Vagueness and Rough Location

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Abstract. This paper deals with the representation and the processing of information about spatial objects with indeterminate location like valleys or dunes (objects subject to vagueness). The indeterminacy of the location of spatial objects is caused by the vagueness of the unity condition provided by the underlying human concepts *valley* and *dune*. We propose the notion of rough, i.e., approximate, location for representing and processing information about indeterminate location of objects subject to vagueness. We provide an analysis of the relationships between vagueness of concepts, indeterminacy of location of objects, and rough approximations using methods of formal ontology.

In the second part of the paper we propose an algebraic formalization of rough location, and hence, a formal method for the representation of objects subject to vagueness on a computer. We further define operations on those representations, which can be interpreted as union and intersection operations between those objects.

The discussion of vagueness of concepts, indeterminacy of location, rough location and the relationships between these notions contributes to the theory about the ontology of geographic space. The formalization presented can provide the foundation for the implementation of vague objects and their location indeterminacy in GIS.

Keywords: Qualitative Spatial Reasoning, Vagueness, Approximations, Formal Ontology, Rough Location, Granular Partitions

1. Introduction

The notion of location is a critical component of geographic information. This paper primarily deals with the notion of location, its ontological status, and its formalization. We distinguish the exact location of a spatial object in its unique region of space and the approximate location of a spatial object within a set of regions forming a partition of the underlying space. We discuss the relationships between these notions of location and the compositional structures of spatial objects and regions of space.

Geographic information is often about natural phenomena, cultural, and human resources (European Commission DG XIII/E, 1997). These domains are often formed by objects with indeterminate boundaries (Burrough and Frank, 1995) such as 'The Ruhr', 'The Paris-Brussels

Axis', 'The Sunshine Coast', 'The Alps' (see figure 1[1]). In this paper we discuss the relations between the vagueness of human concepts, e.g., the concept *the alps*, and the indeterminate character of the boundaries of the objects to which those concepts apply, e.g., the giant formation of rock and where it begins and ends. We further discuss means of representing indeterminacy of location in a GIS.

Natural phenomena, cultural, and human resources are not studied in isolation. They are studied in certain contexts. In the spatial domain context is often provided by regional partitions[2] forming frames of reference. Consider, for example, the location of the spatial object 'The Alps' in figure 1. We are not able to draw the exact boundaries of this object. However, in order to specify its location is often sufficient to say that parts of 'The Alps' are located in South Eastern France, Northern Italy, Southern Germany, and so on. This means that we specify the location of 'The Alps' with respect to the regional partition created by the regions of the European states. This regional partition can be refined by distinguishing northern, southern, eastern, and western parts of countries. It provides a frame of reference and an ordering structure, which is used to specify the location of 'The Alps', and which can be exploited in the representation and reasoning process.

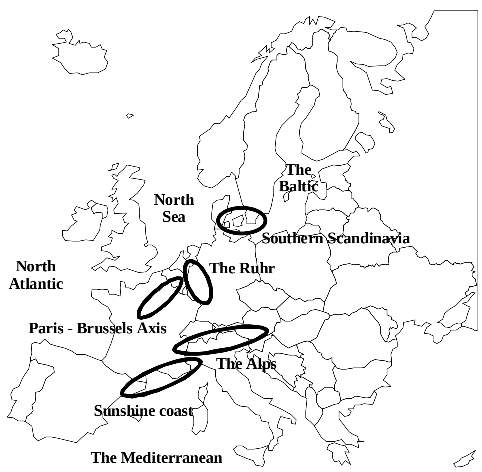

Figure 1. Vague spatial objects in Europe

[1] The figure was published originally in (Burrough and Masser, 1998) (Figure 1.1).

[2] A regional partition is a set of regions that do not overlap and which, as a set, sum up the whole space. Neighboring regions have coincident boundary parts.

This paper has four major points: (1) We discuss the relationships between vagueness of human concepts and the indeterminate location of spatial location of objects to which those concepts apply. In this context we go beyond the analysis of vagueness and indeterminacy provided, for example, in (Cohn and Gotts, 1996), (Erwig and Schneider, 1997b), (Clementini and Di Felice, 1996), or (Roy and Stell, 2001). (2) We introduce the notion of rough location, which characterizes the location of a spatial object within a regional partition of space and show that this notion can be used to deal with locational indeterminacy of vague objects[3]. (3) We show that rough location *approximates* the (indeterminate) location of vague objects with respect to sets of regions which form a regional partition of the surface of Earth. (4) We show that the notion of rough location can be formalized using the location mapping model proposed by Bittner and Stell (1998). There are alternative ways of dealing with vagueness which are based on supervaluation (van Fraassen, 1966)(Fine, 1975). Examples are (Varzi, 2001), (Smith and Brogaard, 2001). Relationships between the supervaluation and the approximation based approaches are discussed in (Bittner and Smith, 2001a).

The formalization of rough location in the context of GIS has two major aspects: Firstly, the notion of rough location allows one to separate two aspects: (a) The *exact* representation of the location of well defined objects, and (b) the finite *approximation* of the indeterminate location of vague objects in terms of their relations to well defined ones. Secondly, formal representations need to be suitable to define operations which correspond to operations on the objects they are supposed to represent. In this paper we show that the join and meet operations on location mappings defined in (Bittner and Stell, 1998) can be used to represent union and intersection operations between vague objects.

This paper contributes to an analysis of vague objects and the indeterminate character of their location. It provides an abstract mathematical formalization, which builds upon an ontological analysis. The formalization provides a basis for representation and reasoning about vague objects and can be seen as abstract specifications of data structures and operations, which can be implemented and incorporated into GIS.

[3] We use the notion 'vague object' to refer to the members of the class of spatial objects to which concepts with vague unity conditions apply. Unity conditions specify which parts belong to a whole and which do not (section 2.2.3). Our usage of the term 'vague object' is, thus, consistent with the view of vagueness as a property of the relation between human names and concepts and reality, i.e., vagueness *de dicto* in the sense of (Varzi, 2001).

This paper is structured as follows: It starts with a discussion of related work about the Ontology of geographic objects and their location in geography space (section 2). In section 3 we introduce the notion of rough location and show how this notion can be used to specify the indeterminate location of vague objects. In sections 4 and 5 a formal model for vague objects is discussed and its algebraic structures is explored. The algebraic structure provides the basis for the formalization of union and intersection operations between vague objects. The conclusions are presented in section 6.

2. Background and Related Work

In this section we discuss the compositional structure of spatial objects and the relations between geographic objects and regions of space. We start by reviewing the notions of part and whole and of exact and part location based on (Simons, 1987) and (Casati and Varzi, 1995). These notions relate the compositional structure of spatial objects to the compositional structure of spatial regions. We further discuss the relationships between vagueness the unity-condition of concepts and the indeterminate character of location of objects to which those concepts apply based on (Guarino and Welty, 2000). At the end of this section we relate our paper to the literature on partitions.

2.1. Parts and wholes

The compositional structure of an object is characterized by the relationships between the *whole* object and the different *parts* comprising the object. An extensive discussion of what spatial objects are and how they are made up of parts can be found in (Simons, 1987; Casati and Varzi, 1994; Casati and Varzi, 1997). Formally we use the predicate $P(x, y)$, which means that x is a part of y. We assume extensional mereology (Simons, 1987) as the formal theory axiomatizing the part-of relation. The predicate $P(x, y)$ is axiomatized to be antisymmetric, reflexive, and transitive, i.e., a partial ordering. In terms of the part-of relation product and sum operations are defined and their properties are axiomatized corresponding to union and intersection operations between objects.

A whole is mereologically characterized as the sum of its parts. In order to define wholes as sums of parts, unity-condition are needed that specify what sums of parts constitute wholes. Unity conditions are based on a unifying relation that holds among all parts of the whole but not between a part of the whole and a part of any other whole

(Simons, 1987). The unifying relation allows distinguishing the parts of an object from the rest of the world. It binds wholes together and prevents them from containing anything else but their parts. Guarino and Welty (2000) distinguish several classes of unity conditions: topological unity (a piece of coal, a lump of coal), morphological unity (a ball, a constellation), functional unity (a hammer, a bikini).

Wholes can be made up of parts in different ways. Gerstl and Pribbenow (1995) distinguish three kinds of wholes: Wholes of homogeneous structural kind (masses), wholes of uniform structural kind (collections), and wholes of heterogeneous structural kind (complexes). In this paper we consider spatial wholes of homogeneous and uniform structural kind, i.e., spatial masses and collections of spatial objects. Spatial masses do not have internal structure. They may be partitioned arbitrarily into parts. Geographic fields, forests, oceans are examples for homogeneous wholes of geographic kind. Important for this paper are collections of geographic objects which form a partition of space. Examples are the collection of European states or the collection of Federal States of the United States. We do not consider complexes like cars, buildings, and human artifacts in general.

2.2. Location

2.2.1. *Exact Location*

Exact location is a binary relation between spatial objects that exist and regions of space. A theory about spatial things and their corresponding regions was proposed in Casati and Varzi (1995). This theory is based on extensional mereology and the the additional primitive *exact location*, $L(x, y)$. The predicate $L(x, y)$ is interpreted as "a relation whose second term, y, is always a region in space ... the first term of the location relation, x, can be whatever sort of entity you have in your spatial ontology - spatial regions included ..." (Casati and Varzi, 1995, p.208). Exact location of a spatial object is the region of space taken up by the object. For example, "John ... is exactly located in the space 'carved out' of the air, or of whatever medium he might be in (water if he is swimming ...)" (Casati and Varzi, 1995, p. 280).

During its existence every spatial object is located in[4] a single region of space at each moment of time. A single object cannot be exactly located in different regions at the same moment in time. In the domain

[4] We say that the object x is located *in* the region y in order to stress the exact fit of object and region (the object matches the region). It is important to distinguish the exact match from the case of an object being located *within* a region which intuitive meaning allows the region to be bigger than the object and the case of the object *covering* a region which intuitively implies the region to be smaller than the object.

of spatial objects $L(x, y)$ is a functional relation. In the remainder of this paper the phrase 'the region of x' is used to refer in natural language to the region at which the spatial object x is exactly located. On the formal level we use the notion $r(x)$ in order to refer to the exact region of x. Spatial change causes spatial objects may be located in different regions at different moments in time.

The exact region of a spatial object may be a simple region of three dimensional space, think of your body and the region of space it carves out of the air. It may be a complex region, consisting of multiple simple regions of three dimensional space, as in the case of the exact region of the Hawaiian islands. The exact region may be a complex region, consisting of multiple simple regions of two dimensional space, as in the case of the representation of the Hawaiian islands on a paper map. Notice that even in this case we distinguish between the spatial object 'Map representation of the Hawaiian islands' consisting of several layers and blends of paint and the region of space it carves out of the map space. Geographic objects are often two dimensional (Egenhofer and Mark, 1995), i.e., they are located in regions which are parts of the surface of Earth. In the remainder we concentrate on such two dimensional objects.

2.2.2. *Part Location*

Exact location refers to the region of space in which the *whole* spatial object is exactly located. Spatial wholes have a compositional structure, i.e., they consist of parts. Spatial regions have parts, which are spatial regions themselves. The notion of exact location relates *spatial wholes* to *regional wholes*. In this section, the notion of part location is discussed. Part location relations relate *parts of spatial objects* to *parts of regions of space*. There is a single relation characterizing exact location but there are multiple relations characterizing relationships between parts of objects and parts of regions of space.

There are multiple relations characterizing multiple ways how parts of objects can be located in parts of regions of space since: (1) Spatial objects and spatial regions consist of different kinds of parts, e.g., object or parts vs. boundary parts (Smith, 1997). Relations characterizing part location can be defined, for example, by taking boundary parts into account or ignoring them. This results in boundary sensitive and boundary insensitive relations. In this paper we mainly concentrate on boundary insensitive relations. Boundary sensitive relations were discussed in Bittner (1997) and Bittner and Stell (1998). (2) There are multiple ways how (object) parts of spatial objects can be related to regional parts of spatial regions corresponding to binary topological relations between regions (Randall et al., 1992; Egenhofer and Franzosa, 1991).

Casati and Varzi (1995) introduce several notions of part location. Examples are the notions of *wholly located* (WL), *partly located*[5] (PL), and *generically located* (GL) in order to capture the notion of part location: $WL(o,r) =_{def} \exists z(P(z,r) \wedge L(o,z))$, $PL(o,r) =_{def} \exists z(P(z,o) \wedge L(z,r))$, and $GL(o,r) =_{def} \exists z \exists w(P(z,o) \wedge P(w,r) \wedge L(z,w))$. An object, o, is wholly located in a region, $WL(o,r)$, if r is a part of the exact region of o. It is partly located in a region, $PL(o,r)$, if there exists a part of o, which is exactly located in r. An object, o, is generically located in a region, $GL(o,r)$, if there exists a part w of o, which is exactly located in a part z of r.

2.2.3. *Vagueness and indeterminate location*

Consider geographic objects like mountains, hills, valleys, ridges, or capes. " ... we can all agree that they are real, and that it is obvious where the top of a mountain or the end of a cape is to be found. But where is the boundary of Cape Flattery on the inland side? Where is the boundary of Mount Blanc among its foothills?" (Smith and Mark, 1998, p. 316) The human concepts and descriptions do not specify the location of their boundaries. Any human being can (within a certain range of freedom) complete the definitions and create her or his boundary of 'Mount Blanc' or 'Cape Flattery'. The vagueness of the underlying human concepts causes indeterminacy of location. "... if you point to an irregularly shaped protuberance in the sand and say 'dune', then the correlate of your expression is a ... object whose constituent unary parts are comprehended (articulated) through the concept dune. The vagueness of the concept itself is responsible for the vagueness with which the referent of your expression is picked out. Each one of a large verity of slightly different and precisely determinate aggregates of molecules has an equal claim to being such a referent." (Smith and Mark, 1998, p. 315)

A concept describes a class of objects (things, particulars) using a set of properties. Concepts provide identity and unity conditions (Guarino and Welty, 2000). We use identity and unity conditions to make judgments concerning identity and unity for a certain class of things depending on properties holding for these things. "Identity is related to the problem of distinguishing a specific instance of a certain class from other instances by means of a *characteristic property*, which is unique for *it* (that *whole* instance). Unity, on the other hand is related

[5] The notion of 'part location' refers a set of location relations. The predicate $PL(o,r)$, with the interpretation 'o is partly located in r', refers to a specific relation belonging to this set.

to the problem of distinguishing the *parts* of an instance from the rest of the world by means of an *unifying relation* that binds them together (not involving anything else)" (Guarino and Welty, 2000, p.3).

In this context we need to distinguish two different kinds of vagueness of concepts: (1) the vagueness of the identity condition and (2) the vagueness of the unity conditions. Vagueness of unity conditions cause indeterminacy of location. Every spatial object is located in a single region of space at every moment of time. This region is the sum of all regions in which parts of the object are located exactly. Due to the vagueness of the unity condition of the underlying concept it is to a certain degree indeterminate which parts form the whole, and, hence there are are multiple candidates for being the exact region of a object, i.e., there are multiple sums of potential object parts. Each of those region of space is an equally good candidate to be the object's exact region (Cohn and Gotts, 1996). This reflects the *location indeterminacy* caused by the vagueness of the unity condition of the underlying human concept.

Given an underlying regional partition then the vague unity conditions are often precise enough to decide whether a particular partition region is *a part of* the object's exact region, whether it *overlaps* the object's exact region, or whether it is *disjoint* from the object's exact region. Consider, for example, the vague object 'The Alps' and the regional partition formed by the exact regions of the European states. There is no region in this partition, which is part of the exact region of the alps. There are several regions in the partition, which overlap the exact region of 'The Alps' like the regions of Germany, Austria, Switzerland, Italy, and France. At the regions of Great Britain, The Netherlands, Belgium and others no parts of 'The Alps' are located.

Often vague unity conditions are precise enough to fix a certain core of the exact region, e.g., the region of the mountain without its foothills, and leave only the exact location of the boundary indeterminate. Those objects are subject to *indeterminacy of boundary location.* In this case we are able to draw boundaries around a 'certain' core and 'certain' exterior. This means that we create a regional partition consisting of three concentric regions: $core, wide - boundary, exterior$. In this regional partition the vague object is located such that its boundary is located somewhere in the $wide{-}boundary$ region. Again, every region of space which satisfies these conditions is an equally good candidate to be the exact region of the object in question. This particular case has been studied widely in the literature, e.g., (Cohn and Gotts, 1996; Erwig and Schneider, 1997b; Clementini and Di Felice, 1996; Roy and Stell, 2001). The formalization provided by these authors is usually based on two concentric regions implicitly assuming the partition structure and the specific rough location described above.

2.3. Regional partitions

Regional partitions have been studied extensively in the GIS literature in particular in the context of categorical coverages, e.g., (Frank et al., 1997), (Erwig and Schneider, 1997a). Categorical coverages are an exhaustive partitioning of a two-dimensional region into arbitrarily shaped zones that are defined by membership in a particular category of a classification system (Chrisman, 1997). Categorical coverages are often used to represent land-use cover data (Frank et al., 1997). A special class of categorical coverages, consisting of a single (possibly scattered) region and its complement, is created by distinguishing a single category from the void. These coverages represent 'isolated objects' (Chrisman, 1997). The general case, based on an exhaustive classification into more than two classes, creates categorical coverage network.

Apart from categorical coverages regional partitions are often created by measurement or observation processes, e.g., a clock partitions the time-line into intervals separated by clock-ticks, geo-referencing remotely sensed images creates a (roughly raster shaped) partition of the surface of Earth, e.g., (Kraus, 1993). Raster shaped partitions have been extensively studied in the context of remote sensing, e.g., (Winter, 1995). We have already discussed the class of partitions consisting of three concentric regions that are used to represent vague objects.

In the context of this paper we are not interested in a classification of different kinds of partitions, their origins and application. For an extended discussion see (Smith and Bittner, 2001; Bittner and Smith, 2001b). For us is important that there are partitions and we are going to show that in general indeterminate location of vague objects can be approximated with respect to suitable regional partitions.

3. Rough Location

In this section we first formally define the notion of rough location in terms of exact and part location. We then propose a classification of rough location and apply the notion of rough location to the representation of indeterminate location of vague objects.

3.1. Patterns of Part Location Relations

The definitions of the part location predicates *wholly located*, $WL(o, r)$, *partly located*, $PL(o, r)$, and *generically located*, $GL(o, r)$, discussed above, have one shortcoming: Taken as a set, these predicates are not jointly exhaustive and pairwise disjoint (JEPD) (Casati and Varzi, 1995). A set of binary predicates is JEPD (Randall et al., 1992) if and only if for all pairs of objects for which the predicates are defined, one and only one predicate in the set holds. Sets of jointly exhaustive and pairwise disjoint binary predicates *partition* the domain of pairs of objects.

Jointly exhaustive and pairwise disjoint sets of relations provide the basis for the formalization of rough location. Based on the definitions of Casati and Varzi (1995) the following sets of JEPD part location predicates can be easily defined:

Name	Intended Meaning	Relation Set
Contained Sensitive	the region of o is either a part of r or not	$\{WL(o, r), \neg WL(o, r)\}$
Containment Sensitive	r is either a part of the region of o or not	$\{PL(o, r), \neg PL(o, r)\}$
Overlap Sensitive	the region of o either overlaps r or not	$\{GL(o, r), \neg GL(o, r)\}$

Consider the set of part location predicates with three elements, $\{FL(o, r), OL(o, r), NL(o, r)\}$, defined as follows: $FL(o, r) =_{def} PL(o, r)$, $OL(o, r) =_{def} GL(o, r) \wedge \neg PL(o, r)$, and $NL(o, r) =_{def} \neg GL(o, r)$. They are obviously JEPD. The intended meaning of the predicates is: An object o is *fully located* within region r, $FL(o, r)$ if the exact region of o is a part of r (this includes exact location of o in r). An object o is *overlapping located* with respect to a region y, $OL(o, r)$, if parts of o are exactly located in parts of r but there are parts of o that are not located in parts of r. An object o is *non overlapping located* with respect to the region r, $NL(o, r)$, if no parts of o are located in parts of r. (This includes the case where the exact region of o and r are externally connected, i.e., share a boundary segment.)

Given a JEPD set of part location predicates then the *rough location* of a spatial object, o, within the set of regions forming a regional partition, G, is characterized by a conjunction of part location predicates. Such formulas characterize the part location of the *single* spatial object, o, with respect to *all* elements, g, of the regional partition, $g \in G$.

3.2. A Classification of Rough Location

There are multiple ways how parts of objects can be located in regions of space. Consequently, multiple sets of JEPD part location relations were defined. We distinguish containment sensitive, overlap sensitive, and overlap & containment sensitive rough location depending on which particular set of relations was chosen to relate the object to the partition regions.

3.2.1. *Overlap Sensitive Rough Location.*

Rough location can be expressed in terms of logic as a conjunction of statements about part location. We use the abbreviation $LOC_G(o)$ to refer to the rough location of the spatial object, o, within the regional partition, G. Let $l_\exists = \{GL, \neg GL\}$ be the set of JEPD predicates characterizing overlap sensitive part location. The overlap sensitive rough location of the object, o, within the regional partition, $G = \{g_1, \ldots, g_n\}$, is characterized by the sequence $(l_1, \cdots, l_n) \in l_\exists \times \cdots \times l_\exists$ such that $\bigwedge_{i=1}^n l_i(o, g_i)$. The notation $\mathrm{LOC}(o)_G^\exists$ will be used to refer to the set of pairs $\{(l_1, g_1), \ldots, (l_n, g_n)\}$.

Consider the location of the object 'German Language dominated regions in Europe' within the regional partition created by the European states (figure 1). There are states in Europe where German language is spoken everywhere: Austria (A), Germany (G), Liechtenstein. There are states in Europe where German language is spoken only in certain parts: Switzerland (CH), Poland, Czech Republic, France, Italy (I), Belgium. German is not spoken, for example, in the United Kingdom (GB). The overlap sensitive rough location of the object 'German language dominated regions in Europe' within the regional partition created by the European states, *Europe*, is characterized by:

$$\begin{aligned}\mathrm{LOC}(GLR)^\exists_{Europe} &= \{(GL, r(G)), (GL, r(CH)), (GL, r(A)), \\ &\quad (\neg GL, r(GB)), \ldots\}.\end{aligned}$$

3.2.2. *Overlap & Containment Sensitive Rough Location.*

Let $l_3 = \{NL, OL, FL\}$ be a set of JEPD predicates characterizing overlap & containment sensitive part location. The overlap & containment sensitive rough location of the object, o, within the regional partition, $G = \{g_1, \ldots, g_n\}$, is characterized by the sequence $(l_1, \cdots, l_n) \in l_3 \times \cdots \times l_3$ such that $\bigwedge_{i=1}^n l_i(o, g_i)$. The notation $\mathrm{LOC}(o)_G^3$ will be used to refer to the set of pairs $\{(l_1, g_1), \ldots, (l_n, g_n)\}$. For example, the overlap & containment sensitive rough location of of the object 'German language dominated regions in Europe' is characterized by the formula:

$$\begin{aligned}\mathrm{LOC}(GLR)^3_{Europe} &= \{(OL, r(CH)), (FL, r(G)), (FL, r(A)),\\ &\quad (NL, r(GB)), \ldots\}.\end{aligned}$$

3.2.3. *Boundary Sensitive Rough Location*

All part and rough location relations discussed so far were defined by postulating the *existence* or *non-existence* of parts of objects and parts of regions with specific relations to each other. We denote these kinds of part and rough location relations *identity insensitive location relations.*

Identity insensitive part and rough location can be refined by taking *specific* parts of regions into account. Specific in this context means that we refer to parts of regions of space which *identity* we know. For example: A specific part of your land property region is the exact region of your house. Boundaries are specific (lower dimensional) parts of regions of space which identity is known if the identity of its hosting region is known. Taking specific parts of regions into account results in refinements of overlapping location, OL. Those part location relations are called *identity sensitive.*

Boundary sensitive location (Bittner, 1997; Bittner and Stell, 1998) is a special instance of identity sensitive location. Relations to specific boundary parts, i.e., boundary segments shared by neighboring partition regions are used to refine overlapping location. Consider for example, the boundary segment, which is shared by the regions of Germany and Austria. A spatial object which is overlap located in the region of Austria can either completely contain this boundary segment, contain parts of it, or not intersect the boundary segment at all. The formal definitions of boundary sensitive location can be found in (Bittner, 1997).

In the remainder of this paper we concentrate on identity insensitive rough location and its formalization. An extensive discussion of identity sensitive location was provided in (Bittner, 1999). Formal models for boundary sensitive rough location were discussed in (Bittner and Stell, 1998). Basically, all discussions about identity insensitive location and its formalization in the remainder of this paper apply to identity sensitive location as well.

3.3. Rough location and indeterminacy of location

Vague unity conditions of spatial concepts cause indeterminacy of location of objects to which they apply. There are many but usually not arbitrarily many candidates for being the object's exact regions. Often, the unity condition of the underlying concepts are precise enough to specify the object's rough location with respect to some regional partition. Consider the overlap and containment sensitive rough location of the vague object 'The Alps', (the giant formation of rock to which the vague concept *the alps* applies) within the regional partition created by the exact regions of the European states, *Europe* (figure 1):

$$\begin{aligned} \mathrm{LOC}(Alps)^3_{Europe} &= \{(OL, r(G)), (OL, r(A)), (OL, r(I)), \\ &\quad (OL, r(CH)), \ldots\}. \end{aligned}$$

The rough location is determinate. It is not affected by the location indeterminacy caused by the vagueness of the unity condition provided by the concept *the alps.* This means that all region candidates which are consistent with the unity condition of the concept *the alps* share the same rough location within the underlying regional partition.

Obviously, the underlying regional partition is very coarse and allows many regions which are definitely inconsistent with the unity condition of the concept *the alps* to share the same rough location. The choice of the appropriate regional partition is an issue of the right level of resolution. At the right level of resolution the underlying regional partition needs to be coarse enough to allow all regions consistent with the vague unity condition of the underlying concept to share the same rough location. The underlying partition needs to be fine enough to prevent regions, which are inconsistent with the unity condition, from sharing the rough location with the consistent regions.

Consider the specification of the the rough location of 'The Alps' within the partition created by the European states. There are regions of space, which have the same rough location as 'The Alps', but extend to the Atlantic Ocean. Those regions are certainly inconsistent with the vague unity condition of *the alps.* Specifying the rough location of 'The Alps' with respect to the regional partition formed by the Federal States of the European countries or with respect to a $50km \times 50km$ raster might be more appropriate. This prevents obviously inconsistent regions from sharing the rough location with 'The Alps'. Another possibility would be to refine the political subdivision partition by distinguishing northern, southern, western, and eastern parts of countries as discussed in the introduction.

A formal approach to granularity, resolution and a formal treatment of relations between regional partitions of different levels of resolution were proposed by Stell and Worboys (1998).

3.4. Vagueness, Indeterminacy, and Uncertainty

So far we argued that the vagueness of unity conditions of human concepts causes indeterminacy of location of the objects to which they apply. Indeterminacy of location means that there exist multiple regions of space, which are equally good candidates to be the object's exact region. These candidates are regions which are sums of regions of parts that consistent with the vague unity condition. The notion of rough location was used to specify indeterminate location. This means that the regions which are consistent with the vague unity-condition of the underlying concept are exactly those regions which share the same rough location.

There are three additional, critical assumptions underlying this chain of thought: (1) The assumption that we can decide for arbitrary regions of space if they are consistent with the vague unity condition or not; (2) The assumption that being consistent with the vague unity condition is a sufficient condition for being a candidate for the object's exact region; (3) The assumption that the decision about consistency of a given region with the vague unity condition can be reduced to the decision about identity of rough location.

These assumptions are very strong and are not in general true in reality. Firstly, it is the very nature of the vagueness of the unity condition that there exist regions of space which are neither obviously consistent nor obviously inconsistent with it. There is no sharp boundary between consistent and inconsistent. This has the consequence that if we are making a judgment about consistency then we are often *uncertain* about the *truth* of this judgment.

Secondly, being a candidate for the exact region implies consistency with the vague unity condition. But, for an arbitrary region, consistency with a vague unity condition does not imply that this region is a candidate for the object's exact region. It only says that the definition does not state anything which explicitly excludes this region. Consequently, for an arbitrary region, consistency with the object definition is a necessary but not a sufficient condition for being an exact-region-candidate. This has the consequence that if we assume that consistency implies being an exact-region-candidate then we *cannot be certain* about the *truth* of this conclusion. This means that there might be regions which are consistent with the object definition but which are not candidates for the exact region. For example, there exists a ellipse shaped region of space which is consistent with our vague definition of 'The Alps' (figure 1). In reality the 'The Alps' are obviously not exactly located in this region.

Thirdly, regional partitions underlying the notion of rough location are often independent of the vague object (except in the case of the rough location in the regional partition formed by the concentric regions $core, wide - boundary, exterior$). Consequently, we cannot expect that the judgment about identity of rough location within this independently formed regional partition necessarily coincides with the judgment about consistency with a vague unity condition. Consequently, we cannot be certain about the truth of conclusions about consistency derived from identity of rough location.

This shows that there are at least three sources for uncertainty caused by making these assumptions. The question is what do we gain by making those assumptions? There are four major points: (1) Using the notion of rough location for handling location indeterminacy of vague objects opens the door for applying the formalisms discussed in the next two sections for representing and reasoning about vague objects and their implementation in GIS. (2) Identity of rough location is easy to decide, consistency with vague unity conditions is not. (3) Roughly speaking, due to the vague character of the definitions - excluding everything what is inconsistent with these definitions and taking whatever is left over is all we can do anyway. (4) Within certain limits we can chose or create (for example by measurement processes such as Remote Sensing) regional partitions that are 'compatible' with the vague unity conditions of the underlying concepts. This relates to the question of choosing the appropriate level of resolution as discussed above.

In the context of this paper it was important to make the sources of the uncertainty explicit. Dealing with the problem of uncertainty goes beyond the scope of this paper. There are multiple ways of handling uncertainty. Possible options are probabilistic approaches, e.g., (Finn, 1993; Winter, 2000) or many valued logics (Hajek, 1998), e.g., (Roy and Stell, 2001; Cheng et al., 1997; Fisher, 1996).

4. Formalization

So far we have implicitly assumed an underlying formal theory of location, which was proposed by Casati and Varzi (1995). This theory is based on first order predicate calculus and extensional mereology. Predicate calculus can be used in this context to prove theorems which follow from certain axioms. We are able to check the correctness chains of reasoning and the consistency of the underlying assumptions about the relationships between spatial objects and regions of space. Predicate calculus was used on the conceptual level. It cannot be used on an operational level to perform actual computation in the sense of operations performed in a GIS.

In order to provide a basis for the implementation of the notions introduced in the previous sections efficiently implementable mathematical structures are needed. For this purpose three basic components need to be expressed formally using finite representable and effective computable mathematical structures: (1) The compositional structure of spatial objects and regions of space; (2) The notion of exact location relating spatial wholes to regional wholes; (3) The notion of rough location relating parts of spatial wholes to parts of regions forming a regional partition. These components will be used to formalize definite and vague objects. As in the previous sections we concentrate on the formalization of aspects of location. We will show that rough location of vague objects can be represented using the notion of location mappings proposed by Bittner and Stell (1998).

4.1. Exact Location

In Section 3 we dealt with parts of objects and parts of regions and their relationships terms of a formal theory about object parts (Mereology) and location proposed by Casati and Varzi (1995). In terms of this theory the notions of exact, part, and rough location were defined. There is a close relationship between Mereology and Boolean algebra (Simons, 1987). Both concepts can be used to express compositional structure at a formal level.

Boolean algebra is characterized in terms of formal properties of the operations meet, $\wedge$, join, $\vee$, and complement, $'$ (Halmos, 1963; Stell and Worboys, 1997). A Boolean algebra is complete if meet and join are defined for arbitrary sets of elements, not just finite sets. In every Boolean algebra, R, a *partial order*, $\preceq$, can be defined: $x \preceq y$ iff $x \wedge y = x$ for $x, y \in R$. In a Boolean algebra there exist a bottom element, $\perp$, and a top element, $\top$. The bottom element is the the least element with respect to the ordering $\preceq$ and a top element is the greatest element with respect to the ordering $\preceq$.

In the remainder these components are used to formalize the compositional structure of spatial objects and regions of space. The Boolean algebra structures O and R model spatial objects and regions of space. The operations $\vee$ and $\wedge$ model the mereological sum and product, i.e., union and intersection of regions and the compositional structure of spatial objects. The top element of the Boolean algebra, $\top$, is the universal region, U, or the mereological sum of all spatial objects. The bottom element of the Boolean algebra, $\perp$, is the empty object or the empty region. Notice, that in Mereology there is no notion of empty objects and empty regions. Empty objects and empty regions are artifacts of the model and support nice algebraic properties. They ensure that the meet operation is defined also for disjoint objects and disjoint regions[6]. The partial order $\preceq$ of the Boolean algebra models the mereological 'part of' relation $P(x, y)$.

Let O be the Boolean algebra modeling the compositional structure of spatial objects and R be the Boolean algebra modeling the compositional structure of regions of space. Since every spatial object is exactly located in a single region of space at each moment of time we can define a function, r, mapping spatial objects onto regions of space: $r : O \to R$. The mapping r is a bijection in the domain of physical objects but not in the domain of non-physical objects like political and administrative units. This is due to the fact that no two physical objects can be located in the same region of space in the same moment of time, but non-physical objects of *different kind* can (Casati and Varzi, 1995). For example, the objects 'City of Vienna' and 'Federal state Vienna' are located in the same region of space.

In the remainder of this section we do not distinguish between colocated objects. We furthermore consider all objects as masses, i.e., assume arbitrary sub-divisibility. This has the advantage that the Boolean algebra structure of spatial objects and regions of space correspond to each other and that we can consider the mapping r as an order-isomorphism. This allows us at the model level to abstract from the distinction between spatial objects and their exact regions of space.

4.2. Rough Location

We now review the notions of relationship mappings proposed by Bittner and Stell (1998) and discuss their application to modeling rough location. This provides the basis for the representation of the indeterminate location of vague objects.

4.2.1. *Relationship Mappings*

Consider Figure 2. A regional partition is formed by the set of regions $G = \{A, B, C, D, E, Ext\}$. With respect to this partition the regions Q, Z, S, T, V are to be approximated. Assume the Boolean algebra structure of regions of space.

The function $p_\exists$ is a relationship function,

$$p_\exists : G \times R \to \Omega_\exists \quad \text{with} \quad p_\exists(g, r) = \begin{cases} po & \text{if } \exists a \preceq g (a \preceq r) \\ no & \text{otherwise} \end{cases},$$

where the set $\Omega_\exists = \{po, no\}$ is a set of values describing how elements g of G can relate to an element r of R. The mapping $p_\exists(g, r)$ returns

[6] Strictly speaking, classical extensional mereology is modeled by complete Boolean algebra without bottom element (Simons, 1987). In the remainder of this paper we assume that formal measures have been taken to make mereology and complete Boolean algebra with bottom and top element compatible.

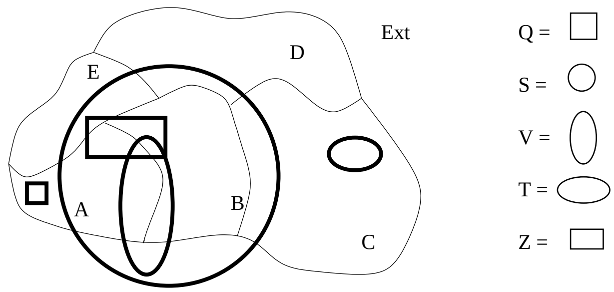

Figure 2. Example Configuration

the value *po* if and only if g and r share parts. The set $\Omega_\exists$ is called the *overlap sensitive* value domain and the function $p_\exists$ the *overlap sensitive* relationship function.

The relationship function $p_\exists$ induces a function $\alpha_\exists : R \to {\Omega_\exists}^G$: $\alpha_\exists\, r \;=_{def}\; g \mapsto p_\exists(g, r)$, where ${\Omega_\exists}^G$ is a set of functions from G to $\Omega_\exists$. The mapping $\alpha_\exists$ assigns to each $r \in R$ a function $(G \to \Omega_\exists) \in {\Omega_\exists}^G$. The function $(\alpha_\exists\, r) : G \to \Omega_\exists$ is called an *approximation function* of r, since it can be seen as an approximation of the set r in terms of its relations to elements of the (regional) partition G.

The graph of a mapping explicitly lists the set of tuples forming the mapping. Consider Figure 2 and the left table below. The columns in the table represent n-tuples in the graph of the approximation function $(\alpha_\exists Z) : G \to \Omega_\exists$.

G	A	B	C	D	E	Ext
$\Omega_\exists$	*po*	*po*	*no*	*no*	*po*	*no*

G	A	B	C	D	E	Ext
$\Omega_\forall$	*no*	*fo*	*no*	*no*	*no*	*no*

Let $\Omega_\forall$ be the set $\{fo, no\}$. The relationship function, $p_\forall$, is defined by:

$$p_\forall : G \times R \to \Omega_\forall;\;\; p_\forall(g, r) = \begin{cases} fo & \text{if } \forall a \preceq g (a \preceq r) \\ no & \text{otherwise} \end{cases}$$

The mapping $p_\forall(g, r)$ returns the value *fo* if and only if g is a part of r. The set $\Omega_\forall = \{fo, no\}$ is called *containment sensitive* value domain and the function $p_\forall$ *containment sensitive* relationship function. The relationship function $p_\forall$ induces a function $\alpha_\forall : R \to (G \to \Omega_\forall$. Consider Figure 2. The graph of the containment sensitive approximation function $(\alpha_\forall S) : G \to \Omega_\forall$, which approximates the region, S, with respect to the partition, G, is shown in the right table above.

Overlap & containment sensitive approximation functions, $\alpha_3 : R \to {\Omega_3}^G$, are based on the value domain Ω_3. The overlap & containment sensitive value domain is defined as pairs elements of the containment and overlap value domains $\Omega_\forall$ and $\Omega_\exists$:

$p(g,r)$	$p_\forall(g,r) = fo$	$p_\forall(g,r) = no$
$p_\exists(g,r) = po$	(fo, po)	(no, po)
$p_\exists(g,r) = no$	-	(no, no)

Assuming the partition structure of G ensures that there are no empty regions in G. Consequently, the case '$p_\forall(g,r) = fo$ and $p_\exists(g,r) = no$' cannot occur since this would cause a contradiction in that sense that $\forall a \preceq g(a \preceq r)$ and $\neg\exists a \preceq g(a \preceq r)$.

The relationship function p_3 induces the approximation function $\alpha_3 : R \rightarrow (G \rightarrow \Omega_3$. Consider Figure 2. The regions Q, Z, S, T, V are approximated with respect to the partition, G. The graphs of the approximation functions $(\alpha_3 r) : G \rightarrow \Omega_3$ symbolically representing approximation of the regions Q, Z, S, T, V with respect to the partition G is given in the table below.

	A	B	C	D	E	Ext
Q	(no, po)	(no, no)	(no, no)	(no, no)	(no, no)	(no, no)
S	(no, po)	(fo, po)	(no, po)	(no, po)	(no, po)	(no, po)
T	(no, no)	(no, no)	(no, po)	(no, no)	(no, no)	(no, no)
V	(no, po)	(no, po)	(no, no)	(no, no)	(no, no)	(no, po)
Z	(no, po)	(no, po)	(no, no)	(no, no)	(no, po)	(no, no)

4.2.2. *Modeling Rough Location*

Above we discussed how spatial regions can be approximated with respect to regional partitions by means of approximation functions. The notion of location refers to relation between spatial objects, $o \in O$, and regions of space, $r \in R$. In order to formalize rough location using approximation functions we define location mappings as follows:

$$loc : O \rightarrow \Omega^G; \quad (\text{loc } o) \quad =_{def} \quad (\alpha \circ r)o$$

The function $r(o)$ refers to the exact region of the object $o \in O$. The operator $\circ$ refers to the composition of α and r defined as $(\alpha \circ r)o = \alpha(r(o))$. Depending on the particular approximation domain, e.g., overlap sensitive, containment sensitive, or overlap & containment sensitive approximation, the resulting location mappings are used to formalize overlap sensitive, containment sensitive, or overlap & containment sensitive rough location.

Without loss of generality we discuss the formalization of overlap & containment sensitive rough location. Location functions of signature $loc_3 : O \to {\Omega_3}^G$ are used to formalize overlap & containment sensitive rough location. Overlap & containment sensitive rough location is characterized by conjunctions of overlap & containment sensitive part location predicates (Section 3).

Let $o \in O$ be a spatial object, $r(o) \in R$ be the region of space at which o is exactly located, and $\{g_1, \ldots, g_n\}$ be the elements of the regional partition, G. Let $(\mathrm{loc}_3\ o)$ be the location mapping characterizing the overlap & containment sensitive rough location of the object, o, within the regional partition, G. In terms of location mappings we can define the overlap & containment sensitive part location relations as follows: $fl(o, g_i) \ =_{def} \ ((\mathrm{loc}\ o)g_i) = (fo, po)$, $ol(o, g_i) \ =_{def} \ ((\mathrm{loc}\ o)g_i) = (no, po)$, and $nl(o, g_i) \ =_{def} \ ((\mathrm{loc}\ o)g_i) = (no, no)$. The graph of the location mapping is $(\mathrm{loc}_3\ o) = \{(g_1, \omega_1), \ldots, (g_n, \omega_n)\}$ if and only if between the the spatial object, o, and the elements of the regional partition, $g_i \in G$, the relations

$$\Delta_G^o = \{\delta_1(o, g_1), \ldots, \delta_n(o, g_n)\}, \delta_i \in \{fl, ol, nl\}$$

hold, where $\delta_i = fl$ iff $((\mathrm{loc}\ o)g_i) = (fo, po)$ and so on. This set of relations corresponds to conjunctions of part location predicates discussed in Section 3. This shows the correspondence between the concepts discussed in the first part and their formalization using location mappings in the second part of this paper.

Consider Figure 2. In the previous examples we implicitly assumed that the labels $A, \ldots, Ext, Q, Z, S, T, V$ refer to regions of space. In fact, these labels refer to graphic *objects* created by a graphic tool on a computer. In order to refer to the corresponding regions we actually had to use the notions $r(A), \ldots, r(V)$. In the previous examples we abstracted from the distinction between objects and their exact regions. In the context of formalization of rough location this distinction matters. For this reason in this example we distinguish between objects, $o \in O$, and their regions, $r(o) \in R$. The graph of the location function $(loc_3 S) : G \to \Omega_3$ is:

G	r(A)	r(B)	r(C)	r(D)	r(E)	r(Ext)
Ω_3	(no, po)	(fo, po)	(no, po)	(no, po)	(no, po)	(no, po)

Corresponding to $(loc_3 S)$ the following set of spatial relations holds:

$$\Delta_G^S = \{ol(S, r(A)), fl(S, r(B)), ol(S, r(C)), ol(S, r(D)), ol(S, r(E)), ol(S, r(Ext))\}$$

The set Δ_G^S corresponds to the formula

$$\begin{aligned}\mathrm{LOC}(S)_G^3 \;=\; & \{(OL, r(A)), (FL, r(B)), (OL, r(C)), (OL, r(D)),\\ & (OL, r(E)), (OL, r(Ext))\}\end{aligned}$$

5. Operations between Vague Objects

Assume that we are modeling vague objects by means of location mappings. Processing on a computer does not merely mean representation but also performing operations, modeling processes and so on. The most basic operations on spatial objects and their exact regions of space are union and intersection operations. Those operations provide the basis for most higher level GIS operations (Laurini and Thompson, 1994).

In this section we discuss union and intersection operations between vague objects. These operations will be defined in terms of operations on location mappings approximating the exact-region-candidates of vague objects. We first discuss the definition of those operations based on (Bittner and Stell, 1998). Then we discuss their relationships to union and intersection operations on the (exact) regions they approximate. It is important to point out that operations on approximation mappings are only defined between approximations with respect to the *same* underlying partition. The definition of operations between approximations with respect to different regional partitions are subject of ongoing research.

Let o_1 and o_2 be two vague objects, which are both roughly located within the regional partition G, such that all exact-region-candidates for the objects o_1 and o_2 are approximated by the approximation mappings (loc o_1) and (loc o_2). In order to compare operations on exact-region-candidates and approximation mappings we apply the following technique: Firstly, we perform operations, $\star$, between two exact-region-candidates x and y with $(\alpha\ x) = (\text{loc}\ o_1)$ and $(\alpha\ y) = (\text{loc}\ o_2)$ and transform the result in the approximation domain, i.e., $(\alpha\ (x \star y))$. Secondly, we compare this result with the result we get when performing the operation, $\otimes$, on approximation mappings, i.e., $(\alpha\ x) \otimes (\alpha\ y)$.

We will show in this section is that union and intersection operations between two vague objects consist of *pairs* of minimal and maximal operations which *constrain* possible results of operations between the exact-region-candidates of both objects.

5.1. Operations

Consider the domain of regions, R, with its mereological structure modeled by Boolean algebra. The $\wedge$ and $\vee$ operations model union and intersection operations on regions. In this sub-section we define operations on approximation mappings, $\otimes = \{\wedge_{\text{loc}}, \vee_{\text{loc}}\}$. In order to define these operation and to compare these operations with the corresponding operations on regions ordering relations between the elements of the value domains and ordering relations between approximation mappings need to be defined. Without loss of generality we concentrate on overlap & containment sensitive approximation mappings and their value domain.

5.1.1. *Ordering Structure*

The overlap & containment sensitive value domain, Ω_3, is formed by pairs of elements of the containment sensitive, $\Omega_\forall = \{fo, no\}$, and the overlap sensitive value domain, $\Omega_\exists = \{po, no\}$. We assume $fo > no$ and $po > no$ and define the order for pairs to be $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. We are now able to define the order structure in the domain of approximation mappings as follows. Assuming that G represents the underlying regional partition then we define: For all $r_1, r_2 \in R$ $(\alpha\, r_1) \leq (\alpha\, r_2)$ if and only if $\forall g \in G((\alpha\, r_1)\, g \leq (\alpha\, r_2)\, g)$.

5.1.2. *Operations on Approximation Functions*

Rough approximation spaces, Ω^G, are formed by mappings of the signature $(G \to \Omega)$. Let $h, k \in \Omega^G$ be approximation mappings representing the rough approximations of the regions $r_1, r_2 \in R$, within the partition G, i.e., $h = (\alpha\, r_1)$, $k = (\alpha\, r_2)$. Operations on approximation mappings are defined in terms of operations on elements of the value domain, $\omega_1 \star_\Omega \omega_2$:

$$\begin{array}{rcll} \otimes & : & (G \to \Omega) \times (G \to \Omega) & \to \quad (G \to \Omega) \\ (h \otimes k)g & =_{def} & (h\, g) \star_\Omega (k\, g), & \quad g \in G \end{array} \tag{1}$$

The meet and join operations in the value domain are defined as:

$$\omega_1 \wedge_\Omega \omega_2 =_{def} \min\{\omega_1, \omega_2\} \quad \omega_1 \vee_\Omega \omega_2 =_{def} \max\{\omega_1, \omega_2\}, \tag{2}$$

where $\omega_1, \omega_2 \in \Omega$ and $\Omega \in \{\Omega_\forall, \Omega_\exists, \Omega_3\}$.

Intuitively this means that we define operations on approximation mappings in terms of operations in the value domain between values referring to the same partition region. Corresponding to the approximation domain the operations are performed either in the containment sensitive, overlap sensitive, or overlap & containment sensitive value domain. Assume, for example, overlap sensitive rough location. The results of the union and intersection between the approximation mappings of the regions Q and S in figure 2 are:

	A	B	C	D	E	Ext
$(\alpha_\exists\, Q)$	*po*	*no*	*no*	*no*	*no*	*no*
$(\alpha_\exists\, S)$	*po*	*po*	*po*	*po*	*po*	*po*
$(\alpha_\exists\, Q) \wedge_{\text{loc}} (\alpha_\exists\, S)$	*po*	*no*	*no*	*no*	*no*	*no*
$(\alpha_\exists\, Q) \vee_{\text{loc}} (\alpha_\exists\, S)$	*po*	*po*	*po*	*po*	*po*	*po*

5.2. Approximating Operations

We are now able to compare operations performed in the domain of regions with operations performed on approximation mappings. In the context of this paper operations on approximation mappings are interpreted as operations between vague objects, i.e., sets of exact-region-candidates sharing the same rough location. Consequently, we compare operations between the (indeterminate but existing) exact regions of vague objects with operations between sets of exact-region-candidates.

5.2.1. *Ideal Approximations*

What one would like to have is the correspondence of operations in the domain of regions and operations on approximation mappings. Consider the left diagram in Figure 3. Correspondence of operations means that the diagram commutes, i.e., that we get the same result independent of whether we first apply operations, $\star$, in the domain of regions and then transform the result in the approximation domain, or first transform the regions into the approximation domain and then apply the operation on approximation mappings. Alternatively we could write: $(\alpha\ x) \otimes (\alpha\ y) = (\alpha\ (x \star y))$ for all $x, y \in R$. One can easily verify that join and meet operations in the containment sensitive approximation domain, defined by applying the definitions 1 and 2 to the containment sensitive value domain, correspond to join and meet operations on regions of space, i.e., make the diagram above commute. One can further verify that the join operation in the overlap sensitive approximation domain corresponds to the join operation on spatial regions in the same sense.

Consider figure 2. If we first approximate regions and then perform the intersection operation in the overlap sensitive approximation domain then we get $(\alpha_\exists\, S) \wedge_{\text{loc}} (\alpha_\exists\, Q) = (\alpha_\exists\, Q)$. (See the table above.) However, if we first apply the intersection operation in the domain of

$$\begin{array}{ccc} R \times R & \xrightarrow{\star} & R \\ {\scriptstyle \alpha \times \alpha} \downarrow & = & \downarrow {\scriptstyle \alpha} \\ \Omega^G \times \Omega^G & \xrightarrow{\otimes} & \Omega^G \end{array} \qquad \begin{array}{ccc} R \times R & \xrightarrow{\star} & R \\ {\scriptstyle \alpha \times \alpha} \downarrow & \underset{\min}{\leq} & \downarrow {\scriptstyle \alpha} \\ \Omega^G \times \Omega^G & \xrightarrow{\overset{\min}{\otimes}} & \Omega^G \end{array} \quad \begin{array}{ccc} R \times R & \xrightarrow{\star} & R \\ {\scriptstyle \alpha \times \alpha} \downarrow & \underset{\max}{\geq} & \downarrow {\scriptstyle \alpha} \\ \Omega^G \times \Omega^G & \xrightarrow{\overset{\max}{\otimes}} & \Omega^G \end{array}$$

Figure 3. Commutative diagrams for ideal (left) and rough (right) approximations.

regions and then transform the result to the overlap sensitive approximation domain then we get different results: $((\alpha_\exists\,(S \wedge Q)) = (\alpha_\exists\,\bot) = \bot) \neq (\alpha_\exists\,Q)$. Obviously, we do not have a correspondence in the sense of the diagram above. It was shown by Bittner and Stell (1998) that in general there do not exist operations in the approximation mapping domain, which always return the *same* result as the operation in the domain of regions.

5.2.2. *Minimal and Maximal Operations*

We are now defining operations on approximation mappings, which results are *always greater than or equal to*, $\overset{\max}{\otimes}$, or *always less than or equal to*, $\overset{\min}{\otimes}$, than $(\alpha\ (x \star y))$. This means that if we first transform the regions into the approximation domain and then apply the operation on approximation mappings, i.e., $(\alpha\ x) \otimes (\alpha\ y)$ then the result is either always *greater than or equal to* (right diagram) or always *less than or equal to* (left diagram) the result of first applying operations, $\star$, in the domain of regions and then transforming the result to the approximation domain, i.e., $(\alpha\ (x \star y))$. This means that the right pair of diagrams in Figure 3 commutes (Bittner and Stell, 1998). The diagrams correspond to the following inequalities:

$$(\alpha\ x) \overset{\min}{\otimes} (\alpha\ y) \ \leq\ (\alpha\ (x \star y)) \qquad (3)$$

$$(\alpha\ (x \star y)) \ \leq\ (\alpha\ x) \overset{\max}{\otimes} (\alpha\ y) \qquad (4)$$

Operations satisfying equation 3 are called *lower operations.* Operations satisfying equation 4 are called *upper operations.*

There are different ways of defining lower and upper operations. We are looking for the greatest lower, $\overset{\min}{\otimes}$, and the least upper, $\overset{\max}{\otimes}$, operations. Bittner and Stell (1998) proved that pairs of *greatest lower* and *least upper* operations on approximation mappings always exist and showed how to construct those operations. Consequently, *pairs* of greatest lower and least upper operations on approximation mappings correspond to *single* operations on regions.

In the next section we discuss the definition of those pairs of operations for approximating meet operations in the overlap sensitive approximation domain as well as the definition of pairs in the overlap & containment sensitive approximation domain approximating join and meet operations.

5.2.3. *Defining Pairs of Operations*

Consider the intersection of regions, $\wedge$, and the corresponding operation $\wedge_{\text{loc}} \; : \; {\Omega_\exists}^G \times {\Omega_\exists}^G \rightarrow {\Omega_\exists}^G$ in the overlap sensitive approximation domain. The *greatest minimal* and the the *least maximal* meet operation in the overlap sensitive approximation domain are defined based on the following operations in the corresponding value domain:

$$\omega_1 \overset{\max}{\wedge}_{\Omega_\exists} \omega_2 =_{def} \min\{\omega_1, \omega_2\} \quad \omega_1 \overset{\min}{\wedge}_{\Omega_\exists} \omega_2 =_{def} \mathit{no}, \quad \forall \omega_1, \omega_2 \in \Omega_\exists$$

Consider the intersection of regions, $\wedge$, and the corresponding operation $\wedge_{\text{loc}} \; : \; {\Omega_3}^G \times {\Omega_3}^G \rightarrow {\Omega_3}^G$ in the overlap & containment sensitive approximation domain. The *greatest minimal* and the the *least maximal* meet operation in the overlap sensitive approximation domain are defined based on the following operations in the corresponding value domain:

$$\begin{aligned} \omega_1 \overset{\max}{\wedge}_{\Omega_3} \omega_2 &=_{def} \min\{\omega_1, \omega_2\} \\ \omega_1 \overset{\min}{\wedge}_{\Omega_3} \omega_2 &=_{def} (\mathit{no}, \mathit{no}) \text{ if } \max\{\omega_1, \omega_2\} \neq (\mathit{fo}, \mathit{po}), \\ &\quad\;\; \text{otherwise } \min\{\omega_1, \omega_2\} \end{aligned}$$

The *greatest minimal* and the the *least maximal* join operation are defined as:

$$\begin{aligned} \omega_1 \overset{\min}{\vee}_{\Omega_3} \omega_2 &=_{def} \max\{\omega_1, \omega_2\} \\ \omega_1 \overset{\max}{\vee}_{\Omega_3} \omega_2 &=_{def} (\mathit{fo}, \mathit{po}) \text{ if } \min\{\omega_1, \omega_2\} \neq (\mathit{no}, \mathit{no}), \\ &\quad\;\; \text{otherwise } \max\{\omega_1, \omega_2\} \end{aligned}$$

This concludes the definition of operations approximating join and meet operations on regions of space in containment, overlap, and overlap & containment sensitive approximation domains. The main point of this section is that union and intersection operations between two vague objects consist of pairs of minimal and maximal operations which constrain possible results of operations between the exact-region-candidates of both objects.

6. Conclusions

The geographic world is populated by spatial objects that are located in regions of space. Exact location characterizes the unique relationship between a spatial object and the region of space it occupies in a particular moment of time. The rough location of a spatial object within a regional partition is characterized by a set of relations between parts of the object and parts of the regions forming the regional partition. Notions of location describe the relationships between the compositional structure of spatial objects and the compositional structure of regions of space. The notion of exact location uniquely relates spatial wholes to regional wholes where the notion of rough location links parts of spatial objects to parts of partition regions. In this paper multiple notions of rough location were introduced and classified.

The vagueness of unity conditions provided by (some) human concepts causes indeterminacy of location of the objects to which they apply. Unity conditions provide the basis for judgments about which parts belong to a whole and which parts do not. Indeterminacy of location means that there exist multiple regions of space, which are equally good candidates to be the object's exact region. Those candidates are regions which are sums of regions of parts that consistent with the vague unity condition of the underlying concept. Vague unity conditions are, however, often exact enough to specify the rough location of the objects to which they apply within some suitable regional partition of space. This means that regions, which are consistent with the vague unity condition, share the same rough location and regions inconsistent with it do not.

In the context of representation of vague objects in GIS the notion of rough location allows one to separate two different aspects: the finite representation of exact objects creating a regional partition and the finite approximation of vague objects in terms of those exact objects.

At the formal level rough location of vague objects was represented formally by location mappings. Location mappings take a single spatial object, o, and return an approximation mapping of signature $(G \to \Omega)$. The interpretation of this mapping is that it is an approximations of the region $r(o) \in R$ in terms of its relationships with respect to the regions in the regional partition, G. The approximation mapping, $G \to \Omega$, returns for each partition region, $g \in G$, its relation, $\omega \in \Omega$, to the region, $r(o)$, to be approximated. This is based on the assumption that for arbitrary vague objects we can always find a regional partition such that all regions consistent with the object definition share the same rough location.

In section 5 it was shown that union and intersection operations between two vague objects consist of pairs of minimal and maximal operations, $\overset{\max}{\otimes}$ and $\overset{\min}{\otimes}$, which constrain possible results of operations between the exact-region-candidates of both objects. Pairs of operations on approximation mappings constrain operations on regions of space, i.e., $(\alpha\, r_1) \overset{\min}{\otimes} (\alpha\, r_2) \;\leq\; (\alpha\, (r_1 \star r_2)) \;\leq\; (\alpha\, r_1) \overset{\max}{\otimes} (\alpha\, r_2)$.

These operations can be used in order to formalize binary topological relations between approximations of vague objects (Bittner and Stell, 2000). Ongoing work deals with the representation of vague temporal objects (Bittner, 2000).

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