Introduction

I am interested in studying number-theoretic questions using the techniques of analysis. Specifically, I am interested in applying techniques from classical analytic number theory to the study of automorphic forms and exponential sums on $GL(3)$.

Many problems from elementary and additive number theory, such as the determination of the number of representations of an integer by the quadratic form $ax^2 + by^2 + cz^2 + dt^2$, lead to the study of bases of the square-integrable functions on the complex upper half-plane modulo $SL(2,\mathbb{Z})$, via the exponential sums of Kloosterman. A distinguished basis is given by the Hecke-Maass forms. These are defined as the eigenfunctions of a collection of differential and arithmetic operators, which give their Fourier coefficients many nice properties. In particular, these Fourier coefficients may be assembled into $L$-functions which generalize the Riemann zeta function and can be studied with techniques dating back to Riemann himself, as well as a large array of more modern methods. The knowledge gained from such studies then leads back to answers for difficult problems from classical number theory, e.g. [12].

The first immediate generalization of these objects are the automorphic forms, $L$-functions, and exponential sums attached to $GL(3)$. By the symmetric-square construction, $GL(2)$ automorphic forms and their $L$-functions can be lifted to a subset of those on $GL(3)$, so knowledge about $GL(3)$ automorphic forms generally brings knowledge about $GL(2)$ forms as well, but the study of exponential sums on $GL(3)$ is desirable for an entirely different reason.

The classical Kloosterman sums on $GL(2)$ are exponential sums over a single variable, and the Kuznetsov formula gives cancellation in smooth averages of Kloosterman sums beyond what is attainable through algebraic geometry. As many exponential sums that arise in number theory are given by sums over multiple variables, there has always been hope that trace formulas generalizing the Kuznetsov formula to higher rank groups would apply to smooth averages of these larger sums.

1 Automorphic forms and $L$-functions

The use of techniques from classical analytic number theory on $GL(3)$ is now gaining traction; the primary difficulties are a lack of tools and a limited understanding of the new special functions that arise. In particular, the weight functions in the spectral Kuznetsov formula are currently extremely difficult to handle due to sheer complexity. In [4], by studying their differential equations, I give simple expressions for these functions, which should make the Kuznetsov formula accessible to more than just the experts, thus enabling the generalization
of the hundreds of results which use the Kuznetsov formula on $GL(2)$.

The first major application of these was to the subconvexity problem for $GL(3)$ $L$-functions, a joint work with Valentin Blomer [2]. Existing subconvexity results on $GL(3)$ involved twisting by a lower-rank form and bounding the result in terms of the parameters of the lower-rank form; the majority of the cancellation then comes from the twist instead of the $GL(3)$ form. We were able to prove subconvexity for $GL(3)$ $L$-functions without any twists, and the cancellation is truly coming from the $GL(3)$ form. The primary tool here is a deep analysis of the weight functions, applied to an amplified fourth moment.

Also joint with Valentin Blomer, we give optimal spectral large sieve inequalities for $GL(3)$ Hecke eigenvalues [1]. On $GL(2)$, the spectral large sieve inequalities are used in the study of moments of $L$-functions and have applications in many equidistribution problems and questions relating directly to $GL(2)$ automorphic forms. The $GL(3)$ large sieve inequalities have been treated in a number of papers, but this latest version is considerably stronger than all of the previous results, and should be sufficient for applications. The primary tools here are a global decomposition of the long-element Kloosterman sum, and continuing analysis of the weight functions.

I hope to generalize these results in the level and rank aspects – that is, to congruence subgroups and $GL(n)$ for $n$ higher than three – and to higher moments. Many of the proofs of the integral formulas involved readily generalize, provided one can prove a small analytic continuation of certain functions (the key technical detail). Additionally, I intend to investigate the subconvexity problem near the self-dual forms on $GL(3)$; this was specifically avoided in [2] for technical reasons, but a subconvexity result for self-dual $GL(3)$ forms implies a quantitative version of the Quantum Unique Ergodicity problem on $GL(2)$.

The strength of many of the results, in particular the large sieve inequalities, on $GL(3)$ are limited by the contribution of the maximal parabolic Eisenstein series on the spectral side. In a paper with Fan Zhou [11], we give integral representations for the weight functions which should prove far more effective than the double Bessel formulas used in [2] and [1]. If one can push these formulas to give asymptotics across large ranges of the parameters, then with the deep understanding of the Kloosterman sums in [1], there is hope to cancel the maximal parabolic Eisenstein contribution with terms on the geometric side. Even on $GL(2)$, such cancellation is astoundingly difficult to see, but a highly desirable application would be to produce optimal spectral large sieve inequalities for the cusp forms independent of the rest of the spectrum.
2 Higher weight and exponential sums

The automorphic forms on $GL(2)$ fall into two categories – the (spherical or weight zero) Maass forms and the holomorphic modular forms, which may be collectively generalized as Maass forms with weight. On $GL(3)$, only the spherical Maass forms have been studied analytically, but, for instance, the symmetric-square lifts of holomorphic modular forms are non-spherical. Some aspects of the higher-weight theory exist in the literature, e.g. [14], but not in the minute detail necessary for analytic number theory.

In my thesis [5], using the $GL(3)$ Kuznetsov formula on spherical forms, I showed that smooth averages of the big-cell Kloosterman sum on $SL(3, \mathbb{Z})$ exhibit the same strong cancellation as the classical Kloosterman sums, but the more arithmetically interesting sums are the small-cell Kloosterman sums, which are the hyper-Kloosterman sums. It is essentially impossible to study these hyper-Kloosterman sums using the techniques of my thesis, as they are invisible behind the big-cell sums. Instead, I can show that a smooth average of the hyper-Kloosterman sums has as its primary asymptotic a sum over $SL(3, \mathbb{Z})$ cusp forms in higher weight. This is already a strong step towards resolving a conjecture of Bump, Friedberg and Goldfeld from 1988 [3]. It remains to study this sum over the cusp forms, which is a strong motivator for the previous problem.

A known hard problem which is similar to the averages of hyper-Kloosterman sums is the cubic equidistribution problem. The problem is to show the equidistribution of the “mod $p$” roots of a cubic polynomial which can be connected to averages of exponential sums on $SL(3, \mathbb{Z})$ by the work of Hooley [13]. This relates to the study of averages of hyper-Kloosterman sums by the structure of the average being considered, as both types of averages can be considered a Poincaré-like series over a subset of $SL(3, \mathbb{Z})$, subject to some constraints on the entries. The subset in this problem is much smaller, hence more difficult, than in the hyper-Kloosterman problem, but it is my hope that the higher-weight approach will also yield results in this situation.

The cubic equidistribution problem is one particular example of a large class of questions that can be phrased as exponential sums over a subset of some lattice in $SL(n, \mathbb{Z})$, subject to some constraints on the entries. For problems not constrained to some subset, we would use a trace formula on the symmetric space $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ to obtain cancellation; the higher-weight approach is to use the compact group $SO(n, \mathbb{R})$ to isolate the constraints. For instance, averages of hyper-Kloosterman sums appear as a sum over $SL(3, \mathbb{Z})$ with the constraint that the lower-left entry be zero. If we wish to detect when $\gamma$ has lower-left entry equal to zero from the Iwasawa decomposition of some $\gamma z$, this is precisely equivalent to the $k$-part of the decomposition having lower-left entry equal to zero, so we may design a function.
on $SO(3, \mathbb{R})$ that approximates the constraint. Constructing a trace formula with non-trivial dependence on $SO(3, \mathbb{R})$ gives the connection to higher-weight automorphic forms.

I have recently completed a string of papers working out the necessary details on the higher-weight automorphic forms:

1. In [6] and [7], I make explicit the Eisenstein series and cuspidal portions of the Langlands spectral expansion for non-spherical forms on $GL(3)$. This introduces several new types of cusp forms, including the symmetric square lifts of $GL(2)$ holomorphic modular forms.

2. The papers [8] and [9] construct generalizations of the Kuznetsov and Petersson trace formulas for these new forms and apply them to arithmetically-weighted Weyl laws. In particular, this demonstrates that such forms exist beyond the symmetric-squares of holomorphic modular forms. These also exhibit a significantly simpler method than [4] for obtaining Kuznetsov-type trace formulas, which should apply in great generality; that is, to groups beyond $GL(n)$.

3. The paper [10] detours back to the central problem of my thesis and, by applying a true inversion formula, gives stronger bounds for smooth sums of the long-element Kloosterman sum, as well as the analytic continuation of the Kloosterman zeta function, in one particular case of the signs of the indices. This derives from combining the spherical and higher-weight Kuznetsov formulas.

As the fundamental work on higher-weight forms is now complete, I have now begun work on the hyper-Kloosterman sum problem. The best-possible outcome in this case would be a Kuznetsov-type formula expressing a smooth average of hyper-Kloosterman sums as a spectral sum over the full $SL(3, \mathbb{Z})$ spectrum (with weight). This would occur by a limiting process applied to the spectral interpretation described above. As trace formulas are capable of obtaining results a step beyond algebraic geometry for smooth sums of exponential sums, so are limits of trace formulas a step beyond the trace formulas themselves. This line of attack has become popular recently with Langlands’ “Beyond Endoscopy” program, and such a solution to the hyper-Kloosterman sum problem would be a strong step forward in that direction.

References


