Rough sets in spatio-temporal data mining*

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Abstract. In this paper I define spatio-temporal regions as pairs consisting of a spatial and a temporal component and I define topological relations between them. Using the notion of rough sets I define approximations of spatio-temporal regions and relations between those approximations. Based on relations between approximated spatio-temporal regions configurations of spatio-temporal objects can be characterized even if only approximate descriptions of the objects forming them are available.

1 Introduction

Rough set theory [Paw82] provides a way of approximating subsets of a set when the set is equipped with a partition or equivalence relation. Rough sets were extensively used in the context of Data Mining, e.g., [Lin95,LC97]. So far, however, they were used mainly in non spatio-temporal contexts, for example, in order to classify and analyze phenomena, like diseases, given a finite number of observations or symptoms, e.g., [NSR92,BNSNT$^+$95]. It is the purpose of this paper to apply rough sets in a spatio-temporal context, i.e., to describe and classify (configurations of) spatio-temporal objects.

An important task in spatio-temporal data mining is to discover characteristic configurations of spatial objects. Characterizing spatial configurations is important, for example, in order to retrieve your new 'ideal' home from a property database such that it has access to a highway, is located by the shore of a lake, within a beautiful forest, and far away from the next nuclear power station. Another important task is to find classes of configurations that characterize molecules like amino-acids and proteins [GFA93]. Spatio-*temporal* relations are often important to identify causal relationships between events in which are spatio-temporal objects are involved: In order to interact with each other things often need to be at the same place at the same time.

There are three major aspects characterizing *spatio-temporal objects*: (1) Aspects characterizing *what* they are, e.g., the class of things they belong to; (2) Aspects characterizing *where* they are, i.e., their spatial location; (3) Aspects characterizing *when* they existed and when they have been, are, or will be where, i.e., their temporal location. Between spatio-temporal objects hold *spatio-temporal relations* such as 'being in the same place at the same time', or 'having been in a place before something else'. Sets of spatio-temporal objects form *spatio-spatial configurations* that are characterized by sets

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of spatial, temporal, and spatio-temporal relations that hold between objects forming the configuration. In this paper I concentrate on topological spatio-temporal relations (like ‘being in the same place at the same time’). Topological relations between regions of space and time play a major role in characterizing spatial and temporal configurations [EFG97,All83].

Today the classification of spatio-temporal configurations is based on relations between objects and on relations between the spatio-temporal regions they occupy. Unfortunately, it is often impossible to identify the region of space and time those objects exactly occupy, i.e., the exact location of spatio-temporal objects is often *indeterminate* [BF95]. [Bit99] argued that often *approximate* location of spatial objects is known. The notion of approximate location is based on the notion of rough sets, i.e., the approximation of (exact) location with respect to a regional partition of space and time. In this paper I discuss how rough sets can be used in order to describe approximate location in space and time and how to derive possible relations between objects given their approximations.

This paper is structured as follows. In Section 2 I define the notions of spatio-temporal object, location, region, and the relationships between them. I define topological relations between spatio-temporal regions in Section 3. The notion of a rough set is used in Section 4 in order to approximate spatio-temporal regions with respect to regional partitions of space and time. In Section 5 binary topological relations between those approximations are defined. These relations can be used to characterize configurations of spatio-temporal objects even if we know only their approximate location. In Section 6 the conclusions are given.

2 Location of spatio-temporal objects

Every spatio-temporal object, o, is located in[1] a unique region of time, $t(o)$, bounded by the begin and the end of its existence. In every moment of time a spatio-temporal object is exactly located in a single region, x^s, of space [CV95]. This region is the exact or precise spatial location of o at the time point t, i.e., $x^s = r_t(o)$ at t. Spatio-temporal wholes have temporal parts, which are located in parts of the temporal regions occupied by their wholes[2]. Consider, for example, the region of time, x^t, where the object, o, is located temporally, while being spatially located in the region x^s. If x^t is a maximal connected temporal region, i.e., o was once spatially located in x^s for a while, left and never came back, then x^t is bounded by the time instances (points) t_1 and t_2. Since time is a totally ordered set of time points (the set of all possible boundaries of time intervals) forming a directed one-dimensional space [Gea66], we have $t_1 < t_2$. In this paper time is modeled as a one dimensional directed line and space is modeled as a

[1] We say that the object x is located *in* the (spatial, temporal or spatio-temporal) region y in order to stress the exact fit of object and region (the object matches the region). It is important to distinguish the exact match from the case of an object being located *within* a region which intuitive meaning allows the region to be bigger than the object and the case of the object *covering* a region which intuitively implies the region to be smaller than the object.

[2] Notice that this implies a four dimensional ontology of spatio-temporal objects [Sim87]

2-dimensional plane. In the remainder I concentrate on *regions* of time and space and topological relations between them.

Spatio-temporal objects may be at rest, i.e., being located in the same region of space for a period of time, or they may change, i.e., being located in different regions of space at each moment of time[3]. Spatial change may be continuous, i.e., regions of consecutive moments of time are topologically close as in the case of change of bona-fide objects [SV97] like cars, planets, and human beings, or discontinuous, as (sometimes) in the case of change of fiat objects [SV97] like land property.

Consider an object at rest. I assume that the exact region of a spatio-temporal object, o, has always a corresponding time interval[4], x^t, which is bounded by the moment of time, t_1, where o 'stopped' at x^s and the moment, t_2, of time where o 'leaves' x^s (or the current moment of time if o is currently resting at x^s), i.e., $\forall t \in x^t : r_t(o) = x^s$. I define the *spatio-temporal region* of the *resting* object o as a pair, $r^r_{st}(o) = (x^t, x^s)$. The region x^t is a part of the exact temporal region of o, i.e., $P(x^t, t(o))$.

Spatial change causes spatio-temporal objects to be located in different regions of space at different moments of time. Consider a changing (moving, growing, shrinking, $\ldots$) spatio-temporal object, o, within the time interval $x^t = [t_b, t_e]$. Let x^{s_m} be the sum, $\vee$, of the regions in which o was located during x^t, i.e., $x^{m_s} = \bigvee\{r_t(o) \mid t_b \leq t \leq t_e\}$. I define the *spatio-temporal region* of the *spatially changing* object o as a pair, $r^c_{st}(o) = (x^t, x^{s_m})$. Notice, that doing this we do not know anymore, where exactly o is located during x^t. It can be everywhere within x^{s_m}, but it cannot be somewhere else. In the special case of continuous movement the region x^{s_m} can be thought of as the *path* of the object's movement during x^t. In the remainder I will use the metaphor 'path of change during x^t' in order to refer to the sum of spatial regions of a spatially changing object during the interval x^t.

3 Binary topological relations between spatio-temporal objects

Binary topological relations between regions such as overlap, contained/containing, disjoint, are well known in the spatial reasoning community, e.g., [EF91,RCC92]. Recently, [BS00] proposed a specific style that allows to define binary topological relations between regions exclusively based on constraints regarding the outcome of the meet (intersection) operation, denoted by $\wedge$, between (one and two dimensional) regions. This is critical for the generalization of these relations to the approximation case in Section 5. In this section I shortly review those definitions based on [BS00] and apply them to temporal and spatio-temporal regions afterwards.

3.1 Relations between regions of (2D) space

Given two regions x and y the boundary insensitive binary topological relation (RCC5 relations [RCC92]) between them can be determined by considering the triple of boolean values [BS00]:

$$(x \wedge y \neq \bot,\ x \wedge y = x,\ x \wedge y = y).$$

[3] This implies an ontology of absolute space and time, i.e., regions do not change.

[4] A maximal connected region of time.

The formula $x \wedge y \neq \perp$ is true if the intersection of y and y is not the empty region; The formula $x \wedge y = x$ is true if the intersection of x and y is identical to x; The formula $x \wedge y = y$ is true if the intersection of x and y is identical to y. The correspondence between such triples of boolean values and the RCC5 classification is given in the table below. Possible geometric interpretations are given in Figure 1 [BS00].

$x \wedge y \neq \perp$	$x \wedge y = x$	$x \wedge y = y$	RCC5
F	F	F	DR
T	F	F	PO
T	T	F	PP
T	F	T	PPi
T	T	T	EQ

The set of triples is partially ordered by defining $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ iff $a_i \leq b_i$ for $i = 1, 2, 3$, where the boolean values are ordered by $\mathsf{F} < \mathsf{T}$. [BS00] refer to the Hasse diagram of the partially ordered set (The right diagram in Figure 1.) as the RCC5 *lattice*.

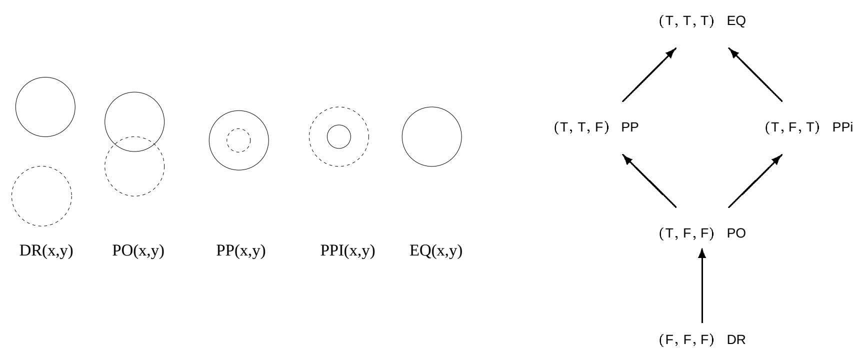


Fig. 1. RCC5 relations and RCC5 lattice

3.2 Relations between temporal regions

Consider the maximal connected one dimensional regions, x and y, i.e., intervals. Boundary insensitive topological relation between *intervals* x and y on a *directed* line (RCC_1^9 relations) can be determined by considering the triple of values belonging to the set $\{\mathsf{FLO}, \mathsf{FLI}, \mathsf{T}, \mathsf{FRI}, \mathsf{FRO}\}$:

$$(x \wedge y \not\sim \perp,\ x \wedge y \sim x,\ x \wedge y \sim y)$$

where

$$x \wedge y \not\sim \perp = \begin{cases} \mathsf{FLO} & \text{if } x \wedge y = \perp \text{ and } x \ll y \\ \mathsf{FRO} & \text{if } x \wedge y = \perp \text{ and } x \gg y \\ T & \text{if } x \wedge y \neq \perp \end{cases}$$

where

$$x \wedge y \sim x = \begin{cases} \mathsf{FLO} & \text{if } x \wedge y \neq x \text{ and } x \wedge y \neq y \text{ and } x \ll y \\ \mathsf{FLI} & \text{if } x \wedge y \neq x \text{ and } x \wedge y = y \text{ and } x \ll y \\ \mathsf{FRO} & \text{if } x \wedge y \neq x \text{ and } x \wedge y \neq y \text{ and } x \gg y \\ \mathsf{FRI} & \text{if } x \wedge y \neq x \text{ and } x \wedge y = y \text{ and } x \ll y \\ \mathsf{T} & \text{if } x \wedge y = x \end{cases}$$

and where

$$x \wedge y \sim y = \begin{cases} \mathsf{FLO} & \text{if } x \wedge y \neq y \text{ and } x \wedge y \neq x \text{ and } x \ll y \\ \mathsf{FLI} & \text{if } x \wedge y \neq y \text{ and } x \wedge y = x \text{ and } y \gg x \\ \mathsf{FRO} & \text{if } x \wedge y \neq y \text{ and } x \wedge y \neq x \text{ and } x \gg y \\ \mathsf{FRI} & \text{if } x \wedge y \neq y \text{ and } x \wedge y = x \text{ and } y \ll x \\ \mathsf{T} & \text{if } x \wedge y = y \end{cases}$$

with

$$x \ll y = \begin{cases} T & \text{if } L(x) \wedge L(y) = L(x) \text{ and } L(x) \wedge L(y) \neq L(y) \\ F & \text{otherwise} \end{cases}$$
$$x \gg y = \begin{cases} T & \text{if } R(x) \wedge R(y) = R(x) \text{ and } R(x) \wedge R(y) \neq R(y) \\ F & \text{otherwise} \end{cases}$$

$L(x)$ ($R(y)$) is the one dimensional region occupying the whole line left (right)[5] of x. The intuition behind $x \wedge y \sim x = \mathsf{FLO}$ ($x \wedge y \sim x = \mathsf{FRO}$) is that "$x \wedge y = x$ is false because of parts of x 'sticking out' to the left (right) of y". The intuition behind $x \wedge y \sim y = \mathsf{FLI}$ ($x \wedge y \sim y = \mathsf{FRI}$) is that "$x \wedge y = y$ is false because of parts of y 'sticking out' to the right (left) x".

The triples formally describe jointly exhaustive and pairwise disjoint relations under the assumption that x and y are intervals in a one dimensional directed space. The correspondence between the triples and the boundary insensitive relations between intervals is given in the table below. Possible geometric interpretations of the defined relations are given in Figure 2.

$x \wedge y \not\sim \bot$	$x \wedge y \sim x$	$x \wedge y \sim y$	RCC_1^9
FLO	FLO	FLO	DRL
FRO	FRO	FRO	DRR
T	FLO	FLO	POL
T	FRO	FRO	POR
T	T	FLI	PPL
T	T	FRI	PPR
T	FLI	T	PPiL
T	FRI	T	PPiR
T	T	T	EQ

For example. The relation $\mathsf{DRL}(x, y)$ holds if x and y do not overlap and x is left of y; $\mathsf{POL}(x, y)$ holds if x and y partly overlap and the non overlapping parts of x are left

[5] I use the spatial metaphor of a line extending from the left to the right rather than the time-line extending from the past to the future in order to focus on the aspects of the time-line as a one-dimensional directed space. Time itself is much more difficult.

of y; $\mathsf{PPL}(x, y)$ holds if x is contained in y but x does not cover the very right parts of y; $\mathsf{PPiL}(x, y)$ holds if y is a part of x and there are parts of x sticking out to the left of y; $\mathsf{PPR}(x, y)$ holds if x is a part of y and x does not cover the very left parts of y; $\mathsf{PPiR}(x, y)$ holds if y is a part of x and there are parts of x sticking out to the right of y.

Assuming the ordering $FLO < FLI < \mathsf{T} < FRI < FRO$ a lattice is formed, which has (FLO, FLO, FLO) as minimal element and (FRO, FRO, FRO) as maximal element. The ordering is indicated by the arrows in Figure 2.

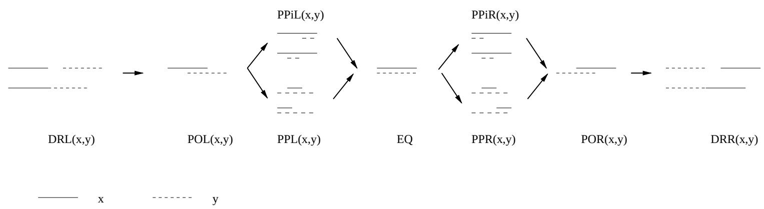


Fig. 2. Possible geometric interpretations of the RCC^9_1 relations.

3.3 Relations between spatio-temporal regions

Let o_1 and o_2 be two spatio-temporal objects at rest with spatio-temporal location $r^r_{st}(o_1) = (x^t, x^s)$ and $r^r_{st}(o_2) = (y^t, y^s)$, where x^t and y^t are time intervals, i.e., maximally connected temporal regions, and x^s and y^s are arbitrary, possibly scattered, 2D regions. The spatio-temporal relation between (x^t, x^s) and (y^t, y^s) can be described using the following pair of triples:

$$((x^t \wedge y^t \not\sim \perp,\ x^t \wedge y^t \sim x^t,\ x^t \wedge y^t \sim y^t),$$
$$(x^s \wedge y^s \neq \perp,\ x^s \wedge y^s = x^s,\ x^s \wedge y^s = y^s))$$

The relationship between those pairs of triples and spatio-temporal relations is given in the following table:

$x \wedge y \not\sim \perp$	$x \wedge y \sim x$	$x \wedge y \sim y$	$x \wedge y \neq \perp$	$x \wedge y = x$	$x \wedge y = y$	(RCC^9_1 ,RCC5)
FLO	FLO	FLO	F	F	F	(DRL,DR)
FLO	FLO	FLO	T	F	F	(DRL,PO)
…	…	…	…	…	…	…
T	FLO	FLO	T	F	F	(POL,PO)
T	FLO	FLO	T	T	F	(POL,PP)
T	FLO	FLO	T	F	T	(POL,PPi)
T	FLO	FLO	T	T	T	(POL,EQ)
T	T	FLI	F	F	F	(PPL,DR)
T	T	FLI	T	F	F	(PPL,PO)
…	…	…	…	…	…	…
T	T	T	T	T	T	(EQ,EQ)
FRO	FRO	FRO	F	F	F	(DRR,DR)
…	…	…	…	…	…	…

The the Hasse diagram of the partially ordered set is called the (RCC_1^9 ,RCC5) *lattice*.

For example. The relation $(\mathsf{DRL},\mathsf{PO})((x^t,x^s),(y^t,y^s))$ is interpreted as follows: The spatial regions x^s and y^s partially overlap and the relation between the time interval x^t when o_1 rested in x^s and the time interval y^t when o_2 rested in y^s is DRL. The scenario, i.e., the sequence of events, could be described as: o_1 changes to location x^s and rests there during x^t. At some time in the future (o_1 has already left x^t), o_2 changes to y^s such that $\mathsf{PO}(x^s,y^s)$[6].

The relation $(\mathsf{POL},\mathsf{PO})((x^t,x^s),(y^t,y^s))$ is interpreted as follows: The spatial regions x^s and y^s partially overlap and the relation between the time interval x^t when o_1 rested in x^s and the time interval y^t when o_2 rested in y^s is POL. The scenario is: o_1 changes to its location x^s and rests there during x^t. While o_1 is resting in x^s, o_2 changes to y^s such that $\mathsf{PO}(x^s,y^s)$ holds. While o_2 is still resting in y^s, o_1 changes to another region. This new region may or may not overlap y^s.

Formally, for changing objects the same style of definition applies. Only the exact regions of o_1 and o_2, x^s and y^s during x^t and y^t, are replaced by the path of change x^{s_m} and y^{s_m}, of o_1 and o_2 during x^t and y^t. In this case we do not describe the relation between the location of rest of o_1 during x^t and the location of rest of o_2 during y^t, but the relation between the path of change of o_1 during x^t and the path of change of o_2 during y^t. The relation $(\mathsf{POL},\mathsf{PO})((x^t,x^s),(y^t,y^s))$ is interpreted as follows: The path of change of o_1 during x^t and the path of change of o_2 during y^t do partially overlap and the relation $\mathsf{POL}(x^t,y^t)$ holds between the time intervals x^t and y^t. The interpretation of $\mathsf{POL}(x^t,y^t)$ is that we started monitoring the path of o_1 earlier than monitoring the path of o_2 and finished monitoring the path of o_1 earlier than monitoring the path of o_2.

4 Rough approximations

Rough set theory [Paw82] provides a way of approximating subsets of a set when the set is equipped with a partition or equivalence relation. Given a set X with a partition $\{a_i \mid i \in \mathcal{I}\}$, an arbitrary subset $b \subseteq X$ can be approximated by a function $\varphi_b : \mathcal{I} \to \{\mathsf{fo},\mathsf{po},\mathsf{no}\}$. The value of $\varphi_b(i)$ is defined to be fo if $a_i \subseteq b$, it is no if $a_i \cap b = \varnothing$, and otherwise the value is po. The three values fo, po, and no stand respectively for 'full overlap', 'partial overlap' and 'no overlap'; they measure the extent to which b overlaps the elements of the partition of X.

4.1 Approximating spatial and temporal regions

[BS00] showed that regions of space and time can be described by specifying how they relate to a partition of space and time into cells which may share boundaries but which do not overlap. A region can then be described by giving the relationship between the region and each cell. Suppose a space $\mathcal{S}$ of precise regions. By imposing a partition, G, on $\mathcal{S}$ we can approximate elements of $\mathcal{S}$ by elements of Ω_3^G. That is, we approximate regions in $\mathcal{S}$ by functions from G to the set $\Omega_3 = \{\mathsf{fo},\mathsf{po},\mathsf{no}\}$. The function which

[6] Remember, regions do not change.

assigns to each region $x \in \mathcal{S}$ its approximation is denoted $\alpha_3 : \mathcal{S} \rightarrow \Omega_3^G$. The value of $(\alpha_3\, x)\, g$ is fo if x covers all the of the cell g, it is po if x covers some but not all of the interior of g, and it is no if there is no overlap between x and g.

Each approximate region $X \in \Omega_3^G$ stands for a set of precise regions, i.e., all those precise regions having the approximation X. This set which will be denoted $[\![X]\!]$ provides a semantics for approximate regions: $[\![X]\!] = \{x \in \mathcal{S} \mid \alpha_3 x = X\}$ [BS00].

4.2 The meet operation

The domain of regions is equipped with a meet operation interpreted as the intersection of regions. In the domain of approximation functions the meet operation between regions is approximated by pairs of greatest minimal, $\underline{\wedge}$, and least maximal, $\overline{\wedge}$, meet operations on approximation mappings [BS98]. Consider the operations $\underline{\wedge}$ and $\overline{\wedge}$ on the set $\Omega_3 = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$ that are defined as follows:

$\underline{\wedge}$	no	po	fo
no	no	no	no
po	no	no	po
fo	no	po	fo

$\overline{\wedge}$	no	po	fo
no	no	no	no
po	no	po	po
fo	no	po	fo

These operations extend to elements of Ω_3^G (i.e. the set of functions from G to Ω_3) by

$$(X \,\underline{\wedge}\, Y)g = (Xg) \,\underline{\wedge}\, (Yg)$$

and similarly for $\overline{\wedge}$.

4.3 Approximating spatio-temporal regions

Spatio-temporal regions are pairs, (x^t, x^s), consisting of a spatial component, x^s, and a temporal component, x^t. Both components can be approximated separately by approximation functions, X^s and X^t with respect to partitions G_T and G_S of time and space, as described above. Consequently, an approximate spatio-temporal regions is a pair (X^t, X^s). Each approximate spatio-temporal region $(X^t, X^s) \in \Omega^{G_T} \times \Omega^{G_S}$ stands for a set of precise spatio-temporal regions, i.e., all those precise regions having the approximation (X^t, X^s). This set which will be denoted $[\![(X^t, X^s)]\!]$ provides a semantics for approximate spatio-temporal regions:

$$[\![(X^t, X^s)]\!] = \{(x^t, x^s) \in \mathcal{T} \times \mathcal{S} \mid \alpha\ x^t = X^t \text{ and } \alpha\ x^s = X^s\},$$

where $\mathcal{T}$ denotes the set of regions of the time-line and $\mathcal{S}$ denotes the regions of the plane. The greatest minimal and least maximal meet operations between approximations of spatial and temporal regions generalize in the natural way to approximations of spatio-temporal regions:

$$(X^t, X^s) \,\underline{\wedge}\, (Y^t, Y^s) = (X^t \,\underline{\wedge}\, Y^t, X^s \,\underline{\wedge}\, Y^s)$$

and similarly for $\overline{\wedge}$.

5 Approximating binary topological relations

I discussed above the importance of qualitative spatial relation for the description of spatial configurations. In this section I define approximate topological relations between approximations of spatial, temporal, and spatio-temporal regions. In this context I apply the specific style of definitions discussed in Section 3. Relations between approximations of spatial have been discussed separately in [BS00]. This will be reviewed and then applied to relations between approximated temporal and spatio-temporal regions.

In order to define relations between approximations of spatial regions and temporal intervals I firstly pursue the syntactic approach: (1) I replace in the definitions of relations between spatial regions and temporal intervals the (variables ranging over) regions by (variables ranging over) approximations, e.g., I replace $x \in \mathcal{S}$ by $X \in \Omega_3^G$; and (2) I replace the meet operation between regions by the greatest minimal and least maximal operations between approximations, i.e., I replace $\wedge$ by $\underline{\wedge}$ and $\overline{\wedge}$. Secondly, I check whether the syntactically generated pairs of relations constrain the appropriate set of relations between the approximated regions (the semantic approach).

5.1 Relations between approximations of spatial regions

The above formulation of the RCC5 relations can be extended to approximate regions. One way to do this is to replace the operation $\wedge$ with an appropriate operation for approximate regions. If X and Y are approximate regions (i.e. functions from G to Ω_3) we can consider the two triples of Boolean values [BS00]:

$$\begin{array}{l}(X \,\underline{\wedge}\, Y \neq \bot,\ X \,\underline{\wedge}\, Y = X,\ X \,\underline{\wedge}\, Y = Y),\\ (X \,\overline{\wedge}\, Y \neq \bot,\ X \,\overline{\wedge}\, Y = X,\ X \,\overline{\wedge}\, Y = Y).\end{array}$$

In the context of approximate regions, the bottom element, $\bot$, is the function from G to Ω_3 which takes the value no for every element of G. Each of the above triples provides an RCC5 relation, so the relation between X and Y can be measured by a pair of RCC5 relations. These relations will be denoted by $\underline{R}(X,Y)$ and $\overline{R}(X,Y)$. The pairs $(\underline{R}(X,Y), \overline{R}(X,Y))$ which can occur are all pairs (a,b) where $a \leq b$ with the exception of $(\mathsf{PP}, \mathsf{EQ})$ and $(\mathsf{PPi}, \mathsf{EQ})$ [BS00].

Consider the ordering of the RCC5 lattice. The relation $\underline{R}(X,Y)$ is the minimal relation and the relation $\overline{R}(X,Y)$ is the maximal relation that can hold between $x \in [\![X]\!]$ and $y \in [\![Y]\!]$. For all relations R, with $\underline{R}(X,Y) \leq R \leq \overline{R}(X,Y)$ there are $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ such that $R(x,y)$ [BS00].

5.2 Syntactic generalization of relations between temporal intervals

In order to generalize the above formulation of RCC_1^9 relations to relations between approximations of temporal intervals we need to define operations $X \ll Y$ and $X \gg Y$ corresponding to operations $x \ll y$ and $x \gg y$. The behavior of $X \ll Y$ is shown in Figure 3. Formally we define $X \ll Y$ as

$$X \ll Y = \begin{cases} \mathsf{T} \ \text{if } L(X) \,\overline{\wedge}\, L(Y) = L(X) \text{and } L(X) \,\overline{\wedge}\, L(Y) \neq L(Y) \\ \mathsf{M} \ \text{if } L(X) \,\overline{\wedge}\, L(Y) = L(X) \text{ and } L(X) \,\overline{\wedge}\, L(Y) = L(Y) \text{ and} \\ \quad L(X) \,\underline{\wedge}\, L(Y) < L(X) \,\overline{\wedge}\, L(Y) \\ \mathsf{F} \ \text{otherwise} \end{cases}$$

and similarly $X \gg Y$ using $R(X)$ and $R(Y)$, where $L(X)$ yields the approximation of the part of the time-line left of $x \in [\![X]\!]$ and $R(Y)$ yields the approximation of the part of the time-line right of $y \in [\![Y]\!]$ respectively. Formally, L and R are defined as follows. Firstly, we define the complement operation $X' \, g_i = (X \, g_i)'$ with $\mathsf{no}' = \mathsf{fo}$, $\mathsf{po}' = \mathsf{po}$, and $\mathsf{fo}' = \mathsf{no}$. Assuming that partition cells g_i are numbered in increasing order in direction of the underlying space, we secondly define $L(X)$ and $R(Y)$ as:

$$(L(X) \, g_i) = \begin{cases} (X \, g_i)' & \text{if } i \leq \min\{k \\ & \mid (X \, g_k) \neq \mathsf{no}\} \\ \mathsf{no} & otherwise \end{cases} \; ; \quad (R(Y) \, g_i) = \begin{cases} (Y \, g_i)' & \text{if } i \geq \max\{k \\ & \mid (Y \, g_k) \neq \mathsf{no}\} \\ \mathsf{no} & otherwise \end{cases} .$$

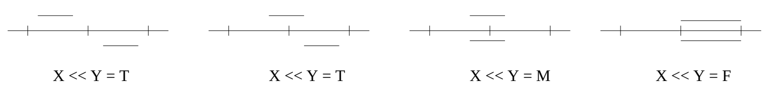


Fig. 3. The behavior of $X \ll Y$, where $x \in [\![X]\!]$ is above the time-line and $y \in [\![Y]\!]$ is below the time-line.

We need two more operations: $X \triangleright Y$ and $X \triangleleft Y$, where $X \triangleright Y = \mathsf{T}$ means that $x \in [\![X]\!]$ is contained in $y \in [\![Y]\!]$ and x does not cover the very right parts of y and $X \triangleleft Y = \mathsf{T}$ is interpreted as $x \in [\![X]\!]$ is contained in $y \in [\![y]\!]$ and x does not cover the very left parts of y. The behavior of $X \triangleright Y$ are shown in Figure 4. Formally we define a set $\Gamma(X, Y) = \{(R(X) \underline{\wedge} Y) \, g_i \mid g_i \in G\}$, containing the elements of the co-domain of $(R(X) \underline{\wedge} Y)$, and the operation

$$X \triangleright Y = \begin{cases} \mathsf{T} & \text{if } \mathsf{fo} \in \Gamma(X,Y) \text{ or } \{\mathsf{po}, \mathsf{po}\} \subseteq \Gamma(X,Y) \\ \mathsf{M} & \text{if } \mathsf{po} \in \Gamma(X,Y) \text{ and } \{\mathsf{po}, \mathsf{po}\} \not\subseteq \Gamma(X,Y) \\ \mathsf{F} & \text{otherwise} \end{cases} .$$

We define $X \triangleleft Y$ respectively by replacing $R(X)$ by $L(X)$ in the definition of $X \triangleright Y$.

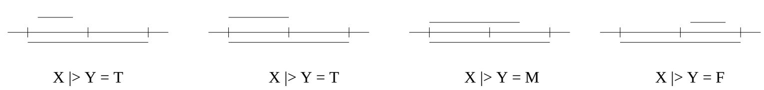


Fig. 4. The behavior of $X \triangleright Y$, where $x \in [\![X]\!]$ is above the time-line and $y \in [\![Y]\!]$ is below the time-line.

We are now able to generalize the above formulation of RCC_1^9 relations to relations between approximations. Let X and Y be boundary insensitive approximations of temporal intervals. We can consider the two triples of values:

$$\begin{array}{l} ((X \underline{\wedge} Y \not\sim \perp, \; X \underline{\wedge} Y \sim X, \; X \underline{\wedge} Y \sim Y), \\ (X \overline{\wedge} Y \not\sim \perp, \; X \overline{\wedge} Y \sim X, \; X \overline{\wedge} Y \sim Y)). \end{array}$$

where

$$X \underline{\triangle} Y \not\sim \perp = \begin{cases} FLO & \text{if } X \underline{\triangle} Y = \perp \text{ and } (X \ll Y) \neq \mathsf{F} \text{ and } (X \ll Y) \geq (X \gg Y) \\ FRO & \text{if } X \underline{\triangle} Y = \perp \text{ and } (X \gg Y) \neq \mathsf{F} \text{ and } (X \gg Y) > (X \ll Y) \\ T & \text{if } X \underline{\triangle} Y \neq \perp \end{cases}$$

where

$$X \underline{\triangle} Y \sim X = \begin{cases} FLO & \text{if } X \underline{\triangle} Y \neq X \text{ and } X \underline{\triangle} Y \neq Y \text{ and } X \ll Y \neq \mathsf{F} \text{ and } X \ll Y \geq X \gg Y \\ FLI & \text{if } X \underline{\triangle} Y \neq X \text{ and } X \underline{\triangle} Y = Y \text{ and } leftCheck(X, Y) \\ FRO & \text{if } X \underline{\triangle} Y \neq X \text{ and } X \underline{\triangle} Y \neq Y \text{ and } X \gg Y \neq \mathsf{F} \text{ and } X \gg Y > X \ll Y \\ FRI & \text{if } X \underline{\triangle} Y \neq X \text{ and } X \underline{\triangle} Y = Y \text{ and } rightCheck(X, Y) \\ T & \text{if } X \underline{\triangle} Y = X \end{cases}$$

and where

$$X \underline{\triangle} Y \sim X = \begin{cases} FLO & \text{if } X \underline{\triangle} Y \neq Y \text{ and } X \underline{\triangle} Y \neq X \text{ and } X \ll Y \neq \mathsf{F} \text{ and } X \ll Y \geq X \gg Y \\ FLI & \text{if } X \underline{\triangle} Y \neq Y \text{ and } X \underline{\triangle} Y = X \text{ and } rightCheck(Y, X) \\ FRO & \text{if } X \underline{\triangle} Y \neq Y \text{ and } X \underline{\triangle} Y \neq X \text{ and } X \gg Y \neq \mathsf{F} \text{ and } X \gg Y > X \ll Y \\ FRI & \text{if } X \underline{\triangle} Y \neq Y \text{ and } X \underline{\triangle} Y = X \text{ and } leftCheck(Y, X) \\ T & \text{if } X \underline{\triangle} Y = X \end{cases}.$$

The functions $leftCheck(X, Y)$ and $rightCheck(X, Y)$ are defined as follows:

$$leftCheck(X, Y) = \begin{cases} T & \text{if } Y \lhd X = T \text{ or } (Y \lhd X = M \text{ and } Y \rhd X = F) \\ F & \text{if } Y \lhd X \neq T \text{ and } Y \rhd X = T \\ X \ll Y \neq \mathsf{F} \text{ and} & \text{otherwise} \\ X \ll Y \geq X \gg Y & \end{cases},$$

$$rightCheck(X, Y) = \begin{cases} T & \text{if } Y \rhd X = T \text{ or } (Y \rhd X = M \text{ and } Y \lhd X = F) \\ F & \text{if } Y \rhd X \neq T \text{ and } Y \lhd X = T \\ X \gg Y \neq \mathsf{F} \text{ and} & \text{otherwise} \\ X \gg Y > X \ll Y & \end{cases}.$$

Both functions assume that $y \in [\![Y]\!]$ is contained in $x \in [\![X]\!]$. The behavior of $leftCheck$ is shown in Figure 5. The definitions of $X \overline{\wedge} Y \not\sim \perp$, $X \overline{\wedge} Y \sim Y$, and $X \overline{\wedge} Y \sim Y$ are obtained by replacing $\underline{\triangle}$ by $\overline{\wedge}$ in the definitions of $X \underline{\triangle} Y \not\sim \perp$, $X \underline{\triangle} Y \sim Y$, and $X \underline{\triangle} Y \sim Y$.

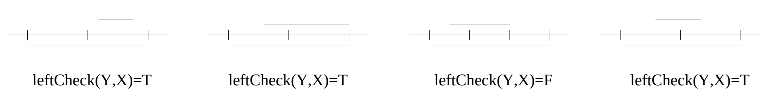


Fig. 5. The behavior of $leftCheck(Y, X)$, where $x \in [\![X]\!]$ is above the time-line and $y \in [\![Y]\!]$ is below the time-line.

Each of the above triples defines an RCC_1^9 relation, so the relation between X and Y can be measured by a pair of RCC_1^9 relations. These relations will be denoted by $\underline{R^9}(X, Y)$ and $\overline{R^9}(X, Y)$.

Theorem 1 *The pairs*

$$(\min\{\underline{R}^9(X,Y), \overline{R^9}(X,Y)\}, \max\{\underline{R}^9(X,Y), \overline{R^9}(X,Y)\})$$

that can occur are all pairs (a,b) *where* $a \leq b \leq \mathsf{EQ}$ *and* $\mathsf{EQ} \leq a \leq b$ *with the exception of* $(\mathsf{PPL}, \mathsf{EQ})$, $(\mathsf{PPR}, \mathsf{EQ})$, $(\mathsf{PPiL}, \mathsf{EQ})$, $(\mathsf{PPiR}, \mathsf{EQ})$, *and* $(\mathsf{EQ}, \mathsf{DRR})$.

Proof The pairs $(\mathsf{PPL}, \mathsf{EQ})$, $(\mathsf{PPR}, \mathsf{EQ})$, $(\mathsf{PPiL}, \mathsf{EQ})$, $(\mathsf{PPiR}, \mathsf{EQ})$ cannot occur since RCC^9_1 relations are refinements of RCC5 relations and the pairs $(\mathsf{PP}, \mathsf{EQ})$ and (PPi, EQ) cannot occur in the RCC5 case [BS00]. The pair $(\mathsf{EQ}, \mathsf{DRR})$ cannot occur due to the non-symmetry of the underlying definitions. In order to generate all remaining pairs approximations of time intervals in regional partitions consisting of at least three elements need to be considered. A Haskell [Tho99] program generating all remaining pairs of relations between approximations with respect to a partition consisting of three intervals can be found at [Bit00b]. $\square$

5.3 Semantic generalization of relations between temporal intervals

At the semantic level we consider how syntactically generated pairs, $(\underline{R}^9(X,Y), \overline{R^9}(X,Y))$[7], relate to relations between the approximated regions $x \in [\![X]\!]$ and $y \in [\![Y]\!]$. The aim is that the syntactically generated pairs constrain the possible relations that can hold between the approximated intervals x and y [BS00]:

$$\{R \mid \underline{R}^9(X,Y) \leq R(X,Y) \leq \overline{R^9}(X,Y)\} = \{\rho(x,y) \mid x \in [\![X]\!], y \in [\![Y]\!]\}$$

We proceed by considering all pairs containing the relation EQ. Consider configuration (a) in Figure 6, which represents the most indeterminate case. The syntactic approach described above yields the pair $(\mathsf{DRL}, \mathsf{EQ})$. Since in this kind of configuration the pair $(\mathsf{DRL}, \mathsf{EQ})$ is consistent with $(\mathsf{EQ}, \mathsf{DRR})$ and $(\mathsf{DRL}, \mathsf{EQ})$ was chosen arbitrarily, $(\mathsf{DRL}, \mathsf{EQ})$ is corrected syntactically to $(\mathsf{DRL}, \mathsf{DRR})$.

Consider configuration (b) in Figure 6. The syntactic approach yields the pair $(\mathsf{DRL}, \mathsf{EQ})$ which is not correct if x and y are intervals as depicted. Notice that the meet operations were originally defined for arbitrary regions not for one-dimensional intervals. Assuming $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ to be (time) intervals the outcome of the minimal meet must not be empty. This needs to be taken into account in the definition of $\underline{\wedge}$. Let X and Y be boundary insensitive approximations of time intervals:

$$(X\,g_i)(\underline{\wedge}')(Y\,g_i) = \begin{cases} \mathsf{PO} & \text{if } ((X\,g_i) = \mathsf{PO} \text{ and } (Y\,g_i) = \mathsf{PO}) \text{ and} \\ & ((X\,g_{i-1}) \geq \mathsf{PO} \text{ and } (Y\,g_{i-1}) \geq \mathsf{PO}) \text{ or} \\ & (X\,g_{i+1}) \geq \mathsf{PO} \text{ and } (Y\,g_{i+1}) \geq \mathsf{PO})) \\ (X\,g_i)\,\underline{\wedge}\,(Y\,g_i) & \text{otherwise} \end{cases}$$

Applying $(\underline{\wedge}')$ to X and Y in Figure 6 (b) yields EQ as minimal relation. But $(\mathsf{EQ}, \mathsf{EQ})$ still does not characterize Figure 6 (b) correctly, since between $x \in [\![X]\!]$ and $y \in [\![Y]\!]$

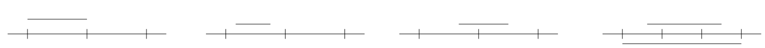
[7] In the remainder of the paper I write $(\underline{R}^9(X,Y), \overline{R^9}(X,Y))$ instead of $(\min\{\underline{R}^9(X,Y), \overline{R^9}(X,Y)\}, \max\{\underline{R}^9(X,Y), \overline{R^9}(X,Y)\})$.

the relations $\{\mathsf{POL}(x,y), \mathsf{EQ}(x,y), \mathsf{POR}(x,y)\}$ can hold. Consider also Figure 6 (c) for which the operations defined above yield $(\mathsf{POL}, \mathsf{EQ})(X,Y)$, but the approximations X and Y are also consistent with $\mathsf{POR}(x,y)$ for $x \in \llbracket X \rrbracket$ and $y \in \llbracket Y \rrbracket$. Consequently, if $\max\{\underline{R^9}(X,Y), \overline{R^9}(X,Y)\} = \mathsf{EQ}$ and $\min\{\underline{R^9}(X,Y), \overline{R^9}(X,Y)\} \neq \mathsf{DRL}$ and the leftmost *or* the rightmost non-empty approximation values of X *and* Y have the value PO then the RCC_1^9 relation between X and Y is $(\mathsf{POL}, \mathsf{POR})(X,Y)$. This also applies to the configuration Figure 6 (d). The corrected relations are denoted $\underline{R_c^9}(X,Y)$ and $\overline{R_c^9}(X,Y)$.

(a) <DRL,EQ>(X,Y) (b) <DRL,EQ>(X,Y) (c) <POL,EQ>(X,Y) (d) <POL,EQ>(X,Y)

Fig. 6. Configurations characterized by pairs containing the relation $\mathsf{EQ}(X,Y)$, where $x \in \llbracket X \rrbracket$ is above the time-line and $y \in \llbracket Y \rrbracket$ is below the time-line.

Finally, consider the configuration (a) in Figure 7. Our definitions yield $(\mathsf{PPiL}, \mathsf{PPiL})(X,Y)$ but the approximations X and Y are also consistent with $\mathsf{PPiR}(x,y)$ but not with $\mathsf{EQ}(x,y)$ for $x \in \llbracket X \rrbracket$ and $y \in \llbracket Y \rrbracket$. The interval y can cover the very right part of x or the very left part of x or some part in the middle. The same holds if we switch X and Y (Figure 7 (b)): Our definitions yield $(\mathsf{PPL}, \mathsf{PPL})(X,Y)$ but the approximations X and Y are also consistent with $\mathsf{PPR}(x,y)$ but not with $\mathsf{EQ}(x,y)$. Consider Figure 7 (c) and (d). These cases are different: Assuming that x and y are intervals x can cover neither the very letf of y nor the very right of y. Consequently, the configuration is consistent with both $(\mathsf{PPL}, \mathsf{PPL})$ and $(\mathsf{PPR}, \mathsf{PPR})$ but in this cases it is o.k. to chose one, since x must cover parts in the middle of y.



(a) <PPiL,PPiL>(X,Y) (b) <PPL,PPL>(X,Y) (c) <PPL,PPL>(X,Y) (d) <PPL,PPL>(X,Y)

Fig. 7. Configurations characterized by pairs $(\mathsf{PPL}, \mathsf{PPL})$ or $(\mathsf{PPiL}, \mathsf{PPiL})$, where $x \in \llbracket X \rrbracket$ is above the time-line and $y \in \llbracket Y \rrbracket$ is below the time-line.

The cases depicted in Figure 7 (a) and (b) need to be handled separately. For them the theorem below does not hold. The problem disappears if boundary sensitive approximations are used. For all other case we state Theorem 2:

Theorem 2 *The relation $\underline{R_c^9}(X,Y)$ is the minimal relation and the relation $\overline{R_c^9}(X,Y)$ is the maximal relation that can hold between $x \in \llbracket X \rrbracket$ and $y \in \llbracket Y \rrbracket$. For all relations R, with $\underline{R_c^9}(X,Y) \leq R \leq \overline{R_c^9}(X,Y)$ there are $x \in \llbracket X \rrbracket$ and $y \in \llbracket Y \rrbracket$ such that $R(x,y)$.*

Proof RCC_1^9 relations are refinements of RCC5 relations. Figure 2 shows that the RCC_1^9 lattice can be separated into the left and the right RCC5 sub-lattices ($\mathsf{DRL} \leq R \leq \mathsf{EQ}$ and $\mathsf{EQ} \leq R \leq \mathsf{DRR}$). Theorem 1 tells us that our syntactic procedure yields minimal and maximal relation pairs that either belong to the left RCC5 sub-lattice or the right RCC5 sublattice. It also tells us that the generated pairs are same pairs occuring in the RCC5 case. Consequently, with the exeption of the special cases discussed above, Theorem 2 of [BS00] applies, stating that the syntactic approach constrains the right set of relations. Consequently, what remains to show is that the theorem holds for the special cases: $(\mathsf{DRL}, \mathsf{DRR})$ and $(\mathsf{POL}, \mathsf{POR})$.

The case $(\mathsf{DRL}, \mathsf{DRR})$ occurs in configurations where the syntactic procedure yields $(\mathsf{DRL}, \mathsf{EQ})$, i.e., in configurations that are equivalent to the configuration in Figure 6 (a). DRL and DRR are trivially minimal and maximal and it is easy to verify that all relation ρ with $\mathsf{DRL}(X, Y) \leq \rho(x, y) \leq \mathsf{DRR}(X, Y)$ can actually occur for $x \in [\![X]\!]$ and $y \in [\![Y]\!]$.

The case $(\mathsf{POL}, \mathsf{POR})$ occurs in configurations where the syntactic procedure yields that $\max\{\underline{R^9}(X, Y), \overline{R^9}(X, Y)\} = \mathsf{EQ}$ and $\min\{\underline{R^9}(X, Y), \overline{R^9}(X, Y)\} \neq \mathsf{DRL}$ and that the leftmost or the rightmost non-empty approximation values of X and Y have the value PO, i.e., in configurations that are similar to the configuration in Figure 6 (b-d). It is easy to verify that exactly the relations ρ with $\mathsf{POL}(X, Y) \leq \rho(x, y) \leq \mathsf{POR}(X, Y)$ can actually occur for $x \in [\![X]\!]$ and $y \in [\![Y]\!]$. $\square$

5.4 Approximating topological relations between spatio-temporal objects

Based on relations between approximations of spatial regions and relations between approximations of temporal regions we now define relations between approximations of spatio-temporal regions. Let o_1 and o_2 be two spatio-temporal objects at rest with spatio-temporal location $r_{st}^c(o_1) = (x^t, x^s)$ and $r_{st}^c(o_2) = (y^t, y^s)$ with approximations (X^t, X^s) and (Y^t, Y^s). Consider the following structure:

$$\begin{array}{c} (((X^t \underline{\wedge} Y^t \not\sim \bot,\ X^t \underline{\wedge} Y^t \sim X^t,\ X^t \underline{\wedge} Y^t \sim Y^t) \\ (X^s \underline{\wedge} Y^s \neq \bot,\ X^s \underline{\wedge} Y^s = X^s,\ X^s \underline{\wedge} Y^s = Y^s)), \\ ((X^t \overline{\wedge} Y^t \not\sim \bot,\ X^t \overline{\wedge} Y^t \sim X^t,\ X^t \overline{\wedge} Y^t \sim Y^t), \\ (X^s \overline{\wedge} Y^s \neq \bot,\ X^s \overline{\wedge} Y^s = X^s,\ X^s \overline{\wedge} Y^s = Y^s))) \end{array}.$$

Each cpmponent of the above pair of pairs of triples defines a spatio-temporal relation, (RCC_1^9 , RCC5). So the relation between (X^t, X^s) and (Y^t, Y^s) can be measured by a pair of spatio-temporal relations: $((\underline{R^9}(X, Y), \underline{R}(X, Y)), (\overline{R^9}(X, Y), \overline{R}(X, Y)))$. The pairs

$$((\min\{\underline{R_c^9}(X, Y), \overline{R_c^9}(X, Y)\}), \underline{R}), (\max\{\underline{R_c^9}(X, Y), \overline{R_c^9}(X, Y)\}, \overline{R}))$$

that can occur are exactly those that can occur in the separate treatment of approximations of RCC5 and RCC_1^9 relations.

Consequently, relations between approximations of spatio-temporal regions, (X^t, X^s) and (Y^t, Y^s), are represented by pairs of minimal and maximal spatio-temporal relations $((\underline{R_c^9}, \underline{R}), (\overline{R_c^9}, \overline{R}))$ such that $(\underline{R_c^9}, \underline{R})((X^s, X^t), (Y^t, Y^s))$ is the least spatio-temporal relation and $(\overline{R^9_c}, \overline{R})((X^s, X^t), (Y^t, Y^s))$ is the largest spatio-temporal relation that can hold between spatio-temporal regions $(x^t, x^s) \in [\![(X^t, X^s)]\!]$ and $(y^t, y^s) \in [\![(Y^t, Y^s)]\!]$.

6 Conclusions

In this paper I defined spatio-temporal regions as pairs consisting of a spatial and a temporal component. I defined topological relations between spatio-temporal regions based on topological relations between the spatial and temporal components. Approximations of spatio-temporal regions were defined using approximations of their spatial and temporal components. I defined topological relations between approximations of spatio-temporal regions based on a specific style that allows to define relations between spatio-temporal regions exclusively based on constraints on the outcome on the meet operation. The proposed framework can be used in order to describe spatial configurations based on approximate descriptions of spatio-temporal objects and relation between those approximations. Those approximate descriptions can be much easier obtained from observations of reality than exact descriptions.

The formalism discussed in this paper deals only with boundary insensitive topological relations between spatio-temporal regions. This can be easily extended to boundary sensitive relations using the formalisms proposed in [BS00] and [Bit00a].

References

[All83] J.F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

[BF95] Peter Burrough and Andrew U. Frank, editors. *Geographic Objects with Indeterminate Boundaries.* GISDATA Series II. Taylor and Francis, London, 1995.

[Bit99] T. Bittner. On ontology and epistemology of rough location. In *Spatial information theory - Cognitive and computational foundations of geographic information science, COSIT 99*, number 1661 in Lecture Notes in Computer Science, Hamburg, Germany, 1999. Springer Verlag.

[Bit00a] T. Bittner. Approximate temporal reasoning. In *Workshop proceedings of the Seventeenth National Conference on Artificial Intelligence, AAAI 2000*, 2000.

[Bit00b] T. Bittner. A Haskell program generating all possible relations between boundary insensitive approximations of time intervals. http://www.cs.queensu.ca/∼bittner, 2000.

[BNSNT+95] J. Bazan, H. Nguyen Son, T. Nguyen Trung, A. Skowron, and J. Stepaniuk. Application of modal logics and rough sets for classifying objects. In M. De Glas and Z. Pawlak, editors, *Proceedings of the Second World Conference on Fundamentals of Artificial Intelligence (WOCFAI'95)*, pages 15–26, Paris, 1995.

[BS98] T. Bittner and J. G. Stell. A boundary-sensitive approach to qualitative location. *Annals of Mathematics and Artificial Intelligence*, 24:93–114, 1998.

[BS00] T. Bittner and J. Stell. Rough sets in approximate spatial reasoning. In *Proceedings of the Second International Conference on Rough Sets and Current Trends in Computing (RSCTC'2000)*. Springer Verlag, 2000.

[CV95] R. Casati and A. Varzi. The structure of spatial localization. *Philosophical Studies*, 82(2):205–239, 1995.

[EF91] Max J. Egenhofer and Robert D. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5(2):161–174, 1991.

[EFG97] M. J. Egenhofer, D.M. Flewelling, and R.K. Goyal. Assessment of scene similarity. Technical report, University of Maine, Department of Spatial Information Science and Engineering, 1997.

[Gea66] P. Geach. Some problems about time. *Proceedings of the British Academy*, 11, 1966.

[GFA93] J. Glasgow, S. Fortier, and F.H. Allen. Molecular scene analysis: Crystal structure determination through imagery. In L. Hunter, editor, *Artificial Intelligence and Molecular Biology*. AAAI/MIT Press, 1993.

[LC97] T.Y. Lin and N. Cercone, editors. *Rough Sets and Data Mining. Analysis of Imprecise Data.* Kluwer Academic Publishers, Boston, Dordrecht, 1997.

[Lin95] T.Y. Lin, editor. *Proceedings of the Workshop on Rough Sets and Data Mining at 23rd Annual Computer Science Conference*, Nashville, Tenessee, 1995.

[NSR92] R. Nowicki, Slowinski, and J. R., Stefanowski. Rough sets analysis of diagnostic capacity of vibroacoustic symptoms. *Journal of Computers and Mathematics with Applications*, 1992.

[Paw82] Z. Pawlak. Rough sets. *Internat. J. Comput. Inform*, 11:341–356, 1982.

[RCC92] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In *3rd Int. Conference on Knowledge Representation and Reasoning.* Boston, 1992.

[Sim87] P. Simons. *Parts, A Study in Ontology*. Clarendon Press, Oxford, 1987.

[SV97] B. Smith and A. Varzi. Fiat and bona fide boundaries: Towards an ontology of spatially extended objects. In S. Hirtle and A. Frank, editors, *Spatial Information TheoryA Theoretical Basis for GIS, International Conference COSIT '97, Laurel Highlands, PA*, volume 1329 of *Lecture Notes in Computer Science*, pages 103–119. Springer-Verlag, Berlin, 1997.

[Tho99] Simon Thompson. *Haskell: The Craft of Functional Programming*. Addison-Wesley, 2 edition, 1999.