Indeterminacy and rough approximation

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Abstract

This paper deals with the problem of merging descriptions of approximate spatial location specified at different levels of granularity. We distinguish between the *roughness* of an approximation at a given level of granularity and the *indeterminacy* that arises when approximations are transformed from one level of granularity to another. Indeterminacy arises in particular when an approximation is transformed (lifted) to a finer level of granularity and when a approximation in a gappy level of granularity (e.g., a map with 'white spaces' of unknown territories) is transformed (generalized) to a coarser level of granularity.

Introduction

Understanding approximate reference to spatial and temporal location is fundamental to the analysis of sentences like

A   John is in Hyde Park.
B   The Rocky Mountains are located in
     Montana, Idaho, Wyoming, Nevada, Colorado, Utah, Arizona, and New Mexico.

In this context it is important to understand (a) the role of frames of reference and their structure and (b) the way objects and frames of reference relate to each other. There is an extended body of literature dealing with these questions (SC78; Tal83; Lev96).

Frames of reference hereby are hierarchical tree-like structures like the subdivision of London into Burroughs such as Westminster, Camden, etc at one level of granularity and Parks and neighborhoods like Hyde Park and Soho at another level of granularity. Within a given frame of reference approximate location can be specified at multiple levels of granularity. For example, as an alternative to (A) one can say

A′   John is in London.

and to (B) one can say

B′   The Rocky Mountains are located in the
      western part of the United States.

Often it is necessary to switch between levels of granularity. This paper focuses on cases where a location is specified at one level of granularity and one wants to derive a description of this location at a coarser or finer level of granularity. This kind of reasoning is required, for example, in order to derive conclusions from descriptions of approximate location which come from different sources or are collected in different contexts or for different purposes.

This paper emphasizes the distinction between the *roughness* of an approximation at a given level of granularity and the *indeterminacy* that arises when the approximation is transformed to a different level. In order to address this issue we start with a discussion of hierarchically organized frames of references and levels of granularity within them. We then discuss the notion of rough approximation with respect to these levels of granularity. Afterwards the notion of approximation is generalized in order to take the indeterminacy into account.

Frames of reference

In this section we review the notions of granular partitions, levels of granularity, and mereological fullness which were originally discussed in (BS02) and (Bit02).

Granular partitions

A granular partition is a triple,

$$G = <(Z, \subseteq), (\Delta, \leq), \pi >.$$

Here $(Z, \subseteq)$ is a *cell structure* with a partial ordering defined by $\subseteq$ which forms a finite tree. $(\Delta, \leq)$ is the *target domain* which is a partial ordering which satisfies the axioms of general extensional mereology (GEM) (Sim87). The projection mapping $\pi : Z \rightarrow \Delta$ is an order-homomorphism from Z into Δ such that for all $z_1, z_2 \in Z$ and for all $o_1, o_2 \in \Delta$ we have:[1]

$z_1 \subseteq z_2 \Rightarrow (\pi \ z_1) \leq (\pi \ z_2)$
$(\pi \ z_1) \leq (\pi \ z_2) \Rightarrow z_1 \subseteq z_2$
$(\exists x)(x \leq (\pi \ z_1) \text{ and } x \leq (\pi \ z_2)) \Rightarrow (z_1 \subseteq z_2 \text{ or } z_2 \subseteq z_1).$

Consider Figures 1 and 2. The left part of Figure 1 shows a tree with minimal cells labeled 'a', 'b', 'c', 'd', 'e', 'f' and

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[1] The notion of granular partitions introduced in (BS02) is more general. Here we focus on granular partitions which are mereologically strong monotonic.

non-minimal cells labeled 'r', 's', 't', 'u'. The targets of the cells are regions which are shown in Figure 2, e.g., $\pi('a') = a$, $\pi('b') = b$, etc. (The target of the root cell which is the plane as a whole is omitted here.) Another example of a granular partition is a map of the states of the United States. Here the cell structure is formed colored regions with labels like 'Montana'. Targets in reality are the actual states like the state *Montana*. For more examples see (BS02).

$$\begin{aligned}\delta_0 &= \{r\}\\ \delta_1 &= \{s, f\}\\ \delta_2 &= \{t, u, e, f\}\\ &\dots\\ \delta_{\min} &= \{a, b, c, d, e, f\}\end{aligned}$$
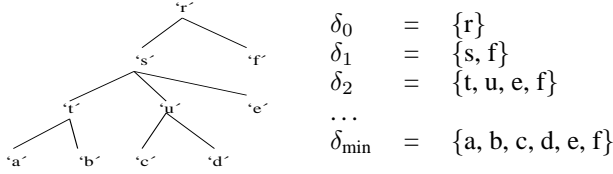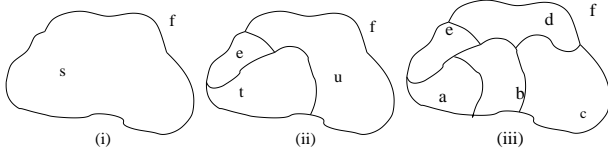
Figure 1: A tree structure with levels of granularity

Figure 2: Depictions of the levels of granularity $\delta_1, \delta_2, \delta_{\min}$.

Since we are only interested in those entities in Δ which are targeted by cells in Z it will be sufficient to refer to the cell structure as a proxy for the granular partition as a whole. Consequently we will write $(Z, \subseteq)$ as an abbreviation for $< (Z, \subseteq), (\Delta, \leq), \pi >$. We then use the names of objects in order to refer to the denoted portions portions of reality as well as to the targeting cells synonymously.

Levels of granularity

Let $G = (Z, \subseteq)$ be a granular partition and let $\mathcal{G}$ the corresponding tree representation. A *level of granularity* δ in G is then a *cut* in the tree-structure, where a cut in G is a subset of Z defined inductively as follows: (1) $\{r\}$ is a cut, where r is the root of the tree; (2) Let $\mathsf{d}(z)$ denote the set of immediate subcells of z and let C be a cut and $z \in C$ where $\mathsf{d}(z) \neq \emptyset$, then $(C - \{z\}) \cup \mathsf{d}(z)$ is a cut. This definition ensures that (i) the elements forming a level of granularity are pair-wise disjoint, i.e., $\neg \exists z_1, z_2 \in C : z_1 \subset z_2$ or $z_2 \subset z_1$; (ii) levels of granularity are exhaustive in the sense that $\forall z \in Z : $ if $z \notin C$ then $\exists z' \in C : z \subseteq z'$ or $z' \subseteq z$.[2] Cuts in the tree structure in the left of Figure 1 are listed in the right part of the figure. For further details see (RS95).

We define a partial order on cuts C and C' of a given tree as: $C \ll C'$ if and only if $\forall x \in C : \exists y \in C'$ such that $x \subseteq$

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[2]Our definition captures only certain necessary conditions that characterize levels of granularity, namely (i) and (ii), which are purely mereological in nature. For a more complete characterization also metrical notions are needed, for example in order to require that objects forming a certain level of granularity have roughly the same size. For the purpose of this paper, however, the presented definition is sufficient.

y. The corresponding lattice is called the *granularity lattice*, $\mathbf{G}$ of G. This lattice has the root cell as maximal element – which has the coarsest level of granularity – and the set of leaf-cells as as minimal element – which has the finest level of granularity. For every pair of cuts in a tree there exists a greatest lower and a least upper bound. In our example above we have the ordering: $\delta_{\min} \ll \delta_2 \ll \delta_1 \ll \delta_0$.

Mereological fullness

We can now distinguish two classes of granular partitions: Partitions which are such that whichever level of cells we choose, cells at that level sum up to cells at the next superordinate level. The partition in Figures 1 and 2 is of this type. We call such granular partitions *full*.

On the other hand there are granular partitions which do not have this property. As an example consider the partition in Figures 1 and 2 without a as a minimal cell but by assuming that the root cell covers the whole plane and that δ_1 and δ_2 sum up to δ_0 respectively. In this case the level $\delta_{\min}$ fails to sum up to the whole plane and in particular the cell b fails to sum up to the cell $t \in \delta_2$. Instead of the cell a we have 'empty space' or a hole (which can be thought of as space we know nothing about). Another example of a non-full granular partition would be a partition which is formed by the cells Hyde Park, Soho, Buckingham Palace, Downtown, London, York, Edinburgh, Glasgow, England, Scotland, Great Britain, Germany, Europe and the corresponding nesting.

Formally we define fullness as follows. Let G be a granular partition with tree representation $\mathcal{G}$. A level of granularity δ in G is full if and only if the mereological sum of its cells is identical to the root-cell in the underlying granular partition:

$$Full(\delta) \equiv \delta = \{z_1, \dots, z_n\} \text{ and } root(\mathcal{G}) = z_1 + \dots + z_n.$$

Obviously, the root cell is always full. A granular partition is full if and only if all of its levels of granularity are full. Otherwise it is non-full. For further discussion see (BS02).

Approximating location

The *exact* location of a spatial object is the region of space taken up by the object. Every spatial object is located at a single region of space at every moment in time. In this paper we consider the world at some fixed time instant t. For the purpose of this paper it will be sufficient to identify spatial objects with the region at which they are located exactly at t.

Approximate location is a binary relation between a the exact region r of a spatial object and cells $z_1, \dots z_n$, which belong to a certain level of granularity δ in a granular partition $G = (Z, \subseteq)$. In the remainder we call G the reference partition or *frame of reference* for the approximation and we write δ in order to refer to a level of granularity in G.

Let R be a set of regions satisfying the axioms of GEM. We approximate regions in $r \in R$ by functions from the cells in the level of granularity δ to the set $\Omega = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$. The function which assigns to each region $r \in R$ its approximation relative to δ will be denoted $\lambda_\delta : R \to (\delta \to \Omega)$

or $\lambda_\delta : R \rightarrow \Omega^\delta$. Here Ω^δ is a set of functions of signature $\alpha : \delta \rightarrow \Omega$ which is called the *approximation space* of the granularity-level δ. The value of $(\lambda r)z$ is fo if r covers all the of the cell z, it is po if r covers some but not all of the interior of z, and it is no if there is no overlap between r and z.[3] For example, in Figure 3 (iii) we have $\delta = \{a, b, c, d, e, f\}$ and $((\lambda\,s)\,a) = $ po, $((\lambda\,s)\,b) = $ fo, and $((\lambda\,q)\,b) = $ no.
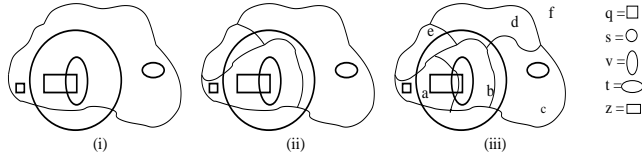
Figure 3: Rough location of the regions q, s, v, t and z within the levels of granularity δ_1, δ_2, and $\delta_{\min}$.

We also write X instead of $(\lambda\,x)$ in order to refer to the rough approximation of x. Each approximation $X \in \Omega^\delta$ stands for a set of approximated regions $[\![X]\!]$ which is defined as: $[\![X]\!] = \{r \in R \mid \lambda r = X\}$. In Figure 3 (iii) we have $z, v \in [\![V]\!]$. In Figure 3 (i) we have $z, v, q, t \in [\![V]\!]$.

Indeterminacy and rough approximation

Approximations are *determinate* in the sense that within a given level of granularity for every x there exists a *unique* approximation X such that $x \in [\![X]\!]$. We how discuss how indeterminacy arises and how it is modeled.

How indeterminacy arises

Consider Figure 4 in which we have regions, x, x_1, x_2, x_3, which are approximated within a granular partition consisting of five cells: z_1, z_2, z_3, z_4 and $z = z_1 + z_2 + z_3 + z_4$. Assume that the region x has an approximation $(\lambda\,x)z = $ po. If this is all we know about the approximate location of x, then there is indeterminacy at the finer level of granularity formed by the cells z_1, z_2, z_3, z_4. In Figure 4 this indeterminacy is shown with respect to the cell z_1: x could be like x_1 and partially overlap z_1; it could be like x_2 and not overlap z_1 at all; or it could be like x_3 which completely covers z_1. The same indeterminacy occurs in the cases of $z_2 \ldots z_4$.
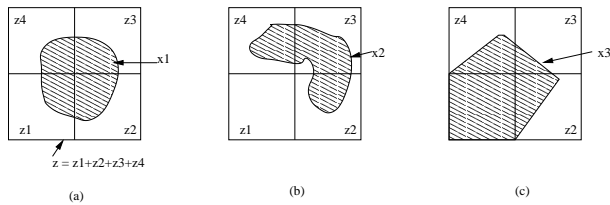

Figure 4: How indeterminacy arises in mereologically full granular partitions.

A similar effect of indeterminacy arises when we derive knowledge about approximation with respect to a coarser

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[3]For details on the presented formalism see (BS98).

level of granularity from knowledge about approximation with respect to a finer but *non-full* level of granularity. Consider Figure 5 (b). Here we have a region x_2 which is approximated within a granular partition consisting of four cells: z_1, z_2, z_3. We want to derive the approximation of x_2 with respect the root cell z. At the fine level of granularity we have $X_2\,z_i = $ fo for $1 \leq i \leq 3$ at the coarser level of granularity. This is consistent with $(X_2\,z) = $ fo as one would normally expect but also it is also consistent with $(X_2\,z) = $ po – the case which is drawn in Figure 5 (b).
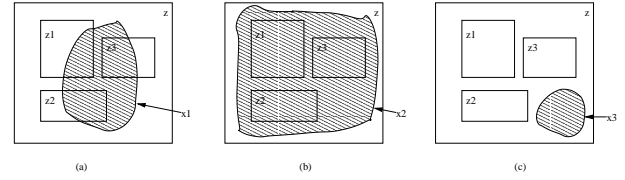

Figure 5: How indeterminacy arises in mereologically *non-full* granular partitions.

Consequently, in the context of rough approximations indeterminacy is characterized by a number of possible approximation values.

Indeterminate approximation

Consider the following subsets of the set $\Omega = \{$fo, po, no$\}$: $\{$fo$\}, \{$po$\}, \{$no$\}, \{$fo, po$\}, \{$po, no$\}, \{$fo, po, no$\}$. We call this set of subsets the *indeterminate* counterpart of Ω, and denote it by $\tilde{\Omega}$. A function which assigns to each region $r \in R$ a mapping of signature $\tilde{\alpha} : Z \rightarrow \tilde{\Omega}$, i.e., an element of $\tilde{\Omega}^\delta$, will be called an *indeterminate approximation function*, denoted by $\tilde{\lambda}_\delta : Z \rightarrow \tilde{\Omega}^\delta$. Correspondingly we call a mapping $\tilde{\alpha} : Z \rightarrow \tilde{\Omega}$ an *indeterminate approximation* and a set of those functions an *indeterminate approximation space* denoted by $\tilde{\Omega}^\delta$. The value of $(\tilde{\lambda} r)z$ is interpreted as a disjunction of possible relations between r and z. For example, the value of $(\tilde{\lambda} r)z$ is $\{$po, no$\}$ if either r covers some but not all of the interior of z or if there is no overlap between r and z. We also write $\tilde{X}$ instead of $(\tilde{\lambda}\,x)$ in order to refer to a rough approximation of x which is subject to this sort of indeterminacy.

Let $\tilde{\lambda} : Z \rightarrow \tilde{\Omega}^\delta$ be an indeterminate approximation function. We now define a binary relation $CR \subset \Omega^\delta \times \tilde{\Omega}^\delta$ interpreted as x is a *crisping* of y:

$$CR(\lambda\,x, \tilde{\lambda}\,x) \equiv \forall z \in \delta((\lambda\,x)z \in (\tilde{\lambda}\,x)z).$$

This means that $(\lambda\,x)$ is a crisping of $(\tilde{\lambda}\,x)$ if and only if for all cells z of the underlying granularity level δ we get $(\lambda\,x)z$ by choosing one element of $(\tilde{\lambda}\,x)z$.

Valid and invalid crispings

There are valid and invalid crispings. To see the difference consider the the following example. In Figure 4 indeterminacy arises when we try to derive an approximation of x with respect to the cells $z_1, \ldots, z_4$ from $(X\,z) = $ po.

This is reflected by the indeterminate approximation function $(\tilde{\lambda}\ x)\ z_i = \{\text{no}, \text{po}, \text{fo}\}$ for $1 \leq i \leq 4$. The configurations (a), (b), and (c) in the figure show a valid crispings of $(\tilde{\lambda}\ x)$. The crispings are valid because they represent ways x could be given what we know from $(X\ z) = \text{po}$.

Now let $\tilde{\lambda}\ x$ be as defined above and consider two cases: (i) Let α be a crisping $CR(\alpha, \tilde{\lambda}\ x)$ with $\alpha\ z_i = \text{fo}$ for $1 \leq i \leq 4$. This α is clearly not a valid crisping of $\tilde{\lambda}\ x$ because α implies that x covers the whole of z which contradicts our assumption $(X\ z) = \text{po}$; (ii) Let α' be a mapping such that with $CR(\alpha', \tilde{\lambda}\ x)$ and $\alpha'\ z_i = \text{no}$ with $1 \leq i \leq 4$. This is not a valid crisping because α' implies that x does not overlap z which contradicts our assumption $(X\ z) = \text{po}$.

In non-full granular partitions, on the other hand, all combinatorially possible crispings, including $(\alpha\ z_i) = \text{fo}$ and $(\alpha'\ z_i) = \text{no}$ with $1 \leq i \leq 4$ are valid as one can see in Figure 5.

Let $\#A$ be the number of elements of the set A. A crisping $(\lambda\ x)$ of an indeterminate approximation $(\tilde{\lambda}\ x)$ of a non-empty region x at a granularity-level δ is *valid* if and only if (a) $(\lambda\ x) \neq \bot$ and (b) $(\forall z \in \delta((\#(\tilde{\lambda}\ x)\ z > 1) \Rightarrow (\lambda\ x)\ z = \text{fo})) \Rightarrow x \in [\![(\lambda\ x)]\!]$. Here (a) rules out case (ii) and (b) rules out case (i).

Stratified approximation spaces

Within a given frame of reference, approximate location can be specified at multiple levels of granularity. As pointed out in the introduction it is often useful to switch between levels of granularity. We now introduce the formal structures that facilitate switching between approximations at different levels of granularity.

We define a *stratified approximation space*[4] as a structure

$$(\mathbf{G}, \tilde{\Omega}^{\delta_1}, \ldots, \tilde{\Omega}^{\delta_n}, Gen, Lift),$$

which consists of a granularity lattice, $\mathbf{G}$; for each level of granularity $\delta_1, \ldots, \delta_n$ a corresponding indeterminate approximation space $\tilde{\Omega}^{\delta_i}$ with $1 \leq i \leq n$; and two transformation functions $Gen[\delta_i, \delta_j]$ and $Lift[\delta_j, \delta_i]$ whenever $\delta_i \ll \delta_j$.

The transformation $Gen[\delta_i, \delta_j] : \tilde{\Omega}^{\delta_i} \to \tilde{\Omega}^{\delta_j}$ transfers by coarsening an approximation $\alpha \in \tilde{\Omega}^{\delta_i}$ from the level of granularity δ_i, to an approximation $\alpha' \in \tilde{\Omega}^{\delta_j}$ on a coarser level δ_j. Consider Figure 3 (ii) and (iii). Given $(\tilde{V}\ a) = \{\text{po}\}$, $(\tilde{V}\ b) = \{\text{po}\}$, $(\tilde{V}\ l) = \{\text{no}\}$ for $l \in \{c, d, e, f\}$ we want the transformation $Gen[\delta_{\min}, \delta_2]$ to yield $(Gen[\delta_{\min}, \delta_2]\ \tilde{V})\ t = \{\text{po}\}$ and $(Gen[\delta_{\min}, \delta_2]\ \tilde{V})\ l = \{\text{no}\}$ for $l \in \{e, u, f\}$. Consider Figure 5 (b). Given $(\tilde{X}_2\ l) = \{\text{fo}\}$ for $l \in \{z_1, z_2, z_3\}$ we want the transformation $Gen[\delta_{z_1, z_2, z_3}, \delta_z]$ to yield $(Gen[\delta_{z_1, z_2, z_3}, \delta_z]\ \tilde{X}_2)\ z = \{\text{po}, \text{fo}\}$.

The transformation $Lift[\delta_j, \delta_i] : \tilde{\Omega}^{\delta_j} \to \tilde{\Omega}^{\delta_i}$ transfers approximations from a coarser level of granularity δ_j to a finer level δ_i in a similar manner.

[4]Stratified approximation spaces are a specific version of the stratified map spaces defined in (SW98).

Generalization

Let $(\lambda\ r) \in \tilde{\Omega}^\delta$ be an approximation of granularity δ_i. A granulation mapping $Gen[\delta_i, \delta_j]$ transforming approximations from the approximation space $\tilde{\Omega}^{\delta_i}$ to the coarser approximation space of level $\tilde{\Omega}^{\delta_j}$ is then defined as follows. Let $\gamma_{g \subseteq h}^{\tilde{\lambda}\ r} = \{(\tilde{\lambda}\ r)g \mid g \subseteq h\} \in \mathcal{P}\tilde{\Omega}$ be the set of sets of relations that under the assumption of indeterminacy may hold between r and the subcells of h at granularity level δ_i.

Consider our five-cell granular partition with z_1, z_2, z_3, z_4 and $z = z_1 + z_2 + z_3 + z_4$ and assume $(\tilde{Y}_1\ z_i) = \{\text{po}\}$ for $1 \leq i \leq 4$. Then we have $\gamma_{z_1, \ldots, z_4}^{\tilde{Y}_1} = \{\{\text{po}\}\}$; If $(\tilde{Y}_2\ z_1) = \{\text{no}\}$ and $(\tilde{Y}_2\ z_i) = \{\text{po}, \text{fo}\}$ for $2 \leq i \leq 4$ then $\gamma_{z_1, \ldots, z_4}^{\tilde{Y}_2} = \{\{\text{no}\}, \{\text{po}, \text{fo}\}\}$; and so on.

The elements of $\tilde{\Omega}$ are ordered by the subset relation. We obtain a lattice structure by taking the lattice operations join and meet to be set union and intersection and by adding the empty set as bottom element: $(\tilde{\Omega} \cup \{\}, \cup, \cap)$. These operations generalize to operations on finite sets $\bigcup A$ and $\bigcap A$ in the standard way. Assuming that the underlying granular partition is mereologically full we can then define the generalization mapping $((Gen[\delta_i, \delta_j]\ \tilde{\lambda})\ r)h$ by distinguishing the following cases:

	$\bigcap \gamma_{g \subset h}^{(\tilde{\lambda}\ r)}$	$\bigcup \gamma_{g \subset h}^{(\tilde{\lambda}\ r)}$	$((Gen[\delta_i, \delta_j]\ \tilde{\lambda})\ r)h$
1	$\{\text{po}\}$	$-$	$\{\text{po}\}$
2	$\{\text{no}\}$	$\{\text{no}\}$	$\{\text{no}\}$
3	$\{\text{no}\}$	$\neq \{\text{no}\}$	$\{\text{no}, \text{po}\}$
4	$\{\text{fo}\}$	$\{\text{fo}\}$	$\{\text{fo}\}$
5	$\{\text{fo}\}$	$\neq \{\text{fo}\}$	$\{\text{po}, \text{fo}\}$
6	$\{\text{po}, \text{fo}\}$	$-$	$\{\text{po}, \text{fo}\}$
7	$\{\text{no}, \text{po}\}$	$-$	$\{\text{no}, \text{po}\}$
8	$\{\text{no}, \text{po}, \text{fo}\}$	$-$	$\{\text{no}, \text{po}, \text{fo}\}$
9	$\{\}$	$-$	$\{\text{po}\}$

Here row (1) is interpreted as: if $\bigcap \gamma = \{\text{po}\}$ then no matter what value $\bigcup \gamma$ has, the result of $((Gen[\delta_i, \delta_j]\ \tilde{\lambda})\ r)h$ is $\{\text{po}\}$. Row (2) is interpreted as: if $\bigcap \gamma = \{\text{po}\}$ and $\bigcup \gamma = \{\text{no}\}$ then $((Gen[\delta_i, \delta_j]\ \tilde{\lambda})\ r)h = \{\text{no}\}$.

For example, given $\tilde{Y}_1$ as defined above on level $\{z_1, z_2, z_3, z_4\}$ then we have $\bigcap \gamma_{z_1, \ldots, z_4}^{\tilde{Y}_1} = \{\text{po}\}$ and therefore $(Gen[\delta_{z_1, \ldots, z_4}, \delta_z]\ \tilde{Y}_1)z = \{\text{po}\}$. Given $\tilde{Y}_1$ as defined above then we have $\bigcap \gamma_{z_1, \ldots, z_4}^{\tilde{Y}_2} = \{\}$ and therefore $(Gen[\delta_{z_1, \ldots, z_4}, \delta_z]\ \tilde{Y}_2)z = \{\text{po}\}$.

The definitions represented in the table assume that the underlying granular partition is mereologically *full*. Otherwise we could not make the conclusions associated with rows (2) and (4). Notice also that under the assumption of mereological fullness the generalization of a crisp approximation yields a crisp approximation since under these conditions only the cases 1, 2, 4 and 9 occur.

For generalization mappings between levels of granularity in mereologically non-full partitions, the rows (2) and (4) need to be revised as follows:

	$\bigcap \gamma_{g \subset h}^{(\tilde{\lambda}\ r)}$	$\bigcup \gamma_{g \subset h}^{(\tilde{\lambda}\ r)}$	$((Gen[\delta_i, \delta_j]\ \tilde{\lambda})\ r)h$
2'	$\{\text{no}\}$	$\{\text{no}\}$	$\{\text{no}, \text{po}\}$
4'	$\{\text{fo}\}$	$\{\text{fo}\}$	$\{\text{po}, \text{fo}\}$

Consider case $(2')$. Since the exact location r of an object is always a non-empty region, and since the underlying granular partition is mereologically non-full it is always possible that the 'bigger' region $h \supseteq g$ is partially overlapped by r even if none of the subcells of h is in δ is overlapped by r (e.g., Figure 5 (c)). For this reason we have $\{\mathsf{no}, \mathsf{po}\}$ as possible relations even if we have $\bigcap \gamma = \bigcup \gamma = \{\mathsf{no}\}$. Consider case $(4')$. Since the underlying partition is mereologically non-full, it is always possible that the cells of finer level of granularity do not sum up to cells at coarser level of granularity (e.g., Figure 5 (b)). For this reason we have $\{\mathsf{po}, \mathsf{fo}\}$ as possible relations even if we have $\bigcap \gamma = \bigcup \gamma = \{\mathsf{fo}\}$.

Notice that in mereologically non-full granular partitions it is not necessarily the case that for all h it holds that $\Sigma\{z_1 \subseteq h \mid z_1 \in \delta\} \neq h$ where ΣA is the mereological sum of all elements of the set A. Therefore there are non-valid crispings in the resulting indeterminate approximation if $\Sigma\{z_1 \subseteq h \mid z_1 \in \delta\} = h$ holds for some h.

Lifting

The lifting transformations are defined as follows:

$$((\tilde{\lambda}\, r) \circ Lift[\delta_j, \delta_i])\, h = \omega_2 \text{ if and only if}$$
$$(\tilde{\lambda}\, r)g = \omega_1 \text{ and } g \subseteq h \text{ and } g \in \delta_i \text{ and } h \in \delta_j \text{ and}$$
$$(i) \quad \text{if } \omega_1 = \{\mathsf{no}\} \text{ then } \omega_2 = \omega_1 \text{ and}$$
$$(ii) \quad \text{if } \mathsf{po} \in \omega_1 \text{ then } \omega_2 = \{\mathsf{no}, \mathsf{po}, \mathsf{fo}\} \text{ and}$$
$$(iii) \quad \text{if } \omega_1 = \{\mathsf{fo}\} \text{ then } \omega_2 = \omega_1$$

Since g is a subcell of h it follows that: (i) if h does not overlap a given region r then neither does g; and (ii) if h is a part of r then so is g. Indeterminacy arises only in cases where r and h partially overlap. This is because in this case r and g may or may not overlap. In fact r may even contain g. For lifting, mereological fullness does not need to be considered, since it is not an additional source for indeterminacy.

Degrees of indeterminacy

Let $\tilde{X}$ and $\tilde{Y}$ be indeterminate approximations then $\tilde{X}$ is of less or equal indeterminacy than $\tilde{Y}$ if and only if every crisping of $\tilde{X}$ is also a crisping of $\tilde{Y}$:

$$\tilde{X} \sqsubseteq \tilde{Y} \equiv \quad \forall X (\text{ if } CR(X, \tilde{X}) \text{ then } CR(X, \tilde{Y})).$$

Here $\sqsubseteq$ is clearly reflexive, transitive, and antisymmetric.

The application of Gen and $Lift$ increases indeterminacy in the sense that if generalization and lifting (lifting and generalization) transformations are applied successively to the approximation $\tilde{X}$ the resulting approximation is of greater or equal indeterminacy than $\tilde{X}$:

$$\tilde{X} \sqsubseteq Gen[\delta_1, \delta_2](Lift[\delta_2, \delta_1](\tilde{X}))$$
$$\tilde{X} \sqsubseteq Lift[\delta_2, \delta_1](Gen[\delta_1, \delta_2](\tilde{X}))$$

Conclusions

People often use systems nested regions (cells) that form hierarchically organized tree structures (granular partitions) in order to specify spatial location in an approximate manner. Approximate location is specified in terms of relations between cells of a certain level of granularity in the granular partition and the object/region in question. We called this rough approximation.

We then discussed the distinction between roughness of approximation the indeterminacy that arises when a rough approximation is generalized or lifted, i.e., transformed to a coarser or finer level of granularity. Rough approximations are crisp no matter how coarse the underlying level of granularity. This means that between the object/region in question and each of the cells of a given level of granularity one and only one relation holds. Indeterminacy, on the other hand, is characterized by the fact that between the object/region in question and each of the cells of a given level of granularity a disjunction of possible relations is specified. This indeterminacy arises when a given (crisp) approximation is transformed (lifted) to a finer level of granularity or an approximation in a mereologically non-full granular partition is transformed (generalized) to a coarser level of granularity.

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