

A MERELOGICAL THEORY OF FRAMES OF REFERENCE

Technical report version

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In this paper a mereological theory of frames of reference is presented. It shows that mereology extended by the notion of granularity and approximation is sufficient to provide a theory for location based features of frames of reference. More complex theories, taking also into account orientation and metric properties can be built as extensions of the presented theory. In order to take the hierarchical organization of frames of reference into account we introduce the notion of stratified approximation to facilitate transformations between different levels of granularity. This paper shows that the ontological grounding of the theory of frames of reference into mereology allows us to give a clear semantics to the notion 'degree of parthood' which is central to the notion of approximation. It also shows how epistemic aspects which affect the use of frames of reference help us to understand the feature of epistemic vagueness and the way it affects the notions of approximation and of degree of parthood.

Keywords: Mereology, approximation, frames of reference, rough sets, vagueness

1. Introduction

This paper proposes a mereological theory of frames of reference. There is a broad body of literature on frames of reference in at least four major areas: (i) in physics, in particular in the contexts of mechanics and relativity^{26,13,11}; (ii) in linguistics, where people are interested in the semantics of spatial prepositions and in the insights into human cognition, which language can provide^{25,34,20}; (iii) in psychology, where the distinction between figure and ground in human cognition is subject of ongoing research³²; and (iv) in artificial intelligence in attempts to formalize human reasoning about space and time^{19,15,30}.

Mereology^{24,23,31}, the formal theory of parthood, has not, hitherto, been held to provide a basis for theorizing about frames of reference. This is probably because the most salient features of frames of reference are characterized by ordering relations and metric properties, features which require theories much stronger than mereology for their formalization. Examples of metric frames of reference are the ones used in physics to measure

distance, weight, temperature, etc. Examples of frames of reference based on ordering relations are those of cardinal directions (N,S,E, and W), the distinction between left and right relative to your body axis, clock and calendar time, etc. However it will turn out that mereology with only some extensions will be sufficient to formally describe locational frames of reference.

This paper will show that mereology can help us to understand the more basic features of frames of reference which are already pointed to by the Gestalt psychologists in their distinction between figure and ground³². Here the ground is seen as a frame of reference in which the figure is located. An important point here is that the ground, if it is to provide a frame of reference, must have a relatively simple structure, which, as we shall show, can be understood in terms of mereology. In order to use mereology as a formal foundation, however, it must be extended by the feature of granularity. We will refer to this theory as 'granular mereology'. (See also ³³ for further arguments along these lines.)

Since frames of reference are often quite coarse, as in the case of the frame of reference which divides your surroundings into the part in front of you and the part behind, the specification of location of the figure within its ground (or of the referenced object within the frame of reference) is often *rough or approximate*. In this respect we will build upon the theory of granular partitions by ². It will turn out that the resulting formalism is quite close to the notions of rough sets ²⁷ and rough mereology ²⁸, which are extensions of set theory and mereology developed in the context of data analysis and data mining.

One important feature of frames of reference is their hierarchical organization. An example is the tree-like structure formed by the subdivision of London into Boroughs such as Westminster, Camden, etc at one level of granularity and parks and neighborhoods like Hyde Park, Soho, etc. at another level of granularity. Within a frame of reference based on this tree structure the relation between entities and the frame of reference, i.e., between figure and ground, can be specified at multiple levels of granularity.

Consider the sentences

A 'John is in Hyde Park'

B 'The Rocky Mountains are in the Western United States'.

Here hierarchically ordered systems of places are used as frames of reference. As an alternative to (A) one can say, for example:

A' John is in London.

And to (B) one can say

B' The Rocky Mountains are partly located in Montana, Idaho, Wyoming, Nevada, Colorado, Utah, Arizona, and New Mexico.

Given formal representations of the location of entities within a frame of reference it is often necessary to transform approximations between different levels of granularity. The notion of stratified approximation introduced below will facilitate these kinds of transformations.

It is important to distinguish between: (a) the study of the nature and the formal structure of frames of reference, and (b) the study of their application in certain contexts. This is because (a) refers to the study of the *formal ontology* of frames of reference. (b), on the other hand, refers to the question of how *epistemic issues*, for example issues pertaining to the limits on human knowledge and the representation of human knowledge, affect the

specification of location in frames of reference and our capability to switch between levels of granularity in the ways illustrated in (A, A') and (B, B'). From the perspective of formal ontology approximations with respect to a frame of reference are rough but *crisp*. Epistemic issues then give, as we shall see, raise to vagueness.

Vagueness hereby is understood in the sense that there are multiple, equally good, approximations which are consistent with the knowledge at hand. This epistemic understanding of vagueness needs to be distinguished from semantic vagueness which affects the ways names like 'Mount Everest' refer to parts of the surface of Earth^{14,36,3}. We refer to the latter as to *semantic* vagueness and to the former as to *epistemic* vagueness. Since in this paper we exclusively focus on epistemic vagueness it will be often sufficient to use the term vagueness.

In order to provide a mereological theory that addresses the points raised above we start by extending mereology in order to take into account the feature of granularity. We then give a very general formal account of levels of granularity. In the following section we introduce the notion of approximation and show how stratified approximations can be defined across multiple levels of granularity. We then discuss epistemic issues that give raise to vagueness and extend the crisp formalism of stratified approximation in order to take epistemic vagueness into account. In the end we discuss related work and give the conclusions. The examples we use throughout the paper will be mostly spatial in nature. Due to its general nature, the underlying theory can easily be extended to domains of other sorts.

2. Frames of references as granular partitions

In this section I give a formal definition of frames of reference as an extension of the theory of granular partitions which was originally introduced in².

2.1. Frames of reference

A frame of reference is a triple,

$$G = \langle (Z, \sqsubseteq), (\Delta, \leq), \pi \rangle .$$

(Z, \sqsubseteq) is a *cell structure* with a partial ordering defined by \sqsubseteq which forms a finite tree. (Δ, \leq) is the *target domain* which is a partial ordering which satisfies the axioms of extensional mereology (EM)³⁵. The projection mapping $\pi : Z \rightarrow \Delta$ is an order-homomorphism from Z into Δ .

We will discuss all three components in detail below but before doing so, let us consider some examples to get an intuitive understanding. In figure 1 some cell structures of frames of reference popular in artificial intelligence are shown. In general, cell structures are cognitive artifacts which are projected by cognitive subjects onto a target domain – portions of physical reality like parts of the surface of the Earth. This projection imposes a structure onto the target domain which then can be used by the cognitive subject for example for reasoning purposes.

Consider part (i) of 1 which shows nine cells of a cell structure with ten cells (the tenth, which has all cells represented in the figure as subcells, is omitted here) are shown. When projected onto the surface of the Earth in such a way that the horizontal lines are in line with lines of longitude, the vertical lines are in line with lines of latitude, and the cell labeled 0 coincides with the location of some entity x , then the regions carved out by projecting of the cells onto the surface of the Earth are called ‘North of x ’, ‘Northeast of x , and so on. The entity x and the portions of Earth carved out in the way described are parts of the target domain Δ . The spatial projection of the cells onto the surface of Earth is a specific form of the projection π . The frame of reference as a whole is formed by the three components: cell structure, target domain, projection. For an extended discussion of this specific family of frames of reference see ¹⁵.

The cell structure in part (ii) of the figure is projected onto its target domain in such a way that a particular entity (often a human being) is located at the intersection of the two diagonal lines in such a way that front, back, left and right are interpreted relative to the entity in the center in the way indicated in the picture. The frame of reference again is formed by three components: the cell structure depicted in the figure (again the maximal cell containing all cells as subcells is omitted), the target domain, and the projection of the cell structure onto the target domain. For an discussion of frames of reference of this kind see ¹⁹ and for a discussion of frames of reference depicted in (iii) see ¹⁶.

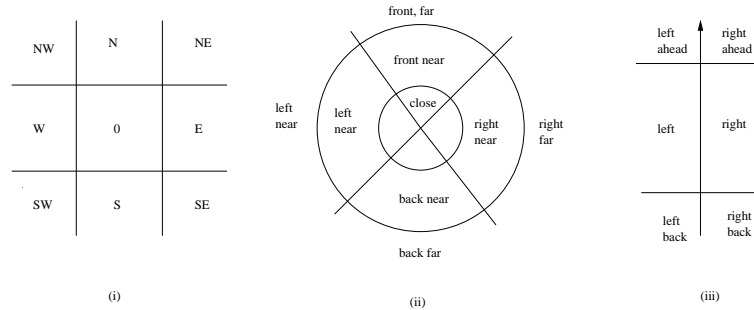


Figure 1: Frames of reference used by Frank (i), by Hernandez (ii), and by Freksa (iii).

Frames of reference are not necessarily based on ordering relations. Consider figures 2 and 3. The left part of Figure 2 shows a tree with minimal cells labeled ‘ a ’, ‘ b ’, ‘ c ’, ‘ d ’, ‘ e ’, ‘ f ’ and non-minimal cells labeled ‘ r ’, ‘ s ’, ‘ t ’ and ‘ u ’. We have ‘ s ’ \sqsubseteq ‘ r ’, ‘ t ’ \sqsubseteq ‘ s ’, etc. The targets of the cells are regions which are shown in Figure 3, e.g., $\pi(‘a’)$ = a , $\pi(‘b’)$ = b , etc, where π is the projection of the cell ‘ a ’ onto the entity a and so on. (The target of the root cell, which is the plane as a whole, is omitted here.)

In the remainder of this section we discuss the components of frames of reference from a mereological perspective in greater detail. For this purpose we assume a sorted first order predicate calculus with identity. We distinguish variables $c, c_1, d, d_1, g, g_1, \dots$ ranging over cells, and variables x, x_1, y, z, \dots ranging over entities of some target domain Δ .

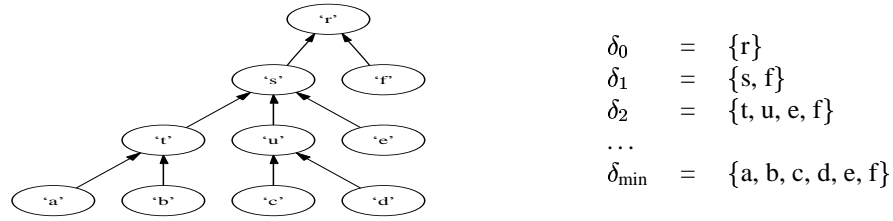


Figure 2: A tree structure (left) with levels of granularity (right).

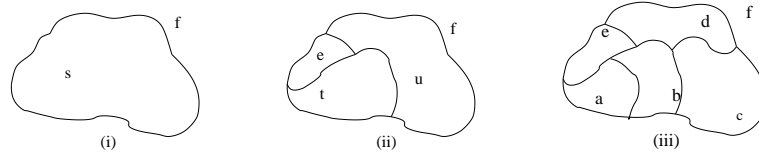


Figure 3: Depictions of the levels of granularity $\delta_1, \delta_2, \delta_{\min}$.

2.2. Cell structures

Let (Z, \sqsubseteq) be a cell structure. Using the primitive subcell relation \sqsubseteq we define the relations proper subcell (D_{\sqsubseteq}), cell-overlap ($D_{O_{\sqsubseteq}}$) and predicates for the root cell (D_{root}) and for atoms (D_{At}):

$$\begin{aligned} D_{\sqsubseteq} & c \sqsubset g \equiv c \sqsubseteq g \ \& \ \neg(c = g) \\ D_{O_{\sqsubseteq}} & O_{\sqsubseteq} cg \equiv (\exists d)(d \sqsubseteq c \ \& \ d \sqsubseteq g) \\ D_{root} & root(c) \equiv (\forall g)(g \sqsubseteq c) \\ D_{At} & At \ c \equiv \neg(\exists g)(g \sqsubset c) \end{aligned}$$

In this subsection the quantification ranges over cells in Z . Here and in the remainder of the paper leading quantifiers are omitted. The proper subcell relation $c \sqsubset g$ holds if c is a subcell of g but c and g are distinct entities. $O_{\sqsubseteq}cg$ is the relation of overlap between cells. The predicates $root$ and At hold if the entity to which they are applied is the root of a tree structure or an atom, i.e., a cell without proper subcells.

The subcell relation \sqsubseteq is governed by the following axioms:

$$\begin{aligned} (ATM1) & c \sqsubseteq c \\ (ATM2) & (c_1 \sqsubseteq c_2 \ \& \ c_2 \sqsubseteq c_1) \Rightarrow c_1 = c_2 \\ (ATM3) & (c_1 \sqsubseteq c_2 \ \& \ c_2 \sqsubseteq c_3) \Rightarrow c_1 \sqsubseteq c_3 \\ (ATM4) & (\exists c)root(c) \\ (ATM5) & O_{\sqsubseteq} c_1 c_2 \Rightarrow (c_1 \sqsubseteq c_2 \ \text{or} \ c_2 \sqsubseteq c_1) \\ (ATM6) & c \sqsubset g \Rightarrow (\exists d)(d \sqsubset g \ \& \ \neg O_{\sqsubseteq} cd) \\ (ATM7) & (\exists g)(At \ g \ \& \ g \sqsubseteq c) \\ (ATM8) & \neg At \ d \Rightarrow (\exists c_1, \dots, c_n)((\bigwedge_{1 \leq i \leq n} c_i \sqsubset d) \ \& \\ & (g)g \sqsubset d \Rightarrow \bigvee_{1 \leq i \leq n} g = c_i) \end{aligned}$$

Here ATM1-3 ensure that \sqsubseteq is a partial ordering. In ATM4 we demand that there is a root cell which has all cells in Z as subcells. Using ATM2 we can then prove that

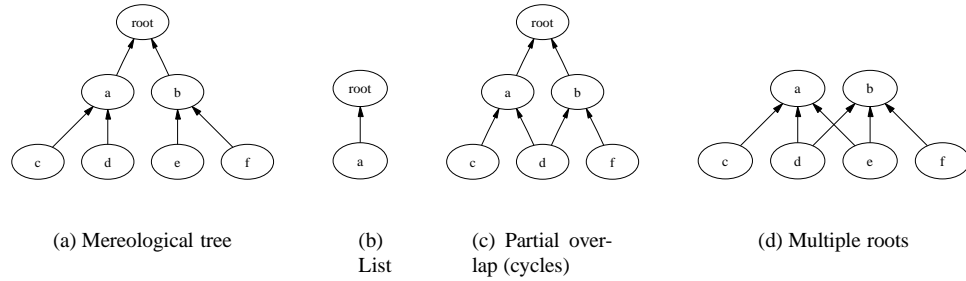


Figure 4: Mereological trees and non-trees.

there exists exactly one root. This rules out the structure in Figure 4(d) to be a tree in the mereological sense. We use the symbol \mathfrak{R} in order to refer to the root cell. ATM5 rules out the possibility of partial overlap of cells. From this it follows that cycles like the one shown in Figure 4(c) cannot occur in mereological trees. ATM6 rules out cases where a cell has only a single proper subcell (Figure 4(b)). It is known in the literature as the weak supplementation principle (WSP)³¹. ATM7 ensures that every cell has at least one atom as subcell. Finally ATM8 is an axiom schema which ensures that every cell is either an atom or has finitely many subcells.

Using ATM1, ATM5, and ATM6 we then can prove that the strong supplementation principle (SSP) holds (TTM1). The SSP tells us that if c is not a subcell of g then there exists a d which is a subcell of c and which does not overlap g . From SSP then immediately follows the extensionality of overlap (TTM2).

$$TTM1 \quad \neg(c \sqsubseteq g) \Rightarrow (\exists d)(d \sqsubseteq c \ \& \ \neg O_{\sqsubseteq} dg)$$

$$TTM2 \quad c = g \Leftrightarrow (w)(O_{\sqsubseteq} wc \Leftrightarrow O_{\sqsubseteq} wg)$$

The extensionality principle (TTM2) tells us that if whatever cell overlaps c also overlaps g then c and g must be identical. From this it follows that there cannot be distinct cells which coincide in the same sense in which the city of Vienna and the Austrian Federal State of Vienna coincide.

One can see that axioms ATM1-ATM8 constrain cell structures to be finite mereological trees similar to the one shown in Figure 4(a).

2.3. Target domain

(Δ, \leq) is the *target domain* which is taken to be a partial ordering which satisfies the axioms of extensional mereology³⁵. Mereologies are formal theories of the binary primitive $x \leq y$ which is interpreted as ‘ x is a part of y ’. For example, your arm is a part of you and California is a part of the United States. In terms of \leq we define the relations of proper parthood and overlap among entities in the target domain:

$$D_{<} \quad x < y \equiv x \leq y \ \& \ \neg(x = y)$$

$$D_O \quad Oxy \equiv (\exists z)(z \leq x \ \& \ z \leq y)$$

In this section quantification ranges over entities in the target domain Δ .

We continue by defining the sum of two entities x and y in Δ as the entity z which is such that anything overlaps z if and only if it overlaps x or y (D_+). Similarly we define the product of two entities x and y as the entity z which is such that anything overlaps z if and only if it overlaps x and y (D_*).

$$\begin{aligned} (D_+) \quad x + y &\equiv (\iota z)(w)(O wz \Leftrightarrow (O wx \text{ or } O wy)) \\ (D_*) \quad x * y &\equiv (\iota z)(w)(w \leq z \Leftrightarrow (w \leq x \ \& \ w \leq y)) \end{aligned}$$

Here (ιz) is the Russellean description operator which is read as ‘the unique z such that ...’ or formally $\Psi[(\iota x)\Phi(x)] \equiv (\exists x)[\Phi(x) \ \& \ (y)(\Phi(y) \Rightarrow x = y) \ \& \ \Psi(x)]$.

In addition to reflexivity, antisymmetry, and transitivity (referred to by AM1-AM3) the parthood relation is characterized by the following axioms:

$$\begin{aligned} AM4 \quad \neg x \leq y &\Rightarrow (\exists z)(z \leq x \ \& \ \neg O zy) \\ AM5 \quad (\exists y)(x) &x \leq y \end{aligned}$$

AM4 is the strong supplementation principle (SSP) for \leq . AM5 ensures that there exists a universe of which everything in Δ is a part. Using AM4 we can then prove that sums, products, and the universe are unique if they exist. One can also prove that summation is associative and hence it is permissible to write finite sums of the form $x = x_1 + \dots + x_n$.

In extensional mereology we can prove that the extensionality principle holds (TM1)*. There might be domains in which this is too strong, dealing with such issues however requires a more elaborate theory, which is beyond the scope of the present paper.

$$TM1 \quad x = y \Leftrightarrow (w)(O wx \Leftrightarrow O wy)$$

2.4. Projection

The relationship between cell structure and target domain is established by the binary relation of projection, Π , which holds between cells and entities. Projection is then governed by the following axioms: (AGM1) If c_1 projects onto x_1 and c_2 projects onto x_2 then c_1 is a subcell of c_2 if and only if x_1 is a part of x_2 ; (AGM2) Every cell projects onto something.

$$\begin{aligned} AGM1 \quad (\Pi c_1 x_1 \ \& \ \Pi c_2 x_2) &\Rightarrow (c_1 \sqsubseteq c_2 \Leftrightarrow x_1 \leq x_2) \\ AGM2 \quad (\exists x)(\Pi cx) & \end{aligned}$$

Using M1 and AGM1 we can prove the theorems TGM1 and TGM2 which tell us that Π is a one-one mapping, i.e., every cell projects onto a single entity in the target domain (TGM1) and no distinct cells project onto the same entity (TGM2).

$$\begin{aligned} TGM1 \quad \Pi cx \ \& \ \Pi cy &\Rightarrow x = y \\ TGM2 \quad \Pi c_1 x \ \& \ \Pi c_2 x &\Rightarrow c_1 = c_2 \end{aligned}$$

*The formal proofs of all theorems are included in the technical report version of this paper and can be obtained from the author.

AGM2 makes Π a total function which maps the domain of cells into the target domain Δ . Consequently we are justified to introduce a functional notation for Π (D_π).

$$D_\pi \quad x = \pi c \equiv \Pi cx \qquad D_{\bar{\pi}} \quad c = \bar{\pi} x \equiv \Pi cx$$

For convenience we define the partial mapping $\bar{\pi}$ ($D_{\bar{\pi}}$). From D_π and $D_{\bar{\pi}}$ then immediately follows that π and $\bar{\pi}$ behave like inverse functions wherever $\bar{\pi}$ is defined (TGM3 and TGM4). Using AGM1 we then prove (TGM5) which tells us that π is indeed an order homomorphism.

$$\begin{aligned} TGM3 \quad & c = \bar{\pi}(\pi c) \\ TGM4 \quad & (\exists c)(\Pi cx) \Rightarrow x = \pi(\bar{\pi} x) \\ TGM5 \quad & ((\exists c)(\Pi cx_1) \ \& \ (\exists c)(\Pi cx_2)) \Rightarrow (x_1 \leq x_2 \Leftrightarrow \bar{\pi} x_1 \sqsubset \bar{\pi} x_2) \\ TGM6 \quad & \neg c_1 \sqsubset c_2 \Rightarrow (\exists z)(z \leq (\pi c_1) \ \& \ \neg O z(\pi c_2)) \\ TGM7 \quad & ((\exists c)(\Pi cx_1) \ \& \ (\exists c)(\Pi cx_2)) \Rightarrow \\ & \{ \neg x_1 \leq x_2 \Rightarrow (\exists d)(d \sqsubset (\bar{\pi} x_1) \ \& \ \neg O d(\bar{\pi} x_2)) \} \end{aligned}$$

We then prove (TGM6) and (TGM7) using AM4, TTM1, AGM1, and TGM5. These theorems tell us that both π and $\bar{\pi}$ preserve the tree structure. Theorems TGM5-TGM7 also tell us that if we are only interested in those entities in Δ which are targeted by cells in Z and mereological relations between those entities then it is sufficient to refer to the cell structure as a proxy for the more complex target domain.

3. Levels of granularity

3.1. Collections

We are not only interested in individual things but also in *collections* of individuals. In particular we need to consider collections of cells which which form a level of granularity (to be defined below). We therefore add a third sort of variables $\delta, \delta_1, \delta_2$ ranging over collections of cells to our language.

Collections consist of individual things – cells in our case. Between individual things and collections the member-of or element-of relation holds. We use the notation \in in order to refer to this relation. Since collections and individuals belong to disjoint categories it follows that \in is irreflexive, asymmetric, and intransitive.

We then demand that two collections are identical if and only if they have the same members (AC1) and define the relation of subcollection to hold between the collections x and y iff every member of x is also a member of y (D_{\subseteq}).

$$\begin{aligned} AC1 \quad & \delta_1 = \delta_2 \Leftrightarrow (g)(g \in \delta_1 \Leftrightarrow g \in \delta_2) \\ D_{\subseteq} \quad & \delta_1 \subseteq \delta_2 \equiv (g)(g \in \delta_1 \Rightarrow g \in \delta_2) \end{aligned}$$

We then can prove that \subseteq is reflexive, antisymmetric, and transitive (TC1-3).

$$\begin{aligned} TC1 \quad & \delta \subseteq \delta \\ TC2 \quad & \delta_1 \subseteq \delta_2 \ \& \ \delta_2 \subseteq \delta_1 \Rightarrow \delta_1 = \delta_2 \\ TC3 \quad & \delta_1 \subseteq \delta_2 \ \& \ \delta_2 \subseteq \delta_3 \Rightarrow \delta_1 \subseteq \delta_3 \end{aligned}$$

If δ is a collection with a single element c then we write $\delta = \{c\}$. If δ is a collection with two elements c and d then we write $\delta = \{c, d\}$, and so on. The empty collection, signified by $\{\}$ or \emptyset , is a collection with no member.

3.2. Properties of levels of granularity

Let $G = \langle (Z, \sqsubseteq), (\Delta, \leq), \pi \rangle$ be a frame of reference with cell structure (Z, \sqsubseteq) . We define a level of granularity as a collection δ of cells which is such that (i) if two cells $g_1, g_2 \in \delta$ overlap then they are identical, i.e., the elements of δ are mereologically disjoint; and (ii) if c is a cell which is not an element of δ then there exists a cell $g \in \delta$ such that c and g overlap, i.e., the elements of δ are exhaustive (D_{GL}).

$$D_{GL} \quad GL \delta \equiv (g_1)(g_2)(g_1 \in \delta \ \& \ g_2 \in \delta \ \& \ O_{\sqsubseteq} g_1 g_2 \Rightarrow g_1 = g_2) \ \& \\ (c)(\neg c \in \delta \Rightarrow (\exists g)(g \in \delta \ \& \ O_{\sqsubseteq} c g))$$

It follows that levels of granularity are non-empty collections (TGL0).

$$TGL0 \quad GL \delta \Rightarrow (\exists c)(c \in \delta)$$

Levels of granularity can be constructed as follows²⁹⁶: (a) The collection containing only the root cell is a level of granularity (δ_0 in Figure 2.); (b) If δ is a level of granularity and g is an element of δ then the collection δ' in which g is replaced by its immediate subcells is a level of granularity. Consider Figure 2. Here we have the levels of granularity $\delta_0 = \{ 'r' \}$ and $\delta_1 = \{ 's', 'f' \}$ where 's' and 'f' are immediate subcells of 'r'.

Definition D_{GL} captures only certain necessary conditions that characterize levels of granularity at the most basic level: pairwise disjointness and weak exhaustiveness, two properties which are purely mereological in nature. More specific characterizations of levels of granularity may include constraints on the type of entities that form the level of granularity. For example consider a cell structure targeting a human being. One level of granularity might contain only biomolecules where another might contain only tissues or only organs. Definition D_{GL} in its basic form admits any kind of mixture of different kinds of things to forming a level of granularity provided only that they satisfy the properties listed in definition D_{GL} . In other cases it might be useful to include also metrical notions, for example in order to require that objects forming a certain level of granularity fall into a certain range of magnitudes¹². For the purpose of this paper, however, the presented definition will be sufficient.

3.3. The granularity ordering

Let δ_1 and δ_2 be levels of granularity in a given granularity tree. We define an ordering relation $\delta_1 \ll \delta_2$ as follows:

$$D_{\ll} \quad \delta_1 \ll \delta_2 \equiv GL \delta_1 \ \& \ GL \delta_2 \ \& \ (c)(c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ c \sqsubseteq g))$$

If $\delta_i \ll \delta_j$ holds then we say that δ_i is a refinement of δ_j , or δ_i refines δ_j , or δ_j is refined

by δ_i . Using ATM1-3, D_{GL} and AC1 we can prove that \ll is a partial ordering (TGL1-3).

$$\begin{aligned} TGL1 \quad & GL \delta \Rightarrow \delta \ll \delta \\ TGL2 \quad & \delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_1 \Rightarrow \delta_1 = \delta_2 \\ TGL3 \quad & \delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_3 \Rightarrow \delta_1 \ll \delta_3 \end{aligned}$$

This ordering includes at one extreme the level of granularity which consists only of the root cell as maximal element and at the other extreme the level of granularity formed by atomic (or leaf) cells as minimal element. The former is the coarsest level of granularity and the latter is the finest level of granularity. The corresponding structure is called the *granularity ordering* of the cell structure (Z, \sqsubseteq) .

The granularity ordering now insures that, given levels of granularity $\delta_i \ll \delta_j$, then there exists a mapping $l_{ij} : \delta_i \rightarrow \delta_j$ such that:

$$D_l \quad l_{ij} c = d \equiv \delta_i \ll \delta_j \ \& \ c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d$$

Using definitions D_l , D_{\ll} , D_{GL} together with axiom ATM1 we then can prove that l_{ij} is indeed a total and surjective mapping (TGL4-TGL6)[†].

$$\begin{aligned} TGL4 \quad & l_{ij} c = d_1 \ \& \ l_{ij} c = d_2 \Rightarrow d_1 = d_2 \\ TGL5 \quad & (\delta_i \ll \delta_j \ \& \ c \in \delta_i) \Rightarrow (\exists d)(d = l_{ij} c) \\ TGL6 \quad & (\delta_i \ll \delta_j \ \& \ d \in \delta_j) \Rightarrow (\exists c)(d = l_{ij} c) \end{aligned}$$

3.4. Mereological cumulativeness

We can now distinguish two classes of levels of granularity: cumulative and non-cumulative. The former are levels of granularity which are such that the targets of their cells sum up to the target of the root cell. The levels of granularity in Figures 2 and 3 are of this type. We call such levels of granularity *cumulative*².

On the other hand there are levels of granularity which do not have this property. An example of a non-cumulative frame of reference which is formed by the cells Hyde Park, Soho, Buckingham Palace, Downtown, London, York, Edinburgh, Glasgow, England, Scotland, Great Britain, Germany, Europe and the corresponding nesting, as given in Figure 5.

As another example, consider the frame of reference yielded by taking Figures 2 and 3 but deleting a as minimal cell and by assuming that the root cell covers the whole plane and that δ_1 and δ_2 sum up to δ_0 . In this case the level δ_{\min} fails to sum up to the whole plane and in particular the cell b fails to sum up to the cell $t \in \delta_2$. Instead of the cell a we have ‘empty space’ or a hole (which can be thought of as space we know nothing about).

Formally we define cumulativeness as follows. Let G be a frame of reference with cell structure (Z, \sqsubseteq) . A level of granularity δ in (Z, \sqsubseteq) is cumulative if and only if anything, say w , overlaps the entity targeted by the root cell ($\pi \mathfrak{R}$) if and only if there exists a cell in δ which target also overlaps w (D_{Com}).

$$D_{Com} \quad Com \delta \equiv GL \delta \ \& \ (w)(O w(\pi \mathfrak{R}) \Leftrightarrow (\exists c)(c \in \delta \ \& \ O w(\pi c)))$$

[†]See also ⁶ for a similar approach.

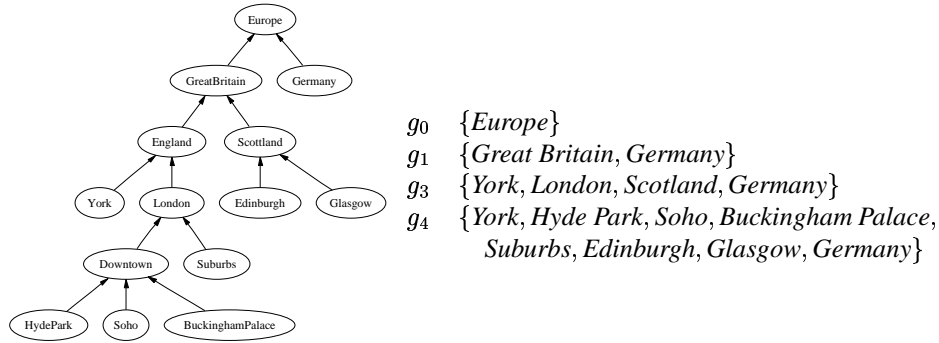


Figure 5: A place-based frame reference with non-cumulative levels of granularity.

We then can prove that if δ_i is a refinement of δ_j and δ_i is cumulative then δ_j is commulative (TGL7) and that δ is cumulative if and only if the projection of the root cell \mathfrak{R} is identical to to the sum of the projections of the cells in δ (TGL8).

$$\begin{aligned}
 TGL7 \quad & \delta_i \ll \delta_j \ \& \ Com \ \delta_i \Rightarrow Com \ \delta_j \\
 TGL8 \quad & Com \ \delta \Leftrightarrow (\exists c_1 \dots c_n)(c_1 \in \delta \ \& \ \dots \ \& \ c_n \in \delta \ \& \\
 & (c)(c \in \delta \Rightarrow c = c_1 \ or \ \dots \ or \ c = c_n) \ \& \ (\pi \ \mathfrak{R}) = (\pi \ x_1 + \dots + \pi \ x_n))
 \end{aligned}$$

Obviously, the root cell is always cumulative. A frame of reference is cumulative if and only if all of its levels of granularity are cumulative. Otherwise it is non-cumulative. For further discussion see ².

4. Approximate location in a frame of reference

4.1. Exact location

Let $G = \langle (Z, \sqsubseteq), (\Delta, \leq), \pi \rangle$ be a frame of reference and let x be an entity of the target domain Δ . We say that the *location* of x in G is the cell c if and only if c projects onto x (D_L). This means that location is just another name for the converse of projection:

$$D_L \quad L \ x \ c \equiv \Pi \ c \ x$$

It then follows from TGM2 and AGM1 that L satisfies the main axioms of a location relation as discussed by Casati and Varzi in ⁹:

$$\begin{aligned}
 TL1 \quad & L \ x \ c \ \& \ L \ x \ d \Rightarrow c = d \\
 TL2 \quad & x \leq y \ \& \ L \ x \ c \ \& \ L \ y \ d \Rightarrow c \sqsubseteq d
 \end{aligned}$$

From our definitions it immediately follows that location is a *partial* relation. In fact most of the entities of the target domain Δ will not be located in the frame of reference in the way described above. This is hardly surprising since otherwise the frame of reference would be as complex as the target domain itself. It follows that we need a more general

relation in order to characterize the relationship between entities in the target domain and the relevant frame of reference.

4.2. Rough approximation

We start by defining additional relations between entities in Δ :

$$\begin{aligned} D_{DR} \quad DR \, xy &\equiv \neg O \, xy \\ D_{PO} \quad PO \, xy &\equiv O \, xy \ \& \ \neg x \leq y \ \& \ \neg y \leq x \\ D_{NSO} \quad NSO \, xy &\equiv PO \, xy \ \text{or} \ x < y \end{aligned}$$

Two entities are disjoint if and only if they do not overlap (D_{DR}). Two entities overlap partially if and only if they overlap but neither is part of the other (D_{PO}). Two entities are in the relation of non-symmetric overlap if and only if they overlap partially or the first is a proper part of the second (D_{NSO}).

We now can prove that, for arbitrary entities x and y in the target domain, one and only one of the relations $DR \, xy$, $NSO \, xy$, or $y \leq x$ holds:

$$\begin{aligned} TRA1 \quad DR \, xy \ \text{or} \ NSO \, xy \ \text{or} \ y \leq x \\ TRA2 \quad \neg(DR \, xy \ \& \ NSO \, xy) \\ TRA3 \quad \neg(DR \, xy \ \& \ y \leq x) \\ TRA4 \quad \neg(NSO \, xy \ \& \ y \leq x) \end{aligned}$$

Let $G = \langle (Z, \sqsubseteq), (\Delta, \leq), \pi \rangle$ be a frame of reference. *Coarsening* is a binary relation between an entity x in the target domain Δ and a mapping X of signature $X : \delta \rightarrow \Omega$, where the co-domain of the mapping, Ω , is a totally ordered set of three values $fo > po > no$ and the domain of the mapping, δ , is a level of granularity in the underlying frame of reference. We write X^δ in order to symbolize the domain of the approximation mapping X .[‡]

We call fo , po , and no *approximation values* and X^δ an *approximation mapping*. Intuitively approximation values can be understood as different degrees of overlap between entities in Δ : fo means ‘full overlap’, po means ‘partial overlap’, and no means ‘no overlap’. Consider the approximation X^δ and let δ be the underlying level of granularity and let $c \in \delta$. The mapping $(X^\delta c)$ yields fo if and only if the entity which is targeted by the projection of the cell c , (πc) , is a part of the entity x . The mapping $(X^\delta c)$ yields the approximation value po if and only if between x and the entity targeted by the projection of the cell c the relation of non-symmetric overlap ($NSO \, x(\pi c)$) holds. The mapping yields no if there is no overlap between x and πc . Formally we express these ideas in (D_{Appr}).

$$\begin{aligned} D_{Appr} \quad Appr \, xX^\delta &\equiv (c)(c \in \delta \Rightarrow ((X^\delta c) = fo \Leftrightarrow (\pi c) \leq x \ \& \\ &\quad (X^\delta c) = po \Leftrightarrow NSO \, x(\pi c) \ \& \ (X^\delta c) = no \Leftrightarrow DR \, x(\pi c))) \end{aligned}$$

[‡]At this point, strictly speaking, we leave the realm of first order logic since we quantify over mappings, i.e., higher order entities. However those mapping are descriptions of entities in our target domain which will turn out to be unique for a given level of granularity. Moreover due to the finite character of the underlying cell structure these descriptions are finite and there can be only finitely many of them. These descriptions can be enumerated and included into the language for any particular cell structure.

Consider, for example, the approximations S^δ and Q^δ of the entities s and q in Figure 6 (iii). Given the level of granularity $\delta = \{a, b, c, d, e, f\}$ we have $(S^\delta a) = \text{po}$, $(S^\delta b) = \text{fo}$, and $(Q^\delta b) = \text{no}$.

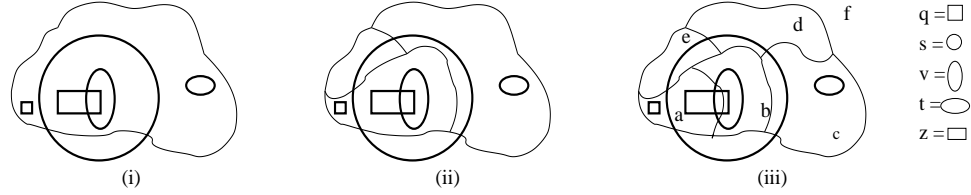


Figure 6: Rough location of the regions q , s , v , t and z within the levels of granularity δ_1 , δ_2 , and δ_{\min} .

Let X^δ be an approximation mapping. We call X^δ the empty or *null approximation* if and only if it yields the approximation value *no* for all cells in δ (D_{Null}). We call a frame of reference full if and only if for every non-empty approximation there is an entity which is approximated. If we wanted to constrain our theory to full frames of references we needed an axiom enforcing fullness as indicated in (**).

$$D_{Null} \quad Null X^\delta \equiv (c)(c \in \delta \Rightarrow (X^\delta c = \text{no}))$$

$$** \quad \neg Null X^\delta \Rightarrow (\exists x)(Appr x X^\delta)$$

We then add an axiom to the effect that every entity in our target domain Δ is a part of the projection root cell, $(\pi \mathfrak{R})$, of the underlying frame of reference (ARA1). From this it follows that (i) for every entity in the target domain there is a non-null approximation at any level of granularity (TRA5); and (ii) in any cumulative level of granularity there is for any x in Δ a cell which projection overlaps x (TRA6).

$$ARA1 \quad x \leq (\pi \mathfrak{R})$$

$$TRA5 \quad (\exists X^\delta)(Appr x X^\delta \ \& \ \neg Null X^\delta)$$

$$TRA6 \quad Com \ \delta \Rightarrow (x)(\exists c)(c \in \delta \ \& \ O \ x(\pi c))$$

Let X^δ and Y^δ be approximation mappings. Two mappings are identical if and only if their domains and co-domains are identical and identical entities in the domain are mapped to identical entities in the co-domain. Every approximation mapping is defined with respect to a fixed level of granularity which determines its domain and all approximation mappings have the co-domain Ω . Formally we then demand that X^δ and Y^δ are identical if they yield the same results for whatever input from their target domain (ARA2).

$$ARA2 \quad X^\delta = Y^\delta \Leftrightarrow (c)(c \in \delta \Rightarrow (X^\delta c = \text{fo} \Leftrightarrow Y^\delta c = \text{fo} \ \& \\ X^\delta c = \text{po} \Leftrightarrow Y^\delta c = \text{po} \ \& \\ X^\delta c = \text{no} \Leftrightarrow Y^\delta c = \text{no}))$$

Using ARA2 we then can prove that coarsening with respect to a given level of granularity is unique (TRA7).

$$TRA7 \quad Appr x X^\delta \ \& \ Appr x Y^\delta \Rightarrow X^\delta = Y^\delta$$

Since by axiom ARA1 for any x in the target domain Δ there is an approximation X^δ and since coarsening is unique within a fixed level of granularity (TRA7) we can define granularity-indexed coarsening functions (D_{appr}).

$$D_{appr} \quad appr_\delta x = X^\delta \equiv Appr x X^\delta$$

Due to ARA1 coarsening functions are total. If in addition also (**) holds then they are also surjective.

Often we consider only a fixed level of granularity. We then will omit the subscript (i.e., we write $appr x = X$ instead of $appr_\delta x = X^\delta$). If we apply the convention to use non-capitalized variables x, y, z for entities and capitalized variables X, Y, Z for the corresponding approximation mappings then we can omit the reference to the coarsening function $appr$ completely and write X or X^δ in order to refer to the rough approximation of the entity x .

Based on the notion of coarsening we now define an equivalence relation between entities in the target domain Δ in terms of identity of approximation:

$$D_\sim \quad x \sim_\delta y \equiv appr_\delta x = appr_\delta y$$

From D_{Appr} and D_\sim it follows immediately that \sim_δ is reflexive, symmetric, and transitive (TRA8-TRA10)

$$\begin{aligned} TRA8 \quad & x \sim_\delta x \\ TRA9 \quad & x \sim_\delta y \Rightarrow y \sim_\delta x \\ TRA10 \quad & x \sim_\delta y \ \& \ y \sim_\delta z \Rightarrow x \sim_\delta z \end{aligned}$$

In Figure 6 (iii) we have $z \sim_{\delta_{min}} v$. In Figure 6 (i) we have $z \sim_{\delta_1} v \sim_{\delta_1} q \sim_{\delta_1} t$. In the remainder we omit the subscript where there is no danger of confusion.

An important feature of frames of reference is the fact that they provide means for a certain sort of abstracting or economizing information. That is, we can talk about entities in the target domain not in terms of their exact location but rather in terms of approximations. Entities with identical approximations cannot be distinguished. This significantly simplifies the structure of the target domain and facilitates approximate reasoning^{5,1,4}.

4.3. Stratified approximations

Frames of reference have a hierarchical structure and therefore entities can be approximated at different levels of granularity. We saw examples in natural language in sentences A and A' and B and B' in the introduction and we also saw this in the example depicted in figure 6. We now introduce the notion of stratified approximations in order to capture these features of frames of reference formally.

Let $G = \langle (Z, \sqsubseteq), (\Delta, \leq), \pi \rangle$ be a frame of reference with levels of granularity $\delta_1, \dots, \delta_n$; let G be mereologically cumulative, i.e., $Com \delta_1 \ \& \ \dots \ \& \ Com \delta_n$; and let x be an entity of the target domain Δ . We have unique approximations $X^{\delta_1}, \dots, X^{\delta_n}$ at different levels of detail (ARA1, TRA7).

Using the strong supplementation principle (AM4), transitivity of parthood (AM3), theorems TRA5, TRA6 and definitions D_{Appr} and D_{appr} we now can prove the following:

Let δ_i be a level of granularity and cumulative and let δ_j be a refinement of δ_i (which is then cumulative by theorem TGL7). Furthermore let $(appr_{\delta_i} x)$ be the approximation of the entity x with respect to δ_i and let $(appr_{\delta_j} x)$ be the approximation of x with respect to δ_j . It holds for every cell $c \in \delta_j$ that the approximation value $((appr_{\delta_j} x) c)$ is full overlap (fo) if and only if the approximation value $((appr_{\delta_i} x) d)$ for all subcells $d \sqsubseteq c$ with $d \in \delta_i$ is full overlap (fo) (TSA1).

$$TSA1 \quad Com \delta_i \& \delta_i \ll \delta_j \Rightarrow \\ (c)(c \in \delta_j \Rightarrow [(d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = fo)] \Leftrightarrow (appr_{\delta_j} x) c = fo)]$$

This tells us that that under the assumption that the underlying levels of granularity are cumulative we do not lose information about full overlap if we approximate with respect to a coarser level of granularity (given, of course, that the approximation with respect to all relevant cells at the finer approximation yields full overlap).

Under the same assumptions we can prove a similar theorem for the approximation value no (non-overlap or disjointness): It holds for every cell $c \in \delta_j$ that the approximation value $((appr_{\delta_j} x) c)$ is non-overlap if and only if the approximation value $((appr_{\delta_i} x) d)$ for all subcells $d \sqsubseteq c$ with $d \in \delta_i$ is non-overlap (TSA2).

$$TSA2 \quad Com \delta_i \& \delta_i \ll \delta_j \Rightarrow \\ (c)(c \in \delta_j \Rightarrow [(d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no)] \Leftrightarrow (appr_{\delta_j} x) c = no)]$$

TSA2 tells us that that under the assumption that the underlying levels of granularity are cumulative we do not lose information about non-overlap if we approximate with respect to a coarser level of granularity (given that the approximation with respect to all relevant cells at the finer approximation yields non-overlap).

It is important to stress the assumption of cumulateness in theorems TSA1 and TSA2. Without this assumption neither of them is provable. This is because the proofs heavily depend on theorem TRA6. We will discuss the consequences of giving up the assumption of cumulateness in the next section.

Let $\omega_{ij}^p = \{X^{\delta_i} k \mid (\ell_{ij} k) = p\}$ be the collection of approximation values under (X^{δ_i}) with respect to the subcells $k \in \delta_i$ of the cell $p \in \delta_j$. Whenever $\delta_i \ll \delta_j$ we define a generalization mapping $\alpha_{ij} : \Omega \rightarrow \Omega$ with

$$D_{\alpha_{ij}} \quad (\alpha_{ij} (X^{\delta_i})) k = \begin{cases} fo & \text{iff } \omega_{ij}^{(\ell_{ij} k)} = \{ fo \} \\ no & \text{iff } \omega_{ij}^{(\ell_{ij} k)} = \{ no \} \\ po & \text{otherwise} \end{cases}$$

and $\{(f k) \mid Q k\} = \{u\} \Leftrightarrow (k)(Q k \Rightarrow (f k) = u)$.

Using TSA1, TSA2 and ARA2 we can then prove the following:

$$TSA3 \quad (Com \delta_i \& \delta_i \ll \delta_j) \Rightarrow (c)(c \in \delta_i \Rightarrow ((\alpha_{ij} (appr_{\delta_i} x)) c = ((appr_{\delta_j} x) \ell_{ij} c)))$$

Theorem (TSA3) tells us that under the assumption of cumulateness of the underlying levels of granularity the following two ways of computing generalizations of approximations yield the same result:

1. We first map the cell c in δ_i to its supercell in δ_j by means of ℓ_{ij} and then apply X^{δ_j} , i.e., we perform the composition of the mapping ℓ_{ij} and X^{δ_j} .
2. We first map the cell c onto its approximation value and then generalize the approximation by means of α_{ij} , i.e., we perform the composition of the mapping X^{δ_i} and α_{ij} .

In the first case we perform the generalization on the level of cells and in the second case we perform the generalization on the level of approximation values. This is equivalent to saying that the following diagram commutes:

$$\begin{array}{ccc}
 \delta_j & \xrightarrow{X^{\delta_j}} & \Omega \\
 \uparrow \ell_{ij} & & \uparrow \alpha_{ij} \\
 \delta_i & \xrightarrow{X^{\delta_i}} & \Omega
 \end{array}$$

Following the authors of ³⁷ and ⁶ we now define:

Definition 1 A stratified approximation is a family of approximations $X^{\delta_1}, \dots, X^{\delta_n}$, such that whenever $\delta_i \ll \delta_j$ we have $(k)(k \in \delta_i \Rightarrow (\alpha_{ij}(X^{\delta_i})k = ((X^{\delta_j})\ell_{ij})k)$.

Stratified approximations allow us to represent entities in the target domain at different levels of granularity and to transform fine approximations into coarser ones.

5. Epistemic vagueness and rough approximation

Approximations are *crisp* in the sense that within a given level of granularity for every x there exists a *unique* approximation X (TRA7). For every approximation there exists a unique generalization mapping to every coarser level of granularity (TSA3). Moreover, it is completely determinate which entities can be distinguished with respect to the underlying approximation and which are indistinguishable, i.e., equivalent in the sense of \sim_δ .

However frames of reference are artifacts of human cognition and their *application* in concrete situations is subject to limitations of human knowledge. In this section we will discuss three different kinds of limitations of human knowledge:

1. Limitations of knowledge about relations between an entity x and entities targeted by the cell tree (Z, \sqsubseteq) . This can be caused by limitations of our observations or by limitation of knowledge about the projection.
2. Limitations of granularity of knowledge: The approximation at hand is too coarse.
3. Limitations of knowledge about the underlying frame of reference: The underlying frame of reference might be non-cumulative in the sense defined in (\mathbb{D}_{Com}) .

These limitations of human knowledge give rise to what we call *epistemic vagueness*. Epistemic vagueness hereby is understood in the sense that there are multiple, equally good, approximations which are consistent with the knowledge at hand. Epistemic vagueness

needs to be distinguished from the semantic vagueness of terms like ‘Mount Everest’ where there are multiple equally good *candidate referents* for the name ‘Mount Everest’^{36,3}. Epistemic vagueness can be caused by semantic vagueness. However epistemic vagueness can also apply to entities which are not subject to vagueness in the semantic sense at all. This is because epistemic vagueness is rooted in the underlying frame of reference and its application rather than in the objects which are approximated.

5.1. *Epistemic vagueness from observations*

When using frames of references in a concrete situation knowledge about relations between an entity targeted by the cell tree, πc , and an entity x is typically gained by observations. Here it might be impossible to observe which of the relations holds: $DR(\pi c)x$ or $NSO(\pi c)x$ or $x \leq (\pi c)$. The property of joint exhaustiveness and pairwise disjointness of these relations (TRA1-4) supports the specification of knowledge in a coarser way by means of disjunctions of relations.

We can collapse the distinction of three relations into the distinction between two relations of which one is a disjunction. For example we can say that the entities (πc) and x overlap or do not overlap, which means that we collapse the relations $NSO(\pi c)x$ and $x \leq (\pi c)$ into a disjunction. Another way of collapsing the three relations mentioned about into two is to say that either (πc) is a part of x or not, which means that we collapse the distinction between $NSO(\pi c)x$ and $DR(\pi c)x$ into a disjunction.

A third way is to make no distinction at all, i.e., by collapsing all three relations into a disjunction of the form $DR(\pi c)x$ or $NSO(\pi c)x$ or $x \leq (\pi c)$. This means that the relation between the entity in question and the projection (πc) is completely unknown.

Notice, that in all these cases we represent knowledge, i.e., true justified beliefs, rather than mere (false) belief.

5.2. *Epistemic vagueness from lack of resolution*

Knowledge about approximation is often limited in the sense that for a given entity x of the target domain Δ only the approximation X^{δ_i} is known, where δ_i is relatively coarse level of granularity.

Consider Figure 7. Here we have entities, x_1, x_2, x_3 in our target domain Δ , which are approximated within a frame of reference consisting of five cells: z, z_1, z_2, z_3, z_4 such that $\delta_0 = \{z\}$ and $\delta_1 = \{z_1, z_2, z_3, z_4\}$. The level of granularity δ_1 is mereologically cumulative, i.e., $(\pi z) = (\pi z_1) + (\pi z_2) + (\pi z_3) + (\pi z_4)$. The entity x_1 has the approximation $X_1^{\delta_0} z = \text{po}$. One can easily verify in Figure 7 that we have $X_1^{\delta_0} = X_2^{\delta_0} = X_3^{\delta_0}$ but $X_1^{\delta_1} \neq X_2^{\delta_1}$, $X_2^{\delta_1} \neq X_3^{\delta_1}$ and $X_1^{\delta_1} \neq X_3^{\delta_1}$.

Now choose some x with approximation $X^{\delta_0} z = \text{po}$ and assume that all we know about x is this approximation. At the level of granularity δ_0 we cannot say whether x is distinct from x_1, x_2 , and x_3 since all entities are equivalent in the sense that they cannot be distinguished. At a finer level of granularity we might potentially be in a position to distinguish x from some of the x_i since the x_i have distinct and incompatible approximations at this level of granularity. However if all we know is the coarse approximation X^{δ_0} then

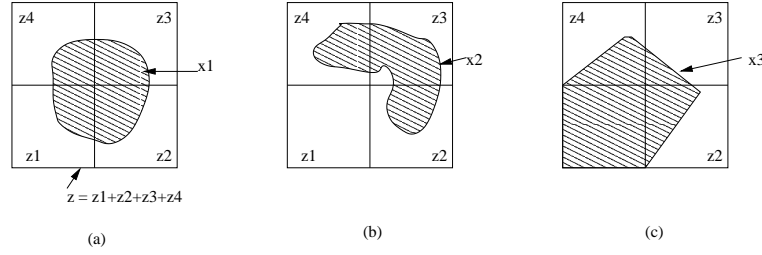


Figure 7: How vagueness arises in cumulative frames of reference.

the approximation of the entities x_1 , x_2 , and x_3 at the level of granularity δ_1 are consistent with X^{δ_0} . Therefore, for example, x could be like x_1 or like x_2 or like x_3 , in the sense that from all we know about x it could have the approximations $X^{\delta_1} = X_1^{\delta_1}$, or $X^{\delta_1} = X_2^{\delta_1}$, or $X^{\delta_1} = X_3^{\delta_1}$.

Consequently, if we transform approximations from a coarse level of granularity to a finer then epistemic vagueness arises. That is, disjunctions of possible relations that can hold between the targets of the cells of the finer level of granularity and the entity to be approximated arise.

5.3. Vagueness from non-cumulativeness

Above we have shown that given an approximation X^{δ_i} with respect to a mereologically cumulative level of granularity δ_i then there exists always a unique generalization mapping to coarser levels of granularity δ_j with $\delta_i \ll \delta_j$. If we give up the assumption that δ_i is cumulative, then unique generalization mappings do not necessarily exist any more. This is because theorem TSA3 depends on the theorems TSA1 and TSA2. We will continue by discussing two examples which show that the consequents of theorems TSA1 and TSA2 do not hold for approximations in non-cumulative levels of granularity.

Consider Figure 8. Here we have entities, x_1, x_2, x_3 in our target domain Δ , which are approximated within a frame of reference consisting of four cells: z, z_1, z_2, z_3 such that $\delta_0 = \{z\}$ and $\delta_1 = \{z_1, z_2, z_3\}$. Here δ_1 is mereologically *non-cumulative*, i.e., $(\pi z_1) + (\pi z_2) + (\pi z_3) + (\pi z_4) < (\pi z)$.

In Figure 8 (b) we have an entity x_2 with approximation $(X_2^{\delta_1} z_i) = \text{fo}$ for $1 \leq i \leq 3$. This is consistent with $(X_2^{\delta_0} z) = \text{fo}$ as one would normally expect, but it is also consistent with $(X_2^{\delta_0} z) = \text{po}$ – the case depicted in Figure 8 (b). Consequently, given the approximation $(X_2^{\delta_1} z_i) = \text{fo}$, a disjunction of possible generalizations exist. Similar is the situation in Figure 8 (c) where we have an entity x_3 with approximation $(X_3^{\delta_1} z_i) = \text{no}$ for $1 \leq i \leq 3$. This approximation is consistent with $(X_3^{\delta_0} z) = \text{no}$ as one would normally expect, but it is also consistent with $(X_3^{\delta_0} z) = \text{po}$ – the case depicted in Figure 8 (b). Consequently, given the approximation $(X_3^{\delta_1} z_i) = \text{no}$ a disjunction of possible generalizations exist.

6. Vague approximation

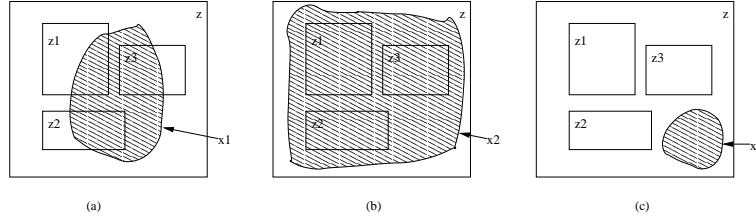


Figure 8: How vagueness arises in *non-cumulative* frames of reference.

Limits of human knowledge cause epistemic vagueness regarding the approximation of entities within a frame of reference. Epistemic vagueness hereby means that the relationships between entities in the target domain entities (also in the target domain) targeted by cells of the frames of reference are characterized by disjunctions of possible relations rather than by one of the crisp relations specified in D_{appr} . We now introduce the notion of vague approximation in order to reflect this phenomenon on the formal level. It follows that this notion will be applicable to deal with the different sources of vagueness discussed in the previous section.

The presented formalism is a generalization of the crisp approach discussed above. In what follows we will use the notion of collections of collections. Strictly speaking and continuing the style of formalization started in section 3.1 collections of collections had to be introduced as a separate sort with axioms and definitions similar to the ones in section 3.1. Due to space limitations this has to be omitted here. We therefore borrow corresponding notions from set theory and assume that the reader can imagine how this can be expressed within the present formalism along the lines set out above.

6.1. Vague approximation and crisping

Let $G < (Z, \sqsubseteq), (\Delta, \leq), \pi >$ be a frame of reference. Vague approximation, $VAppr$ is a ternary relation between an entity x in the target domain Δ , a level of granularity δ in the tree structure (Z, \sqsubseteq) of the underlying frame of reference, and a mapping \tilde{X} of signature $\tilde{X} : \delta \rightarrow \tilde{\Omega}$, where $\tilde{\Omega}$ is the collection $\{\{ fo \}, \{ po \}, \{ no \}, \{ fo, po \}, \{ po, no \}, \{ fo, po, no \}\}$. Sets of values here represent disjunctions of possible approximation values. To get an intuition assume that $VAppr$ holds for the entity x , the level of granularity δ , and let \tilde{X} be a mapping of signature $\tilde{X} : \delta \rightarrow \tilde{\Omega}$. Then the result of applying \tilde{X} to a cell $c \in \delta$ is interpreted as a disjunction of possible relations between x and (πc) . For example, the value of $(\tilde{X} c)$ is $\{ po, no \}$ if either x covers some but not all of the interior of (πc) or if there is no overlap between x and (πc) .

The elements of $\tilde{\Omega}$ are ordered by the subset relation. We obtain a lattice structure by taking the lattice operations join and meet to be set union and intersection and by adding the empty set as bottom element: $(\tilde{\Omega} \cup \emptyset, \cup, \cap)$. These operations generalize to operations on finite sets $\bigcup A$ and $\bigcap A$ in the standard way.

Let $CR \sqsubseteq (\delta \rightarrow \Omega) \times (\delta \rightarrow \tilde{\Omega})$ be a binary relation between crisp and vague approxi-

mations interpreted as ‘ X^δ is a *crisping* of \tilde{X}^δ ’. We define:

$$D_{CR} \quad CR X^\delta \tilde{X}^\delta \equiv (c)(c \in \delta \Rightarrow (X^\delta c) \in (\tilde{X}^\delta c)).$$

This means that X^δ is a crisping of \tilde{X}^δ if and only if for all cells c of the underlying granularity level δ we obtain $(X^\delta c)$ by choosing one element of $(\tilde{X}^\delta c)$. For example, if we have $CR X \tilde{X}$ and $(\tilde{X} c) = \{ \text{po}, \text{no} \}$ then $(X c) = \text{po}$ or $(X c) = \text{no}$. We say that the vague approximation \tilde{X} is crisp if and only if it has only a single crisping:

$$D_{crisp} \quad crisp \tilde{X} \equiv (x)(y)(CR (appr x) \tilde{X} \ \& \ CR (appr y) \tilde{X} \Rightarrow (appr x) = (appr y))$$

From the definitions D_{crisp} , D_{appr} and D_{\sim} it immediately follows that a vague approximation is crisp if and only if the entities approximated by is crispings are equivalent in the sense of \sim (TVA1).

$$TVA1 \quad crisp \tilde{X} \Leftrightarrow (x)(y)(CR (appr x) \tilde{X} \ \& \ CR (appr y) \tilde{X} \Rightarrow x \sim y)$$

Vague coarsening then is a binary relation, $VAppr x \tilde{X}^\delta$, which holds between an entity x of the target domain and a mapping \tilde{X}^δ if and only if the approximation $(appr_\delta x)$ is a crisping of \tilde{X}^δ :

$$D_{VAppr} \quad VAppr x \tilde{X}^\delta \equiv CR (appr_\delta x) \tilde{X}^\delta$$

It is important to notice that (a) $VAppr$ is not a function, i.e., there may be multiple coarsenings for the same entity; that (b) not every crisping of a vague approximation of x is an approximation of x ; and that therefore (c) identity of vague approximation does not imply equivalence (in the sense of \sim_δ) between the entities approximated by the crispings.

Nevertheless, we continue to use non-capitalized variables x, y, z for entities and capitalized variables $\tilde{X}, \tilde{Y}, \tilde{Z}$ for vague approximations and write \tilde{X}^δ in order to refer to *some* vague approximation of the entity x at the granularity level δ . Sometimes we will also omit the superscript.

6.2. Stratified vague approximations

Let δ_i be a *non-cumulative* level of granularity and let $\omega_{ij}^d = \{(\tilde{X}^{\delta_i} c) \mid (\ell_{ij} c) = d\} \sqsubset \mathcal{P}\tilde{\Omega}$ be the collection containing the collection of approximation values under \tilde{X}^{δ_i} with respect to the subcells of d in δ_i , $\{c \in \delta_i \mid (\ell_{ij} c) = d\}$. Whenever $\delta_i \ll \delta_j$ we define a mapping $\tilde{\alpha}_{ij} : \tilde{\Omega} \rightarrow \tilde{\Omega}$ with:

$$D_{\tilde{\alpha}_{ij}} \quad (\tilde{\alpha}_{ij} \tilde{X}^{\delta_i}) k = \begin{cases} \{ \text{po} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{po} \} \\ \{ \text{no}, \text{po} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{no} \} \\ \{ \text{fo}, \text{po} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{fo} \} \\ \{ \text{po}, \text{fo} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{po}, \text{fo} \} \\ \{ \text{no}, \text{po} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{no}, \text{po} \} \\ \{ \text{no}, \text{po}, \text{fo} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \text{no}, \text{po}, \text{fo} \} \\ \{ \text{po} \} & \text{if } \bigcap \omega_{ij}^{(\ell_{ij} k)} = \{ \} \end{cases}$$

Here $\bigcap \omega_{ij}^{(\ell_{ij} k)}$ is the intersection of the sets of approximation values in $\omega_{ij}^{(\ell_{ij} k)}$. Consider, again, Figure 8 (b). Assume that we have $(\tilde{X}_2^{\delta_1} z_i) = \{ \text{fo} \}$ for $1 \leq i \leq 3$ and $\delta_1 \ll \delta_0$. In this case $\omega_{10}^{(\ell_{10} z_i)}$ is the set $\{ \{ \text{fo} \} \}$. Consequently we have $(\tilde{\alpha}_{10} \tilde{X}_2^{\delta_1}) z = \{ \text{fo}, \text{po} \}$ which corresponds to our intuitions discussed above and the fact that the underlying level of granularity is non-cumulative.

We then can prove that $\tilde{\alpha}$ preserves the crisping relation, i.e., if X is a crisping of \tilde{X} then (αX) is a crisping of $(\tilde{\alpha} \tilde{X})$ (TVA2)

$$TVA2 \quad CR X \tilde{X} \Rightarrow CR (\alpha X) (\tilde{\alpha} \tilde{X})$$

This is obviously much weaker than the result we had in the crisp case in form of theorem (TSA3). However theorem (TVA2) tells us that if \tilde{X} is a vague approximation of the entity x then so is the generalization of \tilde{X} .

In mereologically *cumulative* levels of granularity we are able to define a slightly stronger version of the generalization mapping $\tilde{\alpha}$. This is because, if δ_i is cumulative and \tilde{X}^{δ_i} is crisp in the sense of D_{crisp} , then the vague generalization coincides with the (crisp) generalization mapping α defined in D_α . We then define $\hat{\alpha}$ as the vague generalization mapping which takes the distinction between cumulativeness and non-cumulativeness of the underlying level of granularity into account:

$$D_{\hat{\alpha}_{ij}} \quad (\hat{\alpha}_{ij} (\tilde{X}^{\delta_i})) k = \begin{cases} (\alpha X^{\delta_i}) k & \text{if } Com \delta_i \ \& \ crisp \ \tilde{X}^{\delta_i} \ \& \ CR \ X^{\delta_i} \ \tilde{X}^{\delta_i} \\ (\tilde{\alpha} \tilde{X}^{\delta_i}) k & \text{otherwise} \end{cases}$$

Whenever δ_i is cumulative and \tilde{X}^{δ_i} is crisp we apply crisp generalization mapping α to the single crisping of \tilde{X}^{δ_i} . Otherwise we apply we apply $\tilde{\alpha}$ as usual.

Following the authors of ⁶ we now define:

Definition 2 A vague stratified approximation is a family of approximations $\tilde{X}^{\delta_1}, \dots, \tilde{X}^{\delta_n}$, such that, whenever $\delta_i \ll \delta_j$, we have $(k) (k \in \delta_i \Rightarrow (\hat{\alpha}_{ij} (\tilde{X}^{\delta_i})) k = ((\tilde{X}^{\delta_j}) \ell_{ij}) k)$.

Vague stratified approximations are such that approximations at a finer level of granularity can be transformed to an approximation at a coarser level of granularity in such a way that the following diagram commutes.

$$\begin{array}{ccc} \delta_j & \xrightarrow{\tilde{X}^{\delta_j}} & \tilde{\Omega} \\ \uparrow \ell_{ij} & & \uparrow \tilde{\alpha}_{ij} \\ \delta_i & \xrightarrow{\tilde{X}^{\delta_i}} & \tilde{\Omega} \end{array}$$

Notice, however, that for a single entity of the target domain there might exist multiple vague stratified approximations.

6.3. Refinement

The notion of vague approximation allows us to define refinement from coarser to finer levels of granularity. Obviously, if all we have is knowledge about approximation at a coarse

level of granularity then refinement adds vagueness in the sense defined above. Formally we define refinement as a mapping $\beta : \tilde{\Omega} \rightarrow \tilde{\Omega}$ as follows:

$$D_{\beta} \quad (\beta \tilde{X}) h = \omega_2 \equiv (\tilde{X}g) = \omega_1 \ \& \ g \sqsubseteq h \ \& \ g \in \delta_i \ \& \ h \in \delta_j \ \& \\ (i) \quad \text{if } \omega_1 = \{ \text{no} \} \text{ then } \omega_2 = \omega_1 \ \& \\ (ii) \quad \text{if } \text{po} \in \omega_1 \text{ then } \omega_2 = \{ \text{no}, \text{po}, \text{fo} \} \ \& \\ (iii) \quad \text{if } \omega_1 = \{ \text{fo} \} \text{ then } \omega_2 = \omega_1$$

Since g is a subcell of h it follows that: (i) if h does not overlap a given region r then neither does g ; and (ii) if h is a part of r then so is g . Vagueness arises only in cases where r and h partially overlap. This is because in this case r and g may or may not overlap. In fact r may even contain g . For refinement, mereological cumulativeness does not need to be considered, since it is not an additional source for vagueness.

6.4. Degrees of vagueness

Let \tilde{X} and \tilde{Y} be vague approximations then \tilde{X} is of less or equal degree of vagueness than \tilde{Y} if and only if every crisping of \tilde{X} is also a crisping of \tilde{Y} :

$$D_{\preceq} \quad \tilde{X} \preceq \tilde{Y} \equiv (X)(CR(X, \tilde{X}) \Rightarrow CR(X, \tilde{Y})).$$

Here \preceq is clearly reflexive, transitive, and antisymmetric.

The application of generalization in non-cumulative levels of granularity and refinement increases vagueness in the sense that if generalization and refinement (refinement and generalization) transformations are applied successively to the approximation \tilde{X} the resulting approximation is of greater or equal vagueness than \tilde{X} (TVA3 and TVA4)

$$\begin{aligned} TVA3 \quad \tilde{X} \preceq (\hat{\alpha}_{ij} \beta_{ji}) \tilde{X} \\ TVA4 \quad \tilde{X} \preceq (\beta_{ji} \hat{\alpha}_{ij}) \tilde{X} \end{aligned}$$

7. Related work

Related work comes from two major sources: literature about rough sets and rough mereology and literature about frames of reference .

7.1. Rough sets

Crisp approximations as introduced above are a generalization of rough sets ²⁷. Rough sets are an extension of set theory and are based on an indiscernibility relation \approx on a set of objects S . Indiscernibility is an equivalence relation and creates a partition of the underlying set into a set of jointly exhaustive and pairwise disjoint subsets. A given subset $X \subseteq S$ can be approximated with respect to the partition of S by means of lower and upper approximations:

$$\begin{aligned} D_{\underline{X}} \quad \underline{X} &\equiv \{x \mid [x] \subseteq X\} \\ D_{\overline{X}} \quad \overline{X} &\equiv \{x \mid [x] \cap X \neq \emptyset\} \end{aligned}$$

The lower approximation of a set X is the set of all x such that the equivalence class of x with respect to \approx is a subset of X . The upper approximation of X is the set of all x such that its equivalence class $[x]$ has a non-empty intersection with X .

In the above we have used a mereological framework but we can interpret the target domain Δ as a set and entities as subsets of Δ . The difference between our approach and the rough set approach is that we start with a tree structure and levels of granularity within this tree. A level of granularity is the counterpart of the partition induced by the indiscernibility relation in the rough set approach. Sets forming levels of granularity are pairwise disjoint and exhaustive in a weak sense (D_{GL}). They are, however, not necessarily exhaustive in the strong sense in which a partition induced by an equivalence relation exhausts a set. This is because levels of granularity are not necessarily cumulative. For cumulative levels of granularity one can prove the equivalence of stratified (i.e., hierarchically ordered) rough sets³⁷ and approximations within frames of reference⁶.

The approach presented in this paper allows us to take into account more explicitly the nature of frames of reference, their hierarchical structure, and the epistemic issues that come up when frames of reference are applied in specific situations. In particular we discussed several sources of epistemic vagueness which can be identified within the presented. We then further extended the notion of rough sets in order to take vagueness into account.

7.2. Rough mereology

Rough mereology²⁸ extends rough set theory by incorporating it into the mereological framework of Lesniewski²⁴. On the other hand it extends mereology by allowing degrees of parthood. Rough parthood, $\mu xy = z$, is a ternary functional relation which is interpreted as ‘ x is a part of y to the degree of at least z ’.

There are no further assumptions on the structure of the domain of values specifying degrees of parthood beyond the fact that it needs to form a complete lattice with minimal and maximal element. For this purpose the unit interval $[0, 1]$ of rational or real numbers is often used. Given a set of entities, the prototypical way of defining degrees of parthood among two classes formed by those entities is to count instances that fall into each of them and then to compute a ratio:

$$D_{\mu_1} \quad \mu_1 XY \equiv \begin{cases} \frac{|X \cap Y|}{|X|} & \text{if } X \neq \emptyset \\ 1 & \text{if } X = \emptyset \end{cases}$$

A closely related approach, which brings the indiscernibility relation \approx underlying rough sets into the picture, is to define degrees of parthood as

$$D_{\mu_2} \quad \mu_2 xY \equiv \frac{|[x] \cap Y|}{|[x]|}.$$

Here $[x]$ is the equivalence class of entities in X which are indiscernible from x , i.e., $[x] \equiv \{y \in X \mid x \approx y\}$.

There are different variants of this approach (including fuzzy sets¹⁸) which have proven to be quite successful for example in data mining applications. For an overview see²¹. In

the remainder we refer to this approach to rough mereology as ‘the data or observation driven approach’.

We now can show that the approach presented in this paper is in fact a special case of rough mereology. We define:

$$\begin{aligned} \text{DRP1} \quad & RP \ xy \ fo \equiv x \leq y \\ \text{DRP2} \quad & RP \ xy \ po \equiv NSO \ yx \\ \text{DRP3} \quad & RP \ xy \ no \equiv DR \ yx \end{aligned}$$

Clearly, $fo > po > no$ form a lattice of the kind demanded above. From this perspective the relations $x \leq y$, $NSO \ yx$, and $DR \ yx$ represent different degrees of parthood between x and y . We can then prove that all but one of the axioms of rough mereology are theorems in our framework. The one axiom of rough mereology that is not a theorem is an axiom postulating a *null entity* an entity which is a maximal part of any entity in the domain at hand. Since we assume that the target domains Δ in the presented framework satisfy the axioms of extensional mereology there does not exist a null entity in them. It follows that the presented formalism is a specific version of rough mereology. We call it the ontology driven approach to rough mereology (modulo the existence of the null-entity).

Now compare the data driven and the ontology driven approach. Rough mereology was developed in the context of data analysis and is used in order to discover structures in data sets. In other words its aim is to extract conceptual (and thus to a certain degree ontological) structure from observations. Hereby no specific assumptions are made about the nature of observations, sources of epistemic vagueness, the role of frames of reference, and the role of the granular structure of the underlying domain. This has the consequence that the semantics of the domain describing the degree of parthood tends to be rather abstract and often mixes ontological and epistemic aspects.

This is reflected by a criticism that has often been raised against the data driven approach, namely that the semantics of the values describing degrees of membership is often not clear. What does $\mu \ XY = 0.7$ mean, beyond sequences of occurrences in data sets? (A parallel criticism can be made in relation to many applications of fuzzy set theory¹⁷.)

On the approach pursued in this paper, making explicit the underlying assumptions about the nature of observations, sources of epistemic vagueness, the role of frames of reference, and the role of the granular structure of the underlying domain of entities helps to give a clear semantics for the notion ‘degree of parthood’. As we have seen in (DRP1-3), it is very easy to give a specific interpretation of degrees of parthood as disjointness, non-symmetric partial overlap, and parthood. We have seen how, starting out from there one can explore in a systematic fashion the effects of such epistemic factors as limitations of observations, deficiencies of frames of reference.

7.3. Frames of reference

As argued above, research on frames of reference has been pursued in areas like physics, linguistics, psychology, and artificial intelligence. In the context of this paper we focus on linguistic frames of reference, the figure ground distinction, and AI approaches to frames of reference.

In linguistics it is well recognized that schematization, a process that involves the systematic selection of certain aspects of a referent scene to represent the whole while disregarding the remaining aspects, plays a fundamental role in descriptions of spatial phenomena³⁴. More specifically, schematization is characterized by a distinction between focal or primary object and reference or secondary object. This distinction is closely related to the opposition between figure and ground in Gestalt psychology³². “The figure is a moving or conceptually movable object whose site, path, or orientation is conceived as a variable the particular value of which is the salient issue. The ground is a reference object (itself having a stationary setting within a reference frame) with respect to which the figure’s site, path, or orientation receives characterization.”³⁴

The formalism presented in this paper reflects the distinction between figure and ground in terms of the distinction between levels of granularity (the ground) and entities (the figures) which are approximated or whose location is specified with respect to the ground. In language localizing an object (determining its location) is critical and “involves processes of dividing a space into subregions or segmenting it along its contours, so as to narrow in on an object’s immediate environment.”³⁴. This corresponds exactly to the approximation (approximate location) is defined in Definition D_{Appr} .

In the AI community there is a wide range of attempts to formalize reasoning that is supported by specific kinds of frames of reference. Consider the approach to cardinal directions by Frank¹⁵, or Hernandez’ cone-shaped reference frame, or Freksa’ a vector based frame of reference. All are defined by a partition of space at a certain level of granularity and the localization of (point-like) entities within those regions. Each formalism gains its reasoning powers from (a) the partition structure of the frame of reference, (b) the location information of the entity or entities in the foreground, and (c) additional specific assumptions about the structure of the frame of reference such as: number of cells, shape of cells, the embedding of the cells in the plane. The assumption that the entities are non-extended points is often made to simplify the representation and reasoning.

In the presented framework we are able to deal with (a) and (b) in a general and unified fashion. Moreover the presented formalism applies to extended entities rather than to idealized points. The proposed stratifications allows us to do justice also to the hierarchical character of frames of references. The importance of exploiting hierarchical structures for reasoning purposes was pointed out for example in^{22,7}. Moreover, it was shown that the mereological union and intersection operations (D_+ and D_*) have corresponding pairs of operations at the level of approximations and it was demonstrated in how these operations can be used for reasoning purposes^{5,4,1}.

In the presented framework we do not have the resources to deal with (c) since this requires theories which are stronger than mereology. Ways of extending mereology by topological and morphological principles were discussed for example in^{35,8,10}. Extending the presented theory along those lines is subject of ongoing research.

8. Conclusions

In this paper a mereological theory of frames of reference was presented. We showed that

mereology can help us to understand the more basic features of frames of reference which are related to the distinction between figure and ground in gestalt psychology. The ground is seen as frame of reference in which the figure is located. We showed that mereology extended by the notion of granularity and approximation is sufficient to provide a theory for location based features of frames of reference. More complex theories, taking also into account orientation and metric properties of frames of reference can be built as extensions of the presented theory.

Their hierarchical organization is an important feature of frames of reference. Given approximations of entities within a frame of reference it is often necessary to transform them between different levels of granularity. We introduced the notion of stratified approximation to facilitate those kinds of transformations.

We showed that in the attempt to understand frames of reference there are two important aspects that need to be distinguished: (a) the study of the formal ontology of frames of reference, and (b) epistemic issues that arise when frames of reference are used in specific contexts.

The study of the first aspect helps to understand the role of frames of reference and the importance of their granular structure. The ontological grounding allows us to give a clear semantics to the notion ‘degree of parthood’ which is central to the notion of approximation. We then showed how making explicit the nature and limitations of observations and properties of the embedding of the frame of reference into the target domain (cumulativity) help us to understand the feature of epistemic vagueness and the way it affects the notion of degree of parthood.

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9. Bibliography

- [1] T. Bittner. Approximate qualitative temporal reasoning. *Annals of Mathematics and Artificial Intelligence*, 35(1–2), 2002.
- [2] T. Bittner and B. Smith. A theory of granular partitions. In M. Duckham, M. F. Goodchild, and M. F. Worboys, editors, *Foundations of Geographic Information Science*, pages 117–151. London: Taylor & Francis, 2003.
- [3] T. Bittner and B. Smith. Vague reference and approximating judgments. *Spatial Cognition and Computation*, 3(2), 2003.
- [4] T. Bittner and J. Stell. Approximate qualitative spatial reasoning. *Spatial Cognition and Computation*, 2(4):435–466, 2002.
- [5] T. Bittner and J. G. Stell. A boundary-sensitive approach to qualitative location. *Annals of Mathematics and Artificial Intelligence*, 24:93–114, 1998.
- [6] T. Bittner and J.G. Stell. Stratified rough sets and vagueness. In W. Kuhn, M. Worboys, and S. Timpf, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science. International Conference COSIT’03*, pages 286–303. Springer,

2003.

- [7] Adrijana Car and A.U. Frank. Formalization of conceptual models for gis using gofer. *Comput. Environ. and Urban Systems*, 19(2):89–98, 1995.
- [8] R. Casati and A.C. Varzi. *Holes and Other Superficialities*. MIT Press, Cambridge, Mass., 1994.
- [9] R. Casati and A.C. Varzi. The structure of spatial localization. *Philosophical Studies*, 82(2):205–239, 1995.
- [10] A.G. Cohn, B. Bennett, J. Goodday, and N. Gotts. Qualitative spatial representation and reasoning with the region connection calculus. *geoinformatica*, 1(3):1–44, 1997.
- [11] Robert DiSalle. Space and time: Inertial frames. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. 2002.
- [12] M. Donnelly. Introducing granularity-dependent qualitative distance and diameter measures in common-sense reasoning contexts. In C. Welty and B. Smith, editors, *Formal Ontology in Information Systems*, pages 321–332. ACM Press, 2001.
- [13] A. Einstein. *Relativity: The Special and the General Theory*. New York: Crown Publishers Inc., 1951.
- [14] K. Fine. Vagueness, truth and logic. *Synthese*, 30:265–300, 1975.
- [15] Andrew Frank. Qualitative spatial reasoning about distances and directions in geographic space. *Journal of Visual Languages and Computing*, 3:343–371, 1992.
- [16] C. Freksa. Using orientation information for qualitative spatial reasoning. In A.U. Frank, I. Campari, and U. Formentini, editors, *Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*. Springer-Verlag, 1992.
- [17] C. Freksa. Fuzzy systems in AI. In R. Kruse, J. Gebhardt, and R. Palm, editors, *Fuzzy Systems in Computer Science*. Vieweg, Braunschweig, 1994.
- [18] P. Hajek. *Metamathematics of Fuzzy Logic*. Trends in Logic. Kluwer, 1998.
- [19] D. Hernandez. *Qualitative Spatial Reasoning*. Springer-Verlag, 1994.
- [20] A. Herskowitz. *Language and Spatial Cognition - An Interdisciplinary Study of the Propositions in English*. Studies in natural language processing. Cambridge University Press, Cambridge [Cambridgeshire] ; New York, 1986.
- [21] J. Komorowski, L. Polkowski, and A. Skowron. Rough sets: a tutorial, 1998.
- [22] B. Kuipers. Modeling spatial knowledge. *Cognitive Science*, 2:129–154, 1978.
- [23] H.S. Leonard and N. Goodman. The calculus of individuals and its uses. *Journal of Symbolic Logic*, 5:45–55, 1940.
- [24] S. Lesniewski. Foundations of the general theory of sets. In S. Szrednicki and B Rickey, editors, *Stanislaw Lesniewsk, Collected Works*. Kluwer, Dordrecht, 1992.
- [25] S.C. Levinson. Frames of reference and Molyneux’s question: Crosslinguistic evidence. In P. Bloom, M.A. Peterson, L. Nadel, and M.F. Garrett, editors, *Language and Space*, pages 109–170. Cambridge: MIT Press, 1996.
- [26] I. Newton. *The Principia: Mathematical Principles of Natural Philosophy*. Berkeley and Los Angeles: University of California Press, 1999, 1726.
- [27] Z. Pawlak. *Rough sets : theoretical aspects of reasoning about data*. Theory and decision library. Series D, System theory, knowledge engineering, and problem solving ; v. 9. Kluwer Academic Publishers, Dordrecht ; Boston, 1991.
- [28] L. Polkowski and A. Skowron. Rough mereology: A new paradigm for approximate reasoning. *Journal of Approximate Reasoning*, 1996.
- [29] P. Rigaux and M. Scholl. Multi-scale partitions: Application to spatial and statistical databases. In M. Egenhofer and J. Herring, editors, *Advances in Spatial Databases (SSD’95)*, number 951 in Lecture Notes in Computer Science. Springer-Verlag, Berlin, 1995.
- [30] C. Schlieder. Reasoning about ordering. In A.U. Frank and W. Kuhn, editors, *Spatial Information Theory - A Theoretical basis for GIS*, volume 988 of LNCS, Semmering, Austria, 1995. Springer-Verlag.

- [31] P. Simons. *Parts, A Study in Ontology*. Clarendon Press, Oxford, 1987.
- [32] B. Smith, editor. *Foundations of Gestalt Theory*. Philosophia, Muenchen, 1985.
- [33] B. Smith and B. Brogaard. Quantum mereotopology. *Annals of Mathematics and Artificial Intelligence*, 35(1–2), 2002.
- [34] L. Talmy. How language structures space. In H. Pick and L. Acredolo, editors, *Spatial Orientation: Theory, Research, and Application*. Plenum Press, New York, NY, 1983.
- [35] A. Varzi. Parts, wholes, and part-whole relations: The prospects of mereotopology. *Data and Knowledge Engineering*, 20(3):259–86, 1996.
- [36] A. Varzi. Vagueness in geography. *Philosophy and Geography*, 2001.
- [37] Y.Y. Yao. Stratified rough sets and granular computing. In R.N. Dave and Sudkamp. T., editors, *Proceedings of the 18th International Conference of the North American Fuzzy Information Processing Society*, pages 800–804. IEEE Press, 1999.

10. Proofs

<i>TTM1</i>	$\neg(c \sqsubseteq g) \Rightarrow (\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	<i>SSP</i>
0	$c = g \Rightarrow c \sqsubseteq g$	<i>ass</i>
1	$\neg(c \sqsubseteq g)$	<i>ass</i>
2	$\neg(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	<i>ass</i>
3	$(d)\neg(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	2 <i>QN</i>
4	$\neg(c \sqsubseteq c \ \& \ \neg O \sqsubseteq cg)$	3 <i>UI</i>
5	$\neg(c \sqsubseteq c) \text{ or } \neg\neg O \sqsubseteq cg$	4 <i>DeM</i>
6	$\neg\neg O \sqsubseteq cg$	<i>ATM1, 5 DS</i>
7	$O \sqsubseteq cg$	6 <i>DN</i>
8	$c \sqsubseteq g \text{ or } g \sqsubseteq c$	7, <i>ATM5 MP</i>
9	$g \sqsubseteq c$	1, 8 <i>DS</i>
16	$(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq gd)$	15, <i>TM6(WSP)</i>
17	$f \sqsubseteq c \ \& \ \neg O \sqsubseteq gf$	
18	$f \sqsubseteq c$	17 <i>simp</i>
19	$f \sqsubseteq c \text{ or } f = c$	18 <i>add</i>
20	$f \sqsubseteq c$	19 <i>D\sqsubseteq</i>
21	$\neg O \sqsubseteq gf$	17 <i>simp</i>
22	$f \sqsubseteq c \ \& \ \neg O \sqsubseteq gf$	20, 21 <i>conj</i>
23	$(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq gd)$	22 <i>EG</i>
24	$(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq gd)$	17 – 23 <i>EI</i>
25	$(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq gd) \ \& \ \neg(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	24, 2 <i>conj</i>
26	$(\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	2 – 25 <i>IP</i>
27	$\neg(c \sqsubseteq g) \Rightarrow (\exists d)(d \sqsubseteq c \ \& \ \neg O \sqsubseteq dg)$	1 – 27 <i>CP</i>

<i>TGM1</i>	$z\Pi x \ \& \ z\Pi y \Rightarrow x = y$	
1	$z\Pi x \ \& \ z\Pi y$	<i>ass</i>
2	$z\sqsubseteq z$	<i>ATM1</i>
3	$z\sqsubseteq z \Leftrightarrow x \leq y$	1, <i>AGM1 MP</i>
4	$x \leq y$	3, 2 <i>MP</i>
5	$z\Pi y \ \& \ z\Pi x$	1 <i>comm</i>
6	$z\sqsubseteq z \Leftrightarrow y \leq x$	5, <i>AGM1 MP</i>
7	$y \leq x$	6, 5 <i>MP</i>
8	$x \leq y \ \& \ y \leq x$	4, 7 <i>conj</i>
9	$x = y$	8, <i>M2 MP</i>
10	$z\Pi x \ \& \ z\Pi y \Rightarrow x = y$	1 – 9 <i>CP</i>

<i>TGM2</i>	$x\Pi z \ \& \ y\Pi z \Rightarrow x = y$	
1	$x\Pi z \ \& \ y\Pi z$	<i>ass</i>
2	$z \leq z$	<i>M1</i>
3	$z \leq z \Leftrightarrow x\sqsubseteq y$	1, <i>AGM1 MP</i>
4	$x\sqsubseteq y$	3, 2 <i>MP</i>
5	$y\Pi z \ \& \ x\Pi z$	1 <i>comm</i>
6	$z \leq z \Leftrightarrow y\sqsubseteq x$	5, <i>AGM1 MP</i>
7	$y\sqsubseteq x$	6, 5 <i>MP</i>
8	$x\sqsubseteq y \ \& \ y\sqsubseteq x$	4, 7 <i>conj</i>
9	$x = y$	8, <i>M2 MP</i>
10	$x\Pi z \ \& \ y\Pi z \Rightarrow x = y$	1 – 9 <i>CP</i>

<i>TGM3</i>	$c = \bar{\pi}(\pi c)$	<i>TGM1</i>
1	$(\exists x)(\Pi cx)$	<i>AGM2</i>
2	Πcx	
3	$c = \bar{\pi} x$	2 <i>D$\bar{\pi}$</i>
4	$x = \pi c$	2 <i>Dπ</i>
5	$c = \bar{\pi}(\pi c)$	3, 4 <i>Id</i>
6	$c = \bar{\pi}(\pi c)$	2 – 5 <i>EI</i>

<i>TGM4</i>	$(\exists c)(\Pi cx) \Rightarrow x = \pi(\bar{\pi} x)$	<i>TGM1a</i>
1	$(\exists c)(\Pi cx)$	<i>ass</i>
2	Πcx	
3	$c = \bar{\pi} x$	2 <i>D$\bar{\pi}$</i>
4	$x = \pi c$	2 <i>Dπ</i>
5	$x = \pi(\bar{\pi} x)$	3, 4 <i>Id</i>
6	$x = \pi(\bar{\pi} x)$	2 – 5 <i>EI</i>
7	$(\exists c)(\Pi cx) \Rightarrow x = \pi(\bar{\pi} x)$	1 – 6 <i>CP</i>

<i>TGM5</i>	$((\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2)) \Rightarrow (P x_1x_2 \Leftrightarrow \bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2)$	<i>TGM5</i>
0	$(\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2)$	<i>ass</i>
1	$(\exists c)(\Pi cx_1)$	0 <i>simp</i>
1a	Πcx_1	
1b	$x_1 = \pi c$	1a D_π
1d	$c = \bar{\pi} x_1$	1a $D_{\bar{\pi}}$
2	$(\exists c)(\Pi cx_2)$	0 <i>simp</i>
2a	Πdx_2	
2b	$x_2 = \pi d$	2a D_π
2d	$d = \bar{\pi} x_2$	2a $D_{\bar{\pi}}$
2e	$\Pi cx_1 \& \Pi dx_2$	1a, 2a <i>conj</i>
3	$P x_1x_2$	<i>ass</i>
4	$P (\pi c)(\pi d)$	3, 1b, 2b <i>Id</i>
4b	$\Pi c(\pi c) \& \Pi d(\pi d)$	2e, 1b, 2b <i>Id</i>
4d	$c \sqsubseteq d \Leftrightarrow P (\pi c)(\pi d)$	4b, (AGM1 UI) <i>MP</i>
5	$c \sqsubseteq d$	4, 4d <i>MP</i>
6	$\bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2$	5, 1d, 2d <i>Id</i>
7	$\bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2$	2a – 6 <i>EI</i> , 1a – 6 <i>EI</i>
8	$P x_1x_2 \Rightarrow \bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2$	3 – 7 <i>CP</i>
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9	$\bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2$	<i>ass</i>
10	$\Pi (\bar{\pi} x_1)x_1 \& \Pi (\bar{\pi} x_2)x_2$	2e, 1d, 2d <i>Id</i>
11	$\bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2 \Leftrightarrow P x_1x_2$	10, (AGM1 UI) <i>MP</i>
12	$P x_1x_2$	9, 11 <i>MP</i>
13	$\bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2 \Rightarrow P x_1x_2$	9 – 12 <i>CP</i>
14	$P x_1x_2 \Leftrightarrow \bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2$	(8, 13 <i>conj</i>) <i>Eq</i>
15	$((\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2)) \Rightarrow$ $(P x_1x_2 \Leftrightarrow \bar{\pi} x_1 \sqsubseteq \bar{\pi} x_2)$	0 – 14 <i>CP</i>

<i>TGM6</i>	$\neg c_1 \sqsubseteq c_2 \Rightarrow (\exists z)(P z(\pi c_1) \& \neg O z(\pi c_2))$	<i>TGM6</i>
1	$(\exists x_1)(\Pi c_1x_1)$	<i>AGM2</i>
2	Πc_1x_1	
2a	$x_1 = \pi c_1$	2 D_π
3	$(\exists x_2)(\Pi c_2x_2)$	<i>AGM2</i>
4	Πc_2x_2	
4a	$x_2 = \pi c_2$	4 D_π
5	$\Pi c_1x_1 \& \Pi c_2x_2$	2, 4 <i>conj</i>
6	$c_1 \sqsubseteq c_2 \Leftrightarrow P x_1x_2$	5, (AGM1 UI) <i>MP</i>
7	$P x_1x_2 \Rightarrow c_1 \sqsubseteq c_2$	(6 <i>Eq</i>) <i>simp</i>
8	$\neg c_1 \sqsubseteq c_2$	<i>ass</i>
9	$\neg P x_1x_2$	8, 7 <i>MT</i>
10	$(\exists z)(P z x_1 \& \neg O z x_2)$	9, AM4 <i>MP</i>
11	$(\exists z)(P z(\pi c_1) \& \neg O z(\pi c_2))$	10, 2a, 4a <i>Id</i>
12	$(\exists z)(P z(\pi c_1) \& \neg O z(\pi c_2))$	4 – 12 <i>EI</i> , 2 – 12 <i>EI</i>
13	$\neg c_1 \sqsubseteq c_2 \Rightarrow (\exists z)(P z(\pi c_1) \& \neg O z(\pi c_2))$	1 – 12 <i>CP</i>

<i>TGM7</i>	$((\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2))\{\neg P x_1x_2 \Rightarrow$	<i>TGM6a</i>
	$(\exists d)(d \sqsubseteq (\bar{\pi} x_1) \& \neg O \sqsubseteq d(\bar{\pi} x_2))\}$	
1	$((\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2))$	<i>ass</i>
2	$P x_1x_2 \Leftrightarrow (\bar{\pi} x_1) \sqsubseteq (\bar{\pi} x_2)$	1, <i>TGM5 MP</i>
2a	$(\bar{\pi} x_1) \sqsubseteq (\bar{\pi} x_2) \Rightarrow P x_1x_2$	(2 <i>Eq</i>) <i>simp</i>
3	$\neg P x_1x_2$	<i>ass</i>
4	$\neg(\bar{\pi} x_1) \sqsubseteq (\bar{\pi} x_2)$	3, 2a <i>MT</i>
5	$(\exists d)(d \sqsubseteq (\bar{\pi} x_1) \& \neg O d(\bar{\pi} x_2))$	4, <i>TTM1 MP</i>
6	$\neg P x_1x_2 \Rightarrow (\exists d)(d \sqsubseteq (\bar{\pi} x_1) \& \neg O d(\bar{\pi} x_2))$	3 – 5 <i>CP</i>
7	$((\exists c)(\Pi cx_1) \& (\exists c)(\Pi cx_2)) \Rightarrow$	
	$\{\neg P x_1x_2 \Rightarrow (\exists d)(d \sqsubseteq (\bar{\pi} x_1) \& \neg O d(\bar{\pi} x_2))\}$	1 – 7 <i>CP</i>

<i>TC1</i>	$\delta \subseteq \delta$	
1	$\neg\delta \subseteq \delta$	<i>ass</i>
2	$\neg(g)(g \in \delta \Rightarrow g \in \delta)$	1 <i>D_⊆</i>
3	$(\exists g)\neg(g \in \delta \Rightarrow g \in \delta)$	1 <i>D_⊆</i>
4	$\neg(g \in \delta \Rightarrow g \in \delta)$	
5	$\neg(\neg g \in \delta \text{ or } g \in \delta)$	4 <i>Impl</i>
6	$\neg\neg g \in \delta \& \neg g \in \delta)$	5 <i>DeM</i>
7	$g \in \delta \& \neg g \in \delta)$	6 <i>DN</i>
8	$\delta \subseteq \delta$	1 – 7 <i>IP</i>

<i>TC2</i>	$\delta_1 \subseteq \delta_2 \& \delta_2 \subseteq \delta_1 \Rightarrow \delta_1 = \delta_2$	
1	$\delta_1 \subseteq \delta_2 \& \delta_2 \subseteq \delta_1$	<i>ass</i>
2	$\delta_1 \subseteq \delta_2$	1 <i>simp</i>
3	$(g)(g \in \delta_1 \Rightarrow g \in \delta_2)$	2 <i>D_⊆</i>
4	$g \in \delta_1 \Rightarrow g \in \delta_2$	3 <i>UI</i>
5	$\delta_2 \subseteq \delta_1$	1 <i>simp</i>
6	$(g)(g \in \delta_2 \Rightarrow g \in \delta_1)$	5 <i>D_⊆</i>
7	$g \in \delta_2 \Rightarrow g \in \delta_1$	6 <i>UI</i>
8	$g \in \delta_1 \Rightarrow g \in \delta_2 \& g \in \delta_2 \Rightarrow g \in \delta_1$	4, 7 <i>conj</i>
9	$g \in \delta_1 \Leftrightarrow g \in \delta_2$	8 <i>Equiv</i>
10	$(g)(g \in \delta_1 \Leftrightarrow g \in \delta_2)$	9 <i>UG</i>
11	$\delta_1 = \delta_2$	10, <i>AC1 MP</i>
12	$\delta_1 \subseteq \delta_2 \& \delta_2 \subseteq \delta_1 \Rightarrow \delta_1 = \delta_2$	1 – 12 <i>CP</i>

<i>TC3</i>	$\delta_1 \subseteq \delta_2 \ \& \ \delta_2 \subseteq \delta_3 \Rightarrow \delta_1 \subseteq \delta_3$	
1	$\delta_1 \subseteq \delta_2 \ \& \ \delta_2 \subseteq \delta_3$	<i>ass</i>
2	$\delta_1 \subseteq \delta_2$	1 <i>simp</i>
3	$(g)(g \in \delta_1 \Rightarrow g \in \delta_2)$	2 D_{\subseteq}
4	$g \in \delta_1 \Rightarrow g \in \delta_2$	3 <i>UI</i>
5	$\delta_2 \subseteq \delta_3$	1 <i>simp</i>
6	$(g)(g \in \delta_2 \Rightarrow g \in \delta_3)$	5 D_{\subseteq}
7	$g \in \delta_2 \Rightarrow g \in \delta_3$	6 <i>UI</i>
8	$g \in \delta_1 \Rightarrow g \in \delta_3$	4, 7 <i>HS</i>
9	$(g)(g \in \delta_1 \Rightarrow g \in \delta_3)$	8 <i>UG</i>
10	$\delta_1 \subseteq \delta_3$	9 D_{\subseteq}
11	$\delta_1 \subseteq \delta_2 \ \& \ \delta_2 \subseteq \delta_3 \Rightarrow \delta_1 \subseteq \delta_3$	1 – 10 <i>CP</i>

<i>TGL0</i>	$GL \ \delta \Rightarrow (\exists c)(c \in \delta)$	
1	$GL \ \delta$	<i>ass</i>
2	$\neg(\exists c)(c \in \delta)$	<i>ass</i>
3	$(c)\neg(c \in \delta)$	2 <i>QN</i>
4	$\neg(\mathfrak{R} \in \delta)$	3 <i>UI</i>
5	$(g_1)(g_2)(g_1 \in \delta \ \& \ g_2 \in \delta \ \& \ O_{\subseteq} g_1 g_2 \Rightarrow g_1 = g_2) \ \& \ (c)(\neg c \in \delta \Rightarrow (\exists g)(g \in \delta \ \& \ O_{\subseteq} cg))$	1 D_{GL}
6	$(c)(\neg c \in \delta \Rightarrow (\exists g)(g \in \delta \ \& \ O_{\subseteq} cg))$	5 <i>simp</i>
7	$\neg \mathfrak{R} \in \delta \Rightarrow (\exists g)(g \in \delta \ \& \ O_{\subseteq} \mathfrak{R}g)$	6 <i>UI</i>
8	$(\exists g)(g \in \delta \ \& \ O_{\subseteq} \mathfrak{R}g)$	4, 7 <i>MP</i>
9	$g \in \delta \ \& \ O_{\subseteq} \mathfrak{R}g$	
10	$g \in \delta$	9 <i>simp</i>
11	$(\exists c)(c \in \delta)$	10 <i>EG</i> , 9 – 11 <i>EI</i>
12	$(\exists c)(c \in \delta) \ \& \ \neg(\exists c)(c \in \delta)$	11, 2 <i>conj</i>
13	$(\exists c)(c \in \delta)$	2 – 12 <i>IP</i>
14	$GL \ \delta \Rightarrow (\exists c)(c \in \delta)$	1 – 13 <i>CP</i>

<i>TGL1</i>	$GL \delta \Rightarrow \delta \ll \delta$	
0	$GL \delta$	<i>ass</i>
1	$\neg \delta \ll \delta$	<i>ass</i>
2	$\neg(GL \delta \& GL \delta \& (c)(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g)))$	1 <i>D</i> \ll
2a	$\neg(GL \delta \& GL \delta) \text{ or } \neg(c)(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g))$	2 <i>DeM</i>
2b	$\neg GL \delta \text{ or } \neg(c)(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g))$	2b <i>taut</i>
2c	$\neg(c)(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g))$	0, 2b <i>DS</i>
3	$(\exists c)\neg(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g))$	2 <i>QN</i>
4	$\neg(c \in \delta \Rightarrow (\exists g)(g \in \delta \& c \sqsubseteq g))$	
5	$\neg(\neg c \in \delta \text{ or } (\exists g)(g \in \delta \& c \sqsubseteq g))$	4 <i>Impl</i>
6	$\neg\neg c \in \delta \& \neg(\exists g)(g \in \delta \& c \sqsubseteq g)$	5 <i>DeM</i>
7	$c \in \delta \& \neg(\exists g)(g \in \delta \& c \sqsubseteq g)$	5 <i>DeM</i>
8	$\neg(\exists g)(g \in \delta \& c \sqsubseteq g)$	7 <i>simp</i>
9	$(g)\neg(g \in \delta \& c \sqsubseteq g)$	8 <i>QN</i>
10	$\neg(c \in \delta \& c \sqsubseteq c)$	9 <i>UI</i>
11	$\neg c \in \delta \text{ or } \neg c \sqsubseteq c$	10 <i>DeM</i>
12	$\neg c \in \delta$	11, (<i>ATM1 UI</i>) <i>DS</i>
13	$c \in \delta \& \neg c \in \delta$	(7 <i>simp</i>), 12 <i>conj</i>
14	$\delta \ll \delta$	1 – 13 <i>IP</i>
15	$GL \delta \Rightarrow \delta \ll \delta$	0 – 14 <i>CP</i>

<i>TGL2</i>	$\delta_1 \ll \delta_2 \& \delta_2 \ll \delta_1 \Rightarrow \delta_1 = \delta_2$	
0	$c \sqsubseteq e \Rightarrow O_{\sqsubseteq} ce$	<i>Th</i>
1	$\delta_1 \ll \delta_2 \& \delta_2 \ll \delta_1$	<i>ass</i>
2	$\delta_1 \ll \delta_2$	1 <i>simp</i>
3	$GL \delta_1 \& GL \delta_2 \& (c)(c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \& c \sqsubseteq g))$	2 <i>D</i> \ll
3a	$(c)(c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \& c \sqsubseteq g))$	3 <i>simp</i>
3b	$GL \delta_1$	3 <i>simp</i>
3c	$(g_1)(g_2)(g_1 \in \delta_1 \& g_2 \in \delta_1 \& O_{\sqsubseteq} g_1 g_2 \Rightarrow g_1 = g_2) \& (c)(\neg c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_1 \& O_{\sqsubseteq} cg))$	3b <i>D_{GL}</i>
3d	$(g_1)(g_2)(g_1 \in \delta_1 \& g_2 \in \delta_1 \& O_{\sqsubseteq} g_1 g_2 \Rightarrow g_1 = g_2)$	3c <i>simp</i>
4	$c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \& c \sqsubseteq g)$	3a <i>UI</i>
5	$\delta_2 \ll \delta_1$	1 <i>simp</i>
6	$GL \delta_1 \& GL \delta_2 \& (c)(c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_1 \& c \sqsubseteq g))$	5 <i>D</i> \ll
6a	$(c)(c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_1 \& c \sqsubseteq g))$	6 <i>simp</i>
7	$c \in \delta_1$	<i>ass</i>
8	$(\exists g)(g \in \delta_2 \& c \sqsubseteq g)$	7, 4 <i>MP</i>
9	$g \in \delta_2 \& c \sqsubseteq g$	
10	$g \in \delta_2$	9 <i>simp</i>
11	$g \in \delta_2 \Rightarrow (\exists e)(e \in \delta_1 \& g \sqsubseteq e)$	6a <i>UI</i>
12	$(\exists e)(e \in \delta_1 \& g \sqsubseteq e)$	10, 11 <i>MP</i>
...		

...	<i>TGL2</i>	
13	$e \in \delta_1 \ \& \ g \sqsubseteq e$	
14	$c \sqsubseteq g \ \& \ g \sqsubseteq e$	(9 <i>simp</i>), (13 <i>simp</i>) <i>conj</i>
15	$c \sqsubseteq e$	14, <i>ATM3 MP</i>
16	$O_{\sqsubseteq} ce$	15, 0 <i>MP</i>
17	$c \in \delta_1 \ \& \ e \in \delta_1 \ \& \ O_{\sqsubseteq} ce$	7, 13, 16 <i>conj</i>
18	$c = e$	17, (3 <i>d UI</i>) <i>MP</i>
19	$c \sqsubseteq g \ \& \ g \sqsubseteq e$	14
20	$c \sqsubseteq g \ \& \ g \sqsubseteq c$	19, 18 <i>Id</i>
21	$c = g$	20, <i>ATM2 MP</i>
22	$c \in \delta_2$	10, 21 <i>Id</i>
23	$c \in \delta_1 \Rightarrow c \in \delta_2$	7 – 22 <i>CP</i>
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24	$c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_1 \ \& \ c \sqsubseteq g)$	6 <i>a UI</i>
25	$c \in \delta_2$	<i>ass</i>
26	$(\exists f)(f \in \delta_1 \ \& \ c \sqsubseteq f)$	24, 25 <i>MP</i>
27	$f \in \delta_1 \ \& \ c \sqsubseteq f$	
28	$f \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ f \sqsubseteq g)$	3 <i>a UI</i>
29	$(\exists h)(h \in \delta_2 \ \& \ f \sqsubseteq h)$	27, 28 <i>MP</i>
30	$h \in \delta_2 \ \& \ f \sqsubseteq h$	
31	$c \sqsubseteq f \ \& \ f \sqsubseteq h$	(27 <i>simp</i>), (30 <i>simp</i>) <i>conj</i>
32	$c \sqsubseteq h$	31, <i>AM3 MP</i>
33	$O_{\sqsubseteq} ch$	32, 0 <i>MP</i>
34	$c \in \delta_2 \ \& \ h \in \delta_2 \ \& \ O_{\sqsubseteq} ch$	25, (30 <i>simp</i>), 33 <i>conj</i>
35	$GL \ \delta_2$	3 <i>simp</i>
36	$(g_1)(g_2)(g_1 \in \delta_2 \ \& \ g_2 \in \delta_2 \ \& \ O_{\sqsubseteq} g_1g_2 \Rightarrow g_1 = g_2) \ \& \ (c)(\neg c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ O_{\sqsubseteq} cg))$	35 <i>D_{GL}</i>
37	$(g_1)(g_2)(g_1 \in \delta_2 \ \& \ g_2 \in \delta_2 \ \& \ O_{\sqsubseteq} g_1g_2 \Rightarrow g_1 = g_2)$	36 <i>simp</i>
38	$c = h$	34, (37 <i>UI</i>) <i>MP</i>
39	$c \sqsubseteq f \ \& \ f \sqsubseteq h$	31
40	$c \sqsubseteq f \ \& \ f \sqsubseteq c$	39, 38 <i>Id</i>
41	$c = f$	40, <i>ATM2 MP</i>
42	$f \in \delta_1$	27 <i>simp</i>
43	$c \in \delta_1$	41, 42 <i>Id</i>
43	$c \in \delta_2 \Rightarrow c \in \delta_1$	25 – 43 <i>CP</i>
44	$c \in \delta_1 \Rightarrow c \in \delta_2 \ \& \ c \in \delta_2 \Rightarrow c \in \delta_1$	23, 24 <i>conj</i>
45	$c \in \delta_1 \Leftrightarrow c \in \delta_2$	44 <i>Eqiv</i>
46	$(c)(c \in \delta_1 \Leftrightarrow c \in \delta_2)$	45 <i>UG</i>
47	$\delta_1 = \delta_2$	46, <i>AC1 MP</i>
48	$\delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_1 \Rightarrow \delta_1 = \delta_2$	1 – 47 <i>CP</i>

<i>TGL3</i>	$\delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_3 \Rightarrow \delta_1 \ll \delta_3$	
1	$\delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_3$	<i>ass</i>
2	$\delta_1 \ll \delta_2$	1 <i>simp</i>
3	<i>GL</i> $\delta_1 \ \& \ \delta_2 \ \& \ (c)(c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ c \sqsubseteq g))$	2 D_{\ll}
4	$(c)(c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ c \sqsubseteq g))$	3 <i>simp</i>
5	$\delta_2 \ll \delta_3$	1 <i>simp</i>
6	<i>GL</i> $\delta_2 \ \& \ \delta_3 \ \& \ (c)(c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_3 \ \& \ c \sqsubseteq g))$	5 D_{\ll}
7	$(c)(c \in \delta_2 \Rightarrow (\exists g)(g \in \delta_3 \ \& \ c \sqsubseteq g))$	6 <i>simp</i>
8	$c \in \delta_1$	<i>ass</i>
9	$c \in \delta_1 \Rightarrow (\exists g)(g \in \delta_2 \ \& \ c \sqsubseteq g)$	4 <i>UI</i>
10	$(\exists g)(g \in \delta_2 \ \& \ c \sqsubseteq g)$	8, 9 <i>MP</i>
11	$g \in \delta_2 \ \& \ c \sqsubseteq g$	
12	$g \in \delta_2$	11 <i>simp</i>
13	$g \in \delta_2 \Rightarrow (\exists e)(e \in \delta_3 \ \& \ g \sqsubseteq e)$	7 <i>UI</i>
14	$(\exists e)(e \in \delta_3 \ \& \ g \sqsubseteq e)$	12, 13 <i>MP</i>
15	$e \in \delta_3 \ \& \ g \sqsubseteq e$	
16	$c \sqsubseteq g \ \& \ g \sqsubseteq e$	(11 <i>simp</i>), (15 <i>simp</i>) <i>conj</i>
17	$c \sqsubseteq e$	16, <i>ATM3 MP</i>
18	$e \in \delta_3 \ \& \ c \sqsubseteq e$	(15 <i>simp</i>), 17 <i>conj</i>
19	$(\exists e)(e \in \delta_3 \ \& \ c \sqsubseteq e)$	18 <i>EG</i> , 15 – 19 <i>EI</i> , 11 – 19 <i>EI</i>
20	$c \in \delta_1 \Rightarrow (\exists e)(e \in \delta_3 \ \& \ c \sqsubseteq e)$	8 – 19 <i>CP</i>
21	$(c)(c \in \delta_1 \Rightarrow (\exists e)(e \in \delta_3 \ \& \ c \sqsubseteq e))$	20 <i>UG</i>
22	<i>GL</i> $\delta_1 \ \& \ \delta_3 \ \& \ (c)(c \in \delta_1 \Rightarrow (\exists e)(e \in \delta_3 \ \& \ c \sqsubseteq e))$	(3 <i>simp</i>), (6 <i>simp</i>), 21 <i>conj</i>
23	$\delta_1 \ll \delta_3$	22 D_{\ll}
24	$\delta_1 \ll \delta_2 \ \& \ \delta_2 \ll \delta_3 \Rightarrow \delta_1 \ll \delta_3$	1 – 23 <i>CP</i>
<i>TGL4</i>	$l_{ij} \ c = d_1 \ \& \ l_{ij} \ c = d_2 \Rightarrow d_1 = d_2$	
0	$\delta_i \ll \delta_j \Rightarrow \text{GL } \delta_i$	<i>Th</i>
0a	$\delta_i \ll \delta_j \Rightarrow \text{GL } \delta_j$	<i>Th</i>
1	$l_{ij} \ c = d_1 \ \& \ l_{ij} \ c = d_2$	<i>ass</i>
2	$\delta_i \ll \delta_j \ \& \ c \in \delta_i \ \& \ d_1 \in \delta_j \ \& \ c \sqsubseteq d_1$	(1 <i>simp</i>) D_ℓ
2a	$c \in \delta_i \ \& \ d_1 \in \delta_j \ \& \ c \sqsubseteq d_1$	2 <i>simp</i>
2b	<i>GL</i> δ_j	(2 <i>simp</i>), 0a <i>MP</i>
2c	$(g_1)(g_2)(g_1 \in \delta_j \ \& \ g_2 \in \delta_j \ \& \ O_{\sqsubseteq} g_1 g_2 \Rightarrow g_1 = g_2) \ \& \ (c)(\neg c \in \delta_j \Rightarrow (\exists g)(g \in \delta_j \ \& \ O_{\sqsubseteq} c g))$	2b D_{GL}
2d	$(g_1)(g_2)(g_1 \in \delta_j \ \& \ g_2 \in \delta_j \ \& \ O_{\sqsubseteq} g_1 g_2 \Rightarrow g_1 = g_2)$	2c <i>simp</i>
3	$\delta_i \ll \delta_j \ \& \ c \in \delta_i \ \& \ d_2 \in \delta_j \ \& \ c \sqsubseteq d_2$	(1 <i>simp</i>) D_ℓ
3a	$c \in \delta_i \ \& \ d_2 \in \delta_j \ \& \ c \sqsubseteq d_2$	3 <i>simp</i>
4	$c \sqsubseteq d_1 \ \& \ c \sqsubseteq d_2$	(2a <i>simp</i>), (3a <i>simp</i>) <i>conj</i>
5	$O_{\sqsubseteq} d_1 d_2$	4 D_O
6	$d_1 \in \delta_j \ \& \ d_2 \in \delta_j \ \& \ O_{\sqsubseteq} d_1 d_2$	(2 <i>simp</i>), (2 <i>simp</i>), 5 <i>conj</i>
7	$d_1 \in \delta_j \ \& \ d_2 \in \delta_j \ \& \ O_{\sqsubseteq} d_1 d_2 \Rightarrow d_1 = d_2$	2d <i>UI</i>
8	$d_1 = d_2$	7, 6 <i>MP</i>
9	$l_{ij} \ c = d_1 \ \& \ l_{ij} \ c = d_2 \Rightarrow d_1 = d_2$	1 – 8 <i>CP</i>

<i>TGL5</i>	$(\delta_i \ll \delta_j \ \& \ c \in \delta_i) \Rightarrow (\exists d)(d = \ell_{ij} \ c)$	
1	$(\delta_i \ll \delta_j \ \& \ c \in \delta_i)$	<i>ass</i>
2	$\delta_i \ll \delta_j$	1 <i>simp</i>
3	$GL \ \delta_i \ \& \ GL \ \delta_j \ \& \ (c)(c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g))$	2 D_{\ll}
3a	$(c)(c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g))$	3 <i>simp</i>
4	$c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g)$	3a <i>UI</i>
5	$c \in \delta_i$	1 <i>simp</i>
6	$(\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g)$	4,5 <i>MP</i>
7	$d \in \delta_j \ \& \ c \sqsubseteq d$	
8	$\delta_i \ll \delta_j \ \& \ c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d$	2,5,7 <i>conj</i>
9	$\ell_{ij} \ c = d$	8 D_{ℓ}
10	$(\exists d)(\ell_{ij} \ c = d)$	9 <i>EG</i>
11	$(\exists d)(\ell_{ij} \ c = d)$	7 – 10 <i>EI</i>
12	$(\delta_i \ll \delta_j \ \& \ c \in \delta_i) \Rightarrow (\exists d)(\ell_{ij} \ c = d)$	1 – 11 <i>CP</i>

<i>TGL6</i>	$(\delta_i \ll \delta_j \ \& \ d \in \delta_j) \Rightarrow (\exists c)(d = \ell_{ij} \ c)$	
0	$O_{\sqsubseteq} \ cd \Rightarrow O_{\sqsubseteq} \ dc$	<i>Th</i>
0a	$c \sqsubseteq d \ \& \ O_{\sqsubseteq} \ cg \Rightarrow O_{\sqsubseteq} \ dg$	<i>Th</i>
1	$(\delta_i \ll \delta_j \ \& \ d \in \delta_j)$	<i>ass</i>
2	$\delta_i \ll \delta_j$	1 <i>simp</i>
3	$GL \ \delta_i \ \& \ GL \ \delta_j \ \& \ (c)(c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g))$	2 D_{\ll}
4	$GL \ \delta_i$	3 <i>simp</i>
5	$(g_1)(g_2)(g_1 \in \delta_i \ \& \ g_2 \in \delta_i \ \& \ O_{\sqsubseteq} \ g_1 g_2 \Rightarrow g_1 = g_2) \ \& \ (c)(\neg c \in \delta_i \Rightarrow (\exists g)(g \in \delta_i \ \& \ O_{\sqsubseteq} \ cg))$	4 D_{GL}
6	$d \in \delta_i \ \text{or} \ \neg d \in \delta_i$	<i>ExMid</i>
7	$d \in \delta_i$	<i>ass</i>
8	$d \in \delta_i \ \& \ d \in \delta_j \ \& \ d \sqsubseteq d$	(1 <i>simp</i>), 7, <i>ATM1 conj</i>
9	$(\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	3 <i>EG</i>
10	$d \in \delta_i \Rightarrow (\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	7 – 9 <i>CP</i>
11	$\neg d \in \delta_i$	<i>ass</i>
12	$\neg d \in \delta_i \Rightarrow (\exists c)(c \in \delta_i \ \& \ O_{\sqsubseteq} \ dc)$	(5 <i>simp</i>) <i>UI</i>
13	$(\exists c)(c \in \delta_i \ \& \ O_{\sqsubseteq} \ dc)$	11,12 <i>MP</i>
14	$c \in \delta_i \ \& \ O_{\sqsubseteq} \ dc$	
15	$O_{\sqsubseteq} \ dc$	14 <i>simp</i>
16	$O_{\sqsubseteq} \ cd$	15,0 <i>MP</i>
19	$c \in \delta_i$	14 <i>simp</i>
20	$c \in \delta_i \Rightarrow (\exists h)(h \in \delta_j \ \& \ c \sqsubseteq h)$	3 <i>simp</i>
...		

...	<i>TGL6</i>	
21	$(\exists h)(h \in \delta_j \ \& \ c \sqsubseteq h)$	19, 20 <i>MP</i>
22	$h \in \delta_j \ \& \ c \sqsubseteq h$	
23	$c \sqsubseteq h \ \& \ O \sqsubseteq cd$	(22 <i>simp</i>), 16 <i>conj</i>
24	$O \sqsubseteq hd$	23, 1a <i>MP</i>
25	$h \in \delta_j \ \& \ d \in \delta_j \ \& \ O \sqsubseteq hd$	(22 <i>simp</i>), (1 <i>simp</i>), 24 <i>conj</i>
26	<i>GL</i> δ_j	3 <i>simp</i>
27	$(h)(d)(h \in \delta_j \ \& \ d \in \delta_j \ \& \ O \sqsubseteq hd \Rightarrow h = d) \ \& \ (c)(\neg c \in \delta_j \Rightarrow (\exists g)(g \in \delta_j \ \& \ O \sqsubseteq cg))$	26 <i>D_{GL}</i>
28	$h \in \delta_j \ \& \ d \in \delta_i \ \& \ O \sqsubseteq hd \Rightarrow h = d$	(27 <i>simp</i>) <i>UI</i>
29	$h = d$	25, 28 <i>MP</i>
30	$c \sqsubseteq h$	23 <i>simp</i>
31	$c \sqsubseteq d$	29, 30 <i>Id</i>
32	$c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d$	19, (1 <i>simp</i>), 31 <i>conj</i>
33	$(\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	32 <i>EG</i>
34	$(\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	14 – 33 <i>EI</i>
35	$\neg d \in \delta_i \Rightarrow (\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	11 – 34 <i>CP</i>
36	$(d \in \delta_i \Rightarrow (\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)) \ \& \ (\neg d \in \delta_i \Rightarrow (\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d))$	10, 35 <i>conj</i>
37	$(\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d) \ \& \ (\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	6, 37 <i>CD</i>
38	$(\exists c)(c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d)$	37 <i>simp</i>
38a	$c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d$	
38b	$\delta_i \ll \delta_j \ \& \ c \in \delta_i \ \& \ d \in \delta_j \ \& \ c \sqsubseteq d$	2, 38a <i>conj</i>
38c	$d = l_{ij} c$	38b <i>D_l</i>
39	$(\exists c)(d = l_{ij} c)$	38v <i>EG</i>
40	$(\delta_i \ll \delta_j \ \& \ d \in \delta_j) \Rightarrow (\exists c)(d = l_{ij} c)$	1 – 40 <i>CP</i>

<i>TGL7</i>	$\delta_i \ll \delta_j \ \& \ Com \ \delta_i \Rightarrow Com \ \delta_j$	
0	$c \sqsubseteq g \Rightarrow (\pi c) \leq (\pi g)$	<i>Th</i>
0a	$x \leq y \ \& \ O \ xz \Rightarrow O \ yz$	<i>Th</i>
0b	$g \sqsubseteq \mathfrak{R}$	<i>Th</i>
1	$\delta_i \ll \delta_j \ \& \ Com \ \delta_i$	<i>ass</i>
2	$\delta_i \ll \delta_j$	1 <i>simp</i>
3	$GL \ \delta_i \ \& \ GL \ \delta_j \ \& \ (c)(c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g))$	2 D_{\ll}
4	$Com \ \delta_i$	1 <i>simp</i>
5	$(w)(O \ w(\pi \mathfrak{R}) \Leftrightarrow (\exists c)(c \in \delta_i \ \& \ O \ w(\pi c)))$	4 D_{Com}
6	$O \ w(\pi \mathfrak{R})$	<i>ass</i>
7	$(\exists c)(c \in \delta_i \ \& \ O \ w(\pi c))$	5, 6 <i>MP</i>
8	$c \in \delta_i \ \& \ O \ w(\pi c)$	
9	$c \in \delta_i$	8 <i>simp</i>
10	$c \in \delta_i \Rightarrow (\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g)$	3 <i>simp UI</i>
11	$(\exists g)(g \in \delta_j \ \& \ c \sqsubseteq g)$	9, 10 <i>MP</i>
12	$g \in \delta_j \ \& \ c \sqsubseteq g$	
13	$c \sqsubseteq g$	12 <i>simp</i>
14	$(\pi c) \leq (\pi g)$	13, 0 <i>MP</i>
15	$(\pi c) \leq (\pi g) \ \& \ O \ w(\pi c)$	14, (8 <i>simp</i>) <i>conj</i>
16	$O \ w(\pi g)$	15, 0a <i>MP</i>
17	$g \in \delta_j \ \& \ O \ w(\pi g)$	(12 <i>simp</i>), 16 <i>conj</i>
18	$(\exists g)(g \in \delta_j \ \& \ O \ w(\pi g))$	17 <i>EG</i> , 12 – 18 <i>EI</i> , 8 – 18 <i>EI</i>
19	$O \ w(\pi \mathfrak{R}) \Rightarrow (\exists g)(g \in \delta_j \ \& \ O \ w(\pi g))$	6 – 18 <i>CP</i>
20	$(\exists g)(g \in \delta_j \ \& \ O \ w(\pi g))$	<i>ass</i>
21	$g \in \delta_j \ \& \ O \ w(\pi g)$	
22	$O \ w(\pi g)$	21 <i>simp</i>
23	$(\pi g) \sqsubseteq (\pi \mathfrak{R})$	0b, 0 <i>MP</i>
24	$(\pi g) \sqsubseteq (\pi \mathfrak{R}) \ \& \ O \ w(\pi g)$	23, 22 <i>conj</i>
25	$O \ w(\pi \mathfrak{R})$	24, 0a <i>MP</i> , 21 – 25 <i>EI</i>
26	$(\exists g)(g \in \delta_j \ \& \ O \ w(\pi g)) \Rightarrow O \ w(\pi \mathfrak{R})$	20 – 24 <i>CP</i>
27	$O \ w(\pi \mathfrak{R}) \Leftrightarrow (\exists g)(g \in \delta_j \ \& \ O \ w(\pi g))$	(19, 26 <i>conj</i>) <i>Equiv</i>
28	$(w)(O \ w(\pi \mathfrak{R}) \Leftrightarrow (\exists g)(g \in \delta_j \ \& \ O \ w(\pi g)))$	27 <i>UG</i>
29	$Com \ \delta_j$	28 D_{Com}
30	$\delta_i \ll \delta_j \ \& \ Com \ \delta_i \Rightarrow Com \ \delta_j$	1 – 29 <i>CP</i>

TRA1	$(x)(y)(DR\ xy\ or\ NSO\ xy\ or\ P\ yx)$	
2	$\neg(x)(y)(DR\ xy\ or\ NSO\ xy\ or\ P\ yx)$	<i>ass</i>
3	$(\exists x)\neg(y)(DR\ xy\ or\ NSO\ xy\ or\ P\ yx)$	2 <i>QN</i>
4	$(\exists x)(\exists y)\neg(DR\ xy\ or\ NSO\ xy\ or\ P\ yx)$	3 <i>QN</i>
5	$\neg(DR\ xy\ or\ (NSO\ xy\ or\ P\ yx))$	
6	$\neg DR\ xy\ \&\ \neg(NSO\ xy\ or\ P\ yx)$	5 <i>DeM</i>
7	$\neg(NSO\ xy\ or\ P\ yx)$	6 <i>simp</i>
8	$\neg NSO\ xy\ \&\ \neg P\ yx$	7 <i>DeM</i>
9	$\neg NSO\ xy$	8 <i>simp</i>
10	$\neg(PO\ xy\ or\ PP\ xy)$	9 <i>DNSO</i>
11	$\neg PO\ xy\ \&\ \neg PP\ xy$	10 <i>DeM</i>
12	$\neg PO\ xy$	11 <i>simp</i>
13	$\neg(O\ xy\ \&\ (\neg P\ xy\ \&\ \neg P\ yx))$	12 <i>DPO</i>
14	$\neg O\ xy\ or\ \neg(\neg P\ xy\ \&\ \neg P\ yx)$	13 <i>DeM</i>
15	$\neg DR\ xy$	6 <i>simp</i>
16	$O\ xy$	(15 <i>DDR</i>) <i>DN</i>
17	$\neg(\neg P\ xy\ \&\ \neg P\ yx)$	14, 16 <i>DS</i>
18	$\neg\neg P\ xy\ or\ \neg\neg P\ yx$	17 <i>DeM</i>
19	$P\ xy\ or\ P\ yx$	18 <i>DN</i>
20	$\neg P\ yx$	8 <i>simp</i>
21	$P\ xy$	19, 20 <i>DS</i>
22	$PP\ xy\ or\ x = y$	21, <i>Th MP</i>
23	$\neg PP\ xy$	11 <i>simp</i>
24	$x = y$	22, 23 <i>DS</i>
25	$\neg P\ xx$	20, 24 <i>Id</i>
26	$P\ xx$	<i>M1</i>
27	$P\ xx\ \&\ \neg P\ xx$	25, 26 <i>conj</i>
28	$(\exists x)(P\ xx\ \&\ \neg P\ xx)$	27 <i>EG</i>
29	$(\exists x)(P\ xx\ \&\ \neg P\ xx)$	6 – 28 <i>EI</i> , 5 – 28 <i>EI</i>
30	$(x)(y)(DR\ xy\ or\ NSO\ xy\ or\ P\ yx)$	2 – 29 <i>IP</i>

TRA2	$\neg(DR\ xy \ \& \ NSO\ xy)$	
1	$DR\ xy \ \& \ NSO\ xy$	<i>ass</i>
2	$DR\ xy$	1 <i>simp</i>
3	$\neg O\ xy$	2 D_{DR}
4	$\neg(\exists z)(P\ zx \ \& \ P\ zy)$	3 D_O
5	$(z)\neg(P\ zx \ \& \ P\ zy)$	4 QN
6	$\neg(P\ xx \ \& \ P\ xy)$	5 UI
7	$\neg P\ xx \ \text{or} \ \neg P\ xy$	6 DeM
8	$\neg P\ xy$	AM1, 7 DS
9	$\neg(PP\ xy \ \text{or} \ x = y)$	8 D_{PP}
10	$\neg PP\ xy \ \& \ \neg x = y$	9 DeM
11	$\neg PP\ xy$	
12	$NSO\ xy$	1 <i>simp</i>
13	$PO\ xy \ \text{or} \ PP\ xy$	12 D_{NSO}
14	$PO\ xy$	11, 12 DS
15	$O\ xy \ \& \ \neg P\ xy \ \& \ \neg P\ yx$	14 D_{PO}
16	$O\ xy$	15 <i>simp</i>
17	$O\ xy \ \& \ \neg O\ xy$	16, 3 <i>conj</i>
18	$\neg(DR\ xy \ \& \ NSO\ xy)$	1 – 17 IP

TRA3	$\neg(DR\ xy \ \& \ P\ yx)$	
0	$P\ xy \ \Rightarrow \ O\ xy$	<i>Th</i>
0a	$O\ xy \ \Rightarrow \ O\ xy$	<i>Th</i>
1	$DR\ xy \ \& \ P\ yx$	<i>ass</i>
2	$DR\ xy$	1 <i>simp</i>
3	$\neg O\ xy$	2 D_{DR}
4	$P\ yx$	1 <i>simp</i>
5	$O\ yx$	4, 0 MP
6	$O\ xy$	5, 1a MP
7	$O\ xy \ \& \ \neg O\ xy$	6, 3 <i>conj</i>
8	$\neg(DR\ xy \ \& \ P\ yx)$	1 – 7 IP

TRA4	$\neg(NSO\ xy \ \& \ P\ yx)$	
0	$P\ yx \ \Rightarrow \ \neg PP\ xy$	<i>Th</i>
1	$NSO\ xy \ \& \ P\ yx$	<i>ass</i>
2	$NSO\ xy$	1 <i>simp</i>
3	$PO\ xy \ \text{or} \ PP\ xy$	2 D_{NSO}
4	$P\ yx$	1 <i>simp</i>
5	$\neg PP\ xy$	4, 0 MP
6	$PO\ xy$	5, 3 DS
7	$O\ xy \ \& \ \neg P\ xy \ \& \ \neg P\ yx$	6 D_{PO}
8	$\neg P\ yx$	7 <i>simp</i>
9	$P\ yx \ \& \ \neg P\ yx$	4, 8 <i>conj</i>
10	$\neg(NSO\ xy \ \& \ P\ yx)$	1 – 9 IP

<i>TRA7</i>	$Appr xX^\delta \ \& \ Appr xY^\delta \Rightarrow X^\delta = Y^\delta$	
1	$Appr xX^\delta \ \& \ Appr xY^\delta$	<i>ass</i>
2	$Appr xX^\delta$	1 <i>simp</i>
3	$c \in \delta \Rightarrow ((Xc) = fo \Leftrightarrow (\pi c) \leq x \ \& \ (Xc) = po \Leftrightarrow NSO x(\pi c) \ \& \ (Xc) = no \Leftrightarrow DR x(\pi c))$	(2 <i>D_{Appr}</i>) <i>UI</i>
4	$Appr xY^\delta$	1 <i>simp</i>
5	$c \in \delta \Rightarrow ((Yc) = fo \Leftrightarrow (\pi c) \leq x \ \& \ (Yc) = po \Leftrightarrow NSO x(\pi c) \ \& \ (Yc) = no \Leftrightarrow DR x(\pi c))$	(4 <i>D_{Appr}</i>) <i>UI</i>
6	$c \in \delta$	<i>ass</i>
7	$(Xc) = fo \Leftrightarrow (\pi c) \leq x \ \& \ (Xc) = po \Leftrightarrow NSO x(\pi c) \ \& \ (Xc) = no \Leftrightarrow DR x(\pi c)$	6, 3 <i>MP</i>
8	$((Yc) = fo \Leftrightarrow (\pi c) \leq x \ \& \ (Yc) = po \Leftrightarrow NSO x(\pi c) \ \& \ (Yc) = no \Leftrightarrow DR x(\pi c))$	6, 5 <i>MP</i>
9	$Xc = fo$	<i>ass</i>
10	$(Xc) = fo \Leftrightarrow (\pi c) \leq x$	7 <i>simp</i>
11	$(\pi c) \leq x$	9, 10 <i>MP</i>
12	$(Yc) = fo \Leftrightarrow (\pi c) \leq x$	8 <i>simp</i>
13	$(Yc) = fo$	11, 12 <i>MP</i>
14	$Xc = fo \Rightarrow (Yc) = fo$	9 – 13 <i>CP</i>
15	$(Yc) = fo$	<i>ass</i>
16	$(\pi c) \leq x$	15, 12 <i>MP</i>
17	$(Xc) = fo$	16, 10 <i>MP</i>
18	$(Yc) = fo \Rightarrow (Xc) = fo$	15 – 17 <i>CP</i>
19	$Xc = fo \Rightarrow (Yc) = fo \ \& \ (Yc) = fo \Rightarrow (Xc) = fo$	14, 18 <i>conj</i>
20	$Xc = fo \Leftrightarrow (Yc) = fo$	19 <i>Equiv</i>
...		
21	$Xc = po \Leftrightarrow (Yc) = po$	like 9-19
...		
22	$Xc = no \Leftrightarrow (Yc) = no$	like 9-19
23	$Xc = fo \Leftrightarrow (Yc) = fo \ \& \ Xc = po \Leftrightarrow (Yc) = po \ \& \ Xc = no \Leftrightarrow (Yc) = no$	21, 22, 23 <i>conj</i>
24	$c \in \delta \Rightarrow (Xc = fo \Leftrightarrow (Yc) = fo \ \& \ Xc = po \Leftrightarrow (Yc) = po \ \& \ Xc = no \Leftrightarrow (Yc) = no)$	6 – 23 <i>CP</i>
25	$X_\delta = Y_\delta$	24, <i>ARA2 MP</i>
26	$Appr xX^\delta \ \& \ Appr xY^\delta \Rightarrow X^\delta = Y^\delta$	1 – 25 <i>CP</i>

<i>TRA8</i>	$x \sim_\delta x$	
1	$(\exists X^\delta)(Appr xX^\delta \ \& \ \neg Null \ \delta)$	<i>TRA5</i>
2	$Appr xX^\delta \ \& \ \neg Null \ \delta$	
2a	$Appr xX^\delta$	2 <i>simp</i>
3	$X^\delta = appr x$	2a <i>D_{appr}</i>
4	$appr x = appr x$	3 <i>Id</i>
5	$x \sim_\delta x$	4 <i>D_{~δ}</i>

<i>TRA9</i>	$x \sim_\delta y \Rightarrow y \sim_\delta x$	
0	$x = y \Rightarrow y = x$	<i>Th</i>
1	$x \sim_\delta y$	<i>ass</i>
2	$\text{appr } x = \text{appr } y$	1 D_{\sim_δ}
3	$\text{appr } y = \text{appr } x$	2, 0 <i>MP</i>
4	$y \sim_\delta x$	3 D_{\sim_δ}
5	$x \sim_\delta y \Rightarrow y \sim_\delta x$	1 – 4 <i>CP</i>

<i>TRA10</i>	$x \sim_\delta y \ \& \ y \sim_\delta z \Rightarrow x \sim_\delta z$	
0	$x = y \ \& \ y = z \Rightarrow x = z$	<i>Th</i>
1	$x \sim_\delta y \ \& \ y \sim_\delta z$	<i>ass</i>
2	$\text{appr } x = \text{appr } y \ \& \ \text{appr } y = \text{appr } z$	1 D_{\sim_δ}
3	$\text{appr } x = \text{appr } z$	2, 0 <i>MP</i>
4	$x \sim_\delta z$	3 D_{\sim_δ}
5	$x \sim_\delta y \ \& \ y \sim_\delta z \Rightarrow x \sim_\delta z$	1 – 4 <i>CP</i>

<i>TSA1a</i>	$\text{Com } \delta_i \ \& \ \delta_i \ll \delta_j \Rightarrow$ $(c)[(c \in \delta_j \ \& \ (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) d = \text{fo})) \Rightarrow (\text{appr}_{\delta_j} x) c = \text{fo}]$	
0	$d \sqsubseteq c \Rightarrow (\pi d) \leq (\pi c)$	<i>Th</i>
0a	$x \leq y \ \& \ O \ xz \Rightarrow O \ yz$	<i>Th</i>
0b	$O (\pi c)(\pi g) \Rightarrow O_{\sqsubseteq} cg$	<i>Th</i>
0c	$\delta_i \ll \delta_j \ \& \ g \in \delta_i \ \& \ c \in \delta_j \ \& \ O_{\sqsubseteq} cg \Rightarrow g \sqsubseteq c$	<i>Th</i>
1	$\text{Com } \delta_i \ \& \ \delta_i \ll \delta_j$	<i>ass</i>
2	$c \in \delta_j \ \& \ (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) d = \text{fo})$	<i>ass</i>
3	$\delta_i \ll \delta_j \ \& \ c \in \delta_j$	(1 <i>simp</i>), (2 <i>simp</i>) <i>conj</i>
4	$(d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) d = \text{fo})$	2 <i>simp</i>
5	$\neg(\pi c) \leq x$	<i>ass</i>
6	$(\exists z)(z \leq (\pi c) \ \& \ \neg O \ zx)$	5, <i>SSP MP</i>
7	$z \leq (\pi c) \ \& \ \neg O \ zx$	
8	$(w)(\exists g)(g \in \delta_i \ \& \ O \ w(\pi g))$	(1 <i>simp</i>), <i>TRA6 MP</i>
9	$(\exists g)(g \in \delta_i \ \& \ O \ z(\pi g))$	8 <i>UI</i>
10	$g \in \delta_i \ \& \ O \ z(\pi g)$	
11	$z \leq (\pi c) \ \& \ O \ z(\pi g)$	(7 <i>simp</i>), (10 <i>simp</i>) <i>conj</i>
12	$O (\pi c)(\pi g)$	11, 0a <i>MP</i>
13	$O_{\sqsubseteq} cg$	12, 1b <i>MP</i>
14	$\delta_i \ll \delta_j \ \& \ g \in \delta_i \ \& \ c \in \delta_j \ \& \ O_{\sqsubseteq} cg$	3, (10 <i>simp</i>), 13 <i>conj</i>
15	$g \sqsubseteq c$	14, 0c <i>MP</i>
16	$g \in \delta_i \ \& \ g \sqsubseteq c$	(10 <i>simp</i>), 15 <i>conj</i>
17	$g \in \delta_i \ \& \ g \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) g = \text{fo}$	4 <i>UI</i>
18	$(\text{appr}_{\delta_i} x) g = \text{fo}$	16, 17 <i>MP</i>
19	$\text{Appr } x X^{\delta_i} \ \& \ X^{\delta_i} g = \text{fo}$	18 D_{appr}
20	$\text{Appr } x X^{\delta_i}$	19 <i>simp</i>
...		

...	<i>TSA1a</i>	
21	$g \in \delta_i \Rightarrow ((X^{\delta_i} g) = \text{fo} \Leftrightarrow (\pi g) \leq x \ \& \ (X^{\delta_i} g) = \text{po} \Leftrightarrow \text{NSO } x(\pi g) \ \& \ (X^{\delta_i} g) = \text{no} \Leftrightarrow \text{DR } x(\pi g))$	(20 <i>D_{Appr}</i>) <i>UI</i>
22	$g \in \delta_i$	10 <i>simp</i>
23	$(X^{\delta_i} g) = \text{fo} \Leftrightarrow (\pi g) \leq x \ \& \ (X^{\delta_i} g) = \text{po} \Leftrightarrow \text{NSO } x(\pi g) \ \& \ (X^{\delta_i} g) = \text{no} \Leftrightarrow \text{DR } x(\pi g)$	21, 22 <i>MP</i>
24	$(X^{\delta_i} g) = \text{fo} \Leftrightarrow (\pi g) \leq x$	23 <i>simp</i>
25	$X^{\delta_i} g = \text{fo}$	19 <i>simp</i>
26	$(\pi g) \leq x$	24, 25 <i>MP</i>
27	$(\pi g) \leq x \ \& \ O z(\pi g)$	18, (10 <i>simp</i>) <i>conj</i>
28	$O xz$	27, 0a <i>MP</i>
29	$O xz \ \& \ \neg O zx$	28, (7 <i>simp</i>) <i>conj</i>
30	$(\pi c) \leq x$	5 – 29 <i>IP</i>
<hr/>		
31	$(\exists X^{\delta_j})(\text{Appr } xX^{\delta_j} \ \& \ \neg \text{Null } X^{\delta_j})$	<i>TRA5</i>
32	$\text{Appr } xX^{\delta_j} \ \& \ \neg \text{Null } X^{\delta_j}$	
33	$\text{Appr } xX^{\delta_j}$	32 <i>simp</i>
34	$c \in \delta_j \Rightarrow ((X^{\delta_j} c) = \text{fo} \Leftrightarrow (\pi c) \leq x \ \& \ (X^{\delta_j} c) = \text{po} \Leftrightarrow \text{NSO } x(\pi c) \ \& \ (X^{\delta_j} c) = \text{no} \Leftrightarrow \text{DR } x(\pi c))$	(33 <i>D_{Appr}</i>) <i>UI</i>
35	$(X^{\delta_j} c) = \text{fo} \Leftrightarrow (\pi c) \leq x$	34 <i>simp</i>
35a	$(X^{\delta_j} c) = \text{fo}$	35, 30 <i>MP</i>
36	$\text{Appr } xX^{\delta_j} \ \& \ X^{\delta_j} c = \text{fo}$	33, 35a <i>conj</i>
37	$(\text{appr}_{\delta_j} x) c = \text{fo}$	36 <i>D_{appr}</i>
38	$c \in \delta_j \ \& \ (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) d = \text{fo}) \Rightarrow (\text{appr}_{\delta_j} x) c = \text{fo}$	2 – 27 <i>CP</i>
39	$(c)(c \in \delta_j \ \& \ (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) d = \text{fo}) \Rightarrow (\text{appr}_{\delta_j} x) c = \text{fo})$	38 <i>UG</i>
40	$\text{Com } \delta_i \ \& \ \delta_i \ll \delta_j \Rightarrow (c)[c \in \delta_j \ \& \ (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (\text{appr}_{\delta_i} x) c = \text{fo})]$	

<i> TSA2a</i>	$Com \delta_i \& \delta_i \ll \delta_j \Rightarrow$	
	$(c)[(c \in \delta_j \& (d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no)) \Rightarrow (appr_{\delta_j} x) c = no]$	
0	$d \sqsubseteq c \Rightarrow (\pi d) \leq (\pi c)$	<i>Th</i>
0a	$(\delta_i \ll \delta_j \& d \in \delta_i \& c \in \delta_j \& (\exists z)(z \leq (\pi c) \& O z(\pi d))) \Rightarrow d \sqsubseteq c$	<i>Th</i>
0b	$x \leq y \& \neg O yz \Rightarrow \neg O xz$	<i>Th</i>
<i>lc</i>	<i>Th</i>	
1	$Com \delta_i \& \delta_i \ll \delta_j$	<i>ass</i>
2	$c \in \delta_j \& (d)(d \in \delta_j \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no)$	<i>ass</i>
3	$\delta_i \ll \delta_j \& c \in \delta_j$	(1 <i>simp</i>), (2 <i>simp</i>) <i>conj</i>
4	$(d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no)$	2 <i>simp</i>
5	$\neg DR x(\pi c)$	<i>ass</i>
6	$\neg \neg O x(\pi c)$	5 <i>D_{DR}</i>
7	$O x(\pi c)$	6 <i>DN</i>
8	$(\exists z)(z \leq x \& z \leq (\pi c))$	7 <i>D_O</i>
9	$z \leq x \& z \leq (\pi c)$	
10	$(w)(\exists g)(g \in \delta_i \& O w(\pi g))$	1, <i>TRA6 MP</i>
11	$(\exists g)(g \in \delta_i \& O z(\pi g))$	10 <i>UI</i>
12	$g \in \delta_i \& O z(\pi g)$	
13	$z \leq (\pi c) \& O z(\pi g)$	(9 <i>simp</i>), (12 <i>simp</i>) <i>conj</i>
14	$(\exists z)(z \leq (\pi c) \& O z(\pi g))$	13 <i>EG</i>
15	$\delta_i \ll \delta_j \& g \in \delta_i \& c \in \delta_j \& (\exists z)(z \leq (\pi c) \& O z(\pi g))$	(1 <i>simp</i>), (2 <i>simp</i>), (12 <i>simp</i>) <i>conj</i>
16	$g \sqsubseteq c$	15, 0a <i>MP</i>
17	$g \in \delta_i \& g \sqsubseteq c$	(12 <i>simp</i>), 16 <i>conj</i>
18	$g \in \delta_i \& g \sqsubseteq c \Rightarrow (appr_{\delta_i} x) g = no$	4 <i>UI</i>
19	$(appr_{\delta_i} x) g = no$	17, 18 <i>MP</i>
...	unforld definitions of <i>Appr</i> and <i>appr</i>	
20	$DR (\pi g)x$	like 18-26 in <i>LSA1a</i>
21	$\neg O (\pi g)x$	20 <i>D_{DR}</i>
22	$z \leq x \& \neg O (\pi g)x$	(9 <i>simp</i>), 16 <i>conj</i>
23	$\neg O (\pi g)z$	22, 0b <i>MP</i>
24	$O z(\pi g) \& \neg O (\pi g)z$	(12 <i>simp</i>), 23 <i>conj</i>
25	$DR x(\pi c)$	5 – 24 <i>IP</i>
31	$(\exists X^{\delta_i})(Appr xX^{\delta_i} \& \neg Null X^{\delta_i})$	<i>TRA5</i>
32	$Appr xX^{\delta_i} \& \neg Null X^{\delta_i}$	
33	$Appr xX^{\delta_i}$	32 <i>simp</i>
34	$c \in \delta_j \Rightarrow ((X^{\delta_i} c) = fo \Leftrightarrow (\pi c) \leq x \& (X^{\delta_i} c) = po \Leftrightarrow NSO x(\pi c) \& (X^{\delta_i} c) = no \Leftrightarrow DR x(\pi c))$	(33 <i>D_{Appr}</i>) <i>UI</i>
35	$(X^{\delta_i} c) = no \Leftrightarrow DR (\pi c)x$	34 <i>simp</i>
35a	$(X^{\delta_i} c) = fo$	35, 25 <i>MP</i>
36	$Appr xX^{\delta_i} \& X^{\delta_i} c = no$	33, 35a <i>conj</i>
37	$(appr_{\delta_i} x) c = no$	36 <i>D_{appr}</i>
38	$c \in \delta_j \& (d)(d \in \delta_j \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no) \Rightarrow (appr_{\delta_j} x) c = no$	2 – 27 <i>CP</i>
39	$(c)(c \in \delta_j \& (d)(d \in \delta_j \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no) \Rightarrow (appr_{\delta_j} x) c = no)$	38 <i>UG</i>
40	$Com \delta_i \& \delta_i \ll \delta_j \Rightarrow$	
	$(c)[(c \in \delta_j \& (d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = no)) \Rightarrow (appr_{\delta_j} x) c = no]$	1 – 40 <i>CP</i>

<i>TSA2b</i>	$\delta_i \ll \delta_j \Rightarrow$		
	$(c)[(c \in \delta_j \ \& \ (appr_{\delta_j} \ x) \ c = \text{no}) \Rightarrow (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no})]$		
0	$d \sqsubseteq c \Rightarrow (\pi \ d) \leq (\pi \ c)$		<i>Th</i>
0a	$x \leq y \ \& \ \neg O \ yz \Rightarrow \neg O \ xz$		<i>Th</i>
1	$\delta_i \ll \delta_j \text{ ass}$		
2	$c \in \delta_j \ \& \ (appr_{\delta_j} \ x) \ c = \text{no}$		<i>ass</i>
3	$(appr_{\delta_j} \ x) \ c = \text{no}$		2 <i>simp</i>
4	$Appr \ xX^{\delta_j} \ \& \ X^{\delta_j} \ c = \text{no}$		3 <i>D_{appr}</i>
5	$Appr \ xX^{\delta_j}$		4 <i>simp</i>
6	$c \in \delta_j \Rightarrow ((X^{\delta_j} \ c) = \text{fo} \Leftrightarrow (\pi \ c) \leq x \ \& \ (X^{\delta_j} \ c) = \text{po} \Leftrightarrow NSO \ x(\pi \ c) \ \& \ (X^{\delta_j} \ c) = \text{no} \Leftrightarrow DR \ x(\pi \ c))$		(5 <i>D_{Appr}</i>) <i>UI</i>
7	$c \in \delta_j$		2 <i>simp</i>
8	$((X^{\delta_j} \ c) = \text{fo} \Leftrightarrow (\pi \ c) \leq x \ \& \ (X^{\delta_j} \ c) = \text{po} \Leftrightarrow NSO \ x(\pi \ c) \ \& \ (X^{\delta_j} \ c) = \text{no} \Leftrightarrow DR \ x(\pi \ c))$		6, 7 <i>MP</i>
9	$(X^{\delta_j} \ c) = \text{no} \Leftrightarrow DR \ (\pi \ c)x$		8 <i>simp</i>
10	$X^{\delta_j} \ c = \text{no}$		4 <i>simp</i>
11	$DR \ (\pi \ c)x$		9, 10 <i>MP</i>
12	$d \in \delta_j \ \& \ d \sqsubseteq c$		<i>ass</i>
13	$d \sqsubseteq c$		12 <i>simp</i>
14	$(\pi \ d) \leq (\pi \ c)$		13, 0 <i>MP</i>
15	$(\pi \ d) \leq (\pi \ c)DR \ (\pi \ c)x$		14, (12 <i>simp</i>) <i>conj</i>
16	$DR \ (\pi \ d)x$		15, 0a <i>MP</i>
17	$(\exists X^{\delta_i})(Appr \ xX^{\delta_i} \ \& \ \neg Null \ X^{\delta_i})$		<i>TRA5</i>
18	$Appr \ xX^{\delta_i} \ \& \ \neg Null \ X^{\delta_i}$		
19	$Appr \ xX^{\delta_i}$		17 <i>simp</i>
20	$d \in \delta_i \Rightarrow ((X^{\delta_i} \ d) = \text{fo} \Leftrightarrow (\pi \ d) \leq x \ \& \ (X^{\delta_i} \ d) = \text{po} \Leftrightarrow NSO \ x(\pi \ d) \ \& \ (X^{\delta_i} \ d) = \text{no} \Leftrightarrow DR \ x(\pi \ d))$		(19 <i>D_{Appr}</i>) <i>UI</i>
21	$d \in \delta_i$		12 <i>simp</i>
22	$(X^{\delta_i} \ d) = \text{fo} \Leftrightarrow (\pi \ d) \leq x \ \& \ (X^{\delta_i} \ d) = \text{po} \Leftrightarrow NSO \ x(\pi \ d) \ \& \ (X^{\delta_i} \ d) = \text{no} \Leftrightarrow DR \ x(\pi \ d)$		20, 21 <i>MP</i>
23	$(X^{\delta_i} \ d) = \text{no} \Leftrightarrow DR \ (\pi \ d)x$		22 <i>simp</i>
24	$X^{\delta_i} \ d = \text{no}$		16, 23 <i>MP</i>
25	$Appr \ xX^{\delta_i} \ \& \ X^{\delta_i} \ d = \text{no}$		19, 24 <i>conj</i>
26	$(appr_{\delta_i} \ x) \ d = \text{no}$		25 <i>D_{appr}</i>
27	$d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no}$		12 – 26 <i>CP</i>
28	$(d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no})$		27 <i>UG</i>
29	$(c \in \delta_j \ \& \ (appr_{\delta_j} \ x) \ c = \text{no}) \Rightarrow (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no})$		2 – 28 <i>CP</i>
30	$(c)(c \in \delta_j \ \& \ (appr_{\delta_j} \ x) \ c = \text{no}) \Rightarrow (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no})$		29 <i>UG</i>
31	$\delta_i \ll \delta_j \Rightarrow$		
	$(c)[(c \in \delta_j \ \& \ (appr_{\delta_j} \ x) \ c = \text{no}) \Rightarrow (d)(d \in \delta_i \ \& \ d \sqsubseteq c \Rightarrow (appr_{\delta_i} \ x) \ d = \text{no})]$		1 – 30 <i>CP</i>

<i>TSA3</i>	$(Com \delta_i \& \delta_i \ll \delta_j) \Rightarrow (c)(c \in \delta_i \Rightarrow ((\alpha_{ij} (appr_{\delta_i} x)) c = ((appr_{\delta_j} x) l_{ij}) c))$	
0	$\{(appr_{\delta_i} x) d \mid d \sqsubseteq (l_{ij} c)\} = \{u\} \Leftrightarrow$ $(d)((d \in \delta_i \& d \sqsubseteq (l_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = u), \quad u \in \{fo, no\}$	<i>Th</i>
0a	$c \in \delta_i \Rightarrow (l_{ij} c) \in \delta_j$	<i>Th</i>
0b	$\neg((appr_{\delta_j} x) l_{ij}) c = po \Rightarrow (((appr_{\delta_j} x) l_{ij}) c = fo \text{ or } ((appr_{\delta_j} x) l_{ij}) c = no)$	<i>Th</i>
1	<i>Com</i> $\delta_i \& \delta_i \ll \delta_j$	<i>ass</i>
1a	$c \in \delta_i$	<i>ass</i>
1b	$(l_{ij} c) \in \delta_j$	1a, 0a <i>MP</i>
2	$(\alpha_{ij} (appr_{\delta_i} x)) c = fo$	<i>ass</i>
3	$\omega_{ij}^{(l_{ij} c)} = \{fo\}$ $\omega_{ij}^p = \{(appr_{\delta_i} x) d \mid d \sqsubseteq p\}$	2, $D_{\alpha_{ij}}$ $D_{\omega_{ij}^p}$
4	$\{(appr_{\delta_i} x) d \mid d \sqsubseteq (l_{ij} c)\} = \{fo\}$	$D_{\omega_{ij}^p}$
5	$(d)((d \in \delta_i \& d \sqsubseteq (l_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = fo)$	4, 0 <i>MP</i>
6	$(l_{ij} c) \in \delta_j \& (d)((d \in \delta_i \& d \sqsubseteq (l_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = fo)$	1b, 6 <i>conj</i>
7	<i>Com</i> $\delta_i \& \delta_i \ll \delta_j \Rightarrow$ $(g)[(g \in \delta_j \& (c)(c \in \delta_i \& c \sqsubseteq g \Rightarrow (appr_{\delta_i} x) c = fo)) \Rightarrow (appr_{\delta_j} x) g = fo]$	<i>TSA1a</i>
8	$(g)[(g \in \delta_j \& (c)(c \in \delta_i \& c \sqsubseteq g \Rightarrow (appr_{\delta_i} x) c = fo)) \Rightarrow (appr_{\delta_j} x) g = fo]$	1, 7 <i>MP</i>
9	$((l_{ij} c) \in \delta_j \& (d)(d \in \delta_i \& d \sqsubseteq (l_{ij} c) \Rightarrow (appr_{\delta_i} x) d = fo)) \Rightarrow (appr_{\delta_j} x) (l_{ij} c) = fo$	8 <i>UI</i>
10	$(appr_{\delta_j} x) (l_{ij} c) = fo$	9, 6 <i>MP</i>
10a	$((appr_{\delta_j} x) l_{ij}) c = fo$	10 fkt appl
11	$(\alpha_{ij} (appr_{\delta_i} x)) c = fo \Rightarrow ((appr_{\delta_j} x) l_{ij}) c = fo$	2 – 10a <i>CP</i>
12	$((appr_{\delta_j} x) l_{ij}) c = fo$	<i>ass</i>
12a	$(appr_{\delta_j} x) (l_{ij} c) = fo$	12 fkt app
13	$\delta_i \ll \delta_j \Rightarrow$ $(c)[(c \in \delta_j \& (appr_{\delta_j} x) c = fo) \Rightarrow (d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = fo)]$	<i>TSA1b</i>
14	$(c)[(c \in \delta_j \& (appr_{\delta_j} x) c = fo) \Rightarrow (d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = fo)]$	(1 <i>simp</i>), 13 <i>M</i>
15	$[(l_{ij} c) \in \delta_j \& (appr_{\delta_j} x) (l_{ij} c) = fo] \Rightarrow (d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = fo)$	14 <i>UI</i>
16	$(l_{ij} c) \in \delta_j \& (appr_{\delta_j} x) (l_{ij} c) = fo$	1b, 12a <i>conj</i>
17	$(d)(d \in \delta_i \& d \sqsubseteq c \Rightarrow (appr_{\delta_i} x) d = fo)$	15, 16 <i>MP</i>
18	$\{(appr_{\delta_i} x) d \mid d \sqsubseteq (l_{ij} c)\} = \{fo\}$	17, 0 <i>MP</i>
19	$\omega_{ij}^{(l_{ij} c)} = \{fo\}$	18, $D_{\omega_{ij}^p}$
20	$(\alpha_{ij} (appr_{\delta_i} x)) c = fo$	19 $D_{\alpha_{ij}}$
21	$((appr_{\delta_j} x) l_{ij}) c = fo \Rightarrow (\alpha_{ij} (appr_{\delta_i} x)) c = fo$	12 – 20 <i>CP</i>
22	$(\alpha_{ij} (appr_{\delta_i} x)) c = fo \Leftrightarrow ((appr_{\delta_j} x) l_{ij}) c = fo$	(11, 21 <i>conj</i>) <i>E</i>
...		
23	$(\alpha_{ij} (appr_{\delta_i} x)) c = no \Leftrightarrow ((appr_{\delta_j} x) l_{ij}) c = no$ like 2-22 using <i>TSA1a</i> and <i>TSA1b</i>	
...		

...	<i>TVAI</i>	
24	$(\alpha_{ij} (appr_{\delta_i} x)) c = \text{po}$	<i>ass</i>
25	$\neg\omega_{ij}^{(\ell_{ij} c)} = \{ \text{fo} \} \ \& \ \neg\omega_{ij}^{(\ell_{ij} c)} = \{ \text{no} \}$	24, $D_{\alpha_{ij}}$
26	$\neg\omega_{ij}^{(\ell_{ij} c)} = \{ \text{fo} \}$	25 <i>simp</i>
27	$\neg(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{fo})$	26, 0 <i>MT</i>
28	$\neg\omega_{ij}^{(\ell_{ij} c)} = \{ \text{no} \}$	25 <i>simp</i>
29	$\neg(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{no})$	28, 0 <i>MT</i>
30	$\neg((appr_{\delta_j} x) \ell_{ij}) c = \text{po}$	<i>ass</i>
31	$((appr_{\delta_j} x) \ell_{ij}) c = \text{fo} \ \text{or} \ ((appr_{\delta_j} x) \ell_{ij}) c = \text{no}$	30, 0b <i>MP</i>
32	$((appr_{\delta_j} x) \ell_{ij}) c = \text{fo}$	<i>ass</i>
...		
33	$(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{fo})$	like 2-5
34	$((appr_{\delta_j} x) \ell_{ij}) c = \text{no}$	31, 33 <i>DS</i>
...		
35	$(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{no})$	like 2-5
36	$(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{no}) \ \& \ \neg(d)((d \in \delta_i \ \& \ d \sqsubseteq (\ell_{ij} c)) \Rightarrow (appr_{\delta_i} x) d = \text{no})$	35, 29 <i>conj</i>
37	$((appr_{\delta_j} x) \ell_{ij}) c = \text{po}$	30 – 36 <i>IP</i>
38	$(\alpha_{ij} (appr_{\delta_i} x)) c = \text{po} \Rightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{po}$	24 – 37 <i>CP</i>
39	$((appr_{\delta_j} x) \ell_{ij}) c = \text{po}$	<i>ass</i>
...		
40	$(\alpha_{ij} (appr_{\delta_i} x)) c = \text{po}$	like 38-24
41	$((appr_{\delta_j} x) \ell_{ij}) c = \text{po} \Rightarrow (\alpha_{ij} (appr_{\delta_i} x)) c = \text{po}$	39 – 40 <i>CP</i>
42	$(\alpha_{ij} (appr_{\delta_i} x)) c = \text{po} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{po}$	(38, 41 <i>conj</i>) <i>Equiv</i>
43	$(\alpha_{ij} (appr_{\delta_i} x)) c = \text{fo} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{fo} \ \& \ (\alpha_{ij} (appr_{\delta_i} x)) c = \text{po} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{po} \ \& \ (\alpha_{ij} (appr_{\delta_i} x)) c = \text{no} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{no}$	22, 23, 42 <i>conj</i>
44	$(c)(c \in \delta_i \Rightarrow (\alpha_{ij} (appr_{\delta_i} x)) c = \text{fo} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{fo} \ \& \ (\alpha_{ij} (appr_{\delta_i} x)) c = \text{po} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{po} \ \& \ (\alpha_{ij} (appr_{\delta_i} x)) c = \text{no} \Leftrightarrow ((appr_{\delta_j} x) \ell_{ij}) c = \text{no})$	(1a – 43 <i>CP</i>) <i>UG</i>
45	$(c)(c \in \delta_i \Rightarrow (\alpha_{ij} (appr_{\delta_i} x)) c = ((appr_{\delta_j} x) \ell_{ij}) c)$	44 <i>ARA2</i>
46	$Com \ \delta_i \ \& \ \delta_i \ll \delta_j \Rightarrow (c)(c \in \delta_i \Rightarrow (\alpha_{ij} (appr_{\delta_i} x)) c = ((appr_{\delta_j} x) \ell_{ij}) c)$	1 – 45 <i>CP</i>

<i>TVA1</i>	$crisp \tilde{X} \Leftrightarrow (x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim y)$	
1	$crisp \tilde{X}$	<i>ass</i>
2	$(x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow (appr x) = (appr y))$	1 <i>D_{crisp}</i>
3	$CR (appr x)\tilde{X} \& CR (appr y)\tilde{X}$	<i>ass</i>
4	$(appr x) = (appr y)$	2, 3 <i>MP</i>
5	$x \sim_{\delta} y$	4, <i>D_~</i>
6	$CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim_{\delta} y$	3 – 5 <i>CP</i>
7	$(x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim_{\delta} y)$	6 <i>UG</i>
8	$crisp \tilde{X} \Rightarrow (x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim y)$	1 – 7 <i>CP</i>
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9	$(x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim y)$	<i>ass</i>
10	$CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim y$	9 <i>UI</i>
11	$x \sim y \Leftrightarrow (appr x) = (appr y)$	<i>D_~</i>
12	$CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow (appr x) = (appr y)$	10, 11 <i>HS</i>
13	$(x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow (appr x) = (appr y))$	10, 11 <i>HS</i>
14	$crisp \tilde{X}$	13 <i>D_{crisp}</i>
15	$(x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow crisp \tilde{X})$	9 – 14 <i>CP</i>
16	$crisp \tilde{X} \Leftrightarrow (x)(y)(CR (appr x)\tilde{X} \& CR (appr y)\tilde{X} \Rightarrow x \sim y)$	

<i>TV A2</i>	$CR X^{\delta} \tilde{X}^{\delta} \Rightarrow CR (\alpha X^{\delta})(\tilde{\alpha} \tilde{X}^{\delta})$	
1	$CR X^{\delta} \tilde{X}^{\delta}$	<i>ass</i>
2	$c \in \delta$	<i>ass</i>
3	$(\alpha X^{\delta}) c = fo$	<i>ass</i>
4	$\omega_{ij}^{l_{ij} c} = \{ fo \}$	3 <i>D_{\alpha}</i>
5	$(d)(d \sqsubseteq (l_{ij} c) \Rightarrow X^{\delta} d = fo)$	
6	$d \sqsubseteq (l_{ij} c) \Rightarrow X^{\delta} d = fo$	
7	$d \sqsubseteq (l_{ij} c)$	<i>ass</i>
8	$X^{\delta} d = fo$	6, 7 <i>MP</i>
9	$fo \in (\tilde{X}^{\delta} d)$	8 <i>CR X^{\delta} \tilde{X}^{\delta}</i>
10	$d \sqsubseteq (l_{ij} c) \Rightarrow fo \in (\tilde{X}^{\delta} d)$	7 – 9 <i>CP</i>
11	$(d)(d \sqsubseteq (l_{ij} c) \Rightarrow fo \in (\tilde{X}^{\delta} d))$	10 <i>UG</i>
12	$fo \in \bigcap \omega_{ij}^{l_{ij} c}$	11 <i>D_{\omega_{ij}^{l_{ij} c}, D_{\cap}}</i>
13	$fo \in (\tilde{\alpha} \tilde{X}^{\delta})$	12 <i>D_{\tilde{\alpha}}</i>
14	$(\alpha X^{\delta}) c = fo \Rightarrow fo \in (\tilde{\alpha} \tilde{X}^{\delta})$	3 – 13 <i>CP</i>
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...		

...	<i>TVA2</i>	
15	$(\alpha X^\delta) c = \text{no}$	<i>ass</i>
16	$\omega_{ij}^{\ell_{ij} c} = \{ \text{no} \}$	15 D_α
17	$(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{no})$	
18	$d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{no}$	17 <i>UI</i>
19	$d \sqsubseteq (\ell_{ij} c)$	<i>ass</i>
20	$X^\delta d = \text{no}$	18, 19 <i>MP</i>
21	$\text{no} \in (\tilde{X}^\delta d)$	20 <i>CR</i> $X^\delta \tilde{X}^\delta$
22	$d \sqsubseteq (\ell_{ij} c) \Rightarrow \text{no} \in (\tilde{X}^\delta d)$	19 – 21 <i>CP</i>
23	$(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow \text{no} \in (\tilde{X}^\delta d))$	22 <i>UG</i>
24	$\text{no} \in \bigcap \omega_{ij}^{\ell_{ij} c}$	23 $D_{\omega_{ij}^{\ell_{ij} c}}, D_{\bigcap}$
25	$\text{no} \in (\tilde{\alpha} \tilde{X}^\delta)$	24 $D_{\tilde{\alpha}}$
26	$(\alpha X^\delta) c = \text{no} \Rightarrow \text{no} \in (\tilde{\alpha} \tilde{X}^\delta)$	15 – 25 <i>CP</i>
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27	$(\alpha X^\delta) c = \text{po}$	<i>ass</i>
28	$\neg \omega_{ij}^{\ell_{ij} c} = \{ \text{fo} \} \ \& \ \neg \omega_{ij}^{\ell_{ij} c} = \{ \text{no} \}$	27 D_α
29	$\neg(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{fo}) \ \& \ \neg(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{no})$	
30	$\neg(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{fo})$	29 <i>simp</i>
31	$(\exists d) \neg(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{fo})$	30 <i>QN</i>
32	$\neg(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{fo})$	
33	$\neg(\neg d \sqsubseteq (\ell_{ij} c) \text{ or } X^\delta d = \text{fo})$	32 <i>Imp</i>
34	$\neg \neg d \sqsubseteq (\ell_{ij} c) \ \& \ \neg X^\delta d = \text{fo}$	33 <i>DeM</i>
35	$d \sqsubseteq (\ell_{ij} c) \ \& \ \neg X^\delta d = \text{fo}$	34 <i>DN</i>
36	$(\exists d)(d \sqsubseteq (\ell_{ij} c) \ \& \ \neg X^\delta d = \text{fo})$	35 <i>EG, 32 – 36 EI</i>
37	$\neg \text{fo} \in \bigcap \omega_{ij}^{\ell_{ij} c}$	37 $D_{\omega_{ij}^{\ell_{ij} c}}, D_{\bigcap}$
38	$\neg(d)(d \sqsubseteq (\ell_{ij} c) \Rightarrow X^\delta d = \text{no})$	29 <i>simp</i>
...		
39	$\neg \text{no} \in \bigcap \omega_{ij}^{\ell_{ij} c}$	like 30-37
40	$\neg \text{fo} \in \bigcap \omega_{ij}^{\ell_{ij} c} \ \& \ \neg \text{no} \in \bigcap \omega_{ij}^{\ell_{ij} c}$	36, 39 <i>conj</i>
41	$\text{fo} \notin \bigcap \omega_{ij}^{\ell_{ij} c} \ \& \ \text{no} \notin \bigcap \omega_{ij}^{\ell_{ij} c}$	36, 39 <i>conj</i>
42	$\text{po} \in (\tilde{\alpha} \tilde{X}^\delta) c$	21 $D_{\tilde{\alpha}}$ cases { <i>po</i> } and { }
43	$(\alpha X^\delta) c = \text{po} \Rightarrow \text{po} \in (\tilde{\alpha} \tilde{X}^\delta) c$	27 – 42 <i>CP</i>
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44	$(\alpha X^\delta) c \in (\tilde{\alpha} \tilde{X}^\delta) c$	14, 26, 43 ...
45	$c \in \delta \Rightarrow (\alpha X^\delta) c \in (\tilde{\alpha} \tilde{X}^\delta) c$	2 – 44 <i>CP</i>
46	$(c)(c \in \delta \Rightarrow (\alpha X^\delta) c \in (\tilde{\alpha} \tilde{X}^\delta) c)$	45 <i>UG</i>
47	<i>CR</i> $(\alpha X^\delta)(\tilde{\alpha} \tilde{X}^\delta)$	46 D_{CR}
48	<i>CR</i> $X^\delta \tilde{X}^\delta \Rightarrow \text{CR}(\alpha X^\delta)(\tilde{\alpha} \tilde{X}^\delta)$	1 – 47 <i>CP</i>