ECONOMIC STRENGTH IN COOPERATIVE GAMES[*]

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Abstract. This paper revisits the concept of power indices to evaluate the economic strength of individual players in a characteristic function game. A strength index is axiomatically derived and is shown to be uniquely defined. The strength index satisfies individual rationality for all characteristic function games, including games which are not superadditive. Furthermore, a relationship between the index and the core solution concept is illustrated.

1. Introduction. The distribution of economic strength among the players of a cooperative game has been studied for many years (Shapley 1951, Maschler 1963, Coleman 1971, Monderer, Samet, and Shapley 1992, and Iñarra and Usategui 1993). The best-known index of economic power is the Shapley value, which has many desirable properties. For example, every characteristic function game has a unique value which satisfies Shapley's axioms. In addition, if the cooperative game is superadditive, the Shapley value satisfies individual rationality (i.e., each singleton player is allocated an amount at least equal to the value of that player in the game).

The core, another solution concept, represents the set of solutions that distributes the wealth in such a way that satisfies all subsets of players (i.e., each coalition of players is assigned an amount at least equal to the value of the coalition in the game). This property of satisfying the aspiration level of each player makes the core concept an intuitively desirable solution. However, some cooperative games have cores consisting of an infinite number of solutions, while others have cores which are empty.

This paper axiomatically derives an index of economic strength with some properties not found in its predecessors. In particular, individual rationality is guaranteed for all cooperative games. Moreover, a direct relationship will be shown to exist between the strength index and the core.

2. Preliminaries. A *n-person game* consists of players whose individual payoffs are dependent on the decisions made by all n players. In a *cooperative game*, subsets of players are allowed to combine resources in an effort to increase their individual payoffs (Shenoy 1978). The partition of the player set N into subsets of players is referred to as the *coalition structure* of the game.

DEFINITION 1. *A game in* **characteristic function form** *is a real-valued function v defined on the subsets of the player set N satisfying $v(\varnothing) = 0$.*

Note that a coalition of players may be assigned a value less than the sum of the assigned values of some partition of that coalition. Thus, the payoff $v(S)$ to a coalition S should not be confused as the potential economic worth of the players in S.

DEFINITION 2. *Given a game v in characteristic function form, define the* **superadditive cover** *of v as*

$$\tilde{v}(S) \equiv \max_{\mathcal{P}_S} \sum_{R \in \mathcal{P}_S} v(R),$$

where $\mathcal{P}_S$ is a coalition structure of the set of players in S.

Also, define $\mathcal{P}_S^\star$ as that coalition structure for which $\check{v}(S)$ is obtained. Notice that if the game v is superadditive, then $\mathcal{P}_S^\star = \{N - S, S\}$.

DEFINITION 3. *A characteristic function game v is* **superadditive** *if, for all coalitions $P \subseteq N$, $v(P) = \check{v}(P)$.*

3. Other definitions. This paper will relax the often made assumption of superadditivity. At first glance, it would appear reasonable that one need only consider superadditive games. However, some important extensive form games naturally induce non-superadditive characteristic function games when coalitions are allowed to form. For example, hierarchical organizations modeled as Stackelberg games (Stackelberg 1952) induce characteristic function games that are not superadditive (Bialas and Chew 1980, and Bialas 1989).

Before stating the axioms, we need to introduce a few more definitions.

DEFINITION 4. *Let v be a game and α be a scalar. Then the game $v \boxplus \alpha$ is defined as:*

$$[v \boxplus \alpha](P) = \begin{cases} \check{v}(N) + \alpha & \text{if } P = N \\ \check{v}(P) & \text{otherwise.} \end{cases}$$

Note that $v \boxplus \alpha$ is always a superadditive game for any α. Also, for any game v, $v \boxplus 0 = \check{v}$, the superadditive cover of v.

DEFINITION 5. *Let v be a game and γ be a scalar. Then the game γv is defined as:*

$$[\gamma v](P) = \gamma v(P) \quad \text{for all } P \subseteq N$$

DEFINITION 6. *Let u and v be games over the same player set N. Then the game $u + v$ is defined as:*

$$[u + v](P) = u(P) + v(P) \quad \text{for all } P \subseteq N.$$

DEFINITION 7. *Let N_1 and N_2 represent disjoint player sets with game v being composed of players from N_1 and game u being composed of players from N_2. Then, by $v \oplus u$, we mean the game w such that for any $P \subseteq N_1 \cup N_2$, $w(P) = v(P_1) + u(P_2)$ where $P_1 = P \cap N_1$ and $P_2 = P \cap N_2$.*

DEFINITION 8. *The game w_P is said to be* **elementary** *with respect to coalition $P \subseteq N$ if for all coalitions $R \subseteq N$,*

$$w_P(R) = \begin{cases} 0 & \text{if } R \not\supseteq P \\ 1 & \text{if } R \supseteq P. \end{cases}$$

4. The foundations. Consider any game v in characteristic function form and let $\rho[v]$ denote an n-vector satisfying the following axioms:

AXIOM 1. $\rho[v \boxplus \alpha] = \rho[v] + \alpha$ *for any α.*

AXIOM 2. $\rho[\gamma v] = \gamma \rho[v]$ *for any γ.*

AXIOM 3. *Let u and v be games with disjoint player sets N_1 and N_2, respectively. Then,*

$$\rho_i[u \oplus v] = \left\{ \begin{array}{ll} \rho_i[u] & \text{if } i \in N_1 \\ \rho_i[v] & \text{if } i \in N_2 \end{array} \right. .$$

AXIOM 4. *Let u and v be superadditive games. Then,*

$$\rho[u + v] = \rho[u] + \rho[v].$$

The first axiom deserves some discussion. Consider a game v with $v(N) = x$. If $v(N)$ is incrementally changed to $x + \alpha$, then any player in the grand coalition, N, has the capability of undermining the change by leaving coalition N. Axiom 1 accounts for this incremental change in economic power for every player in N.

In addition, since $v \boxplus 0 = \tilde{v}$, Axiom 1 implies that $\rho[v] = \rho[\tilde{v}]$. In other words, the power index for a non-superadditive game is the same as if the players behaved in a superadditive fashion. For example, suppose $v(S \cup T) < v(S) + v(T)$. Then the players in $S \cup T$ could behave as if the union did not form, collect the proceeds $v(S) + v(T)$, and apportion $v(S) + v(T)$ among all the members of $S \cup T$.

We will show that for the family of characteristic function games, there exists a unique vector $\rho[v]$ satisfying Axioms 1–4. In order to do this, we first need a few preliminary results.

LEMMA 9. *Let v be a game of the following form:*

$$\begin{array}{rcl} v(N) & = & 1 \\ v(P) & = & 0 \quad \text{for all } P \subset N. \end{array}$$

Then, $\rho_i[v] = 1$ for all $i \in N$.

Proof. Let v be a game with $v(N) = 1$ and $v(P) = 0$ for all $P \subset N$. Consider the game $u = 0v$; $u(P) = 0$ for all $P \subseteq N$. From Axiom 2, for all $i \in N$,

$$\rho_i[u] = \rho_i[0v] = 0\rho_i[v] = 0.$$

Notice that $v = u \boxplus 1$. Therefore, from Axiom 1, for all $i \in N$,

$$\rho_i[v] = \rho_i[u] + 1 = 1.$$

■

LEMMA 10. *Let w_P be an elementary game with respect to P. Then,*

$$\rho_i[w_P] = \left\{ \begin{array}{ll} 1 & \text{if } i \in P \\ 0 & \text{if } i \notin P. \end{array} \right.$$

Proof. The player set N can be partitioned into two sets; $N_1 = P$ and $N_2 = N - P$. Let u be the game defined over the player set N_1 with

$$\begin{array}{rcl} u(N_1) & = & 1 \\ u(R) & = & 0 \quad \text{for all } R \subset N_1. \end{array}$$

Furthermore, let w be the game defined over the player set N_2 with $w(Q) = 0$ for all $Q \subseteq N_2$.

From Lemma 9, we have $\rho_i[u] = 1$ for all $i \in N_1$ and $\rho_i[w] = 0$ for all $i \in N_2$. Furthermore, notice that $v = u \oplus w$. Thus, from Axiom 3, we obtain the desired result. ∎

LEMMA 11. *If v is any game, v can be uniquely written as a sum (or difference) of elementary games, $v = \sum_{P \subseteq N} c_P w_P$, where w_P is the elementary game with respect to P and $c_P = \sum_{R \subseteq P}(-1)^{|P|-|R|}v(R)$ is a scalar.*

Proof. See Shapley (1951). ∎

LEMMA 12. *Let the games u and v be superadditive and both games have the same player set. Then the game $u + v$ is superadditive.*

Proof. Let games u and v be superadditive with both games having the same player set N and consider an arbitrary coalition $P \subseteq N$. From the definitions and the fact that the games are superadditive, we obtain the following:

$$
\begin{aligned}
[u+v](P) &= u(P) + v(P) \\
[u+v](P) &\geq \sum_{R \in \mathcal{P}_P} u(R) + \sum_{R \in \mathcal{P}_P} v(R) && \forall \mathcal{P}_P \\
[u+v](P) &\geq \sum_{R \in \mathcal{P}_P} \big(u(R) + v(R)\big) && \forall \mathcal{P}_P \\
[u+v](P) &\geq \sum_{R \in \mathcal{P}_P} [u+v](R) && \forall \mathcal{P}_P.
\end{aligned}
$$

Hence, $[u + v]$ is superadditive. ∎

LEMMA 13. *Let $v = \sum_{P \subseteq N} c_P w_P$ be a superadditive game, where the w_P are elementary games and $c_P = \sum_{R \subseteq P}(-1)^{|P|-|R|}v(R)$. Then, for all $i \in N$,*

$$
\rho_i[v] = \sum_{\substack{P \subseteq N \\ i \in P}} c_P.
$$

Proof. Let v be a superadditive game with $v = \sum_{P \subseteq N} c_P w_P$. We then have

$$
\begin{aligned}
v &= \sum_{P \subseteq N} c_P w_P \\
&= \sum_{\substack{P \subseteq N \\ c_P \geq 0}} c_P w_P + \sum_{\substack{P \subseteq N \\ c_P < 0}} c_P w_P \\
&= \sum_{\substack{P \subseteq N \\ c_P \geq 0}} c_P w_P - \sum_{\substack{P \subseteq N \\ c_P < 0}} |c_P| w_P.
\end{aligned}
$$

Therefore, we have

$$
v + \sum_{\substack{P \subseteq N \\ c_P < 0}} |c_P| w_P = \sum_{\substack{P \subseteq N \\ c_P \geq 0}} c_P w_P.
$$

Since v and $\sum_{\substack{P \subseteq N \\ c_P < 0}} |c_P| w_P$ are superadditive (from the Lemma 12), we can use Axiom 4

to get:

$$
\rho[v] + \rho\left[\sum_{\substack{P\subseteq N \\ c_P<0}} |c_P| w_P\right] = \rho\left[v + \sum_{\substack{P\subseteq N \\ c_P<0}} |c_P| w_P\right]
$$

$$
= \rho\left[\sum_{\substack{P\subseteq N \\ c_P\geq 0}} c_P w_P\right].
$$

Thus, for each $i \in N$,

$$
\begin{aligned}
\rho_i[v] &= \rho_i\left[\sum_{\substack{P\subseteq N \\ c_P\geq 0}} c_P w_P\right] - \rho_i\left[\sum_{\substack{P\subseteq N \\ c_P<0}} |c_P| w_P\right] \\
&= \rho_i\left[\sum_{\substack{P\subseteq N \\ c_P\geq 0}} c_P w_P\right] + \rho_i\left[\sum_{\substack{P\subseteq N \\ c_P<0}} c_P w_P\right] \\
&= \sum_{\substack{P\subseteq N \\ c_P\geq 0}} c_P \rho_i[w_P] + \sum_{\substack{P\subseteq N \\ c_P<0}} c_P \rho_i[w_P] \\
&= \sum_{P\subseteq N} c_P \rho_i[w_P] \\
&= \sum_{\substack{P\subseteq N \\ i\in P}} c_P(1) + \sum_{\substack{P\subseteq N \\ i\notin P}} c_P(0) \\
&= \sum_{\substack{P\subseteq N \\ i\in P}} c_P.
\end{aligned}
$$

∎

LEMMA 14. *Let v be a n-person game. Then*

$$
\sum_{\substack{P\subseteq N \\ i\in P}} \sum_{R\subseteq P} (-1)^{|P|-|R|} v(R) = v(N) - v(N - \{i\}).
$$

Proof. First notice that by changing the order of summation,

$$
\sum_{\substack{P\subseteq N \\ i\in P}} \sum_{R\subseteq P} (-1)^{|P|-|R|} v(R) = \sum_{R\subseteq N} \sum_{\substack{P\ni i \\ R\subseteq P\subseteq N}} (-1)^{|P|-|R|} v(R).
$$

For a given coalition R, the inner sum,

$$
\sum_{\substack{P\ni i \\ R\subseteq P\subseteq N}} (-1)^{|P|-|R|},
$$

is a function of the number of coalitions P that satisfy the conditions: $R \subseteq P \subseteq N$ and $i \in P$. The number of ways of forming coalition P of a given size $|P|$ that satisfy the

above conditions when $i \in R$ is $\binom{n-|R|}{x}$, where $x = |P| - |R|$ and $0 \le x \le n - |R|$. This is equivalent to the number of ways of choosing $|P| - |R|$ players from the remaining $n - |R|$ players in set N. When $i \notin R$, we need to add i to R when forming P.

Therefore, the number of ways of forming P of a given size $|P|$ is $\binom{n-|R|-1}{x}$, where $x = |P| - |R| - 1$ and $0 \le x \le n - |R| - 1$. Now we can see that

$$
\sum_{\substack{R \subseteq N \\ R \subseteq P \subseteq N}} \sum_{\substack{P \ni i}} (-1)^{|P|-|R|} v(R) = \sum_{\substack{R \subseteq N \\ i \in R}} \sum_{x=0}^{n-|R|} (-1)^x \binom{n-|R|}{x} v(R)
$$
$$
+ \sum_{\substack{R \subseteq N \\ i \notin R}} \sum_{x=0}^{n-|R|-1} (-1)^{x+1} \binom{n-|R|-1}{x} v(R).
$$

Note that

$$
\sum_{x=0}^{m} (-1)^x \binom{m}{x} = \begin{cases} 1 & \text{when } m = 0 \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore,

$$
\sum_{\substack{R \subseteq N \\ i \in R}} \sum_{x=0}^{n-|R|} (-1)^x \binom{n-|R|}{x} v(R) = v(N),
$$

and

$$
\sum_{\substack{R \subseteq N \\ i \notin R}} \sum_{x=0}^{n-|R|-1} (-1)^{x+1} \binom{n-|R|-1}{x} v(R) = -v(N - \{i\}).
$$

Thus,

$$
\sum_{\substack{R \subseteq N \\ R \subseteq P \subseteq N}} \sum_{\substack{P \ni i}} (-1)^{|P|-|R|} v(R) = v(N) - v(N - \{i\}).
$$

$\blacksquare$

5. Main results. Using the above lemmas, we can now establish several properties of the strength index $\rho[\cdot]$.

5.1. Uniqueness. THEOREM 15. *Let v be any game. There exists a unique function ρ, defined on v, satisfying Axioms 1–4.*

Proof. Let $\tilde{v}$ be the superadditive cover of v. From Axiom 1, when $\alpha = 0$, we have that $\rho[v] = \rho[\tilde{v}]$. Furthermore, from Lemma 11, $\tilde{v}$ can be written as a unique sum of elementary games. Therefore,

$$
\begin{aligned}
\rho[v] &= \rho[\tilde{v}] \\
&= \rho[\sum_{P \subseteq N} c_P w_P].
\end{aligned}
$$

Since $\tilde{v}$ is superadditive, from Lemma 13 we obtain, for each $i \in N$,

$$\rho_i[\tilde{v}] = \sum_{\substack{P \subseteq N \\ i \in P}} c_P.$$

From Lemma 11, the value of each c_P is specified as $\sum_{R \subseteq P} (-1)^{|P|-|R|} \tilde{v}(R)$. Thus,

$$\rho_i[v] = \sum_{\substack{P \subseteq N \\ i \in P}} \sum_{R \subseteq P} (-1)^{|P|-|R|} \tilde{v}(R).$$

From Lemma 14,

$$\sum_{\substack{P \subseteq N \\ i \in P}} \sum_{R \subseteq P} (-1)^{|P|-|R|} \tilde{v}(R) = \tilde{v}(N) - \tilde{v}(N - \{i\}).$$

Thus, for all $i \in N$,

$$\rho_i[v] = \tilde{v}(N) - \tilde{v}(N - \{i\}).$$

∎

Several results are now obtained for this function.

5.2. Individual Rationality. It is reasonable for a player to demand a payoff that is at least as great as the amount that the player can guarantee himself independent of the actions of the other players. This is referred to as *individual rationality*.

DEFINITION 16. *A vector* $(x_1, \ldots, x_n)$ *satisfies* **individual rationality** *for game* v *if for all* $i \in N$,

$$x_i \geq v(\{i\}).$$

THEOREM 17. *The function* ρ *satisfying Axioms 1–4, satisfies individual rationality for games in characteristic function form.*

Proof. Let v be a game in characteristic function form. We have

$$\tilde{v}(N) \geq v(\{i\}) + \tilde{v}(N - \{i\}).$$

From the previous theorem, the function ρ satisfying Axioms 1–4 for game v can be computed for each $i \in N$ as follows:

$$\rho_i[v] = \tilde{v}(N) - \tilde{v}(N - \{i\}).$$

Substituting for $\tilde{v}(N)$, we get

$$\rho_i[v] + \tilde{v}(N - \{i\}) \geq v(\{i\}) + \tilde{v}(N - \{i\}).$$

Thus, for each $i \in N$,

$$\rho_i[v] \geq v(\{i\}).$$

∎

5.3. Relation with the Core. If the amount available to the grand coalition of a game is distributed among the players in such a way that every coalition of players obtain an amount at least as great as the value of the coalition, then no coalition would have a reasonable objection to the distribution. The *core* is the set of all such distributions.

DEFINITION 18. *The* **core** *of a game v in characteristic function form is the set of n-dimensional vectors x satisfying*

1. $\sum_{i \in P} x_i \geq v(P)$ *for all $P \subset N$,*
2. $\sum_{i \in N} x_i = \check{v}(N)$.

Also, define $\mathcal{P}^{\star}_{N-\{i\}}$ as that coalition structure for which $\check{v}(N - \{i\})$ is obtained. Notice that if the game v is superadditive, $\mathcal{P}^{\star}_{N-\{i\}} = \{N - \{i\}\}$.

LEMMA 19. *If $(x_1, x_2, \ldots, x_n) \in \mathcal{C} \neq \emptyset$, then $\check{v}(N - \{i\}) \leq \sum_{j \neq i} x_j$ for all $i \in N$.*

Proof. Let v be an n-person game with $\mathcal{C} \neq \emptyset$ and let $(x_1, x_2, \ldots, x_n) \in \mathcal{C}$. Then, from the definition of the core, it follows that

$$\sum_{j \in P} x_j \geq v(P)$$

for all $P \in N$. By summing both sides of the above inequality over the elements of coalition structure $\mathcal{P}^{\star}_{N-\{i\}}$, we get

$$\sum_{j \neq i} x_j = \sum_{P \in \mathcal{P}^{\star}_{N-\{i\}}} \sum_{j \in P} x_j \geq \sum_{P \in \mathcal{P}^{\star}_{N-\{i\}}} v(P) = \check{v}(N - \{i\}),$$

which yields the desired result. ∎

THEOREM 20. *If imputation x is an element of the core of game v, then $x_i \leq \rho_i[v]$.*

Proof. Assume imputation $x = (x_1, x_2, \ldots, x_n)$ is an element of the core of game v. From Lemma 19, for each player i, $\sum_{j \neq i} x_j - \check{v}(N - \{i\}) \geq 0$. Therefore,

$$
\begin{aligned}
\rho_i[v] &= \check{v}(N) - \check{v}(N - \{i\}) \\
&= \sum_{i \in N} x_i - \check{v}(N - \{i\}) \\
&= x_i + \sum_{j \neq i} x_j - \check{v}(N - \{i\}) \\
&\geq x_i.
\end{aligned}
$$

∎

From the definition of the core of a game, any imputation x that is an element of the core of game v must satisfy the following equation:
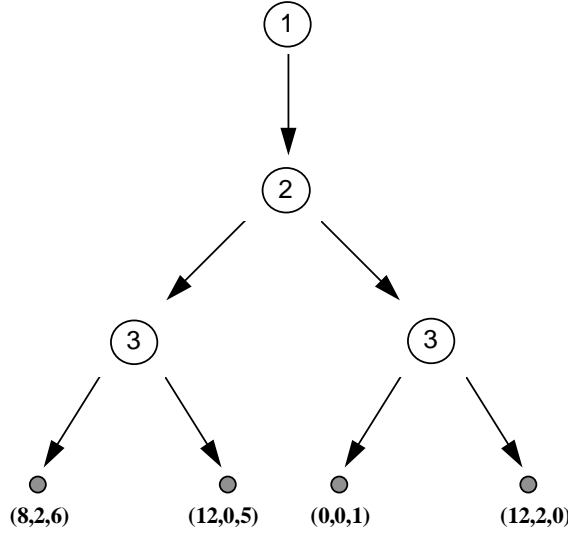
$$\sum_{i \in N} x_i = \check{v}(N).$$

This fact combined with the previous theorem directly leads to the following result:

COROLLARY 21. *Let v be any n-person game in characteristic function form.*

1. *If $\sum_{i \in N} \rho_i[v] < \check{v}(N)$, then $\mathcal{C} = \emptyset$.*
2. *If $\sum_{i \in N} \rho_i[v] = \check{v}(N)$, then $\mathcal{C} \in \{\emptyset, (\rho_1[v], \rho_2[v], \ldots, \rho_n[v])\}$.*
3. *If $\sum_{i \in N} \rho_i[v] > \check{v}(N)$, then*

$$\mathcal{C} \subseteq \left\{ (\rho_1[v] - \alpha_1, \ldots, \rho_n[v] - \alpha_n) \left| \sum_{i=N} \alpha_i = \sum_{i=N} \rho_i[v] - \check{v}(N), \alpha_i \geq 0 \quad \forall i \right. \right\}.$$

Fig. 1. *Example game in extensive form.*

6. Example. Consider the extensive form game in Figure 1. In this game, the three players make sequential decisions based on the decisions previously made and the rational decisions that will be made in reaction to each of their decisions. Once the players have made their decisions, a payoff, (x_1, x_2, x_3), is assigned. Each player's objective is to maximize his own payoff. Perfect information is assumed, and the players know each other's objective function. The sequential decisions, once made, are announced to everyone, and no secretive coalitions are allowed to form.

If the players act independently, the solution to the game will result in the payoff (8,2,6). However, allowing the formation of coalitions results in a different outcome. For example, consider a coalition between player 1 and player 3. This new game is played exactly as the previous game with only a change of the players' objective functions. Instead of maximizing their individual payoffs, the players jointly maximize the payoff to their coalition. The resulting game is shown in Figure 2. The solution to this game results in the payoff (12,2,12),and the amount allocated to coalition {1,3} is 12. It is up to the coalition to distribute the allocation among its members.

We now consider the payoff generated from each possible coalition structure. For notational convenience, define the coalition structures in the following manner:

$$\begin{aligned}
\mathcal{P}_0 &= \{\{1\}, \{2\}, \{3\}\} & \mathcal{P}_1 &= \{\{1\}, \{2, 3\}\} \\
\mathcal{P}_2 &= \{\{2\}, \{1, 3\}\} & \mathcal{P}_3 &= \{\{1, 2\}, \{3\}\} \\
& \mathcal{P}_4 = \{\{1, 2, 3\}\}.
\end{aligned}$$

The value of the coalitions in their respective coalition structures can be shown to be:

$$\begin{aligned}
v(\{1\}, \mathcal{P}_0) &= 8 & v(\{2\}, \mathcal{P}_2) &= 2 \\
v(\{2\}, \mathcal{P}_0) &= 2 & v(\{1, 3\}, \mathcal{P}_2) &= 12 \\
v(\{3\}, \mathcal{P}_0) &= 6 & v(\{3\}, \mathcal{P}_3) &= 6 \\
v(\{1\}, \mathcal{P}_1) &= 8 & v(\{1, 2\}, \mathcal{P}_3) &= 10 \\
v(\{2, 3\}, \mathcal{P}_1) &= 8 & v(\{1, 2, 3\}, \mathcal{P}_4) &= 17.
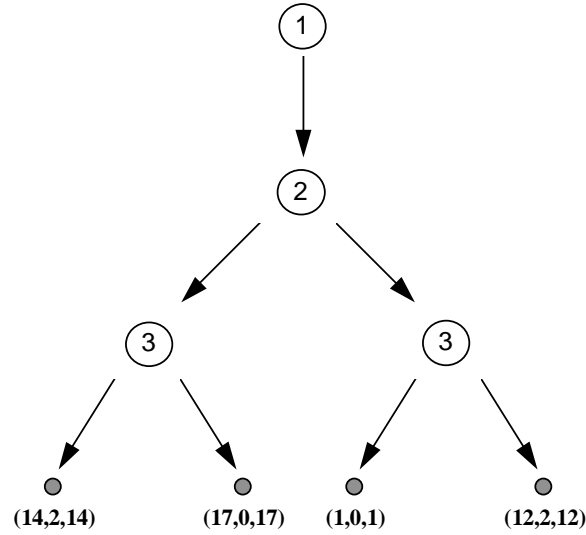\end{aligned}$$

FIG. 2. *Coalition formation between players 1 and 3.*

Since the value of any coalition in this particular game is invariant with respect to the coalition structure, the above game in partition function form (Lucas and Thrall 1963) simplifies to the following non-superadditive characteristic function form game:

$$v(\{1\}) = 8 \qquad v(\{1, 2\}) = 10$$
$$v(\{2\}) = 2 \qquad v(\{1, 3\}) = 12$$
$$v(\{3\}) = 6 \qquad v(\{2, 3\}) = 8$$
$$v(\{1, 2, 3\}) = 17.$$

$$v(\{1\}) = 8 \qquad v(\{1, 2\}) = 10$$
$$v(\{2\}) = 2 \qquad v(\{1, 3\}) = 12$$
$$v(\{3\}) = 6 \qquad v(\{2, 3\}) = 8$$
$$v(\{1, 2, 3\}) = 17.$$

The strength index of this game is $\rho[v] = (9, 3, 7)$. Notice that the sum of the strength of the players is more the value of the grand coalition. Figure 3 illustrates the distribution simplex, the set of possible ways of distributing the amount available to the grand coalition among the players. The strength index is projected onto this distribution simplex forming a triangle as depicted in the diagram. Figure 4 shows that the core of the game lies inside this projection.

7. Conclusions. This paper has offered an alternative axiomatic approach to evaluate the economic strength of individual players in a characteristic function game. The strength index has been shown to be uniquely defined. Moreover, the strength index satisfies individual rationality for all characteristic function games, including games which are not superadditive. In addition, several results have been provided to establish the relationship between the index and the core solution concept.
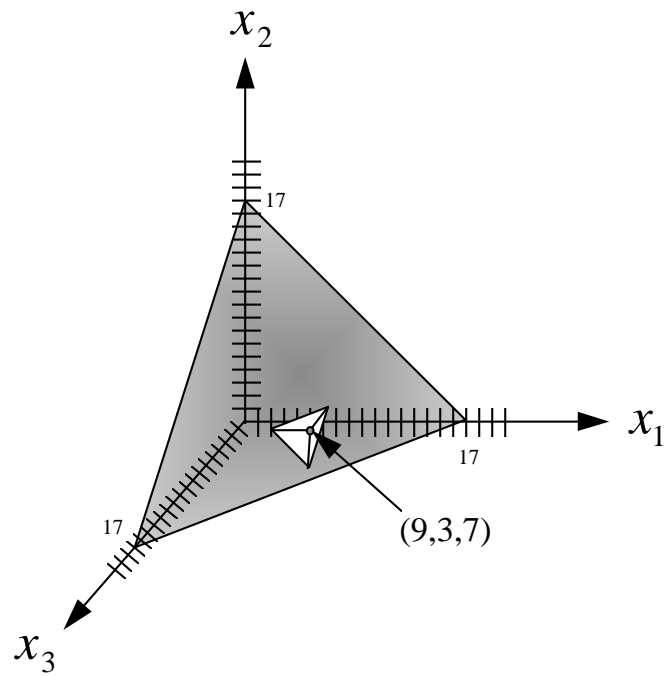
8. Remarks regarding the revisions.

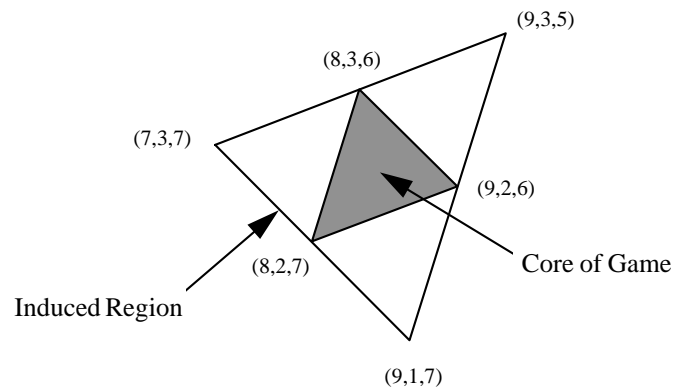FIG. 3. *Projection of the strength index onto the distribution simplex.*

FIG. 4. *Example of the core of the game being a subset of the projection.*

8.1. Original paper. The original paper was produced on September 25, 1995.

8.2. Revision dated August 18, 1998. A new statement of Definition 4 now makes the game, $v \boxplus \alpha$, superadditive. This change now insures that α is added to only the value of the grand coalition N. This corrects an error in the proof of Theorem 15.

Also, as a result of the new definition, Axiom 1 now implies

AXIOM 5. *Let v be any game and let $\tilde{v}$ be the superadditive cover of v. Then,*

$$\rho[v] = \rho[\tilde{v}].$$

which appeared in the earlier version. Axiom 5 is no longer necessary and has been deleted.

9. Acknowledgments. We gratefully acknowledge the assistance of many individuals during the course of this research. In particular, we would like to thank David Hunter, Peter Rogerson, Moises Sudit, Donald K. Wagner, and Chien-Hsin Harris Yang. Of course, any errors or omissions are the sole responsibility of the authors.

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