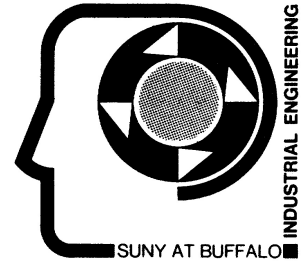


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STATE UNIVERSITY OF NEW YORK AT BUFFALO



MATHEMATICAL METHODS FOR MULTILEVEL PLANNING

by
Wayne F. Bialas
Mark H. Karwan

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Department of Industrial Engineering
State University of New York at Buffalo
Buffalo, New York 14260

Mathematical Methods for Multilevel Planning*

Wayne F. Bialas[†]

Mark H. Karwan[‡]

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Abstract

The general multilevel programming problem is a set of nested optimization problems over a single feasible region. Control over the decision variables is partitioned among the levels, but a decision variable may impact the objective function of several, if not all, levels. This approach is applicable to a variety of water resource planning problems, and will be compared to previous methods of multilevel planning.

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[†]Assistant Professor, Department of Industrial Engineering, State University of New York at Buffalo, Amherst, New York 14260.

[‡]Assistant Professor, Department of Industrial Engineering, State University of New York at Buffalo, Amherst, New York 14260.

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1 Introduction

Many water resource planning problems require compromises among the objectives of several interacting individuals or agencies. Often, these groups are arranged within an administrative or hierarchical structure with independent and perhaps conflicting objectives. For example, the water resource policies set forth by the Federal government affect the objective and options, and hence the strategies, of state officials. This process continues within a hierarchy of decision-makers, including local governments, planning agencies, and basic economic units, such as firms and households. Each unit of the hierarchy wishes to maximize its individual benefit function (or functions) in view of the partial exogenous control exercised at other levels. In the above example, actions at the state level also affect the benefits sought by the Federal government. The Federal government can control this effect by exercising preemptive partial control over the state through budget modifications or regulations.

An important feature of multilevel optimization problems is that a planner at one level of the hierarchy may have his objective function determined, in part, by variables controlled at other levels. However, his control instruments may allow him to influence the policies at other levels, and thereby improve his own objective function. Such policies may include the control of the allocation and use of resources at lower levels, and the control of the benefits conferred upon subordinate levels. Some particular problems are amenable to this framework:

1. **International Water Systems:** An international river basin agency desires to equitably distribute the benefits conferred from a multi-national, multi-reservoir, multi-purpose water system. To do so, the agency will control the operating policies of some, but perhaps not all, reservoirs in the system. In view of these controls, individual nations will determine their water use policies to best meet their individual requirements.
2. **Flood Control:** Seeking to reduce flood risk, government can provide structural flood control measures, implement floodplain zoning programs, and subsidize flood insurance. These policies will influence floodplain development based on the individual objectives and benefits of land users. If not carefully conceived, the governmental programs can be rendered useless if they inadvertently encourage expanded floodplain development.

Note that the above problems have the following characteristics in common:

1. The systems have interacting decision-making units within a predominantly hierarchical or multiechelon structure.
2. Each decision-making unit maximizes net benefits independently of other units, but is affected by the actions of other units as an externality.
3. The external effect on a decision-maker's problem is reflected in both his objective function *and* his set of feasible decisions.

This paper will present an approach for solving a large class of such multilevel planning problems which can be adequately represented as constrained optimization problems.

2 Previous work

Decentralized planning has been long recognized as an important decision-making problem. Mathematical programming methods to solve such problems trace back early in the development of linear programming. The decomposition method of Dantzig and Wolfe [16] for the solution of certain large-scale linear programming problems has served well as the underpinning for much of this study (see, for example, Balas [2], Cooper [12], Geoffrion [19, 20, 21, 23], Lasdon [29]). Such a formulation partitions the decision space among several planning divisions. These units interact through a set of “corporate” constraints involving the decision variables of all divisions. The remaining constraints can be apportioned to each of the divisions, with each constraint a function of the decision variables under the jurisdiction of only a single division.

The subproblem solved by a division maximizes that portion of the overall objective function controlled by the division, subject to the divisional constraints. However, an individual division does not account for interdivisional interactions. This may result in suboptimal (and perhaps infeasible) behavior for the overall problem. The Dantzig-Wolfe method can then be viewed as providing inducements to the division to encourage overall optimal behavior of the corporation. This interpretation rests heavily on the concepts of Koopmans [28] who associated the notions of dual prices and decentralization. A more detailed discussion is provided in an excellent paper by Baumol and Fabian [5].

These techniques were furthered with the work of Charnes, *et al.* [10], who recognized that when subdivisions have alternative solutions to their individual optimization problems, they must receive information from the master planner in order to operate coherently. To provide this information, a preemptive goal was applied to the division objective function forcing it to choose a decision in harmony with the overall system.

The decomposition approach has been successfully applied by Haimes and his associates to a wide range of multilevel planning problems. For example, Haimes, Foley and Yu [25] effectively solve a large model for the control of water quality by a central planning agency with a single overall system objective. The dual variables are interpreted to determine prices (taxes) to be charged to each subproblem (polluter) for violating pollution standards.

Generally, the decomposition approach includes a coordinating mechanism of dual prices preventing the various divisions or agencies from working against the goals of a master planner. However, as pointed out by Baumol and Fabian [5], in practical situations, decentralized decisions may not take into account the benefits or costs incurred by its activities on other subdivisions. When coordination is absent, the system may inherently exhibit suboptimal behavior. Each division can have goals distinct and independent of the others, optimizing its position in view of the external effects imposed by their decisions.

Cassidy, *et al.* [9] proposed a model and solution procedure for a specific case where such a coordinating mechanism does not exist. In analyzing the distribution of a federal budget among several states, they offered a model with some fundamental distinctions. Let T denote the overall budget of the federal government to be distributed to N states, with s_i units to be allocated to state i . Each state, i , has K_i cities and each city has J_{ik} projects from which to choose to fund. Let

$$x_{ijk} = \begin{cases} 1 & \text{if project } j \text{ is undertaken in state } i, \text{ city } k \\ 0 & \text{otherwise} \end{cases}$$

Associated with each project is a cost, c_{ijk} , and a conferred benefit, w_{ijk} . The problem each state wishes to solve is

$$\begin{aligned} \max \quad & \sum_{jk} w_{ijk} x_{ijk} \\ \text{st:} \quad & \sum_{jk} c_{ijk} x_{ijk} \leq s_i & \forall i \\ & \sum_j x_{ijk} \geq 1 & \forall ik \\ & x_{ijk} \in \{0, 1\} & \forall ijk \end{aligned}$$

Let

$$r_i = \frac{\sum_{jk} w_{ijk} - \sum_{jk} w_{ijk} x_{ijk}}{\sum_{jk} w_{ijk}}$$

which represents the relative regret of state i with its budget. The problem to be solved by the federal level is

$$\begin{aligned} \min \quad & [\max_i r_i - \min_i r_i] \\ \text{st:} \quad & \sum_i s_i \leq T \\ & s_i \geq 0 & \forall i \end{aligned}$$

Using a parametric analysis, the optimal x_{ijk} 's resulting from the states' subproblems were determined as a function of the s_i 's. Here, a strict two-level structure was employed. A state maximizes its individual objective given the resource level s_i and impacts directly on the objective function of the master (federal) level.

In the study of a Dutch milk cooperative, Candler and Norton [8] sought a solution to a generalization of the problem posed by Cassidy, *et al.* The government could not influence the dairy prices directly. However, using the mechanism of a subsidy on liquid milk sales and an import duty on butter, the cooperative and its members would vary their production mix of dairy products and thereby alter the price index. This problem required a continuous decision space, and an objective function for each level more general than those proposed by Cassidy, *et al.* Unfortunately, the general problem of Candler and Norton was imprecisely defined, and they did not recognize that the effective feasible regions resulting from the parametric problems at the lower levels were nonconvex sets. This property causes the Candler and Norton algorithm to fail since it requires convexity.

The use of multilevel control theory for the analysis of some multilevel economic systems has been fruitful (see, for example, the work of Meserovic [32]). Often these techniques view the system as a Stackelberg game, a highly structured n -person game (see, for example, Başar [4], Cruz [13, 14], Simaan [37]). Within a broad definition of such games, a static Stackelberg game with fixed leaders and a continuous (or discrete) control space could be defined to encompass multilevel programming problems. However, current methodology does not consider the activity space of one player to be a function of the strategies of other players, a feature necessary for most constrained problems. Such an extension of Stackelberg games would require the payoff function of one level to have discontinuities dependent on the decisions of other players. This formulation is, at best, unwieldy and perhaps intractable.

It must also be noted that multiobjective optimization techniques have been developed to permit a more faithful analysis of the tradeoffs among competing goals (see, for examples, [6, 22, 26, 27, 33, 34, 36, 38, 40, 41, 42]), and assist a planner in reaching an acceptable compromise. Such approaches assume that all objectives are those of a single planner, impacting directly on his state of well-being. Hence, these methods are largely inappropriate

for multilevel optimization with incoherent objectives. The multilevel approach raises and answers questions regarding the assignment of control over certain variables to various levels. In some cases, coalitions of levels could improve the objective functions of all levels. Hence, the multiobjective planner could be viewed as the one, and only, coalition. Furthermore, because of the structure of some multilevel problems, a single level could exercise complete control over the actions of all levels, although controlling only a proper subset of the decision variables.

3 General definition

Let the decision variable space (Euclidean n -space), $\mathbb{R}^n \ni x = (x_1, x_2, \dots, x_n)$, be partitioned among r levels,

$$\mathbb{R}^{n_k} \ni x^k = (x_1^k, x_2^k, \dots, x_{n_k}^k) \quad \text{for } k = 1, \dots, r,$$

where $\sum_{k=1}^r n_k = n$. Denote the maximization of a function $f(x)$ over \mathbb{R}^n by varying only $x^k \in \mathbb{R}^{n_k}$ given fixed $x^{k+1}, x^{k+2}, \dots, x^r$ in $\mathbb{R}^{n_{k+1}} \times \mathbb{R}^{n_{k+2}} \times \dots \times \mathbb{R}^{n_r}$ by

$$\max\{f(x) : (x^k | x^{k+1}, x^{k+2}, \dots, x^r)\}.$$

Let the full set of system constraints for all levels be denoted by S . Then the problem at the lowest level of the hierarchy, level one, is given by

$$(P^1) \begin{cases} \max & \{f_1(x) : (x^1 | x^2, \dots, x^r)\} \\ \text{st:} & x \in S^1 = S \end{cases}$$

The feasible region, $S = S^1$, is defined as the **level-one feasible region**. The solutions to P^1 in \mathbb{R}_1^n for each fixed x^2, x^3, \dots, x^r form a set,

$$S^2 = \{\hat{x} \in S^1 : f_1(\hat{x}) = \max\{f_1(x) : (x^1 | \hat{x}^2, \hat{x}^3, \dots, \hat{x}^r)\}\},$$

called the **level-two feasible region**¹ over which $f_2(x)$ is then maximized by varying x^2 for fixed x^3, x^4, \dots, x^r .

Thus the problem at level two is given by

$$(P^2) \begin{cases} \max & \{f_2(x) : (x^2 | x^3, x^4, \dots, x^r)\} \\ \text{st:} & x \in S^2 \end{cases}$$

In general, the **level- k feasible region** is defined as

$$S^k = \{\hat{x} \in S^{k-1} | f_{k-1}(\hat{x}) = \max\{f_{k-1}(x) : (x^{k-1} | \hat{x}^k, \dots, \hat{x}^r)\}\},$$

Note that \hat{x}^{k-1} is a function of $\hat{x}^k, \dots, \hat{x}^r$. Furthermore, the problem at each level can be written as

$$(P^k) \begin{cases} \max & \{f_k(x) : (x^k | x^{k+1}, \dots, x^r)\} \\ \text{st:} & x \in S^k \end{cases}$$

¹In the terminology of differential games, this is equivalent to the *rational reaction set* of level one. (see Cruz [13]).

which is a function of x^{k+1}, \dots, x^r , and

$$(P^r) : \max_{x \in S^r} f_r(x)$$

defines the entire problem. This establishes a collection of nested mathematical programming problems $\{P^1, \dots, P^r\}$.

Note that the objective at level k , $f_k(x)$, is defined over the decision space of all levels. Thus, the level- k planner may have his objective function determined, in part, by variables controlled at other levels. However, by controlling x^k , after decisions from levels $k + 1$ to r have been made, level k may influence the policies at level $k - 1$ and hence all lower levels to improve his own objective function.

4 Two examples

4.1 The two-level linear resource control problem

The two-level linear resource control problem is the multilevel programming problem of the form

$$\begin{aligned} \max \quad & c^2 x \\ \text{st:} \quad & x \in S^2 \end{aligned}$$

where

$$S^2 = \{\hat{x} \in S^1 : c^1 \hat{x} = \max\{c^1 x : (x^1 | \hat{x}^2)\}\}$$

and

$$S^1 = S = \{x : A^1 x^1 + A^2 x^2 \leq b, x \geq 0\}$$

Here, level 2 controls x^2 which, in turn, varies the resource space of level one by restricting $A^1 x^1 \leq b - A^2 x^2$. Perhaps the idea of nested optimization can be better seen by writing the problem as follows:

$$(P^2) \left\{ \begin{array}{l} \max \quad \{c^2 x = c^{21} x^1 + c^{22} x^2 : (x^2)\} \\ \text{where } x^1 \text{ solves} \\ (P^1) \left\{ \begin{array}{l} \max \quad \{c^1 x = c^{11} x^1 + c^{12} x^2 : (x^1 | x^2)\} \\ \text{st:} \quad A^1 x^1 + A^2 x^2 \leq b \\ x \geq 0 \end{array} \right. \end{array} \right.$$

Note that, by this definition, the model of Cassidy, *et al.* [9] is a two-level resource control problem. The highest level, level two, represents a federal government with a nonlinear objective based on the integer-valued decisions of the states. In this case, no direct cost is assigned to the decision variables of the federal government, i.e., s_1, s_2, \dots, s_N (this corresponds to $c^{22} = 0$ in the linear model). Level one represents the states whose constraints insure distribution of resources to the cities (which Cassidy, *et al.* denotes as a third level, but which is taken here to be part of the decomposable level-one problem).

4.2 The two-level linear price control problem

The two-level linear price control problem is another special case of the general multilevel programming problem given by

$$(P^2) \left\{ \begin{array}{l} \max \{c^2x = c^{21}x^1 + c^{22}x^2 : (x^2)\} \\ \text{st: } A^2x^2 \leq b^2 \\ \text{where } x^1 \text{ solves} \\ (P^1) \left\{ \begin{array}{l} \max \{(x^2)^t x^1 : (x^1 | x^2)\} \\ \text{st: } A^1x^1 \leq b^1 \\ x^1 \geq 0 \end{array} \right. \end{array} \right.$$

In this problem, level two controls the cost coefficients of level one, a problem important to the analysis of tax and other control programs.

The field of application and variation appears fruitful and is by no means limited to the two cases cited above.

5 Properties of the linear resource control problem

Attention will now focus on the two-level linear resource control problem. Of particular importance in developing a solution procedure is analyzing the structure of the set of feasible solutions from which the level-two planner can choose his optimal solution.

5.1 Alternative optimal solutions

Care must be taken when P^k results in alternative optimal solutions for fixed x^{k+1}, \dots, x^r . Although not affecting the value of the level- k objective function, $f_k(x)$, these solutions can have drastically varying impact on the objectives of other levels. Therefore, control over the choice among alternative optimal solutions may have to be delegated to other levels, or an incentive scheme may be required to induce the level- k planner to choose a solution desirable to other levels. If no such scheme is employed, the problem may be ill-defined. Consider the following example of a two-level linear resource control problem:

$$(P^2) \left\{ \begin{array}{l} \max \{x_1 + \frac{1}{2}x_2 : (x_2)\} \\ \text{where } (x_1, x_3) \text{ solves} \\ (P^1) \left\{ \begin{array}{l} \max \{x_2 + x_3 : (x_1, x_3 | x_2)\} \\ \text{st: } \begin{array}{l} x_2 + x_3 = 4 \\ x_2 \geq 1 \\ -x_1 + 2x_2 \leq 2 \\ x_1 + x_2 \leq 4 \\ x_1, x_2, x_3 \geq 0 \end{array} \end{array} \right. \end{array} \right.$$

For $x_2 = 2$, the unique level-one solution is $(x_1, x_2, x_3) = (2, 2, 2)$ with value 4. The corresponding level-two solution value is 3. However, for $x_2 = 1$, there exist alternate optimal

solutions to P^1 , still with value 4, of the form $x = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq 3, x_2 = 2, x_3 = 3\}$. The corresponding level-two objective for this set of solutions ranges continuously from $\frac{1}{2}$ to $3\frac{1}{2}$. For a most favorable solution to be returned to level two for fixed x_2 , i.e., to induce level one to return $x_1 = 3$ for $x_2 = 1$, a side payment to the level-one objective from level two may be employed. For the example, the level-one objective function, $\max\{x_2 + x_3 + \epsilon(x_1 + \frac{1}{2}x_2)\}$ with $\epsilon > 0$ sufficiently small, is a perturbation which accomplishes this. Given this side payment scheme, $x_2 = 1$ is the optimal decision for level two.

Note that, in general, such a perturbation method may not determine a unique solution since level two may have the same objective for a number of level-one alternate optimal solutions. However, any of these solutions are satisfactory for level two.

A more formal scheme, employing the concepts of preemptive goals developed by Charnes, *et al.* [10], may be suggested. Although developed for a problem [that] fits the structure of a large-scale linear programming model solvable by Dantzig-Wolfe decomposition, the same concepts can be employed in the fundamentally different problem presented here. This may be accomplished by introducing a preemptive goal into the functional of level one as follows:

$$(P^1) \begin{cases} \max & \{c^1x + M|c^2x^* - c^2x| : (x^1 | x^2)\} \\ \text{st:} & A^1x^1 \leq b - A^2x^2 \\ & x \geq 0 \end{cases}$$

where $x^* = (x_1^*, x_2)$ is a solution to P^2 with highest level-two objective value, and M is a positive, arbitrarily large number.

5.2 Nonconvexity

In the two-level linear resource control problem,

$$S^1 = \{x \geq 0 : A^1x^1 + A^2x^2 \leq b\},$$

is a convex set. However,

$$S^2 = \{\hat{x} \in S^1 : c^1\hat{x} = \max\{c^1x : (x^1 | \hat{x}^2)\}\}$$

need not be. Therefore, P^2 , which can be written as

$$\begin{aligned} \max & \quad c^2x \\ \text{st:} & \quad x \in S^2 \end{aligned}$$

involves the optimization of a linear function over a nonconvex region. This is a characteristic not found in decentralized planning models [that] fit a large-scale linear programming format. Consider the example shown in Figure 1. The set S^2 for the problem in Figure 1 is a subset of the edges of the boundary of S^1 . In problems of higher dimension, S^2 is composed of edges

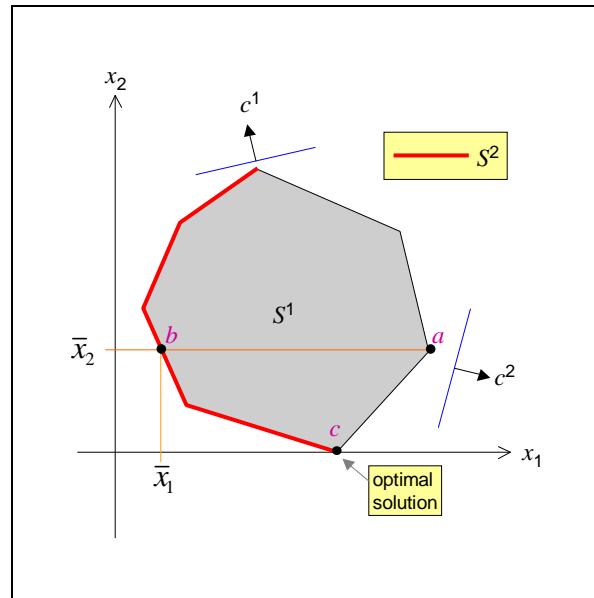


Figure 1: Example of Nonconvexity of S^2

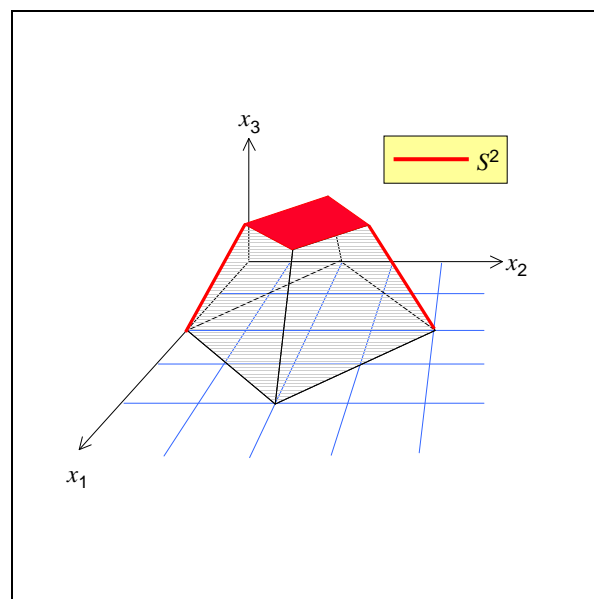


Figure 2: Example of S^2 in Three Dimensions

and faces of the boundary of S^1 . In the following three dimensional example:

$$(P^2) \left\{ \begin{array}{l} \max \{x_2 + x_3 : (x_2)\} \\ \text{where } (x_1, x_3) \text{ solves} \\ (P^1) \left\{ \begin{array}{l} \max \{x_3 : (x_1, x_3 | x_2)\} \\ \text{st: } x_1 + x_2 - x_3 \geq 2 \\ x_1 - x_2 + x_3 \leq 2 \\ x_1 + x_2 + x_3 \leq 6 \\ x_1 + x_2 + x_3 \geq 2 \\ x_3 \leq 1 \end{array} \right. \end{array} \right.$$

S^2 is the red region shown in Figure 2.

5.3 Further characterization of S^2 and P^2

The following theorem and its corollaries help to characterize both S^2 and the optimal solution for P^2 in the two-level linear resource control problem. The proofs for these results can be found in Appendix A.

Theorem 5.1 Suppose $S^1 = \{x : Ax = b, x \geq 0\}$ is bounded. Let

$$S^2 = \{\hat{x} = (\hat{x}^1, \hat{x}^2) \in S^1 : c^1 \hat{x}^1 = \max\{c^1 x^1 : (x^1 | \hat{x}^2)\}\}$$

then the following hold:

- (i) $S^2 \subseteq S^1$
- (ii) Let $\{y_t\}_{t=1}^r$ be any r points of S^1 , such that $x = \sum_t \lambda_t y_t \in S^2$ with $\lambda_t \geq 0$ and $\sum_t \lambda_t = 1$. Then $\lambda_t > 0$ implies $y_t \in S^2$.

Hence, any point in S^1 which positively contributes in any convex combination forming a point in S^2 , must also be in S^2 . Since this is true of any $y_t \in S^1$, including y_t which are extreme points of S^1 , the following corollary results:

Corollary 5.1 If x is an extreme point of S^2 , then x is an extreme point of S^1 .

Using the results above, one can conclude that S^2 is a very special portion of the boundary of S^1 .

Recalling that P^2 may be formulated as $\max_{x \in S^2} c^2 x$ and noting the correspondence of extreme points in S^2 and S^1 , the following result is easily derived:

Corollary 5.2 An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables (S^1).

This is an important result since it justifies extreme point search procedures as a basis for algorithmic approaches to solving the two-level linear resource control problem.

5.4 Cooperation of levels

Optimal solutions to the multilevel programming problem may not be Pareto optimal. While cooperation might improve the objective function at every level, the order and independence with which decisions are made prevent such cooperation. This rules out any algorithmic approach which seeks only Pareto optimal solutions and is one of the distinguishing characteristics between multiobjective and multilevel programming.

The model of Cassidy *et al.* [9] discussed previously can possess optimal solutions which are not Pareto optimal. Recall that it is viewed here as a two-level resource control problem. While models exist to achieve Pareto optimal solutions, they are inappropriate for applications such as Cassidy's.

For a specific example of this behavior, consider Figure 1. Both levels have higher objective values at point (a). However, for x_2 fixed at \bar{x}_2 , level one will choose $x_1 = \bar{x}_1$ (point (b)), thus point (a) is not in S^2 . This leads to the best choice of x_2 to be $x_2 = 0$ with the optimal solution at point (c).

5.5 A complete control theorem for the two-level linear resource control problem

Consider the two-level linear resource control problem. Given any basis $B \subseteq A$ for the set of constraints $Ax = b$, one can write the equivalent set of constraints on x :

$$Bx_B + Nx_N = b$$

or, rewriting,

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

When x_N is fixed, x_B is uniquely determined. Thus to have complete control of the solution, the level-two planner need only control the complete set of nonbasic variables corresponding to *any* basis.

6 Algorithmic approaches to the two-level resource control problem

Based on the above characteristics of the set of level-two feasible solutions, S^2 , a number of algorithmic approaches can be suggested. Since the optimal solution has been shown to be an extreme point of S^2 , an equivalent formulation would be the following:

$$(P^2) \begin{cases} \max & c^2x \\ \text{st:} & x \in [S^2] \end{cases}$$

where $[S^2]$ represents the convex hull of S^2 . The set of extreme points for $[S^2]$ are, by definition of convex hull, identical to the set of extreme points of S^2 .

A number of algorithmic approaches for finding optimal solutions to the general problem, by establishing the convex hull of S^2 with a cutting plane method, are being explored. An

approach based on an “outer approximation” appears promising but must be further developed. Another technique based on cutting planes constructed from the level-two objective function is also being examined closely. This approach is based on two nested parametric linear programming problems. All of the methods discussed above possess the ability to generate local and potentially global optimal solutions in S^2 before adding cuts to achieve global optimality.

Any desirable algorithm for the two-level linear resource control problem should exhibit some particular properties.

Consider the solution, $\hat{x} = (\hat{x}^1, \hat{x}^2)$ to the following problem:

$$(\hat{P}) \begin{cases} \max & c^2 x \\ \text{st:} & Ax = b \\ & x \geq 0 \end{cases}$$

In (\hat{P}) , the level-two planner is given full control over all variables. Now fix $x^2 = \hat{x}^2$ and solve the following problem with solution \bar{x} to determine if $\hat{x} \in S^2$:

$$(\bar{P}) \begin{cases} \max & c^1 x \\ \text{st:} & A_1 x^1 = b - A_2 \hat{x}^2 \\ & x^1 \geq 0 \end{cases}$$

If $\bar{x} = \hat{x}$ then $\hat{x} \in S^2$ is an optimal solution to the overall problem. For example, note that in Figure 1, the vector c^2 could be changed to produce a solution to (\hat{P}) at any extreme point of S^2 . The set S^2 does not vary with changes in the second-level objective, and hence quite different choices of c^2 can produce an optimal solution after solving (\hat{P}) . For the example shown in Figure 1, two particular choices of c^2 which lead to such a condition for the level-one objective shown are both $c^2 = c^1$ and $c^2 = -c^1$. Thus both highly complementary and highly conflicting objectives (as well as many in between) may lead to solutions after solving the two linear programming problems (\hat{P}) and (\bar{P}) . Any reasonable algorithm should have the ability to easily solve any problems for which $\hat{x} \in S^2$.

6.1 An algorithm to find local optimal solutions

While the cutting plane and other approaches to be further explored will guarantee *global* optimal solutions, an important step in some of the procedures will consist of first finding a local optimal solution. Consider the following portion of a bounded simplex tableau to be employed in the proposed algorithm to find a local optimal solution:

	x_1^2	x_2^2	\cdots	x_k^2	RHS
	r_1^2	r_2^2	\cdots	r_k^2	z^2
x_{B_1}	y_{11}	y_{12}	\cdots	y_{1k}	\bar{b}_1
x_{B_2}	y_{21}	y_{22}	\cdots	y_{2k}	\bar{b}_2
\vdots		\vdots			\vdots
x_{B_m}	y_{m1}	y_{m2}	\cdots	y_{mk}	\bar{b}_m

The variables x_1^2, \dots, x_k^2 , represent the nonbasic level-two variables which are at nonzero values, and r_1^2, \dots, r_k^2 represent the reduced costs of these variables with respect to the level-two objective function. In terms of the present basis $B \subseteq A$, $\bar{b} = B^{-1}b - \sum_{j=1}^k y_j x_j^2$ where $(y_{1j}, y_{2j}, \dots, y_{mj})^t = y_j = B^{-1}(a_{1j}, a_{2j}, \dots, a_{mj})^t$ and x_{B_i} is the i^{th} basic variable.

The following algorithm guarantees a local optimal solution:

Step 1. Solve the following problem with optimal solution $\hat{x} = (\hat{x}^1, \hat{x}^2)$ and optimal tableau \hat{T} via the simplex method:

$$\begin{array}{ll} \max & c^2 x \\ \text{st:} & Ax = b \\ & x \geq 0 \end{array}$$

Step 2. Set $x^2 = \hat{x}^2$ and solve the following problem via bounded simplex ($l = u = \hat{x}^2$) beginning with tableau \hat{T} :

$$\begin{array}{ll} \max & c^1 x \\ \text{st:} & Ax = b \\ & x^2 = \hat{x}^2 \\ & x^1 \geq 0 \end{array}$$

Let the optimal solution be \bar{x} . If $\bar{x} = \hat{x}$, stop; \hat{x} is a global optimal solution. Otherwise, go to step 3a with current tableau \bar{T} and relax the constraints $x^2 = \hat{x}^2$.

Step 3a. If all nonbasic variables are equal to zero, go to step 4 with current tableau \bar{T} . Otherwise go to step 3b.

Step 3b. If $\bar{b}_i > 0$ for all i , go to step 3c. Otherwise, without loss of generality, consider $\bar{b}_\ell = 0$. Choose $y_{\ell j}$ such that $1 \leq j \leq k$ and $y_{\ell j} \neq 0$. Bring x_j^2 into the basis via a degenerate pivot. Go to step 3a.

Step 3c. Consider any nonbasic variable which is at a strictly positive value, say x_j^2 . If $r_j^2 \leq 0$, increase x_j^2 until it enters the basis. If $r_j^2 > 0$, decrease x_j^2 until it either reaches zero or it must enter the basis. Go to Step 3a.

Step 4. Beginning with tableau \bar{T} solve the following problem via a modified simplex procedure:

$$\begin{array}{ll} \max & c^2 x \\ \text{st:} & Ax = b \\ & x \geq 0 \end{array}$$

The modification is as follows: Given a candidate to enter the basis (one for which $c^2 x$ will increase) only allow it to enter if the resulting basic solution, \tilde{x} , will be contained in S^2 . This is determined by obtaining the solution $\tilde{\tilde{x}}$ to the following problem:

$$\begin{array}{ll} \max & c^1 x \\ \text{st:} & A_1 x^1 \leq b - A_2 x^2 \\ & x^1 \geq 0 \\ & x^2 = \tilde{x}^2 \end{array}$$

via dual simplex on repeated applications of step 4. If $\tilde{\tilde{x}}^1 = \tilde{x}^1$ then enter the candidate into the basis. Repeat step 4 until no more candidates exist which satisfy the above modification, then stop.

6.2 Validation and convergence

The algorithm begins by finding the maximum of the second-level objective over the entire feasible region, S^1 . A check in step 2 is then made to determine if the resulting solution is in S^2 . If so, the algorithm terminates with a global optimal solution and has solved what was previously termed an easy problem. If the termination does not occur in step 2, the resulting solution from step 2 is by definition contained in S^2 . Since the bounded simplex algorithm was employed, a number of nonbasic level-two variables may be at nonzero values corresponding to appropriate components of \hat{x}^2 . Degeneracy may also have been introduced by fixing the components of x^2 from step 1.

The purpose of step 3 is to relax the constraint $x^2 = \hat{x}^2$ and to move to an extreme point x° which satisfies $x^\circ \in S^2$ and $c^2 x^\circ \geq c^2 \bar{x}$. If a right hand side \bar{b}_ℓ , from the current tableau is equal to zero then step 3b is entered to perform a degenerate pivot. Some nonbasic variable, $x_j^2, j = 1, 2, \dots, k$, is then brought into the basis at its current positive level and x_{B_ℓ} becomes nonbasic at its current value of zero. Thus the number of basic variables [equal to] zero is reduced by one. This is repeated until no degeneracy is present. Note that such a pivot is always possible, that is, $y_{\ell_j} \neq 0$ for some $j = 1, \dots, k$. Suppose that $y_{\ell_j} = 0$ for all $j = 1, \dots, k$. Then repeated applications of step 3c would result in a degenerate extreme point of the original feasible region, S^1 , since x_{B_ℓ} will remain zero no matter how x_1^2, \dots, x_k^2 are varied. This contradicts the original nondegeneracy assumption.

If all $\bar{b}_i > 0$ but there are still nonbasic level-two variables, x_1^2, \dots, x_k^2 , at nonzero values, then step 3c is entered. Any variable $x_j^2, j = 1, \dots, k$, is chosen to be increased or decreased depending on its reduced level-two cost, r_j^2 . Since there are no explicit upper bounds on x_j^2 , any increase is limited by a current basic variable reaching zero. The original problem is bounded, so this must occur. If x_j^2 is decreased, again a current basic variable may reach zero or else x_j^2 itself will become zero. In either case, the number of nonzero nonbasic variables is decreased by one.

The points generated in step 3c can be shown to be contained in S^2 which is assumed when step 4 is entered. Recall that $\bar{b}_i > 0$ for all i as a result of step 3b. Thus there exists two scalars, $\theta_1 > 0$ and $\theta_2 > 0$ such that any increase or decrease in x_j^2 by an amount less than or equal to θ_1 and θ_2 respectively results in a feasible solution (i.e., a point in S^1). This implies that the current solution, which is in S^2 , is a convex combination of two feasible points resulting from a strict increase and a strict decrease in x_j^2 . By Theorem 5.1, such points must also be in S^2 . Thus each point resulting from step 3c must be contained in S^2 .

Step 4 is entered when an extreme point of S^2 has been obtained. A modified simplex method is used to take steps in S^2 along which the level-two objective increases. This is accomplished by using the normal simplex rules with objective $c^2 x$ along with a check that no basis change results in leaving S^2 . The algorithm terminates with an extreme point solution in S^2 which has the property that all adjacent extreme points either lead to a decrease in $c^2 x$ or are not in S^2 . Thus a local optimal solution is obtained.

Convergence of the algorithm is established by noting the following facts:

1. The feasible region defined by $S^1 = \{x : Ax = b, x \geq 0\}$ is bounded and each basis is nondegenerate.
2. Steps 1, 2 and 4 are finite since the simplex, bounded simplex and dual simplex procedures are finite under fact (1).

3. Each application of step 3b strictly decreases the number of basic variables [equal to zero] and also the number of nonzero nonbasic variables.
4. Each application of step 3c reduces the number of nonzero nonbasic variables by one.

7 Conclusions

Multilevel mathematical programming problems, if carefully defined, can serve as useful tools in modeling structured economic systems. Such models can predict the inefficiencies of non-Pareto optimal decisions and identify the seats of true control within hierarchical organizations.

This paper has proposed a general mathematical structure for such problems, and specifically characterizes the two-level linear resource control problem. For this problem, Theorem 5.1 illustrates a key property of the nonconvex feasible region viewed by level two. As a foundation, it justifies extreme point solution techniques and [suggests] the need for methods to establish the convex hull of the level-two feasible region. Towards this goal, this paper has offered an adjacent extreme point method which can find local, and sometimes global, optimal solutions to the two-level linear resource control problem.

A Proofs of major results

Theorem A.1 (Theorem 5.1) *Suppose $S^1 = \{x : Ax = b, x \geq 0\}$ is bounded. Let*

$$S^2 = \{\hat{x} = (\hat{x}^1, \hat{x}^2) \in S^1 : c^1 \hat{x}^1 = \max\{c^1 x^1 : (x^1 | \hat{x}^2)\}\}$$

then the following hold:

(i) $S^2 \subseteq S^1$

(ii) *Let $\{y_t\}_{t=1}^r$ be any r points of S^1 , such that $x = \sum_t \lambda_t y_t \in S^2$ with $\lambda_t \geq 0$ and $\sum_t \lambda_t = 1$. Then $\lambda_t > 0$ implies $y_t \in S^2$.*

Proof: (i) $S^2 \subseteq S^1$ by the definition of S^2 .

(ii) (By contradiction) Let $y_1, y_2, \dots, y_r \in S^1$ with

$$x = (x^1, x^2) = \sum_{t=1}^r \lambda_t y_t \in S^2$$

and

$$\lambda_t \geq 0, \quad \lambda_1 > 0, \quad \sum_{t=1}^r \lambda_t = 1.$$

Suppose $y_1 = (y_1^1, y_1^2) \notin S^2$. Then there exists \tilde{y}_1^1 such that $\tilde{y}_1 = (\tilde{y}_1^1, \tilde{y}_1^2) \in S^2$ with $y_1^2 = \tilde{y}_1^2$ and $c^1 \tilde{y}_1^1 > c^1 y_1^1$.

Using (i), $\tilde{y}_1^1 \in S^1$. Therefore,

$$\tilde{x} = (\tilde{x}^1, \tilde{x}^2) = \lambda_1 \tilde{y}_1 + \sum_{t=2}^r \lambda_t y_t \in S^1$$

since S^1 is convex. Noting that $\tilde{x}^2 = x^2$ and $\lambda_1 > 0$, we have

$$c^1 x^1 = c_1^1 \lambda_1 y_1^1 + \sum_{t=2}^r c_1^1 \lambda_t y_t^1 < c_1^1 \lambda_1 \tilde{y}_1^1 + \sum_{t=2}^r c_1^1 \lambda_t y_t^1 = c^1 \tilde{x}^1$$

given, $x = (x^1, x^2) \in S^2$ and $\lambda_1 > 0$, we have established an \tilde{x} with the following properties:

- (a) $\tilde{x} = (\tilde{x}^1, \tilde{x}^2) \in S^1$
- (b) $x^2 = \tilde{x}^2$
- (c) $c^1 x^1 < c^1 \tilde{x}^1$.

This contradicts the definition of S^2 since $x \in S^2$ maximizes $c^1 x^1$ for the fixed value of x^2 . Therefore $\lambda_1 > 0$ implies $y_1 \in S^2$. Since the choice of y_i among the y 's was arbitrary, we have proven that $\lambda_t > 0$ implies $y_t \in S^2$. ■

Corollary A.1 (Corollary 5.1) *If x is an extreme point of S^2 , then x is an extreme point of S^1 .*

Proof: (By contradiction) Let x be an extreme point of S^2 . Suppose x is not an extreme point of S^1 . Then there exist extreme points $y_1, \dots, y_r \in S^1$, and $\lambda_1 > 0, \dots, \lambda_r > 0$, $\sum_{t=1}^r \lambda_t = 1$ such that $x = \sum_{t=1}^r \lambda_t y_t$. From Theorem A.1, this implies $y_1, \dots, y_r \in S^2$ and hence x cannot be an extreme point of S^2 , a contradiction. ■

Corollary A.2 (Corollary 5.2) *An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables (S^1).*

Proof: The two-level linear resource control problem can be written as $\max_{x \in S^2} c^2 x$. Since $c^2 x$ is linear, if a solution exists, one must occur at an extreme point of S^2 (alternative optimal solutions at nonextreme points may exist). By Corollary A.1, this must be an extreme point of S^1 . ■

References

- [1] Abernathy, W.J., B. Nicholas, J.C. Hershey and S. Wandel, "A Three-Stage Manpower Planning and Scheduling Model – A Service-Sector Example," *Operations Research*, Vol. 21, No. 3, 1973, pp. 693–711.
- [2] Balas, E., "An Infeasibility Pricing Decomposition Methods for Linear Programs," *Operations Research*, Vol. 14, No. 5, September-October 1966, pp. 847–873.
- [3] Baker, N.R., W.E. Sanders, C.R. Shumway, P.M. Maher, and A.H. Rubenstein, "A Budget Allocation Model for Large Hierarchical R&D Organizations," *Management Science*, Vol. 23, No. 1, September 1976, pp. 59–70.
- [4] Başar, T., "On the Relative Leadership Property of Stackelberg Strategies," *Journal of Optimization Theory and Applications*, Vol. 11, No. 6, 1973, pp. 655–661.
- [5] Baumol, W.J. and T. Fabian, "Decomposition, Pricing for Decentralization and External Economies," *Management Science*, Vol. 11, September 1964, pp. 1–32.

- [6] Benayoum, R., J. Montgolfier, J. Tergny and O. Laritchev, "Linear Programming with Multiple Objectives: STEP Methods (STEM)," *Mathematical Programming*, Vol. 1, 1971.
- [7] Bracken, J. and J.T. McGill, "Mathematical Programs with Optimization Problems in the Constraints," *Operations Research*, Vol. 21, 1973, pp. 37–44.
- [8] Candler, W. and R. Norton, *Multilevel Programming*, unpublished research memorandum, DRC, World Bank, Washington, D.C., August 1976.
- [9] Cassidy, R., M.J. Kirby and W.M. Raikes, "Efficient Distribution of Resources Through Three Levels of Government," *Management Science*, Vol. 17, No. 8, April, 1972, pp. 462–473.
- [10] Charnes, A., R.W. Clower and K.O. Kortanek, "Effective Control Through Coherent Decentralization with Preemptive Goals," *Econometrica*, Vol. 35, No. 2, April 1967, pp. 294–319.
- [11] Charnes, A. and W. Cooper, *Management Models and Industrial Applications of Linear Programming*, John Wiley & Sons, 1961.
- [12] Cooper W.W., H.J. Leavitt and M.W. Shelly II, *New Perspectives in Organization Research*, John Wiley & Sons, 1964.
- [13] Cruz, J.B., "Stackelberg Strategies for Multilevel Systems," in *Directions in Decentralized Control, Many Person Optimization and Large-Scale Systems*, Y.C. Ho and S.K. Mitter, Eds., New York, Plenum Press, 1976, pp. 139–147.
- [14] Cruz, J.B., "Leader-Follower Strategies for Multilevel Systems," *IEEE Transactions on Automatic Control*, Vol. AC-23, April, 1978, pp. 244–255.
- [15] Davis, W. and J. Talavage, "Three-Level Models for Hierarchical Coordination," *OMEGA, The International Journal of Management Science*, Vol. 5, No. 6, pp. 709–720.
- [16] Dantzig, G.B. and P. Wolfe, "Decomposition Principle for Linear Programs," *Operations Research*, Vol. 8, No. 1, 1960, pp. 101–111.
- [17] Freeland, J. and N. Baker, "Goal Partitioning in a Hierarchical Organization," *The International Journal of Management Science*, Vol. 3, No. 6, 1975, pp. 673–678.
- [18] Freeland, J. and J. Moore, "Some Organizational Properties of Resource Directive Allocation Mechanisms," Research Paper No. 389, Stanford University, October 1975.
- [19] Geoffrion, A., "Primal Resource-Directive Approaches for Optimizing Non-Linear Decomposable Systems," *Operations Research*, Vol. 18, No. 3, May-June, 1970.
- [20] Geoffrion, A., "Elements of Large Scale Mathematical Programming," *Management Science*, Vol. 16, No. 11, July, 1970, pp. 652–691.
- [21] Geoffrion, A., "Large Scale Linear and Nonlinear Programming," in *Optimization Methods for Large Scale Systems*, David A. Wisnes, Ed., McGraw-Hill Book Company, 1971, pp. 75–120.
- [22] Geoffrion, A., J. Dyer and A. Feinberg, "An Interactive Approach for Multi-Criterion Optimization of an Academic Department," *Management Science*, Vol. 19, No. 4, December, 1972, Part 1, pp. 357–368.

- [23] Geoffrion, A. and W. Hogan, "Coordination of Two Level Organization with Multiple Objectives," in Balakrishnan, A.V., Ed., *Techniques of Optimization*, Academic Press, 1972.
- [24] Goreaux, L. M. and A.S. Manne, *Multi-Level Planning: Case Studies in Mexico*, North-Holland, Amsterdam, 1973.
- [25] Haimes, Y.Y., J. Foley and W. Yu, "Computational Results for Water Pollution Taxation Using Multilevel Approach," *Water Resources Bulletin*, Vol. 8, No. 4, August, 1972, pp. 761–771.
- [26] Haimes, W.A. Hall and H.T. Freedman, *Multiobjective Optimization in Water Resources Systems*, Elsevier, Amsterdam, 1975.
- [27] Keeney, R.L. and H. Raiffa, *Decision with Multiple Objectives: Preferences and Value Trade-offs*, John Wiley, New York, 1976.
- [28] Koopmans, T., "Analysis of Production as an Efficient Combination of Activities," in Koopmans, T.C., Ed. *Activity Analysis of Production and Allocation*, John Wiley, New York, 1965, pp. 33–94.
- [29] Lasdon, L., *Optimization Theory for Large Systems*, The MacMillan Company, New York, 1970.
- [30] Maier, S.F. and J.H. VanderWeide, "Capital-Budgeting in the Decentralized Firm," *Management Science*, Vol. 23, No. 4, December, 1976, pp. 433–443.
- [31] Mathews, J.A., "Primal and Dual Game Theoretic Solution with Application to Decentralized Control," Working Paper, University of Rhode Island, June 1978.
- [32] Mesarovic, M., D. Macko, and Y. Takahara, *Theory of Hierarchical Multi-Level Systems*, Academic Press, New York, 1970.
- [33] Osteryoung, J.S., "Multiple Goals in the Capital Budgeting Decision," in Cochrane, J.L. and Zeleny, M., Eds., *Multiple Criteria Decision Making*, University of South Carolina Press, Columbia, S.C., 1972.
- [34] Ruefli, T., "A Generalized Goal Decomposition Model," *Management Science*, Vol. 17, No. 9, 1971.
- [35] Ruefli, T., "Behavioral Externalities in Decentralized Organizations," *Management Science*, Vol. 17, No. 5, 1971.
- [36] Ruefli, T., "Linked Multicriteria Decision Models," in Cochrane, J.L. and Zeleny, M., Eds., *Multiple Criteria Decision Making*, University of South Carolina Press, Columbia, S.C., 1972.
- [37] Simaan, M. and J.B. Cruz, "On the Stackelberg Strategy on Nonzero-Sum Games," *Journal of Optimization Theory and Applications*, Vol. 11, No. 5, 1973, pp. 533–555.
- [38] Steuer, R.E., "Multiple Objective Linear Programming with Interval Criteria Weights," *Management Science*, Vol. 23, 1976.
- [39] Whinston, A., "Pricing Guides in Decentralized Organization," in W. Cooper, *et al.*, Eds., *New Perspectives in Organization Research*, John Wiley & Sons, New York, 1964.

- [40] Yu, P.L. and M. Zeleny, "The Set of all Nondominated Solutions in Linear Cases and a Multicriteria Simplex Method," *Journal of Mathematical Analysis and Applications*, Vol. 49, 1975.
- [41] Zionts, S. and J. Wallenius, "An Interactive Programming Method for Solving the Multiple Criteria Problem," *Management Science*, Vol. 22, 1976.
- [42] Zionts, S., "A Survey of Multiple Criteria Integer Programming Methods," paper presented at the DO77 conference, University of British Columbia, 1977.