The following are two problems you should know how to solve for Test #3.

(i) Find the eigenvalues and corresponding eigenfunctions for

\[ x^2 \varphi'' + x \varphi' + (\beta \lambda) \varphi = 0 \quad \text{in} \quad [0, 1] \]

(i) \( \varphi(a) = \varphi(b) = 0 \) \( \text{and} \) (ii) \( \varphi'(a) = \varphi'(b) = 0 \) \( \text{or} \)

(iii) \( \varphi(a) = \varphi'(b) = 0 \) \( \text{and} \) (iv) \( \varphi'(a) = \varphi(b) = 0 \).

In eq. 1, \( \beta \) is a constant, \( \beta > 0 \).

Let \( \varphi_n(x) \) \( (n = 0, 1, 2, 3, \ldots) \) be the eigenfunctions found in part (i). Find the coefficient \( a_n \) in

\[ f(x) = \sum a_n \varphi_n(x) \].

We outline the procedure for solving (P) (i): if \( a = 1 \) and \( b > 1 \).

Set \( f(x) = x^p \), and substitute in 1 to find

\[ p(p-1) + 2p + \beta = 0 \]

\[ (p^2 + (a-1)p + \beta) = 0 \]

(Indicial equation)

\( \Rightarrow \)

\[ p = \frac{(1-\lambda)}{2} \pm \frac{1}{2} \sqrt{\beta \lambda - \frac{(1-\lambda)^2}{4}} = \text{(Indicial roots)} \]

assuming that \( \lambda > \frac{(1-\lambda)^2}{4\beta} \).