h(p, t, q) = \int_q^0 e^{(x-q)f(x)/dx} - \frac{W - V}{p + s - W} \int_q^0 e^{(p-v)(q-x)f(x)/dx} \) with \( p > 0, q > 0, \) and \( t \in [t_0, \infty) \). There exists a finite \( p_1(q) \) such that if \( p > p_1(q) \), then \( h(p, t, q) < 0 \) for all \( t \in [t_0, \infty) \).

Proof of Lemma 1

\[
h(p, t, q) = \int_q^0 e^{(x-q)f(x)/dx} - \frac{W - V}{p + s - W} \int_q^0 e^{(p-v)(q-x)f(x)/dx} < e^{(p-v)(q-x)f(x)/dx} - \frac{W - V}{p + s - W} \int_q^0 e^{(p-v)(q-x)f(x)/dx}
\]

\[
= e^{(p-v)(q-x)f(x)/dx} \int_q^0 \left\{ F(q) - \frac{W - V}{p + s - W} \int_q^0 e^{(p-v)(q-x)f(x)/dx} \right\} dx
\]

\[
< e^{(p-v)(q-x)f(x)/dx} \int_q^0 \left\{ F(q) - \frac{W - V}{p + s - W} e^{(p-v)(q-x)(q-x)/2} \right\} dx.
\]
Define \( g(p, q, t) = \frac{e^{(p-v)q/2 - s(b-q)}}{p+s-w}F(q/2) \). Then
\[
\frac{dg(p, q, t)}{dp} = (w - v)F(q/2) \frac{e^{(p-v)q/2 - s(b-q)}}{(p+s-w)^2} \left[(p+s-w)qt/2 - 1\right],
\]
and
\[
\frac{dg(p, q, t)}{dt} = \frac{w - v}{p + s - w}F(q/2) \frac{e^{(p-v)q/2 - s(b-q)}}{(p+s-w)^2}. \tag{15}
\]

From Lemma 1, we can rewrite (17) as follows:
\[
\lim_{p \to \infty} \frac{dg(p, q, t)}{dp} = \lim_{p \to \infty} \frac{w - v}{p + s - w}F(q/2) \frac{e^{(p-v)q/2 - s(b-q)}}{(p+s-w)^2} = \infty. \tag{16}
\]

Let \( p_0 = \max(2s(b - q)/v + 2/(t_0) - s + w) \). If \( p > p_0 \), then \((p + s - w)qt/2 - 1 > 0 \) for all \( t \in [t_0, \infty) \), and from (15), it follows that \( \frac{dg(p, q, t_0)}{dp} > 0 \). Furthermore, from L'Hospital's Rule
\[
\lim_{p \to \infty} \frac{dg(p, q, t_0)}{dp} = \lim_{p \to \infty} (w - v)F(q/2) \frac{e^{(p-v)q/2 - s(b-q)}}{(p+s-w)^2} = \infty.
\]

Similarly, from (16) we can verify that if \( p > p_0 \), then \( \frac{dg(p, q, t)}{dt} > 0 \). Therefore, if \( p > p_0 \), then \( g(p, q, t) \geq g(p, q, t_0) \) for all \( t \in [t_0, \infty) \). Thus, for all \( t \in [t_0, \infty) \)
\[
h(t, p, q) < e^{(p-v)q}F(t) - g(p, q, t).
\]

Therefore, if \( F(q) < g(p_0, q, t_0) \), then let \( p_2(q) = p_0 \), it follows that \( h(p, t, q) < 0 \). If \( F(q) > g(p_0, q, t_0) \), since \( \frac{dg(p, q, t_0)}{dp} > 0 \) when \( p > p_0 \) and \( \lim_{p \to \infty} g(p, q, t_0) = \infty \), we can always find a finite \( p_2(q) > p_0 \) such that \( F(q) = g(p_2(q), q, t_0) \). In summary, if \( p > p_2(q) \), then we have \( F(q) < g(p, q, t_0) \leq g(p, q, t) \), i.e., \( h(p, t, q) < 0 \) for all \( t \in [t_0, \infty) \). \( \square \)

**Proof of Theorem 1.** Let \( \gamma = \inf_{x \in (0, \gamma)} F(W) \) and \( U_0(W) = u_0 - e^{-\gamma W} \) for some \( \gamma_0 \in (0, \gamma) \). For utility function \( U_0(W) \), let \( Q_0(p) \) denote the optimal order quantity at selling price \( p \). From Corollary 1, there exists a selling price \( p_2(q) \) such that \( Q_0(p) < q \) for any \( p > p_2(q) \). From Proposition 1, \( Q^*(p) < Q_0(p) \). Therefore, \( Q^*(p) < q \) for any \( p > p_2(q) \). \( \square \)

**Proof of Theorem 2.** For any \( q > 0 \), from (9) and (10), we have:
\[
dU_m(\pi(X, q)) = \int_{u_0}^{q} \left\{ \int_{q}^{b} (p+s-w)e^{-\gamma W_1}f(x)dx - (w-v) \int_{0}^{q} e^{-\gamma W_1}f(x)dx \right\} dG(t).
\]

From Lemma 1, we can rewrite (17) as follows:
\[
dU_m(\pi(X, q)) = \int_{u_0}^{q} (p+s-w)e^{-\gamma W_1}h(p, t, q)dG(t).
\]

From Lemma 1, for any \( q > 0 \), there exists a finite threshold selling price \( p_1(q) \) such that if \( p > p_1(q) \), then \( h(p, t, q) < 0 \) for all \( t \in [t_0, \infty) \). Thus
\[
dU_m(\pi(X, q)) < 0.
\]

Expression (19) implies that \( Q_m(p) < q \) for all \( p > p_2(q) \). \( \square \)

**Proof of Corollary 1.** Since CARA utility function is a special form of mixed utility function (see Caballé and Pomansky, 1996, p. 490), the proof follows immediately from **Theorem 2.** \( \square \)

**Proof of Proposition 1.** Recall that \( Q^* \) denotes the optimal order quantity. Let \( Q^0 \) be the optimal order quantity given utility function \( U_0(W) \), i.e.,
\[
(p+s-w) \int_{q^0}^{b} U_0(\pi_s(X, Q^0))f(x)dx - (w-v) \int_{0}^{q^0} U_0(\pi_s(X, Q^0))f(x)dx = 0.
\]

The value of \( Q^0 \) is an interior point, i.e., there exists \( x_1, x_2 \in I \) such that \( x_1 < Q^0 < x_2 \). Note that \( U_0(\pi_s(X, Q^0)) < U_0(\pi_s(X, Q^0)) \leq U_0(\pi_s(x, Q^0)) \)

and since \( k(\cdot) \) is strictly concave
\[
k'(U_0(\pi_s(X, Q^0))) > k'(U_0(\pi_s(X, Q^0))) \geq k'(U_0(\pi_s(x, Q^0))).
\]

Therefore
\[
\frac{dE[U(\pi(X, Q^0))/dQ} = (p+s-w) \int_{q^0}^{b} k'(U_0(\pi_s(X, Q^0)))U_0(\pi_s(X, Q^0))f(x)dx - (w-v) \int_{0}^{q^0} k'(U_0(\pi_s(X, Q^0)))U_0(\pi_s(X, Q^0))f(x)dx
\]
\[
< k'(U_0(\pi_s(X, Q^0))) \left\{ (p+s-w) \int_{q^0}^{b} U_0(\pi_s(X, Q^0))f(x)dx - (w-v) \int_{0}^{q^0} U_0(\pi_s(X, Q^0))f(x)dx \right\} = 0. \tag{20}
\]

From (20) it follows that \( Q^* < Q^0 \). \( \square \)

**Proof of Theorem 3.**

(i) For utility function \( U(W) \) satisfying A1 and A2, the optimal order quantity \( Q^* \) must satisfy the first-order condition (5). By the implicit function theorem and the concavity of \( E[U(\pi(X, Q))] \) in Q, we have:
From (31), we see that $d\theta$.

**Proof of Proposition 3.**

Let $Q_0(w_1)$ and $Q_0(w_2)$ be the optimal order quantity at the wholesale price $w_1$ and $w_2$, respectively. Then from (21), $Q_0(w_1)$ and $Q_0(w_2)$ satisfy the following first-order conditions:

$$\int_{Q_0(w_1)}^{b} e^{\alpha(s-x)}f(x)dx - (w_1 - v) \int_{0}^{Q_0(w_1)} \frac{e^{\alpha(s-x)}}{p + s - w_1} f(x)dx = 0. \tag{22}$$

$$\int_{Q_0(w_2)}^{b} e^{\alpha(s-x)}f(x)dx - (w_2 - v) \int_{0}^{Q_0(w_2)} \frac{e^{\alpha(s-x)}}{p + s - w_2} f(x)dx = 0. \tag{23}$$

Since the function $\int_{Q_0(w_1)}^{b} e^{\alpha(s-x)}f(x)dx - (w_1 - v) \int_{0}^{Q_0(w_1)} \frac{e^{\alpha(s-x)}}{p + s - w_1} f(x)dx$ is decreasing in $w$, for any $w_2 > w_1$, we have

$$\int_{Q_0(w_2)}^{b} e^{\alpha(s-x)}f(x)dx - (w_2 - v) \int_{0}^{Q_0(w_2)} \frac{e^{\alpha(s-x)}}{p + s - w_2} f(x)dx < 0. \tag{24}$$

By the concavity of $E[U_0(\pi(X, Q))]$, expressions (23) and (24) imply that for any $w_2 > w_1$, we have $Q_0(w_2) < Q_0(w_1)$. Therefore, the optimal order quantity is decreasing in $w$. \hfill \Box

**Proof of Proposition 2.**

Substituting $U_i(W) = \ln(W)$ into the first-order condition (5) yields

$$\int_{0}^{Q_0(w_1)} \frac{p + s - w}{W_0 + pQ_0(w_1) + wQ_0 - s(x - Q_0)} f(x)dx - \int_{0}^{Q_0} \frac{w - v}{W_0 + px + v(Q_0 - x) - wQ_0} f(x)dx = 0. \tag{25}$$

By the implicit function theorem,

$$dQ_0/dp = \frac{d^2E[U_0(\pi(X, Q))] / dQ_0 dp}{-d^2E[U_0(\pi(X, Q))] / dQ_0^2}. \tag{26}$$

By the concavity of $E[U_0(\pi(X, Q))]$ in $Q$, the sign of $dQ_0/dp$ only depends on the numerator of (26). After taking the partial derivative of (25) with respect to $p$, we get

$$\frac{d^2E[U_0(\pi(X, Q))] / dQ_0 dp}{dQ_0 dp} = \int_{0}^{Q_0} \left[ \frac{(W_0 - sX)f(x)}{W_0 + pQ_0 + wQ_0 - s(x - Q_0)} f(x)dx - \int_{0}^{Q_0} \frac{(w - v)xf(x)}{W_0 + px + v(Q_0 - x) - wQ_0} f(x)dx \right]. \tag{27}$$

From A4, we know $W_0 - sX > 0$ for all $X > 0$. Therefore, both terms in (27) are positive. Therefore, we have $d^2E[U_0(\pi(X, Q))] / dQ_0 dp > 0$, which implies $dQ_0/dp > 0$ for all $p > w$. \hfill \Box

**Proof of Proposition 3.**

Substituting $U_p(W) = W^\alpha(0 < \alpha < 1)$ into (5) yields

$$\int_{0}^{Q_0} \frac{p + s - w}{W_0 + pQ_0 + wQ_0 - s(x - Q_0)} f(x)dx - \int_{0}^{Q_0} \frac{w - v}{W_0 + px + v(Q_0 - x) - wQ_0} f(x)dx = 0. \tag{28}$$

Similarly, the sign of $dQ_0/dp$ only depends on $d^2E[U_0(\pi(X, Q))] / dQ_0 dp$. After taking the partial derivative of (28) with respect to $p$, we get

$$d^2E[U_0(\pi(X, Q))] / dQ_0 dp = \int_{0}^{Q_0} \left[ \frac{(W_0 - sX)f(x)}{W_0 + pQ_0 + wQ_0 - s(x - Q_0)} f(x)dx - \int_{0}^{Q_0} \frac{(w - v)xf(x)}{W_0 + px + v(Q_0 - x) - wQ_0} f(x)dx \right]. \tag{29}$$

From A4, we know both terms in (29) are positive. Therefore, $d^2E[U_0(\pi(X, Q))] / dQ_0 dp > 0$, which implies $dQ_0/dp > 0$ for all $p > w$. \hfill \Box

**Proof of Proposition 4.**

We first prove that $H(p|q)$ has at most two roots. After taking the first and second derivative of $H(p|q)$ with respect to $p$, we get:

$$dH(p|q)/dp = (2p - w)(D - q) - wqpe^{pq}, \tag{30}$$

$$d^2H(p|q)/dp^2 = 2(D - q) - wr^2e^{pq}. \tag{31}$$

From (31), we see that $d^2H(p|q)/dp^2$ is strictly decreasing in $p$ and $\lim_{p \to -\infty} d^2H(p|q)/dp^2 < 0$. Therefore, if $d^2H(p|q)/dp^2|_{p \to -\infty} < 0$, then $d^2H(p|q)/dp^2 < 0$ for all $p > w$. This implies $H(p|q)$ is concave in $p$ and $H(p|q)$ has at most two roots. If $d^2H(p|q)/dp^2|_{p \to w} > 0$, then we can always find a $p^* > w$ satisfying $d^2H(p|q)/dp^2 = 0$ so that $d^2H(p|q)/dp^2 > 0$ for all $p < p^*$ and $d^2H(p|q)/dp^2 < 0$ for all $p > p^*$. Therefore,
$H(p,q)$ is first convex then concave in $p$. Since $H(w,q) = -\frac{1}{q} (e^{qw} - 1) < 0$ and $\lim_{p \to -\infty} H(p,q) = -\infty < 0$, this also implies $H(p,q)$ has at most two roots.

From (13), the optimal order quantity $Q^*(p)$ must satisfy the following first-order condition:

$$
\frac{dE[U(\pi(X,Q^*(p)))]}{dQ} = \frac{e^{-p-wQ^*(p)}}{p} - H(p,Q^*(p)) = 0.
$$

Replacing $Q^*(p)$ with $q$ in (32), we have

$$
\frac{dE[U(\pi(X,q))]}{dQ} = \frac{e^{-p-wq}}{p} - H(p,q).
$$

(i) If $H(p)$ has two roots, $p_1$ and $p_2$ with $p_2 > p_1$, then for all $p \in (p_1, p_2)$, we must have $H(p,q) > 0$ and $\frac{dE[U(\pi(X,q))]}{dq} > 0$ (by expression (33)). Since $E[U(\pi(X,Q))]$ is concave, this implies $Q^*(p) > q$ for all $p \in (p_1, p_2)$. Similarly, for all $p > p_2$, we must have $H(p,q) < 0$ and $\frac{dE[U(\pi(X,q))]}{dq} < 0$. This implies $Q^*(p) < q$ for all $p > p_2$. Therefore, the threshold price is $p^*_1(q) = p_2$, i.e., the larger of the two roots of $H(p,q)$.

(ii) The proof is similar to Proposition 4(i).

(iii) If $H(p,q)$ has no root, then since $H(w,q) < 0$ and $\lim_{p \to -\infty} H(p,q) < 0$, we have $H(p,q) < 0$ for all $p > w$. Therefore, $p^*_1(q) = w$. □

Proof of Proposition 5

(i) After taking the first derivative of (14) with respect to $q$ and applying some algebraic manipulations, we get:

$$
\frac{dp_r(q)}{dq} = \frac{p_r(q)(p_r(q) - w) + w p_r(q)e^{p_r(q)q}}{(2p_r(q) - w)(D - q) - wq e^{p_r(q)q}}.
$$

From Proposition 4, if $p_r(q)$ satisfies (14), then it must be the larger of the two roots or the only root of values of $H(p,q)$. Therefore, from (30), we must have

$$
dH(p_r(q)/dq)/dp = (2p_r(q) - w)(D - q) - wq e^{p_r(q)q} \leq 0.
$$

From (34) and (35), we have $dp_r(q)/dq \leq 0$.

(ii) Similarly, after taking the first derivative of (14) with respect to $r$, we get:

$$
\frac{dp_r(q)}{dr} = \frac{w[(r p_r(q) - 1)e^{p_r(q)q} + 1]}{r^2[(2p_r(q) - w)(D - q) - wq e^{p_r(q)q}]}.
$$

From (35) and (36), if $r p_r(q) - 1 \geq 0$, i.e., $p_r(q) \geq \frac{1}{r}$, then $dp_r(q)/dr < 0$. □

References


