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TECHNICAL REPORT

Nash-Stackelberg Equilibrium Solutions for Linear Multidivisional Multilevel Programming Problems

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Abstract

Multidivisional multilevel programming is a methodology for modelling multilevel organizations with more than one decision maker at each level of the hierarchy. A decision maker may affect the feasible decisions and benefit functions of all others in the organization. The problem is analyzed as a multistage Stackelberg game with a Nash game embedded at each level. Under mild conditions, it is proved that a Nash-Stackelberg equilibrium solution must exist at an extreme point for any bounded linear multidivisional multilevel programming problem. For a special case, a solution procedure that searches all extreme points is described and used to solve an example problem.

Keywords: mathematical programming, multilevel optimization, game theory, bilevel programming, equilibrium solutions

1 Introduction

Often planning problems involve a finite number of decision makers organized in a hierarchical decision-making process with more than one decision maker at each level of the hierarchy. This paper considers a class of planning problems called *multidivisional multilevel program problems* where every decision maker may affect the feasible decisions and benefit functions of any other decision makers in the organization.

The concept of multidivisional multilevel programming is based on bilevel (two-level) programming (see, for example, Amouzegar [1], Bard [4, 5], Bard and Falk [6], Bialas and Karwan [8, 9], Candler and Townsely [12], Fortuny-Amat and McCarl [16], Migdalas *et al.* [20], Vicente and Calamai [32], Wen and Hsu [36] and White and Anandalingam [37]). This paper will show these principles can be generalized allowing the analysis of an arbitrary number of levels and an arbitrary number of decision-makers. The resulting problem will be a combination of non-convex programming, game theory and a bit of topology.

The general structure of a multidivisional multilevel programming problem is illustrated in Figure 1. The decision makers (referred to as *divisions*) are organized as a hierarchy of L planning levels. The divisions at the *top level* (Level L) make their decisions first. Then, the divisions at the next level (Level $L-1$) specify their decisions given the decisions made by Level L. Continuing down the hierarchy, decisions are made one level after the other. All of the divisions at any one level react simultaneously to the preemptive decisions from all upper levels. This decision process is completed with a decision by the divisions at the *bottom level* (Level 1).

Multidivisional multilevel programming problems have the following characteristics:

- The decision-making divisions are arranged within a hierarchical multilevel structure with one or more divisions at each level.
- Each division controls only a subset of the decision variables in the system but all variables are subject to global constraints that define the system-wide relationships of the decision-making environment.
- Decisions are implemented sequentially from upper to lower level, and simultaneously within each level.
- Each division acts to maximize (or minimize) its own benefits (or costs), but is affected by the actions of other divisions.

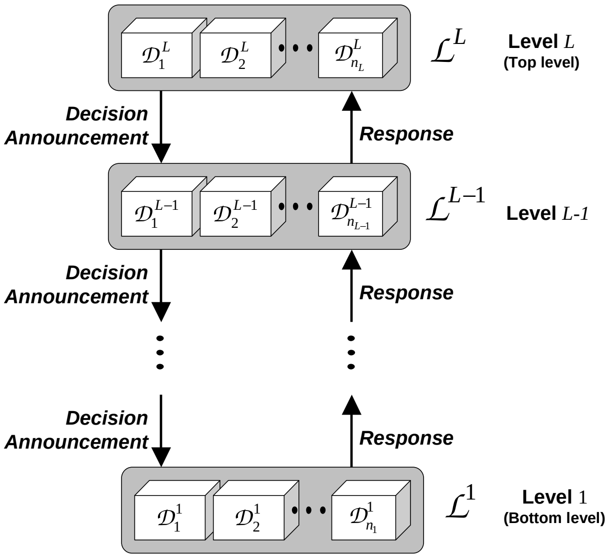


Figure 1: A multidivisional multilevel decision-making system

- The external effect on a division's problem from other divisions can alter both its objective function and its set of feasible decisions.
- Every division has perfect information.
- Cooperation among the divisions is not permitted.
- When making its decision, a division is fully informed about all decisions at upper levels, but provided no information about the decisions of other divisions at its own level or below until the conclusion of the decision process.

When there is only one division at each level, the multidivisional multilevel programming problem is a *multilevel programming problem*.

Multidivisional multilevel programming problems can be analyzed using concepts from game theory. *Within each level*, the divisions play an n-person non-zero sum game similar to those studied and solved by Nash [22]. *Between levels*, the sequential decision process is an n-person leader-follower game similar to those studied and solved by von Stackelberg [34]. Thus, the overall problem can be thought of as a Stackelberg game embedded with "Nash-type" decision problems at each level.

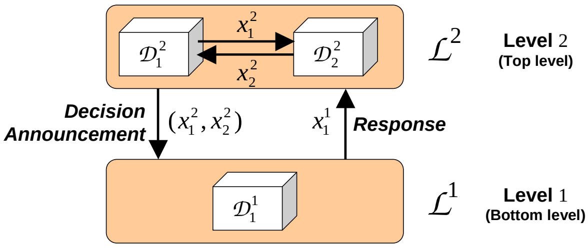


Figure 2: The decision making system for Example One

For this reason, the multidivisional multilevel programming problem is called a **Nash-Stackelberg game**.

The Nash-Stackelberg equilibrium solution concept has appeared in previous papers. For example, Cruz [13] studied *Stackelberg-Nash* equilibria as solutions for dynamic hierarchical systems. Prentice and Sibly [25] and Rob [26] applied the concept of *Nash-Stackelberg-Hybrid Equilibria* to obtain equilibrium price distributions in economic markets.

Similar problems have been investigated by Anandalingam and Apprey [2], Bard [3], Brito and Intriligator [11], Liu [19], Murphy *et al.* [21], Nishizaki *et al.* [24], Sherali *et al.* [27, 28, 29], Sinha *et al.* [31], and Zangwill and Garcia [38, 39]. The existence and characterization of solutions have been established for only a few specific cases.

2 Example one

As an introductory example, consider the decision problem illustrated in Figure 2 with three divisions and two levels. Superscripts are used to index the levels and subscripts are used to index the divisions at a specific level. Therefore, the decision variable x_i^k belongs to division $\mathcal{D}_i^k$, the i-th division at Level k.

The vector of all decision variables is denoted by $x \equiv (x_1^1, x_1^2, x_2^2)$. At Level two (the "top" level):

- Division $\mathcal{D}_1^2$ controls decision variable x_1^2 and seeks to maximize the objective function $c_1^2 x = 2x_1^1 + 0.7x_1^2 - 0.6x_2^2$

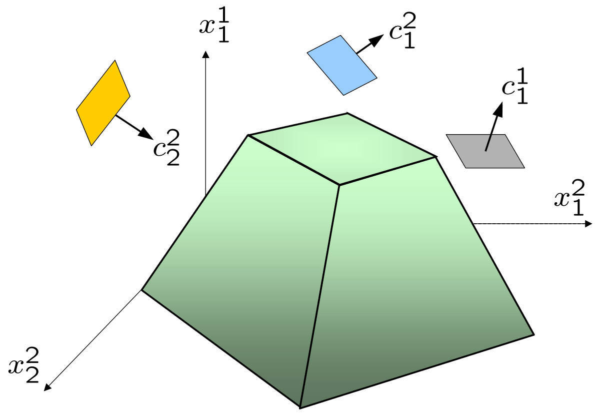


Figure 3: A graphical representation of Example One

- Division $\mathcal{D}_2^2$ controls decision variable x_2^2 and seeks to maximize the objective function $c_2^2 x = -2x_1^1 + 2x_1^2 - 1.5x_2^2$

At Level one (the "bottom" level):

- Division $\mathcal{D}_1^1$controls decision variable x_1^1 and seeks to maximize the objective function $c_1^1 x = x_1^1 + 0.8x_1^2 + 1.2x_2^2$

Furthermore, the values of x_1^1, x_1^2, and x_2^2 must satisfy all of the following constraints:

$$\begin{array}{l} x_1^1 + x_1^2 + x_2^2 \leq 3 \\ -x_1^1 + x_1^2 + x_2^2 \geq 1 \\ x_1^1 + x_1^2 - x_2^2 \leq 1 \\ x_1^1 - x_1^2 + x_2^2 \leq 1 \\ x_1^1 \leq 0.5 \\ x_1^1, x_1^2, x_2^2 \geq 0. \end{array}$$

The feasible values for $x = (x_1^1, x_1^2, x_2^2)$ are represented by the hexahedron shown in Figure 3. First, $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ at Level two separately and simultaneously choose values for x_1^2 and x_2^2. Then, given the value of (x_1^2, x_2^2), the division $\mathcal{D}_1^1$ at Level one selects x_1^1. This results in a point (x_1^1, x_1^2, x_2^2) that is an element of one of the five upper facets of the hexahedron. Therefore, the effective feasible region for this problem is merely the union of these five upper facets.

Figure 4 shows the hexahedron as viewed from above the (x_1^2, x_2^2)-plane. For every

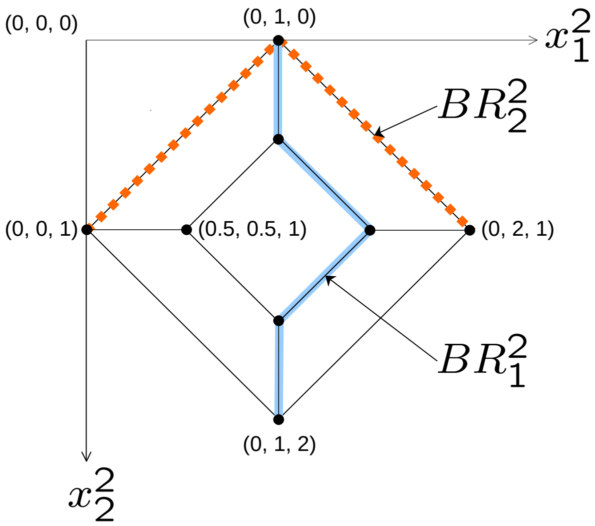


Figure 4: Best response sets for Level two in Example One

possible choice of x_2^2 by $\mathcal{D}_2^2$, the set BR_1^2 contains those values of (x_1^1, x_1^2, x_2^2) such that

- x_1^2 is the "best response" of $\mathcal{D}_1^2$ to the choice of x_2^2, *and*
- x_1^1 is the subsequent response of $\mathcal{D}_1^1$ at Level one to the combined choice of (x_1^2, x_2^2) by both divisions at Level 2.

A similar set, BR_2^2, represents the "best responses" of $\mathcal{D}_2^2$ for each possible choice of x_1^2 by $\mathcal{D}_1^2$.

Any point $\hat{x} = (\hat{x}_1^1, \hat{x}_1^2, \hat{x}_2^2) \in BR_1^2 \cap BR_2^2$ is an equilibrium solution for this problem. It has the property that no division can improve its objective function by unilaterally changing the value of the decision variable that it controls. Here, the intersection consists of a single point, $\hat{x} = (0, 1, 0)$.

For this example, $BR_1^2 \cap BR_2^2$ is nonempty and contains an extreme point of the polytope consisting of all feasible values of x. It will be shown that, under mild conditions, *every linear* multidivisional multilevel programming problem has an equilibrium solution set that exhibits these properties.

3 Nash Equilibrium Response Sets

Consider a single level of the hierarchy composed of m divisions, where x_i is the vector of decision variables controlled by Division i $(i = 1, \ldots, m)$. Let $x \equiv (x_1, \ldots, x_m)$. Suppose that all m divisions know the values of vectors $y^1, \ldots, y^n$. Later, the y^k's will represent the preemptive decisions of divisions at n levels above the current level. For now, simply think of them as parameters for the problem of the m divisions. Let $w \equiv (x, y^1, \ldots, y^n)$.

Define $x_{-i} \equiv (x_1, ..., x_{i-1}, x_{i+1}, ..., x_m)$. Then, given values for $y^1, ..., y^n$, the objective for Division i is

$$\max\{f_i(w) \,:\, (x_i, x_{-i} \,|\, y^1, ..., y^n)\}$$

which denotes the maximization of the bounded real function $f_i(w)$ with Division i choosing a value for x_i as the best response of to the choices x_{-i} of the other $m-1$ divisions at this level. (The notation $x|y$ is to be read "x given y" and was suggested and used by Dantzig [14] to represent matrix games.)

From the point of view of Division i, the values of $y^1, ..., y^n$ are predetermined parameters. The value of x_{-i}, however, may vary as a function of x_i as the other $m-1$ divisions seek to simultaneously maximize their respective objective functions. The set of all feasible w is the compact set S.

For the combined decision problem of the m divisions:

Definition 1 *The set $\Psi_{f_1,\ldots,f_m}(S)$ defined as*

$$\Psi_{f_1,\ldots,f_m}(S) \equiv \left\{ \hat{w} \in S \;\middle|\; \begin{array}{rcl} f_1(\hat{w}) & = & \max\{f_1(w) \,:\, (x_1, \hat{x}_{-1} \,|\, \hat{y}^1, ..., \hat{y}^n)\} \\ f_2(\hat{w}) & = & \max\{f_2(w) \,:\, (x_2, \hat{x}_{-2} \,|\, \hat{y}^1, ..., \hat{y}^n)\} \\ & \vdots & \\ f_m(\hat{w}) & = & \max\{f_m(w) \,:\, (x_m, \hat{x}_{-m} \,|\, \hat{y}^1, ..., \hat{y}^n)\} \end{array} \right\}$$

is the set of **Nash equilibrium responses of $f_1, \ldots, f_m$ over S.**

Assumption 1 *For each fixed value of $\hat{y} = (\hat{y}^1, ..., \hat{y}^n)$, the set $\Psi_{f_1,\ldots,f_m}(S)$ has at most one element.*

Under Assumption 1, when given $\hat{y}^1, ..., \hat{y}^n$, the parametric problem for the m decision makers is not permitted to have multiple equilibrium solutions. Later, for problems with multiple levels, this assumption will insure that the sequential decision problem is well-defined. Such an assumption has been employed previously for similar multi-stage Stackelberg games (see, Bard [5], Bialas and Karwan [8], and Simaan and Cruz [30]).

In Section 5, it will be shown that at least one solution exists when the multidivisional multilevel programming problem is bounded and linear. In those cases, the assumption of *at most one element* implies *exactly one element*.

4 A Definition of the Problem

The concept of a Nash equilibrium response set (Definition 1) provides a way for the multidivisional multilevel programming problem to be defined as a nested collection of Nash equilibrium problems. First, the optimization problem that is solved at each level will be defined. Then that definition will be used to recursively define the multidivisional multilevel programming problem.

4.1 Notation

Let L denote the number of levels and let n_k denote the number of divisions at Level k ($k = 1, \ldots, L$). The quantity $D \equiv \sum_{k=1}^{L} n_k$ is the total number of divisions in the system. The following notation will be used where $k = 1, \ldots, L$ and $i = 1, \ldots, n_k$:

$$\begin{array}{ll} \mathcal{D}_i^k & \text{represents Division } i \text{ at Level } k \\ \mathcal{L}^k & \equiv \{\mathcal{D}_1^k, \ldots, \mathcal{D}_{n_k}^k\} \quad \text{represents Level } k \\ \mathcal{D}_{-i}^k & \equiv \mathcal{L}^k \cap \{\mathcal{D}_i^k\}^c \end{array}$$

Let the decision vector $x_i^k \in X_i^k \subset \mathbb{R}^{m_{k_i}}$ denote the vector of m_{k_i} decision variables that are controlled by $\mathcal{D}_i^k$, for $k = 1, 2, \ldots, L$, and $i = 1, \ldots, n_k$. Let $N \equiv \sum_{k=1}^{L} \sum_{i=1}^{n_k} m_{k_i}$ denote the total number of decision variables for all divisions and all levels. Assume that the **strategy sets**, X_i^k, are compact for all k and i and define

$$\begin{array}{lllll} x^k & \equiv & (x_1^k, x_2^k, \ldots, x_{n_k}^k) & \in & X^k & \equiv & \prod_{i=1}^{n_k} X_i^k \\ x_{-i}^k & \equiv & (x_1^k, \ldots, x_{i-1}^k, x_{i+1}^k, \ldots, x_{n_k}^k) & \in & X_{-i}^k & \equiv & \prod_{j \neq i} X_j^k \\ x & \equiv & (x^1, x^2, \ldots, x^L) & \in & X & \equiv & \prod_{k=1}^{L} \prod_{i=1}^{n_k} X_i^k . \end{array}$$

Let the function $f_i^k : \mathbb{R}^N \to \mathbb{R}^1$ denote the objective function of $\mathcal{D}_i^k$. Then for a compact set $S \subseteq X$, use Definition 1 to define $\Psi^k(S) \equiv \Psi_{f_1^k, \ldots, f_{n_k}^k}(S)$ for each

$k = 1, \ldots, L-1$, as follows:

$$\Psi^k(S) \equiv \left\{ \hat{w} \in S \;\middle|\; \begin{array}{rcl} f_1^k(\hat{w}) & = & \max\{f_1^k(w) \,:\, (x_1^k, \hat{x}_{-1}^k \,|\, \hat{x}^{k+1}, ..., \hat{x}^L)\} \\ f_2^k(\hat{w}) & = & \max\{f_2^k(w) \,:\, (x_2^k, \hat{x}_{-2}^k \,|\, \hat{x}^{k+1}, ..., \hat{x}^L)\} \\ & \vdots & \\ f_{n_k}^k(\hat{w}) & = & \max\{f_{n_k}^k(w) \,:\, (x_{n_k}^k, \hat{x}_{-n_k}^k \,|\, \hat{x}^{k+1}, ..., \hat{x}^L)\} \end{array} \right\}.$$

4.2 The problem at Level one

Let S be a compact subset of $\mathbb{R}^N$. The lowest level of the hierarchy, $\mathcal{L}^1$, is provided predetermined values for $x^2, \ldots, x^L$, and the mathematical programming problem that is simultaneously solved by every $\mathcal{D}_i^1 \in \mathcal{L}^1$ (each controlling x_i^1) is

$$(P^1) \left\{ \begin{array}{rl} \max & \{f_i^1(x) \,:\, (x_i^1, x_{-i}^1 \,|\, x^2, \ldots, x^L)\} \qquad \text{for } i = 1, \ldots, n_1 \\ \text{st:} & x \in S^1 = S \end{array} \right.$$

which is a function of $x^2, \ldots, x^L$. The set of feasible solutions, $S^1 = S$ common to all $\mathcal{D}_i^1$, is called the **level-one feasible region.**

For each $\mathcal{D}_i^1$ $(i = 1, \ldots, n_1)$, the value of x_i^1 is chosen as the best response to x_{-i}^1 as all divisions seek to simultaneously maximize their respective objective functions. If $(x^2, \ldots, x^L)$ is varied over all feasible choices for $\mathcal{L}^2, \ldots, \mathcal{L}^L$, the set that is produced is $S^2 \equiv \Psi^1(S^1)$ representing all Nash equilibrium responses of $\mathcal{L}^1$. Given a specific choice of $(x^3, \ldots, x^L)$, the set S^2 contains the feasible combinations of (x^1, x^2) available to the divisions of $\mathcal{L}^2$ who must consider the reactions of $\mathcal{L}^1$ for every feasible choice of x^2.

4.3 The problem for all levels

In general, the **level-*k* feasible region** for $\mathcal{L}^k$ is defined as

$$S^k \equiv \Psi^{k-1}(S^{k-1}). \tag{1}$$

Note that $S^k \subset \mathbb{R}^N$ for all $k = 1, \ldots, L$. For any given $(x^{k+1}, \ldots, x^L)$, each element of S^k provides a possible choice for x^k and the corresponding response, $(x^1, \ldots, x^{k-1})$, from the decision makers at $\mathcal{L}^1, \ldots, \mathcal{L}^{k-1}$. An equilibrium solution

$$(x^1, \ldots, x^{k-1}, x_i^k, x_{-i}^k, x^{k+1}, \ldots, x^L) \quad \text{for all } \mathcal{D}_i^k \in \mathcal{L}^k$$

must be an element of S^k. If $x \in S^k$, then x satisfies the condition that it will conform to the rational response of levels $\mathcal{L}^1 \ldots, \mathcal{L}^{k-1}$ and x is said to be **Stackelberg-feasible.**

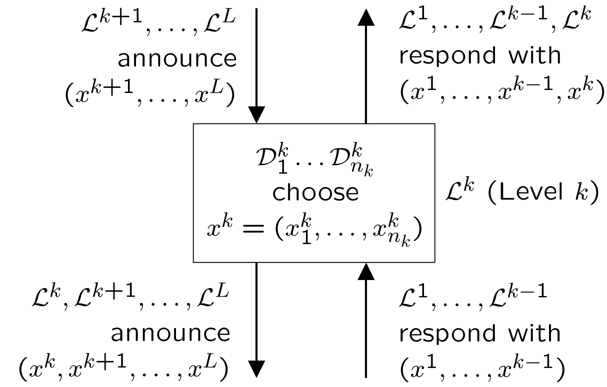


Figure 5: The problem (P^k) at Level k

The mathematical programming problem that is simultaneously solved by every $\mathcal{D}^k_i \in \mathcal{L}^k$ (each controlling x^k_i) is

$$(P^k) \left\{ \begin{array}{rl} \max & \{f^k_i(x) \,:\, (x^k_i, x^k_{-i} \,|\, x^{k+1}, \ldots, x^L)\} \qquad \text{for } i = 1, \ldots, n_k \\ \text{st:} & x \in S^k \end{array} \right.$$

which is a function of $x^{k+1}, \ldots, x^L$ (see Figure 5). In order to maintain Assumption 1, this problem must result in, at most, one solution for any feasible choice of $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$. A solution to (P^k) has the property that it is a Nash equilibrium response and a Stackelberg-feasible solution for $\mathcal{L}^k$. Such a solution is called a **Nash-Stackelberg equilibrium solution**.

Ultimately, the top level, $\mathcal{L}^L$, solves

$$(P^L) \left\{ \begin{array}{rl} \max & \{f^L_i(x) \,:\, (x^L_i, x^L_{-i})\} \qquad \text{for } i = 1, \ldots, n_L \\ \text{st:} & x \in S^L. \end{array} \right.$$

A Nash-Stackelberg equilibrium solution to (P^L) is a stable solution to the collection of nested equilibrium programming problems $\{P^1, \ldots, P^L\}$ representing the decision making process at all levels.

Problem (P^L) is called a ***D*-divisional *L*-level programming problem** and can be written in **explicit form** as:

$$
(P^L)\left\{\begin{array}{l}
\max\,\{f_i^L(x)\,:\,(x_i^L, x_{-i}^L)\}\quad \text{for } i=1,\ldots,n_L\\
\quad \text{where } x^1,\ldots,x^{L-1} \text{ solve}\\
(P^{L-1})\left\{\begin{array}{l}
\max\,\{f_i^{L-1}(x)\,:\,(x_i^{L-1}, x_{-i}^{L-1}\,|\,x^L)\}\quad \text{for } i=1,\ldots,n_{L-1}\\
\quad \text{where } x^1,\ldots,x^{L-2} \text{ solve}\\
\qquad\qquad\ddots\\
(P^2)\left\{\begin{array}{l}
\max\,\{f_i^2(x)\,:\,(x_i^2, x_{-i}^2\,|\,x^3,\ldots,x^L)\}\quad \text{for } i=1,\ldots,n_2\\
\quad \text{where } x^1 \text{ solves}\\
(P^1)\left\{\begin{array}{l}
\max\,\{f_i^1(x)\,:\,(x_i^1, x_{-i}^1\,|\,x^2,\ldots,x^L)\}\quad \text{for } i=1,\ldots,n_1\\
\text{st:}\quad x\in S.
\end{array}\right.
\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

Three questions immediately arise:

1. Do Nash-Stackelberg equilibrium solutions exist for problem (P^L)?
2. What are the mathematical characteristics of Nash-Stackelberg equilibrium solutions for problem (P^L)?
3. How does one find Nash-Stackelberg equilibrium solutions for problem (P^L)?

For the class of bounded linear multidivisional multilevel programming problems, the next sections will answer question (1) and begin to answer questions (2) and (3).

5 Equilibrium Solutions for Linear Problems

The remainder of this paper will be devoted to finding Nash-Stackelberg equilibrium solutions for *linear* multidivisional multilevel programming problems. Specifically, for all $k = 1,\ldots,L$ and $i = 1,\ldots,n_k$, let $c_i^k \in \mathbb{R}^N$, $A \in \mathbb{R}^{(M\times N)}$, and $b \in \mathbb{R}^{(M\times 1)}$ with $S = \{x \in \mathbb{R}^N \,|\, Ax \le b,\ x \ge 0\}$. It is assumed that S is nonempty and bounded so that S is a polytope. Each objective function $f_i^k(x) = c_i^k x \equiv \sum_{j=1}^N c_{ij}^k x_j$ depends on all of the decision variables.

Let $\mathfrak{C}(T)$ denote the convex hull of a set $T \subset \mathbb{R}^N$. Let $\mathfrak{P}(T)$ denote the extreme points of a convex set T. If T is not convex, then let $\mathfrak{P}(T) \equiv \mathfrak{P}(\mathfrak{C}(T))$.

The following sections establish two important facts about linear multidivisional multilevel programming problems when S is nonempty and bounded:

(a) For every level $\mathcal{L}^k$ and each feasible $(\hat{x}^{k+1},\ldots,\hat{x}^L)$, there exists at least one $x \in S^k$ that is a Nash equilibrium response for (P^k), and

(b) At least one Nash-Stackelberg equilibrium solution for (P^L) is an extreme point of S.

These facts permit the development of extreme point search procedures to find Nash-Stackelberg equilibrium solutions for bounded linear multidivisional multilevel programming problems.

Section 5.1 will present several useful properties of Nash equilibrium response sets, S^k, for problem (P^k) *if* solutions exist. In Section 5.2, these characteristics will be used to help prove that, for linear problems, solutions *do* exist.

The proof of existence in Section 5.2 will recursively use the results in Section 5.1 for each S^k as $k = 1, \ldots, L$. In other words, the existence of S^k will imply properties for S^k that will be used to prove the existence of S^{k+1}. This iterative process is the foundation for the proof of the main theorem, Theorem 6 in Section 5.3, which states that at least one extreme point of S is a solution to P^L. Figure 6 provides a flowchart of the overall logical structure of that proof and the relationships among the results in Sections 5.1 and 5.2.

5.1 Properties of Nash equilibrium response sets

This section presents some of the basic properties of the set function $\Psi^k(\cdot)$ for linear multidivisional multilevel programming problems. Let $E \subset \mathbb{R}^N$ be any nonempty, compact convex set. A vector $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^L) \in \Psi^k(E)$ if and only if $\hat{x} \in E$ and

$$\begin{aligned}(c_i^k)(\hat{x}^1, \ldots, \hat{x}^{k-1}, \hat{x}_i^k, \hat{x}_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \geq \\ (c_i^k)(x^1, \ldots, x^{k-1}, x_i^k, \hat{x}_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \\ \text{for all } (x^1, \ldots, x^{k-1}, x_i^k, \hat{x}_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \in E \\ \text{for all } i = 1, \ldots, n_k\end{aligned} \tag{2}$$

This leads to the following theorem:

Theorem 1 *Let $E \subset \mathbb{R}^N$ be a nonempty, compact convex set and let $k \in \{1, \ldots, L-1\}$. Suppose that $\hat{x}, \hat{y}, \hat{z} \in E$. If $\hat{x} \in \Psi^k(E)$ and $\hat{x} = \lambda\hat{y} + (1-\lambda)\hat{z}$ where $0 < \lambda < 1$, then $\hat{y}, \hat{z} \in \Psi^k(E)$.*

PROOF. It will be shown that $\hat{y} \in \Psi^k(E)$ by contradiction. The proof for $\hat{z}$ is similar. Suppose that $\hat{y} \notin \Psi^k(E)$. Then condition (2) must fail for some i. Without loss of generality, suppose that this occurs for $i = 1$. Hence, there exists $\tilde{y} \in E$

with $\tilde{y}^k_{-1} = \hat{y}^k_{-1}$, and $\tilde{y}^j = \hat{x}^j$ for $j = k+1, \ldots, L$ such that

$$(c^k_1)(\hat{y}^1, \ldots, \hat{y}^{k-1}, \hat{y}^k_1, \hat{y}^k_{-1}, \hat{x}^{k+1}, \ldots, \hat{x}^L) < \\ (c^k_1)(\tilde{y}^1, \ldots, \tilde{y}^{k-1}, \tilde{y}^k_1, \hat{y}^k_{-1}, \hat{x}^{k+1}, \ldots, \hat{x}^L).$$

Since $\hat{y}, \tilde{y} \in E$, then $\lambda \hat{y} + (1-\lambda)\tilde{y} \equiv \tilde{x} \in E$. Furthermore, $\tilde{x}^k_{-1} = \hat{x}^k_{-1}$ and $\tilde{y}^j = \hat{x}^j$ for $j = k+1, \ldots, L$. Thus a vector $\tilde{x} \in E$ has been constructed such that

$$(c^k_1)(\hat{x}^1, \ldots, \hat{x}^{k-1}, \hat{x}^k_1, \hat{x}^k_{-1}, \hat{x}^{k+1}, \ldots, \hat{x}^L) < \\ (c^k_1)(\tilde{x}^1, \ldots, \tilde{x}^{k-1}, \tilde{x}^k_1, \hat{x}^k_{-1}, \hat{x}^{k+1}, \ldots, \hat{x}^L).$$

Therefore $\hat{x} \notin \Psi^k(E)$, a contradiction. ■

Theorem 1 is a generalization of a theorem by Bialas and Karwan [8] and provides that any points of E that strictly contribute in a convex combination to form a point in $\Psi^k(E)$ must also be elements of $\Psi^k(E)$. The set $\Psi^k(E)$ possesses a convex-like property with respect to the convex set E. This property is also evident in the rational reaction sets (inducible regions) of two-level (bilevel) linear programming problems (see Bard [5], and Bialas and Karwan [8]).

Corollary 1 *For any nonempty, compact convex set $E \subset \mathbb{R}^N$,*

$$\mathfrak{P}(\Psi^k(E)) \subseteq \mathfrak{P}(E) \quad \textit{for } k = 1, \ldots, L-1. \tag{3}$$

Proof. This will be proved by contradiction. Choose $x \in \mathfrak{P}(\Psi^k(E))$. Suppose that $x \notin \mathfrak{P}(E)$. Then there exists $y, z \in E$ such that $x = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$. Theorem 1 implies $y, z \in \Psi^k(E)$. Therefore $x \notin \mathfrak{P}(\Psi^k(E))$, producing a contradiction. ■

Theorem 1 and Corollary 1 apply when E is a nonempty, compact convex set. If, in addition, the set E is a polytope, one obtains the following:

Theorem 2 *If $E \subset \mathbb{R}^N$ is a polytope, then $\Psi^k(E)$ is the union of a finite collection of polytopes.*

Proof. Let $\mathfrak{P}(E) = \{z_1, \ldots, z_\nu\}$ where $\nu < \infty$ is the cardinality of $\mathfrak{P}(E)$. Since E is a polytope, every $x \in E$ can be expressed as $x = \sum_{i=1}^{\nu} \lambda_i z_i$ where $0 \le \lambda_i \le 1$ for all $i = 1, \ldots, \nu$ and $\sum_{i=1}^{\nu} \lambda_i = 1$. For every $x \in E$, define $M(x)$ as the set of indices $i \in \{1, \ldots, \nu\}$ such that $\lambda_i > 0$. Note that $M(x) \neq \emptyset$ for every $x \in E$. Say that $x \simeq y$ if and only if $M(x) = M(y)$. As a result, $(\simeq)$ is an equivalence relation over the elements of E. This partitions E into at most $2^\nu - 1$ sets, each representing an equivalence class.

From Theorem 1, if $y \in \Psi^k(E)$ and $M(x) \subseteq M(y)$, then $x \in \Psi^k(E)$. For each $y \in \Psi^k(E)$ define $T_y \equiv \{x \in E \mid M(x) \subseteq M(y)\}$. Since there are a finite number of unique $M(x)$'s, there are a finite number of unique T_y's. Also, every $\mathfrak{P}(T_y) \neq \emptyset$ with $\mathfrak{P}(T_y) \subseteq \mathfrak{P}(\Psi^k(E))$, and every $x \in T_y$ can be written as a convex combination of the elements of $\mathfrak{P}(T_y)$. Therefore, each T_y is a polytope with $T_y \subseteq \Psi^k(E)$, and $\mathfrak{P}(T_y) \subseteq \mathfrak{P}(\Psi^k(E))$. Furthermore, $\bigcup_{y \in \Psi^k(E)} T_y = \Psi^k(E)$. The set $\Psi^k(E)$ is an uncountable set, but varying y over $\Psi^k(E)$ results in only a finite number of unique sets T_y. Hence, $\Psi^k(E)$ is the union of a finite number of polytopes. ■

Section 5.2 will prove that S^k must exist for all $k = 1, \ldots, L$. However, for now, suppose that S^k does exist. Note that, in general, $\Psi^k(S^k) \neq \Psi^k(\mathfrak{C}(S^k))$. Specifically, $\mathfrak{C}(S^k)$ may contain choices that are desirable to divisions at $\mathcal{L}^k$ but unattainable because they are not Stackelberg feasible. Also, S^k is not necessarily a convex set. So, Theorems 1 and 2 cannot be used for $E = S^k$. However, Theorems 1 and 2 do imply the following properties of S^k:

Theorem 3 *Suppose that $S^\ell \neq \emptyset$ for $\ell = 1, \ldots, k$. Let $\Psi^k \equiv \Psi_{f_1^k, \ldots, f_{n_k}^k}$ where $f_i^k(x) = c_i^k x \equiv \sum_{j=1}^N c_{ij}^k x_j$. Then $\Psi^k(S^k)$ is the union of a finite collection of polytopes for $k = 1, \ldots, L$, and $\mathfrak{P}(\Psi^k(S^k)) \subseteq \mathfrak{P}(S^k)$.*

PROOF. For $k = 1$, S is a polytope with a finite number of extreme points. Hence, $S^1 = S$ is a polytope and the statement $\mathfrak{P}(S^2) \subseteq \mathfrak{P}(S^1)$ is implied by Corollary 1.

For $k > 1$, suppose that

$$\mathfrak{P}(S^1) \supseteq \cdots \supseteq \mathfrak{P}(S^{k-1}) \supseteq \mathfrak{P}(S^k)$$

and S^ℓ is a finite union of polytopes for $\ell = 1, \ldots, k$. Let $V_1^k, \ldots, V_r^k$ denote the polytopes of S^k with $S^k = \bigcup_{i=1}^r V_i^k$.

It is claimed that $\Psi^k(S^k) = \bigcup_{i=1}^r \Psi^k(V_i^k)$ with the following justification by contradiction: Suppose that $\Psi^k(S^k) \neq \bigcup_{i=1}^r \Psi^k(V_i^k)$. Then there exists

$$x = (x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^L) \in \Psi^k(S^k)$$

and $j \in \{1, \ldots, r\}$ with

$$y = (y^1, \ldots, y^{k-1}, x^k, x^{k+1}, \ldots, x^L) \in \Psi^k(V_j^k)$$

such that $x \neq y$. (In other words, for the choice of $x^k, \ldots, x^L$ by $\mathcal{L}^k, \ldots, \mathcal{L}^L$, the solution x must be a Nash equilibrium response among the solutions in S^k and the solution $y \neq x$ must be a Nash equilibrium response among the solutions in V_j^k for some $j \in \{1, \ldots, r\}$. That is, there exists at least two Nash equilibrium responses for that choice of $x^k, \ldots, x^L$.) But Assumption 1 requires that for any such x and y, the components $x^\ell = y^\ell$ for $\ell = 1, \ldots, k-1$. Hence $x = y$, which produces a contradiction. Therefore, $\Psi^k(S^k) = \bigcup_{i=1}^r \Psi^k(V_i^k)$.

From Theorem 2, since each V_i^k is a polytope, then each $\Psi^k(V_i^k)$ is a finite union of polytopes for every $i = 1, \ldots, r$. Hence, since r is finite, $S^{k+1} \equiv \Psi^k(S^k) = \bigcup_{i=1}^r \Psi^k(V_i^k)$ is a finite union of polytopes.

Using Corollary 1, $\mathfrak{P}(\Psi^k(V_i^k)) \subseteq \mathfrak{P}(V_i^k) \subseteq \mathfrak{P}(S^k)$. Therefore, $\mathfrak{P}(S^{k+1}) = \mathfrak{P}(\Psi^k(S^k)) = \mathfrak{P}\left(\bigcup_{i=1}^r \Psi^k(V_i^k)\right) \subseteq \bigcup_{i=1}^r \mathfrak{P}(\Psi^k(V_i^k)) \subseteq \bigcup_{i=1}^r \mathfrak{P}(V_i^k) \subseteq \mathfrak{P}(S^k)$.

Therefore, $\Psi^k(S^k)$ is the union of a finite collection of polytopes and $\mathfrak{P}(\Psi^k(S^k)) \subseteq \mathfrak{P}(S^k)$. ■

Theorem 3 is similar to a result proved by Wen [35] for linear multilevel programming problems (i.e., linear multidivisional multilevel programming problems with only one division at every level).

5.2 Existence of equilibrium solutions

This section will prove that, for any $k = 1, \ldots, L$ and each feasible $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$, if $S^k \neq \emptyset$, then there exists at least one $x \in S^k$ that is a Nash equilibrium response for the problem (P^k) for $\mathcal{L}^k$. That is, $S^{k+1} \neq \emptyset$. The proof employs Freund's [17] generalization of the *von Neumann Intersection Theorem* [33].

Let $E \subset \prod_{j=1}^n \mathbb{R}^{m_j}$ be a nonempty, compact convex set. For each $i = 1, \ldots, n$, let Y_{-i} denote the projection of E onto $\prod_{j \neq i} \mathbb{R}^{m_j}$. That is,

$$Y_{-i} \equiv \left\{ (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \;\middle|\; \begin{array}{l} \text{there exists } \xi_i \in \mathbb{R}^{m_i} \text{ such that} \\ (y_1, \ldots, y_{i-1}, \xi_i, y_{i+1}, \ldots, y_n) \in E \end{array} \right\}.$$

Let $E_1, \ldots, E_n$ be closed subsets of E with the property that for each $i = 1, \ldots, n$ and each $y_{-i} \in Y_{-i}$ the set

$$[E_i]^{-1}(y_{-i}) \equiv \{ \xi_i \in \mathbb{R}^{m_i} \mid (y_1, \ldots, y_{i-1}, \xi_i, y_{i+1}, \ldots, y_n) \in E_i \} \tag{4}$$

is a nonempty, compact convex set.

Theorem 4 *Under the above conditions,* $\bigcap_{i=1}^n E_i \neq \emptyset$.

PROOF. See Freund [17]. ■

Theorem 4 is a direct consequence the *Fixed Point Theorem* of Kakutani [18]. This version by Freund generalizes von Neumann's theorem in two ways: (1) by allowing the intersection of more than two sets, and (2) by relaxing the condition that the feasible set E must be the cross-product of nonempty, compact convex sets. A similar result, based on a generalized intersection theorem by Fan [15], is provided by Border [10].

Fix $k \in \{1, \ldots, L\}$ and assume a given value for $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$ that is an element of the projection of S onto $\prod_{j=k+1}^{L} X^j$.

Definition 2 *For given $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$, the* **set of best responses** *for $\mathcal{D}_i^k$ against $\mathcal{D}_{-i}^k$ over S^k is*

$$BR_i^k \equiv \left\{ x \in S^k \;\middle|\; \begin{array}{l} (c_i^k)(x^1, \ldots, x^{k-1}, x_i^k, x_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \geq \\ \quad (c_i^k)(\xi^1, \ldots, \xi^{k-1}, \xi_i^k, x_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \\ \quad \text{for all } (\xi^1, \ldots, \xi^{k-1}, \xi_i^k, x_{-i}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \in S^k \end{array} \right\},$$

Corollary 2 *Under Assumption 1, if $S^k \neq \emptyset$, then BR_i^k is nonempty and is the union of a finite collection of polytopes with $\mathfrak{P}(BR_i^k) \subseteq \mathfrak{P}(S^k)$.*

PROOF. Fix the values of $\hat{x}^{k+1}, \ldots, \hat{x}^L$. For any $i = 1, \ldots, n_k$, use Assumption 1 and Definition 1 with a single linear objective function, $f_i(x) = (c_i^k)x$. Then $BR_i^k = \Psi_{f_i}^k(S^k) \neq \emptyset$. (That is, BR_i^k is the Nash equilibrium response set that results from $\mathcal{D}_i^k$ playing a "one-person" game over S^k as x_{-i}^k is varied over all possible choices of $\mathcal{D}_{-i}^k$ given fixed $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$.) Hence, BR_i^k has the properties provided by Theorem 1, Corollary 1 and Theorem 3. In particular, BR_i^k is the union of a finite collection of polytopes and $\mathfrak{P}(BR_i^k) \subseteq \mathfrak{P}(S^k)$. ■

Lemma 1 *Every projection of a polytope is a polytope.*

PROOF. See Nemhauser and Wolsey [23] or Ziegler [40]. ■

The existence proof will use the following set mappings for $E \subseteq X$:

$\perp^k(E)$	denotes the projection of E onto	X^k
$\perp_i^k(E)$	denotes the projection of E onto	X_i^k
$\perp_{-i}^k(E)$	denotes the projection of E onto	X_{-i}^k

To prove the existence of Nash equilibrium responses for the problem (P^k), it is assumed that $S^k \neq \emptyset$ and the nonempty sets $BR_i^k \subseteq S^k \subseteq \mathfrak{C}(S^k)$ are projected onto X^k. Using Theorem 4, it will be shown that the projections of the BR_i^k's have a nonempty intersection. Then under Assumption 1, it will be proved that this further implies that the BR_i^k's have a nonempty intersection. In order to use Theorem 4, the following substitutions are made:

- the set $\perp^k(\mathfrak{C}(S^k))$ takes on the role of E.
- the set $\perp^k_{-i}(\mathfrak{C}(S^k)) = \perp^k_{-i}(\perp^k(\mathfrak{C}(S^k)))$ takes on the role of Y_{-i} for $i = 1\ldots,n_k$.
- the set $\perp^k(BR_i^k)$ takes on the role of E_i for $i = 1\ldots,n_k$.
- as defined by equation (4), the set $\left[\perp^k(BR_i^k)\right]^{-1}(x^k_{-i})$ takes on the role of $[E_i]^{-1}(y_{-i})$ for $i = 1\ldots,n_k$.

Note that an E_i (namely $\perp^k(BR_i^k)$) is provided only for each X_i^k (the subspaces of the divisions of $\mathcal{L}^k$) where $i = 1\ldots,n_k$. The values of $\hat{x}^{k+1},\ldots,\hat{x}^L$ are fixed, and $\mathfrak{C}(S^k)$ is projected onto X^k. As a result, the application of Theorem 4 is restricted to the subspace $X^k = \prod_{i=1}^{n_k} X_i^k$ with $\perp^k(S^k) = \perp^k(\mathfrak{C}(S^k))$.

Theorem 5 *Under Assumption 1, for any given $(\hat{x}^{k+1},\ldots,\hat{x}^L)$, if $S^k \neq \emptyset$, then the set $\bigcap_{i=1}^{n_k} \perp^k(BR_i^k)$ is nonempty.*

PROOF. From Theorem 4, the following are sufficient conditions to insure that $\bigcap_{i=1}^{n_k} \perp^k(BR_i^k) \neq \emptyset$:

(a) **[E is a nonempty, compact convex set]** The set $\mathfrak{C}(S^k) \neq \emptyset$ is a polytope. Therefore, using Lemma 1, $\perp^k(\mathfrak{C}(S^k))$ is a nonempty, compact convex set.

(b) **[For all i, E_i are closed subsets of E]** Under Assumption 1, Corollary 2 provides that BR_i^k, is the union of a finite collection of polytopes. Lemma 1, then implies $\perp^k(BR_i^k)$, is the union of a finite collection of polytopes. Hence, $\perp^k(BR_i^k)$ is a closed subset of $\perp^k(\mathfrak{C}(S^k))$.

(c) **[For all i and all $y_{-i} \in Y_{-i}$, the inverse image $[E_i]^{-1}(y_{-i})$ is a nonempty, compact convex set]** For each $x^k_{-i} \in \perp^k_{-i}(\mathfrak{C}(S^k)) = \perp^k_{-i}(S^k)$, the set $\left[\perp^k(BR_i^k)\right]^{-1}(x^k_{-i})$ is the set of optimal solutions (with fixed values for x^k_{-i}, and $(\hat{x}^{k+1},\ldots,\hat{x}^L)$) to the mathematical programming problem

$$\begin{array}{rl} \max_x & (c_i^k)(x) \\ \text{st:} & x \in S^k \end{array} \tag{5}$$

projected onto X_i^k. Since the objective function is linear, the set of solutions to (5) may be a singleton or (with alternate optima) a polytope. In either case, using Lemma 1, $\left[\perp^k(BR_i^k)\right]^{-1}(x_{-i}^k)$ is a nonempty, compact convex set.

Therefore, $\bigcap_{i=1}^{n_k} \perp^k(BR_i^k) \neq \emptyset$. ■

Corollary 3 *Under Assumption 1, if $S^k \neq \emptyset$, then for any given and feasible $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$, the set $\bigcap_{i=1}^{n_k} BR_i^k$ is nonempty.*

PROOF. Under Assumption 1, for every $\hat{x}^k \in \bigcap_{i=1}^{n_k} \perp^k(BR_i^k)$ there exists unique values $\hat{x}^1, \ldots, \hat{x}^{k-1}$ such that $(\hat{x}^1, \ldots, \hat{x}^{k-1}, \hat{x}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \in BR_i^k$ for all $i = 1, \ldots n_k$. Therefore, $(\hat{x}^1, \ldots, \hat{x}^{k-1}, \hat{x}^k, \hat{x}^{k+1}, \ldots, \hat{x}^L) \in \cap_{i=1}^{n_k} BR_i^k \subseteq S^k$. Therefore, $\bigcap_{i=1}^{n_k} BR_i^k \neq \emptyset$. ■

Corollary 4 *For any linear multidivisional multilevel programming problem, under Assumption 1, if $S^k \neq \emptyset$, then $S^{k+1} = \Psi^k(S^k) \neq \emptyset$ for $k = 1, \ldots, L-1$.*

PROOF. For $k = 1$, $S^1 = S$ is nonempty by definition. For $k = 1, \ldots, L-1$, Corollary 3 implies that, for every feasible $(\hat{x}^{k+1}, \ldots, \hat{x}^L)$, there exists an $x \in S^k$ such that $x \in \bigcap_{i=1}^{n_k} BR_i^k \subseteq S^{k+1}$. Hence, $S^{k+1} \neq \emptyset$. ■

It is important to note that the closure property implied by Corollary 3 was used in part (b) of the proof of Theorem 5. However, Section 5.1 and Corollary 3 *assume* the existence of S^k, which is what is being proved here. To address this issue, the existence S^k is established recursively for $k = 1, \ldots, L$. Specifically, for $k = 1$, the existence of $S^1 = S$ is assumed. Then, for $k = 1, \ldots, L-1$, the closure property for (b) and existence are recursively inherited by S^{k+1} from S^k using Corollary 2 and Corollary 4.

5.3 Existence of extreme point solutions

The preceding sections have shown that Nash-Stackelberg equilibrium solutions must exist for every problem (P^k) where $k = 1, \ldots, L$. Combining these results, it can now be shown that at least one Nash-Stackelberg equilibrium solution to problem (P^L) is an extreme point of S, as follows:

Theorem 6 *Under Assumption 1, a Nash-Stackelberg equilibrium solution to the bounded linear multidivisional multilevel programming problem*

$$(P^L)\begin{cases} \max & \{c_i^L x \,:\, (x_i^L, x_{-i}^L)\} \qquad \textit{for } i = 1, \ldots, n_L \\ \text{st:} & x \in S^L \end{cases}$$

must exist at an extreme point of the set S.

PROOF. Figure 6 presents the iterative argument that establishes the existence of S^L, and the fact that, if x is an extreme point of S^L, then x is an extreme point of S. Specifically, from Corollaries 3 and 4, a Nash-Stackelberg equilibrium solution must exist for

$$(P^L)\begin{cases} \max & \{c_i^L x \,:\, (x_i^L, x_{-i}^L)\} \qquad \text{for } i = 1, \ldots, n_L \\ \text{st:} & x \in S^L \end{cases}$$

Since a solution exists, Theorem 3 implies that a Nash-Stackelberg equilibrium solution x^* to (P^L) must exist such that $x^* \in \mathfrak{P}(S^L) \subseteq \mathfrak{P}(S^{L-1}) \subseteq \cdots \subseteq \mathfrak{P}(S^1) = \mathfrak{P}(S)$. ■

6 Finding Equilibrium Solutions for Linear Problems

Suppose a multidivisional multilevel programming problem has L levels with n_k divisions at Level k. In order to specify the number of divisions at each level, it is sometimes said that the problem is an **$(n_1, n_2, \ldots, n_L)$-divisional L-level programming** problem.

This section presents a procedure to enumerate all extreme point Nash-Stackelberg equilibrium solutions for a *bounded linear* $(1, 2)$-*divisional two-level programming problem* using the results in Section 5. An example using this method is provided in Section 6.2. This procedure can be readily extended to the entire class of linear multidivisional multilevel programming problems.

6.1 An enumeration procedure

Let $x = (x^1, x^2)$, where $x^1 = (x_1^1)$, $x^2 = (x_1^2, x_2^2)$, and

$$\begin{aligned} S^2 &= \Psi^1(S^1) = \{\hat{x} \in S^1 \,|\, c^1\hat{x} = \max\{c^1 x \,:\, (x^1 \,|\, \hat{x}^2)\}\} \\ S^1 &= S = \{x \,|\, A^1x^1 + A^2x^2 \le b,\ x \ge 0\}. \end{aligned}$$

The linear $(1, 2)$-divisional two-level programming problem is written in explicit form as follows:

$$(P^2)\begin{cases} \max & \{c_i^2 x = c_i^{21}x^1 + c_i^{22}x^2 \,:\, (x_i^2, x_{-i}^2)\} \quad \text{for } i = 1, 2 \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{c^1 x = c^{11}x^1 + c^{12}x^2 : (x^1 \,|\, x^2)\} \\ \text{st:} & A^1x^1 + A^2x^2 \le b \\ & x \ge 0. \end{cases} \end{cases}$$

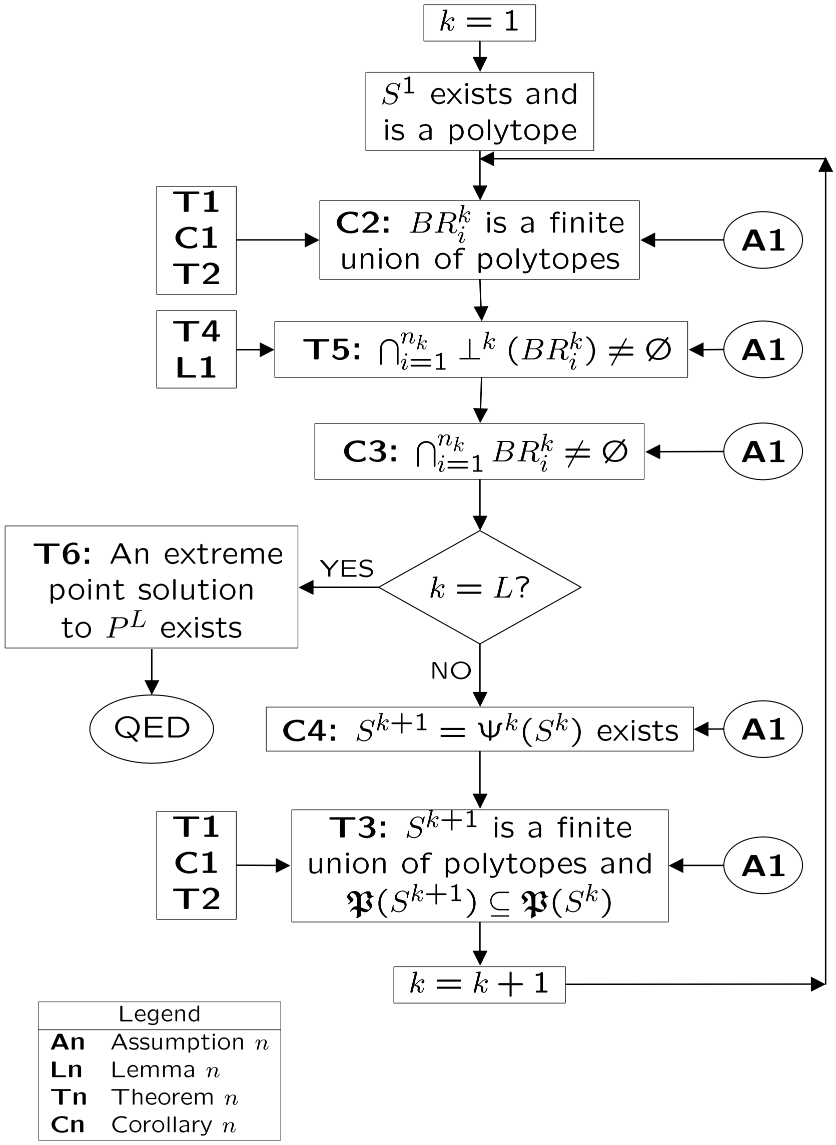


Figure 6: Flowchart for the proof of Theorem 6

Let $\hat{x}_{[1]}, \hat{x}_{[2]}, ..., \hat{x}_{[\mathcal{N}]}$ denote the $\mathcal{N}$ extreme points of S. Their order does not matter in this procedure. Using the results in Section 5, it is known that $\hat{x}_{[i]} = (\hat{x}^1_{1[i]}, \hat{x}^2_{1[i]}, \hat{x}^2_{2[i]})$ solves (P^2) if and only if

1. $\hat{x}_{[i]} \in S^2$. That is, the candidate extreme point solves (P^1), which implies that $\hat{x}_{[i]}$ is Stackelberg feasible, and
2. $\hat{x}_{[i]} \in BR^2_1 \cap BR^2_2$. That is, the candidate extreme point is an Nash equilibrium response for (P^2).

In other words, the goal is to find all extreme points of $\hat{x}_{[i]} \in S$ such that $\hat{x}_{[i]} \in BR^2_1 \cap BR^2_2$.

To test whether or not $\hat{x}_{[i]} \in S^2$, the constraint $(x^2_1, x^2_2) = (\hat{x}^2_{1[i]}, \hat{x}^2_{2[i]})$ is added to S^1, and then P^1 is solved. If $\hat{x}_{[i]}$ solves P^1, then $\hat{x}_{[i]} \in S^2$.

To check if the best response set $BR^2_1 \ni \hat{x}_{[i]}$, the constraint $x^2_2 = \hat{x}^2_{2[i]}$ is added to S^1, and then the following two-level linear programming problem is solved (see Bard [5], and Bialas and Karwan [8, 9]):

$$(P^2_1)\left\{\begin{array}{ll} \max & \{c^2_1 x \,:\, (x^2_1)\} \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\left\{\begin{array}{rl} \max & \{c^1 x \,:\, (x^1_1 \,|\, x^2_1)\} \\ \text{st:} & x \in S^1 \cap \{x^2_2 = \hat{x}^2_{2[i]}\}. \end{array}\right. \end{array}\right.$$

If $\hat{x}_{[i]}$ solves problem (P^2_1), then $\hat{x}_{[i]} \in BR^2_1$. Similarly (P^2_2) is solved and the condition $\hat{x}_{[i]} \in BR^2_2$ is tested.

More formally, the procedure can be stated as follows:

Step 1: Set $i = 1$ and $W = \text{Ø}$.

Step 2: Solve the following linear programming problem using the bounded simplex method:

$$\begin{array}{rl} \max & c^1 x \\ \text{st:} & x \in S^1 \cap \{x^2_1 = \hat{x}^2_{1[i]}\} \cap \{x^2_2 = \hat{x}^2_{2[i]}\}. \end{array} \tag{6}$$

Let $\tilde{x}$ denote the optimal solution to (6). If $\tilde{x} = \hat{x}_{[i]}$, then go to Step 3. Otherwise, go to Step 6.

Step 3: Solve the following two-level linear programming problem:

$$(P_1^2)\begin{cases} \max & \{c_1^2 x \ : \ (x_1^2)\} \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{c^1 x : (x_1^1 \,|\, x_1^2)\} \\ \text{st:} & x \in S^1 \cap \{x_2^2 = \hat{x}_{2[i]}^2\}. \end{cases} \end{cases} \tag{7}$$

Let $\tilde{x}$ denote the optimal solution to (7). If $\tilde{x} = \hat{x}_{[i]}$, then go to Step 4. Otherwise, go to Step 6.

Step 4: Solve the following two-level linear programming problem:

$$(P_2^2)\begin{cases} \max & \{c_2^2 x \ : \ (x_2^2)\} \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{c^1 x : (x_1^1 \,|\, x_2^2)\} \\ \text{st:} & x \in S^1 \cap \{x_1^2 = \hat{x}_{1[i]}^2\}. \end{cases} \end{cases} \tag{8}$$

Let $\tilde{x}$ denote the optimal solution to (8). If $\tilde{x} = \hat{x}_{[i]}$, then go to Step 5. Otherwise, go to Step 6.

Step 5: Set $W = W \cup \{\hat{x}_{[i]}\}$. Go to Step 6.

Step 6: Set $i = i + 1$. If $i > \mathcal{N}$, then stop. $W \neq \emptyset$ contains the extreme point Nash-Stackelberg equilibrium solutions to P^2. Otherwise, go to Step 2.

An alternative solution approach would be to search the extreme points of BR_1^2 for an extreme point that is also in BR_2^2. First, solve the following two-level linear programming problem:

$$(\hat{P}_1^2)\begin{cases} \max & \{c_1^2 x \ : \ (x_1^2)\} \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{c^1 x : (x_1^1 \,|\, x_1^2)\} \\ \text{st:} & x \in S^1. \end{cases} \end{cases} \tag{9}$$

Let $\hat{x} = (\hat{x}_1^1, \hat{x}_1^2, \hat{x}_2^2)$ be the solution to (9). Note that $\hat{x} \in BR_1^2$. If $\hat{x}$ is not an extreme point of S^1, then move to an adjacent extreme point of S^1 using a degenerate simplex pivot and call that new point $\hat{x}$. The result is an extreme point $\hat{x} \in BR_1^2$ (see Bialas and Karwan [8]). Then search among adjacent extreme points of BR_1^2 using simplex pivots, testing each one using (8) to find an extreme point $\hat{x}$ that is also in BR_2^2. Such a point solves (P^2) and must exist as implied by Theorem 6.

6.2 Example one revisited

Example one (Section 2) can be expressed in extensive form as the following linear $(1,2)$-divisional two-level programming problem:

$$(P^2)\left\{\begin{array}{l} \max\ \{2x_1^1+0.7x_1^2-0.6x_2^2 : (x_1^2, x_{-1}^2)\} \\ \max\ \{-2x_1^1+2x_1^2-1.5x_2^2 : (x_2^2, x_{-2}^2)\} \\ \text{where } x^1 \text{ solves} \\ (P^1)\left\{\begin{array}{ll} \max & \{x_1^1+0.8x_1^2+1.2x_2^2 : (x_1^1 \mid x_1^2, x_2^2)\} \\ \text{st:} & x_1^1+x_1^2+x_2^2 \le 3 \\ & -x_1^1+x_1^2+x_2^2 \ge 1 \\ & x_1^1+x_1^2-x_2^2 \le 1 \\ & x_1^1-x_1^2+x_2^2 \le 1 \\ & x_1^1 \le 0.5 \\ & x_1^1, x_1^2, x_2^2 \ge 0. \end{array}\right. \end{array}\right.$$

As before, the feasible region $S^1 = S \ni x = (x_1^1, x_1^2, x_2^2)$ is the entire hexahedron shown in Figure 3. The set $\Psi^1(S^1) \equiv S^2$ is the nonconvex set consisting of the union of the five upper facets of the hexahedron.

Recall that Figure 4 shows the hexahedron as viewed from above the (x_1^2, x_2^2) plane with the best response sets BR_1^2 and BR_2^2 highlighted. Theorem 5 implies that $BR_1^2 \cap BR_2^2 \neq \emptyset$. Since $(0,1,0) \in BR_1^2 \cap BR_2^2$, it is a Nash equilibrium response, and since it is an element of S^2, it is also a Stackelberg feasible solution. Hence, $(0,1,0)$ is a Nash-Stackelberg equilibrium solution to (P^2). Furthermore, $(0,1,0)$ is an extreme point of S as asserted by Theorem 6.

To systematically find the solution to (P^2), Table 1 displays the results of the enumeration procedure described in Section 6.1. Note that $\hat{x}_{[4]} = (0,2,1)$ is an extreme point Pareto-optimal solution to (P^2). However, if $\mathcal{D}_2^2$ chooses $\hat{x}_2^2 = 1$, $\mathcal{D}_1^2$ will choose $x_1^2 = 1.5$ in order to maximize $f_1^2 = 1.45$. The Pareto-optimal solution $\hat{x}_{[4]} = (0,2,1)$ is not a stable solution for this problem.

6.3 Example two

The second example is based on a simplified, hypothetical water quality problem illustrated in Figure 7. Along a river and its tributary, three factories generate a pollutant as a result of their production processes. Two downstream surveillance points monitor the amount of the pollutant in the river. The factories can independently decide to treat any portion of the pollutant they produce. The remaining

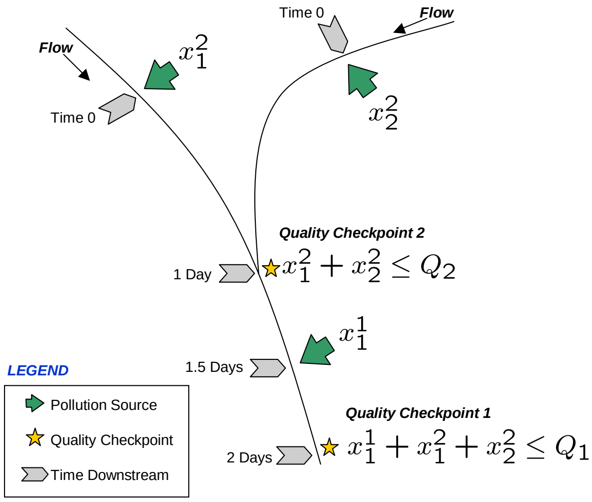


Figure 7: Hypothetical river basin for Examples Two and Three

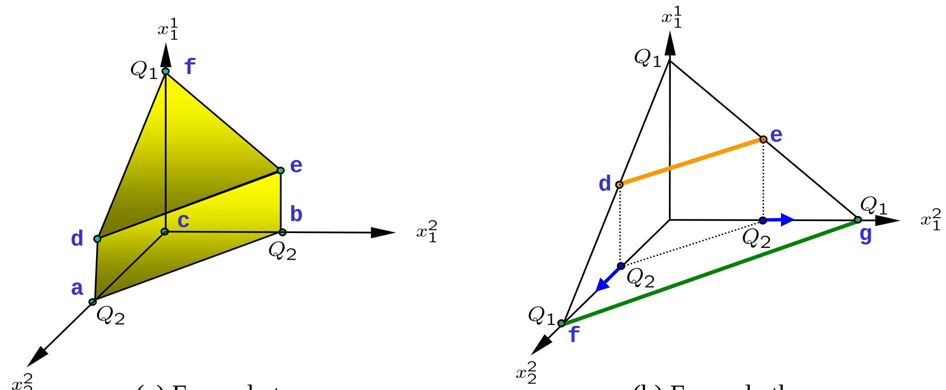
amounts become point sources of pollution in the river. As the pollutant flows downstream, it does not change in composition and when the pollutant from more than one source is mixed within the river, the effect is additive.

Define the following terms (see also Figure 7):

$$\begin{aligned}
x \equiv (x_1^1, x_1^2, x_2^2) &= \text{amounts of pollutant released by the three factories} \\
Q_1, Q_2 &= \text{maximum amount permitted at each checkpoint} \\
c_i^k = ([c_i^k]_1^1, [c_i^k]_1^2, [c_i^k]_2^2) &= \text{vector of benefit coefficients for each factory}
\end{aligned}$$

The scalar coefficient $[c_i^k]_s^r$ is the benefit for each released unit of x_s^r awarded to the factory that controls decision variable x_i^k.

This linear $(1,2)$-divisional two-level programming problem can be written in ex-



(a) Example two (b) Example three

Figure 8: Solutions for Examples Two and Three

plicit form as

$$
(P^2)\left\{\begin{array}{l}
\max\ \{[c_1^2]_1^1 x_1^1 + [c_1^2]_1^2 x_1^2 + [c_1^2]_2^2 x_2^2 : (x_1^2, x_{-1}^2)\} \\
\max\ \{[c_2^2]_1^1 x_1^1 + [c_2^2]_1^2 x_1^2 + [c_2^2]_2^2 x_2^2 : (x_2^2, x_{-2}^2)\} \\
\text{where } x_1^1 \text{ solves} \\
(P^1)\left\{\begin{array}{rl}
\max & \{[c_1^1]_1^1 x_1^1 + [c_1^1]_1^2 x_1^2 + [c_1^1]_2^2 x_2^2 : (x_1^1 \mid x_1^2, x_2^2)\} \\
\text{st:} & x_1^2 + x_2^2 \le Q_2 \\
& x_1^1 + x_1^2 + x_2^2 \le Q_1 \\
& x_1^1, x_1^2, x_2^2 \ge 0.
\end{array}\right.
\end{array}\right.
$$

Using the results of Section 5, the solution to (P^2) can be any one of the extreme points of the polytope $S \equiv S^1$ shown in Figure 8(a) and depends on the values of Q_1, Q_2 and the vectors $c_i^k = ([c_i^k]_1^1, [c_i^k]_1^2, [c_i^k]_2^2)$. When $Q_1 > Q_2 > 0$ and $[c_i^k]_1^1 < 0$ the set $S^2 = \Psi^1(S^1)$ is the facet $\mathfrak{C}(\{\mathtt{a}, \mathtt{b}, \mathtt{c}\})$. When $Q_1 > Q_2 > 0$ and $[c_i^k]_1^1 > 0$, S^2 is the facet $\mathfrak{C}(\{\mathtt{d}, \mathtt{e}, \mathtt{f}\})$.

6.4 Example three

Consider, once again, the water quality problem in Section 6.3, but now suppose that there is an additional decision-maker $\mathcal{D}_1^3$ at a new level, $\mathcal{L}^3$, who controls the water quality standard $x_1^3 \equiv Q_2$ at Quality Checkpoint 2. Furthermore, suppose that the objective of $\mathcal{D}_1^3$ is to maximize Q_2 such that $Q_2 \le Q_1$.

Written in explicit form, the linear $(1,2,1)$-divisional three-level programming problem is:

$$
(P^3)\left\{\begin{array}{l}
\max\ \{Q_2 : (Q_2)\} \\
\text{where } x_1^1, x_1^2, x_2^2 \text{ solve} \\
(P^2)\left\{\begin{array}{l}
\max\ \{[c_1^2]_1^1 x_1^1 + [c_1^2]_1^2 x_1^2 + [c_1^2]_2^2 x_2^2 : (x_1^2, x_{-1}^2)\} \\
\max\ \{[c_2^2]_1^1 x_1^1 + [c_2^2]_1^2 x_1^2 + [c_2^2]_2^2 x_2^2 : (x_2^2, x_{-2}^2)\} \\
\text{where } x_1^1 \text{ solves} \\
(P^1)\left\{\begin{array}{ll}
\max & \{[c_1^1]_1^1 x_1^1 + [c_1^1]_1^2 x_1^2 + [c_1^1]_2^2 x_2^2 : (x_1^1 \mid x_1^2, x_2^2)\} \\
\text{st:} & x_1^2 + x_2^2 - Q_2 \le 0 \\
 & x_1^1 + x_1^2 + x_2^2 \le Q_1 \\
 & Q_2 \le Q_1 \\
 & x_1^1, x_1^2, x_2^2, Q_2 \ge 0.
\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

For each feasible choice of Q_2, $\mathcal{L}^3$ must consider the corresponding response of $\mathcal{L}^2$. Suppose that $[c_i^k]_s^r > 0$ for all i,k,r and s. Then, using the analysis in Section 6.3, under Assumption 1, the response from $\mathcal{L}^2$ (and $\mathcal{L}^1$) will be the point **d** or **e**. Therefore, depending on the values $[c_i^k]_s^r > 0$, a solution to the three-level multidivisional multilevel programming problem will be either the extreme point $(x_1^1, x_1^2, x_2^2, Q_2) = (0, 0, Q_1, Q_1)$ (represented by the point **f** in Figure 8(b)) or the extreme point$(0, Q_1, 0, Q_1)$ (represented by **g**).

For some choices of $[c_i^k]_s^r > 0$, the response of $\mathcal{L}^2$ is not unique and may be *any* point on the line segment $(\mathbf{d}, \mathbf{e})$. In those cases, Assumption 1 is violated. However, for this particular problem, all of these alternative responses (including the extreme points) result in the same objective function value (i.e., Q_1) for $\mathcal{L}^3$. For those multidivisional multilevel programming problems where Assumption 1 is violated, a perturbation of the problem will sometimes permit a unique equilibrium solution to be achieved (see, Bialas and Karwan [9]).

7 Conclusions

This paper has presented a mathematical programming technique called multidivisional multilevel programming to help study decentralized planning problems. For bounded linear multidivisional multilevel programming problems, the existence of extreme point Nash-Stackelberg equilibrium solutions is established under mild conditions. In addition, a simple (but computationally demanding) enumeration procedure has been provided to find Nash-Stackelberg equilibrium solutions for bounded linear problems with two levels and three divisions.

The results provided in this paper also apply to important special cases of the multidivisional multilevel programming problem. When there is only one division at each level, the linear multidivisional multilevel programming problem is a *multilevel linear programming problem.* Therefore, Theorem 6 implies that a solution to a multilevel linear programming problem (see Bialas and Karwan [9]) must exist at an extreme point of the feasible region S, a result previously proved by Wen [35]. When there is only one level, the linear multidivisional multilevel programming problem is an *n-person polytope game* (see Bhattacharjee, *et al.* [7]) and the appropriate results apply. Finally, when there is only one level and one division, Theorem 6 implies the well-known fact that the solution to a bounded *linear programming problem* must exist at an extreme point of the convex polytope S (see Dantzig [14]).

There are readily apparent applications and generalizations of the multidivisional multilevel programming formulation presented here. As a first step, it is hoped that this paper encourages further investigation into multidivisional multilevel decentralized programming problems and their extensions. Perhaps future research will provide a better understanding of multilevel decision-making organizations and their behavior. These are challenging, but tractable, problems.

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	Step 2	Step 3	Step 4	Step 5	Step 6
$\hat{x}_{[1]} =$ $(0,1,0)$	$\in S^2$ $f_1^1 = 0.8$	$\tilde{x} = \hat{x}_{[1]}$ $f_1^2 = 0.7$	$\tilde{x} = \hat{x}_{[1]}$ $f_2^2 = 2$	$W =$ $\{(0,1,0)\}$	Set $i = 2$ Go to Step 2
$\hat{x}_{[2]} =$ $(0.5,1,0.5)$	$\in S^2$ $f_1^1 = 1.9$	$\tilde{x} = \hat{x}_{[2]}$ $f_1^2 = 1.4$	$\tilde{x} = (0,1,0)$ $f_2^2(\tilde{x}) = 2$ $\tilde{x} \neq \hat{x}_{[2]}$ $f_2^2(\hat{x}_{[2]}) = 0.25$ Go to Step 6	—	Set $i = 3$ Go to Step 2
$\hat{x}_{[3]} =$ $(0,0,1)$	$\in S^2$ $f_1^1 = 1.2$	$\tilde{x} = (0.5,1.5,1)$ $f_1^2(\tilde{x}) = 1.45$ $\tilde{x} \neq \hat{x}_{[3]}$ $f_1^2(\hat{x}_{[3]}) = -0.6$ Go to Step 6	—	—	Set $i = 4$ Go to Step 2
$\hat{x}_{[4]} =$ $(0,2,1)$	$\in S^2$ $f_1^1 = 2.8$	$\tilde{x} = (0.5,1.5,1)$ $f_1^2(\tilde{x}) = 1.45$ $\tilde{x} \neq \hat{x}_{[4]}$ $f_1^2(\hat{x}_{[4]}) = 0.8$ Go to Step 6	—	—	Set $i = 5$ Go to Step 2
$\hat{x}_{[5]} =$ $(0.5,0.5,1)$	$\in S^2$ $f_1^1 = 2.1$	$\tilde{x} = (0.5,1.5,1)$ $f_1^2(\tilde{x}) = 1.45$ $\tilde{x} \neq \hat{x}_{[5]}$ $f_1^2(\hat{x}_{[5]}) = 0.75$ Go to Step 6	—	—	Set $i = 6$ Go to Step 2
$\hat{x}_{[6]} =$ $(0.5,1.5,1)$	$\in S^2$ $f_1^1 = 2.9$	$\tilde{x} = \hat{x}_{[6]}$ $f_1^2 = 1.45$	$\tilde{x} = (0,1.5,0.5)$ $f_2^2(\tilde{x}) = 2.25$ $\tilde{x} \neq \hat{x}_{[6]}$ $f_2^2(\hat{x}_{[6]}) = 0.5$ Go to Step 6	—	Set $i = 7$ Go to Step 2
$\hat{x}_{[7]} =$ $(0.5,1,1.5)$	$\in S^2$ $f_1^1 = 3.1$	$\tilde{x} = \hat{x}_{[7]}$ $f_1^2 = 0.8$	$\tilde{x} = (0,1,0)$ $f_2^2(\tilde{x}) = 2$ $\tilde{x} \neq \hat{x}_{[7]}$ $f_2^2(\hat{x}_{[7]}) = -1.25$ Go to Step 6	—	Set $i = 8$ Go to Step 2
$\hat{x}_{[8]} =$ $(0,1,2)$	$\in S^2$ $f_1^1 = 3.2$	$\tilde{x} = \hat{x}_{[8]}$ $f_1^2 = -0.5$	$\tilde{x} = (0,1,0)$ $f_2^2(\tilde{x}) = 2$ $\tilde{x} \neq \hat{x}_{[8]}$ $f_2^2(\hat{x}_{[8]}) = -1$ Go to Step 6	—	Stop

Table 1: Results of the enumeration procedure for Example One.