A NOTE ON MULTI-RESERVOIR SYSTEMS WITH INDEPENDENT OPERATORS AND NO CENTRAL COORDINATOR*

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Abstract. This paper considers a simple water resource system consisting of two independently operated reservoirs and no central coordinator. Even simple multi-reservoir systems exhibit special mathematical properties that affect policy decisions in such cases. When the operators make decisions sequentially and independently, the stable solution of such systems is not necessarily Pareto-optimal and may even be strongly dominated. Conditions to insure solution efficiency are provided.

1 Introduction

For many multi-reservoir water resource systems, there is a central authority managing the overall operation of the system. This coordination allows the reservoirs to be operated considering the objectives of the entire system.

However, sometimes there is no agency available to coordinate operating policies among the reservoirs. Or, even when such an agency exists, it may not want to intervene in every policy decision. For example, Rajabi *et al.* [13] consider the problem of managing a water supply system with independent policy actions.

Much research has already been done on modeling coordinated multi-reservoir systems. Males and McLaughlin [9] provide an approach to developing operating rules and Loucks, Stedinger, and Haith [8] consider the optimal operation of such systems with multiple objectives. In addition, Haimes [5] developed a hierarchical model for a system composed of interacting subsystems. Each subsystem had its own objectives and constraints. However, a higher level coordinator with a system-wide objective function was needed to resolve conflicts among the subsystems.

This paper will show that even the simplest multi-reservoir systems exhibit special mathematical properties when no central coordination is present. We will examine these properties and show how they affect policy decisions.

2 Model Description

To present the methodology, we need only consider a simple, deterministic water resource system. This will allow us to focus directly on the issues that arise from the lack of central coordination. The model can be extended to more realistic multi-reservoir systems using techniques that are now classic in the literature.

Consider a water resource system consisting of two, single-purpose reservoirs. For each of n periods, the operator of each reservoir receives a specified benefit for each unit of water released in that period. The objective of each reservoir operator is to maximize the total benefits individually received in the n analysis periods.

We refer to the upstream reservoir as *reservoir one* and its operator as *operator one*. Similarly, the downstream reservoir will be called *reservoir two* and its operator is *operator two*. The following notation will be used:

n	$\equiv$	number of periods.
$x^j(t)$	$\equiv$	units of water released by operator j in period t.
$S^j(t)$	$\equiv$	units of water stored by operator j in period t.
$c^{jk}(t)$	$\equiv$	profit for operator j for each unit of water released by operator k in period t.
$I^j(t)$	$\equiv$	inflow to reservoir j in period t.
K^j	$\equiv$	capacity of reservoir j.
α	$\equiv$	fixed fraction of water released from reservoir one that becomes inflow to reservoir two.

Each operator has perfect information about the system. Therefore, each operator knows his own objective function as well as that of the other operator. Also, the inflows to reservoir one are assumed known by both operators and are deterministic.

Will will consider the policies resulting from operatiing the two-reservoir system inder different coordination disciplines. The first case has the reservoirs operated independently with an agreement requiring the operator of the upstream reservoir to preannounce his operating rule for all n analysis periods. The second case has independently operated reservoirs with no such agreement. And, third, we consider the operation of a centrally coordinated system.

2.1 Case 1: Preannounced upstream policy

For this case, first consider the problem faced by operator two (downstream). The objective function for operator two can be expressed as:

$$\max_{x^2} Z_P^2 = c^{21}x^1 + c^{22}x^2.$$

However, since x^1 is preemptively determined and known by operator two, $c^{21}x^1$ is a constant. Hence, operator two without loss of generality can use the objective $c^{22}x^2$.

The objective function for operator one can by expressed as:

$$\max_{x^1} Z_P^1 = c^{11}x^1 + c^{12}x^2.$$

In order to maximize Z_P^1, operator one must consider the response of operator two, x^2 given each possible action of operator one, x^1. This

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response is referred to as the *rational reaction* of operator two and is denoted by $\psi(x^1)$.

Both operators must restrict their decisions to $(x^1, x^2) \in B$, where B is a convex polyhedron formed by a set of linear constraints which will be defined in section 3. The problem for operator two is therefore a parametric linear programming problem with parameter x^1.

To analyze this system of problems, we will use a multi-level programming model (Bialas and Karwan [2]). Using their notation, we can write the two-level linear programming problem as:

$$\begin{array}{lll} \max\limits_{x^1} & Z_P^1 = & c^{11}x^1 + c^{12}x^2 \quad \text{where } x^2 \text{ solves} \\ \max\limits_{x^2} & Z_P^2 = & c^{21}x^1 + c^{22}x^2 \\ \text{s.t.} & & (x^1, x^2) \in B. \end{array}$$

Also, we can express the problem for operator one as:

$$\begin{array}{ll} \max\limits_{x^1} Z_P^1 = & c^{11}x^1 + c^{12}x^2 \\ \text{s.t.} & (x^1, x^2) \in B \\ & x^2 = \psi(x^1). \end{array}$$

2.2 Case 2: Independent operation

Consider a two-reservoir system in which the reservoirs are operated independently and the upstream release policy is not preannounced. This problem can be formulated as a Stackelberg model containing $2n$ levels:

$$\begin{array}{lll} \max\limits_{x^1(1)} & Z_I^1 = & c^{11}x^1 + c^{12}x^2 \quad \text{where } x^1(2),\ldots,x^1(n) \text{ and } x^2 \text{ solves} \\ \max\limits_{x^2(1)} & Z_I^2 = & c^{21}x^1 + c^{22}x^2 \quad \text{where } x^1(2),\ldots,x^1(n) \\ & & \qquad \text{and } x^2(2),\ldots,x^2(n) \text{ solves} \\ \vdots & & \\ \max\limits_{x^1(n)} & Z_I^1 = & c^{11}x^1 + c^{12}x^2 \quad \text{where } x^2(n) \text{ solves} \\ \max\limits_{x^2(n)} & Z_I^2 = & c^{21}x^1 + c^{22}x^2 \\ \text{s.t.} & & (x^1, x^2) \in B. \end{array}$$

where $c^{ij}x^j \equiv \sum_t c^{ij}x^j(t)$.

In this $2n$-level formulation, the objective function is repeated for each of the alternating levels. That is, levels $2n, 2n-2, \ldots, 2$ represent the objective function of reservoir one while levels $2n-1, 2n-3, \ldots, 1$ represent the objective function of reservoir two.

2.3 Model with central coordination

In this case, a top-level coordinator controls both reservoirs and wishes to maximize the total profit obtained by the system. The coordinated system can be formulated as:

$$\begin{array}{ll} \max\limits_{x^1, x^2} & Z_C^1 + Z_C^2 = (c^{11} + c^{21})x^1 + (c^{12} + c^{22})x^2 \\ \text{s.t.} & (x^1, x^2) \in B. \end{array}$$

3 Constraint definition

In order to complete the model of the two-reservoir problem, it is necessary to specify the constraints representing the physical characteristics of the system. These constraints can be grouped into four categories: capacity, non-negativity, balance of flow, and hierarchical relationship constraints.

3.1 Capacity Constraints

Since both reservoir one and reservoir two have finite capacities, constraints are needed to insure that the capacity of a reservoir never exceeded. The constraints are:

$$S^j(t) \le K^j \quad \text{for } j = 1, 2;\ t = 1, 2, \ldots, n. \tag{1}$$

3.2 Non-negativity Constraints

Each operator must decide how much water to store in each period, $S^j(t)$, and how many units of water to release, $x^j(t)$. Both of these quantities must be non-negative for all n periods:

$$x^j(t) \ge 0 \quad \text{for } j = 1, 2;\ t = 1, 2, \ldots, n; \tag{2}$$

$$S^j(t) \ge 0 \quad \text{for } j = 1, 2;\ t = 1, 2, \ldots, n. \tag{3}$$

3.3 Balance of Flow Constraints

A set of constraints is required to insure that all water in the system is conserved:

$$x^j(t) + S^j(t) = I^j(t) + S^j(t-1) \quad \text{for } j = 1, 2;\ t = 1, 2, \ldots, n. \tag{4}$$

3.4 Hierarchical Relationship Constraints

The two operators function within a hierarchical decision making structure. A consequence of this relationship is that the decision space for operator two depends on the actions of operator one. This relationship is given by these constraints:

$$I^2(t) = \alpha x^1(t) \quad \text{for } t = 1, 2, \ldots, n;\ 0 \le \alpha \le 1. \tag{5}$$

4 Results

The following definitions will help us describe the policies of the two operators:

Definition 1 (Raiffa [12]) *A solution vector* $(Z^1, Z^2, \ldots, Z^n)$ *is said to be* **Pareto-optimal** *or* **admissible** *if and only if there does not exist another feasible solution* $(\tilde{Z}^1, \tilde{Z}^2, \ldots, \tilde{Z}^n)$ *such that* $Z^j \le \tilde{Z}^j$ *for* $j = 1, 2, \ldots, n$, *with strict inequality holding for at least one* j.

Definition 2 *A solution vector* $(Z^1, Z^2, \ldots, Z^n)$ *is said to* **strongly dominate** *a solution* $(\tilde{Z}^1, \tilde{Z}^2, \ldots, \tilde{Z}^n)$ *if and only if* $Z^j > \tilde{Z}^j$ *for* $j = 1, 2, \ldots, n$.

4.1 The effect of parametric linkage

If $c^{12} = 0$, then only a downward parametric linkage exists. For this case, the problem with *preannounced upstream policy* is equivalent to that with *independent operation* since the actions of the downstream reservoir have no effect on the upper level objective function. The objective function for the upstream reservoir may then be simplified to

$$\max_{x^1} Z^1 = c^{11}x^1.$$

If we let x^{1*} denote the value of x^1 that maximizes Z^1, then

$$Z^{1*} = c^{11}x^{1*} = \max_{\hat{x} \in B} c^{11}\hat{x}.$$

Additionally, let $x^{2*} = \psi(x^{1*})$ and

$$Z^{2*} = c^{21}x^{1*} + c^{22}x^{2*}.$$

Thus, the payoff vector for releases x^{1*} from the upstream reservoir is denoted by (Z^{1*}, Z^{2*}). This results in the following two theorems:

Theorem 1 *For any payoff vector $(\tilde{Z}^1, \tilde{Z}^2)$ obtained from releases $\tilde{x}^1$ by operator one, $\tilde{Z}^1 \leq Z^{1*}$ and the solution (Z^{1*}, Z^{2*}) is therefore not strongly dominated.*

Proof. By contradiction. Assume that there does exist a feasible solution $(\tilde{Z}^1, \tilde{Z}^2)$ such that $\tilde{Z}^1 > Z^{1*}$ and $\tilde{Z}^2 > Z^{2*}$. Therefore, $(\tilde{Z}^1, \tilde{Z}^2)$ would strongly dominate (Z^{1*}, Z^{2*}). However,

$$Z^{1*} = c^{11}x^{1*} = \max_{x^1 \in B} \ c^{11}x^1 \geq c^{11}\tilde{x}^1 = \tilde{Z}^1.$$

Hence $Z^{1*} \geq \tilde{Z}^1$. This is a contradiction to the assumption that $\tilde{Z}^1 > Z^{1*}$. ■

Theorem 2 *If Z^{1*} is the optimal solution for operator one obtained from a unique x^{1*}, then the solution (Z^{1*}, Z^{2*}) is always Pareto-optimal.*

Proof. By contradiction. Assume that there does exist a feasible solution $(\tilde{Z}^1, \tilde{Z}^2)$ such that $\tilde{Z}^1 \geq Z^{1*}$ and $\tilde{Z}^2 \geq Z^{2*}$ with strict inequality holding for at least one.

If $\tilde{x}^1 = x^{1*}$, then $\tilde{Z}^1 = Z^{1*}$. However, $\tilde{Z}^2 = Z^{2*}$ since operator two solves the same problem in both cases. Since a strict inequality does not hold for either objective function value, this is a contradiction to the above assumption.

If $\tilde{x}^1 \neq x^{1*}$ then

$$Z^{1*} = c^{11}x^{1*} = \max_{x \in B} \ c^{11}x > c^{11}\tilde{x}^1 = \tilde{Z}^1.$$

Therefore $Z^{1*} > \tilde{Z}^1$. This is a contradiction to the assumption that $\tilde{Z}^1 \geq Z^{1*}$.

Therefore, no feasible solution exists such that $\tilde{Z}^1 \geq Z^{1*}$ and $\tilde{Z}^2 \geq Z^{2*}$ with strict inequality holding for at least one. Hence, by definition, the solution (Z^{1*}, Z^{2*}) is Pareto-optimal. ■

To summarize, with only a downward parametric linkage, the solution obtained by the two operators will never be strongly dominated. Furthermore, if operator one has a unique optimal solution, then the solution obtained must be Pareto-optimal.

4.2 Other solution characteristics

The models also produce the following general observations about systems without central coordination. First, operator two can, at times, actually improve his objective function value by not requiring preannounced upstream policy from the upstream reservoir. This is stated more formally as:

Characteristic 1 *The solution vectors (Z_P^{1*}, Z_P^{2*}) and (Z_I^{1*}, Z_I^{2*}) for the two-reservoir problem may exist such that $Z_I^{2*} > Z_P^{2*}$.*

In addition, Characteristics 2 and 3 provide that the solution obtained with independent operation or with preannounced releases can be strongly dominated:

Characteristic 2 *For the two-reservoir problem with solution vector (Z_P^{1*}, Z_P^{2*}), there may exist a feasible solution with objectives $(\hat{Z}_P^1, \hat{Z}_P^2)$ such that $\hat{Z}_P^1 > Z_P^{1*}$ and $\hat{Z}_P^2 > Z_P^{2*}$.*

Characteristic 3 *For the two-reservoir problem with solution vector (Z_I^{1*}, Z_I^{2*}), there may exist a feasible solution with objectives $(\hat{Z}_I^1, \hat{Z}_I^2)$ such that $\hat{Z}_I^1 > Z_I^{1*}$ and $\hat{Z}_I^2 > Z_I^{2*}$.*

x^1	$\psi(x^1)$	Z_P^1	Z_P^2
(1,1,1)	(1,0,2)	39	45
(1,0,2)	(0,0,3)	37	50
(0,2,1)	(0,1,2)	36	38
(0,1,2)	(0,0,3)	38	46

Table 1: Integer release possibilities for operator one

4.3 Verifying the characteristics

We will establish the existence of all three solution characteristics by use of an example. Let $n = 3$ and $\alpha = 1$. Suppose an inflow of one unit arrives at the upstream reservoir in each of three periods. Both reservoirs begin with zero active storage and have a capacity of one unit. The objective function coefficients for each operator are:

$$c^{11} = (5, 6, 7)$$
$$c^{12} = (9, 5, 6)$$
$$c^{21} = (9, 5, 7)$$
$$c^{22} = (6, 3, 9)$$

Since there are integer inflows and integer capacities, an optimal solution will exist with integer releases. For the problem with *preannounced upstream policy*, operator one has four possible integer releases. These releases along with the rational reaction of operator two and the corresponding objective function values are shown in Table 1.

From Table 1 we see that the rational choice for operator one is to release one unit of water in each period. With this strategy, operator one receives a benefit of 39 while operator two receives 45.

The system-wide problem under the discipline of *independent operation* can be solved using dynamic programming. For this example, the optimal releases for operator one and operator two are:

$$x^1 = (0, 1, 2);$$
$$x^2 = (0, 0, 3).$$

From these releases, operator one realizes an objective function value of 38 and operator two receives 46.

The objective function value of operator two is greater in the case of independent operation than in the case where preannounced releases are required. Thus, operator two would benefit from not requiring preannounced upstream policy from the upstream reservoir.

Additionally, for releases

$$x^1 = (1, 0, 2)$$
$$x^2 = (1, 0, 2)$$

the objective function value for operator one is 40 and for operator two is 47.

These values are strictly greater than those obtained in the previous cases. Thus, both the solution obtained with *preannounced upstream policy* and the solution obtained with *independent operation* are strongly dominated. That is, a solution exists in which both operators would do strictly better. and a central coordinator could improve the objective function value for each reservoir even without using side payments.

5 Conclusions

In this paper, we have modeled a two-reservoir system and considered its operation under three different disciplines. We have shown that if there are only downward parametric linkages, the solution obtained without coordination will not be strongly dominated. Furthermore, this solution will be Pareto-optimal if it is the result of a unique set of releases from the upstream reservoir.

For the general problem, we have shown that the operator of the downstream reservoir may benefit from not requiring preannounced releases from the upstream reservoir. Additionally, the solutions obtained without coordination may be strongly dominated and thus inadmissible.

This paper has considered a simplified model of a two-reservoir, two-operator system. However, the fundamental economic consequences of coordination and information will be embedded within many real systems.

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