IE675 Game Theory

Lecture Note Set 6

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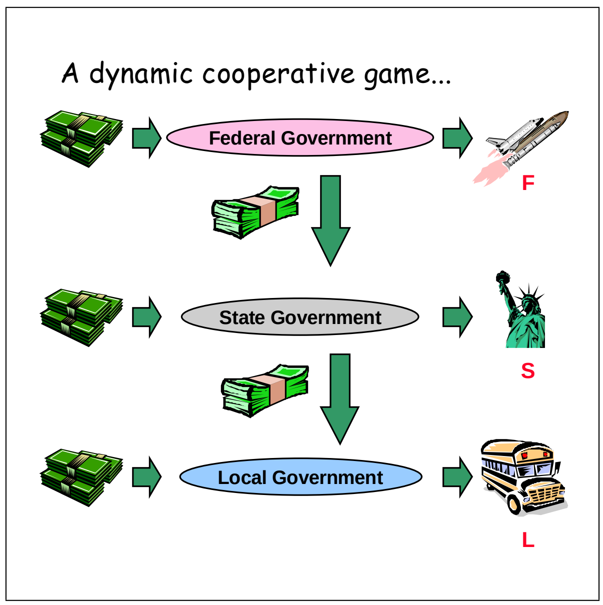
6 DYNAMIC COOPERATIVE GAMES

6.1 Some introductory examples

Consider the following hierarchical game:



In this particular example,

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1. The system has interacting players within a hierarchical structure
2. Each player executes his polices after, and with full knowledge of, the decisions of predecessors.
3. Players might form coalitions in order to improve their payoff.

What do we mean by (3)?

For examples (without coalitions) see Cassidy, *et al.* [12] and Charnes, *et al.* [13].

Without coalitions:

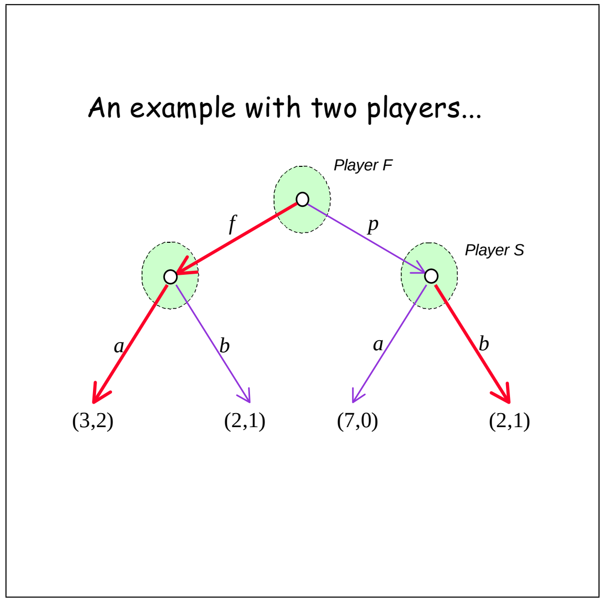
$$\begin{aligned} \text{Payoff to Federal government} &= g_F(F,S,L) \\ \text{Payoff to State government} &= g_S(F,S,L) \\ \text{Payoff to Local government} &= g_L(F,S,L) \end{aligned}$$

A coalition structure of $\{\{F,S\},\{L\}\}$ would result in the players maximizing the following objective functions:

$$\begin{aligned} \text{Payoff to Federal government} &= g_F(F,S,L) + g_S(F,S,L) \\ \text{Payoff to State government} &= g_F(F,S,L) + g_S(F,S,L) \\ \text{Payoff to Local government} &= g_L(F,S,L) \end{aligned}$$

The order of the play remains the same. Only the objectives change.

Here is a two-player game of the same type, but written in extensive form:



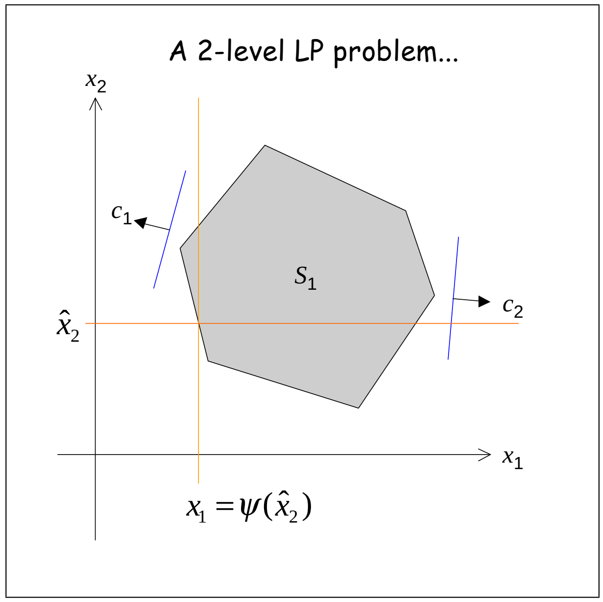
where

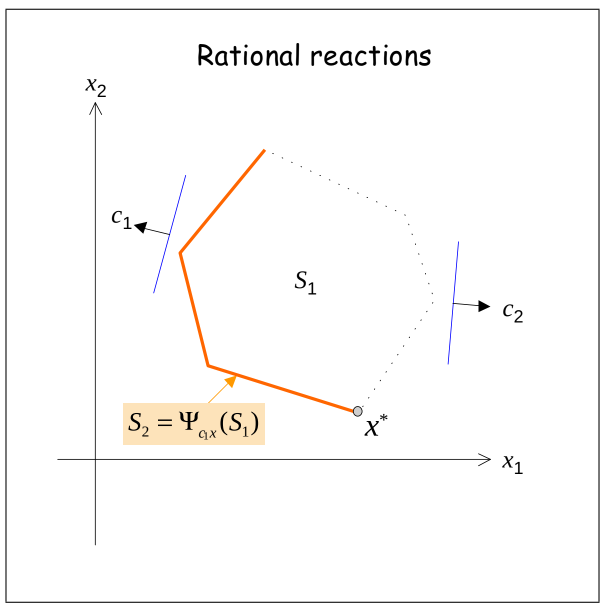
f	Full funding
p	Partial funding
a	Project a
b	Project b

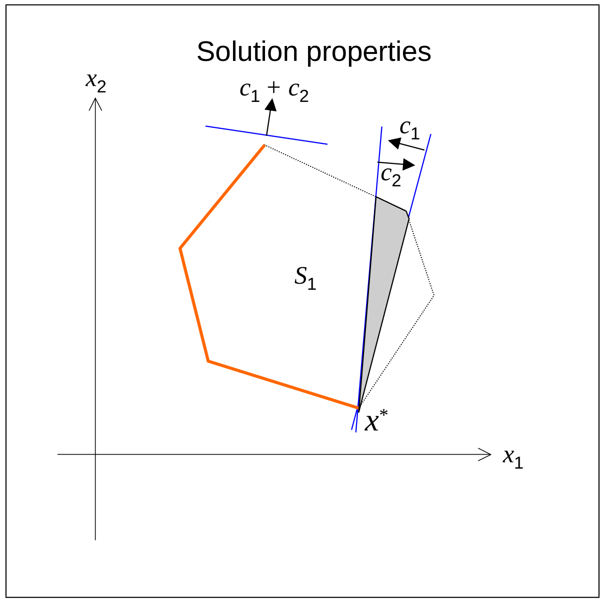
The Stackelberg solution to this game is (f, a) with a payoff of $(3, 1)$. However, if the players cooperated, and utility was transferable, they could get 7 with strategy (p, a).

The key element causing this effect is preemption. A dynamic, cooperative model is needed.

Chew [14] showed that even linear models can exhibit this behavior *and* he developed a dynamic cooperative game model.


A 2-level LP problem...
x2
c1
S1
x̂2
c2
x1
x1 = ψ(x̂2)


Rational reactions
x2
c1
S1
c2
S2 = Ψc1x(S1)
x*
x1



6.1.1 Issues

See Bialas and Karwan [4] for details.

1. alternate optimal solutions
2. nonconvex feasible region

Note 6.1. The cause of inadmissible solutions is not the fault of the optimizers, but, rather, the sequential and preemptive nature of the decision process (i.e., the "friction of space and time").

6.2 Multilevel mathematical programming

The non-cooperative model in this section will serve as the foundation for our cooperative dynamic model. See also Bialas and Karwan [6].

Note 6.2. *Some history:* Sequential optimization problems arise frequently in many fields, including economics, operations research, statistics and control theory. The origin of this class of problems is difficult to trace since it is woven into the fabric of many scientific disciplines.

For the field of operations research, this topic arose as an extension to linear programming (see, for example, Bracken and McGill [8] Cassidy, *et al.* [12], Charnes, *et al.* [13])

In particular, Bracken, *et al.* [8, 7, 9] define a two-level problem where the constraints contain an optimization problem. However, the feasible region of the lower level planner does not depend on the decision variables to the upper-level planner. Removing this restriction, Candler and Norton [10] named this class of problems "multilevel programming." A number of researchers mathematically characterized the geometry of this problem and developed solution algorithms (see, for example, [1, 4, 5, 15]).

For a more complete bibliography, see Vicente and Calamai [18].

Let the decision variable space (Euclidean n-space), $\mathbb{R}^n \ni x = (x_1, x_2, \ldots, x_n)$, be partitioned among r levels,

$$\mathbb{R}^{n_k} \ni x^k = (x_1^k, x_2^k, \ldots, x_{n_k}^k) \quad \text{for } k = 1, \ldots, r,$$

where $\sum_{k=1}^{r} n_k = n$. Denote the maximization of a function $f(x)$ over $\mathbb{R}^n$ by varying only $x^k \in \mathbb{R}^{n_k}$ given fixed $x^{k+1}, x^{k+2}, \ldots, x^r$ in $\mathbb{R}^{n_{k+1}} \times \mathbb{R}^{n_{k+2}} \times \cdots \times \mathbb{R}^{n_r}$ by

(1) $$\max\{f(x) \,:\, (x^k \,|\, x^{k+1}, x^{k+2}, \ldots, x^r)\}.$$

Note 6.3. The value of expression (1) is a function of $x^1, x^2, \ldots, x^{k-1}$.

Let the full set of system constraints for all levels be denoted by S. Then the problem at the lowest level of the hierarchy, level one, is given by

$$(P^1)\begin{cases} \max & \{f_1(x) \,:\, (x^1 \,|\, x^2, \ldots, x^r)\} \\ \text{st:} & x \in S^1 = S \end{cases}$$

Note 6.4. The problem for the level-one decision maker P^1 is simply a (traditional) mathematical programming problem dependent on the given values of $x^2, \ldots, x^r$. That is, P^1 is a parametric programming problem.

The feasible region, $S = S^1$, is defined as the **level-one feasible region.** The solutions to P^1 in $\mathbb{R}_1^n$ for each fixed $x^2, x^3, \ldots, x^r$ form a set,

$$S^2 = \{\hat{x} \in S^1 \,:\, f_1(\hat{x}) = \max\{f_1(x) \,:\, (x^1 \,|\, \hat{x}^2, \hat{x}^3, \ldots, \hat{x}^r)\}\},$$

called the **level-two feasible region** over which $f_2(x)$ is then maximized by varying x^2 for fixed $x^3, x^4, \ldots, x^r$.

Thus the problem at level two is given by

$$(P^2)\begin{cases} \max & \{f_2(x) \,:\, (x^2 \,|\, x^3, x^4, \ldots, x^r)\} \\ \text{st:} & x \in S^2 \end{cases}$$

In general, the **level-k feasible region** is defined as

$$S^k = \{\hat{x} \in S^{k-1} \,|\, f_{k-1}(\hat{x}) = \max\{f_{k-1}(x) \,:\, (x^{k-1} \,|\, \hat{x}^k, \ldots, \hat{x}^r)\}\},$$

Note that $\hat{x}^{k-1}$ is a function of $\hat{x}^k, \ldots, \hat{x}^r$. Furthermore, the problem at each level can be written as

$$(P^k)\begin{cases} \max & \{f_k(x) \,:\, (x^k \,|\, x^{k+1}, \ldots, x^r)\} \\ \text{st:} & x \in S^k \end{cases}$$

which is a function of $x^{k+1}, \ldots, x^r$, and

$$(P^r) : \max_{x \in S^r} f_r(x)$$

defines the entire problem. This establishes a collection of nested mathematical programming problems $\{P^1, \ldots, P^r\}$.

Question 6.1. P^k depends on given $x^{k+1}, \ldots, x^r$, and only x^k is varied. But $f^k(x)$ is defined over all $x^1, \ldots, x^r$. Where are the variables $x^1, \ldots, x^{k-1}$ in problem P^k?

Note that the objective at level k, $f_k(x)$, is defined over the decision space of all levels. Thus, the level-k planner may have his objective function determined, in part, by variables controlled at other levels. However, by controlling x^k, after decisions from levels $k+1$ to r have been made, level k may influence the policies at level $k-1$ and hence all lower levels to improve his own objective function.

6.2.1 A more general definition

See also Bialas and Karwan [5].

Let the vector $x \in \mathbb{R}^N$ be partitioned as (x^a, x^b). Then we can define the following set function over the collection of closed and bounded regions $S \subset \mathbb{R}^N$:

$$\Psi_f(S) = \{\hat{x} \in S \,:\, f(\hat{x}) = \max\{f(x) \,|\, (x^a \,|\, \hat{x}^b)\}\}$$

as the **set of rational reactions** of f over S. This set is also sometimes called the *inducible region.* If for a fixed $\hat{x}^b$ there exists a unique $\hat{x}^a$ which maximizes $f(x^a, \hat{x}^b)$ over all $(x^a, \hat{x}^b) \in S$, then there induced a mapping

$$\hat{x}^a = \psi_f(\hat{x}^b)$$

which provides the rational reaction for each $\hat{x}^b$, and we can then write

$$\mathbf{\Psi}_f(S) = S \cap \{(x^a, x^b) \,:\, x^a = \psi_f(x^b)\}$$

So if $S = S^1$ is the level-one feasible region, the level-two feasible region is

$$S^2 = \mathbf{\Psi}_{f_1}(S^1)$$

and the level-k feasible region is

$$S^k = \mathbf{\Psi}_{f_{k-1}}(S^{k-1})$$

Note 6.5. Even if S^1 is convex, $S^k = \mathbf{\Psi}_{f_{k-1}}(S^{k-1})$ for $k \geq 2$ are typically non-convex sets.

6.2.2 The two-level linear resource control problem

The two-level linear resource control problem is the multilevel programming problem of the form

$$\begin{aligned} \max \quad & c^2 x \\ \text{st:} \quad & x \in S^2 \end{aligned}$$

where

$$S^2 = \{\hat{x} \in S^1 \,:\, c^1\hat{x} = \max\{c^1 x \,:\, (x^1 \,|\, \hat{x}^2)\}\}$$

and

$$S^1 = S = \{x \,:\, A^1x^1 + A^2x^2 \leq b,\ x \geq 0\}$$

Here, level 2 controls x^2 which, in turn, varies the resource space of level one by restricting $A^1x^1 \leq b - A^2x^2$.

The nested optimization problem can be written as:

$$(P^2)\left\{\begin{array}{ll} \max & \{c^2x = c^{21}x^1 + c^{22}x^2 \,:\, (x^2)\} \\ & \text{where } x^1 \text{ solves} \\ & \\ & (P^1)\left\{\begin{array}{ll} \max & \{c^1x = c^{11}x^1 + c^{12}x^2 \,:\, (x^1 \,|\, x^2)\} \\ \text{st:} & A^1x^1 + A^2x^2 \leq b \\ & x \geq 0 \end{array}\right. \end{array}\right.$$

Question 6.2. Suppose someone gives you a proposed solution x^* to problem P^2. Develop an "easy" way to test that x^* is, in fact, the solution to P^2.

Question 6.3. What is the solution to P^2 if $c^1 = c^2$. What happens if c^1 is substituted with $\alpha c^1 + (1-\alpha)c^2$ for some $0 \le \alpha \le 1$?

6.2.3 The two-level linear price control problem

The two-level linear price control problem is another special case of the general multilevel programming problem. In this problem, level two controls the cost coefficients of level one:

$$(P^2)\begin{cases} \max & \{c^2x = c^{21}x^1 + c^{22}x^2 \,:\, (x^2)\} \\ \text{st:} & A^2x^2 \le b^2 \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{(x^2)^{\mathsf{t}}x^1 \,:\, (x^1 \,|\, x^2)\} \\ \text{st:} & A^1x^1 \le b^1 \\ & x^1 \ge 0 \end{cases} \end{cases}$$

In this problem, level two controls the cost coefficients of level one.

6.3 Properties of S^2

Theorem 6.1. *Suppose $S^1 = \{x \,:\, Ax = b,\ x \ge 0\}$ is bounded. Let*

$$S^2 = \{\hat{x} = (\hat{x}^1, \hat{x}^2) \in S^1 \,:\, c^1\hat{x}^1 = \max\{c^1x^1 \,:\, (x^1 \,|\, \hat{x}^2)\}\}$$

then the following hold:

(i) $S^2 \subseteq S^1$

(ii) Let $\{y_t\}_{t=1}^{\ell}$ be any ℓ points of S^1, such that $x = \sum_t \lambda_{t=1}^{\ell} y_t \in S^2$ with $\lambda_t \ge 0$ and $\sum_t \lambda_t = 1$. Then $\lambda_t > 0$ implies $y_t \in S^2$.

Proof: See Bialas and Karwan [4].

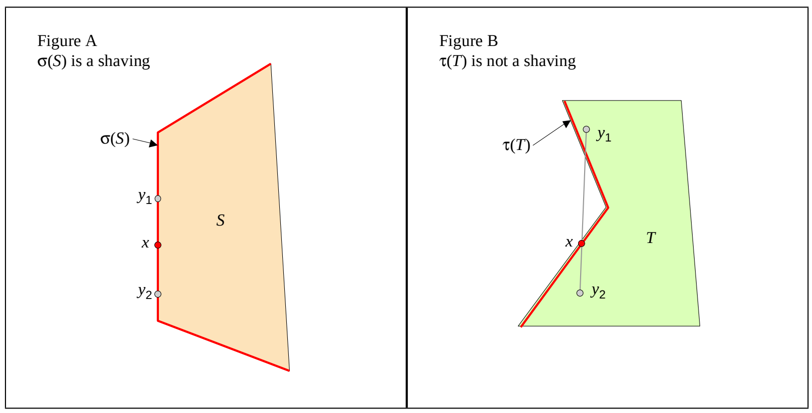
Note 6.6. The following results are due to Wen [19] (Chapter 2).

- a set S^2 with the above property is called a **shaving** of S^1
- shavings of shavings are shavings.
- shavings can be decomposed into convex sets that are shavings

- a convex set is always a shaving of itself.
- a relationship between shavings and the Kuhn-Tucker conditions for linear programming problems.

Definition 6.1. *Let $S \subseteq \mathbb{R}^n$. A set $\sigma(S) \subseteq S$ is a* shaving *of S if and only if for any $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S)$, the statement $\{\lambda_i > 0\}$ implies $y_i \in \sigma(S)$.*

The following figures illustrate the notion of a shaving.



The red region, $\sigma(S)$, in Figure A is a shaving of the set S. However in Figure B, the point $\lambda_1 y_1 + \lambda_2 y_2 = x \in \tau(T)$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 > 0$, $\lambda_2 > 0$. But y_1 and y_2 do not belong to $\tau(T)$. Hence $\tau(T)$ is not a shaving.

Theorem 6.2. *Suppose $T = \sigma(S)$ is a shaving of S and $\tau(T)$ is a shaving of T. Let $\tau \circ \sigma$ denote the composition of the functions τ and σ. Then $\tau \circ \sigma(S)$ is a shaving of S.*

Proof: Let $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S) = T$.

Suppose $\lambda_i > 0$. Since $\sigma(S)$ is a shaving of S then $y_i \in \sigma(S) = T$. Since $\tau(T)$ is a shaving of T, $y_i \in T$, and $\lambda_i > 0$ then $y_i \in \tau(T)$. Therefore $y_i \in \tau(\sigma(S))$ so $\tau \circ \sigma(S)$ is a shaving of S. ■

It is easy to prove the following theorem:

Theorem 6.3. *If S is a convex set, then $\sigma(S) = S$ is a shaving of S.*

Theorem 6.4. *Let $S \subseteq \mathbb{R}^N$. Let $\sigma(S)$ be a shaving of S. If x is an extreme point of $\sigma(S)$, then x is an extreme point of S.*

Proof: See Bialas and Karwan [4].

Corollary 6.1. *An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables (S^1).*

Proof: See Bialas and Karwan [4].

These results were generalized to n-levels by Wen [19]. Using Theorems 6.2 and 6.4, if f_k is linear and S^1 is a bounded convex polyhedron then the extreme points of

$$S^k = \Psi_{f_{k-1}} \circ \Psi_{f_{k-2}} \circ \cdots \circ \Psi_{f_2} \circ \Psi_{f_1}(S^1)$$

are extreme points of S^1. This justifies the use of extreme point search procedures to finding the solution to the n-level linear resource control problem.

6.4 Cooperative Stackelberg games

This section is based on Chew [14], Bialas and Chew [3], and Bialas [2].

6.4.1 An Illustration

Consider a game with three players, named 1, 2 and 3, each of whom controls an unlimited quantity of a commodity, with a different commodity for each player. Their task is to fill a container of unit capacity with amounts of their respective commodities, never exceeding the capacity of the container. The task of filling will be performed in a sequential fashion, with player 3 (the player at the "top" of the hierarchy) taking his turn first. A player cannot remove a commodity placed in the container by a previous player.

At the end of the sequence, a referee pays each player one dollar (or fraction, thereof) for each unit of his respective commodity which has been placed in the container. It is easy to see that, since player 3 has preemptive control over the container, he will fill it completely with his commodity, and collect one dollar.

Suppose, however, that the rules are slightly changed so that, in addition, player 3 could collect five dollars for each unit of *player one's* commodity which is placed in the container. Since player 2 does not receive any benefit from player one's commodity, player 2 would fill the container with his own commodity on his turn, if given the opportunity. This is the *rational reaction* of player 2. For this reason, player 3 has no choice but to fill the container with his commodity and collect only one dollar.

6.4.2 Coalition Formation

In the previous example, there are six dollars available to the three players. Divided equally, each of the three players could improve their payoffs. However, because of the sequential and independent nature of the decisions, such a solution cannot be attained.

The solution to the above problem is, thus, not Pareto optimal (see Chew [14]). However, as suggested by the example, the formation of a coalition among subsets of the players could provide a means to achieve Pareto optimality. The members of each coalition act for the benefit of the coalition as a whole. The question immediately raised are:

- which coalitions will tend to form,
- are the coalitions enforceable, and
- what will be the resulting distribution of wealth to each of the players?

The game in partition function form (see Lucas and Thrall [16] and Shenoy [17]) provides a framework for answering these questions in this Stackelberg setting.

Definition 6.2. *An* **abstract game** *is a pair* (X, dom) *where* X *is a set whose members are called* **outcomes** *and* dom *is a binary relation on* X *called* **domination**.

Let $G = \{1, 2, \ldots, n\}$ denote the set of n players. Let $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$ denote a coalition structure or partition of G into nonempty coalitions, where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{M} R_i = G$.

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \ldots, \{n\}\}$ denote the coalition structure where no coalitions have formed and let $\mathcal{P}_G \equiv \{G\}$ denote the **grand coalition**.

Consider $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$, an arbitrary coalition structure. Assume that utility is additive and transferable. As a result of the coalition formation, the objective function of each player in coalition R_j becomes,

$$f'_{R_j}(x) = \sum_{i \in R_j} f_i(x).$$

Although the sequence of the players' decisions has not changed, their objective functions have. Let $R(i)$ denote the unique coalition $R_j \in \mathcal{P}$ such that player $i \in R_j$. Instead of maximizing $f_i(x)$, player i will now be maximizing $f'_{R(i)}(x)$. Let $\hat{x}(\mathcal{P})$ denote the solution to the resulting n-level optimization problem.

Definition 6.3. *Suppose that S^1 is compact and $\hat{x}(\mathcal{P})$ is unique. The value of (or payoff to) coalition $R_j \in \mathcal{P}$, denoted by $v(R_j, \mathcal{P})$, is given by*

$$v(R_j, \mathcal{P}) \equiv \sum_{i \in R_j} f_i(\hat{x}(\mathcal{P})).$$

Note 6.7. The function v need not be superadditive. Hence, one must be careful when applying some of the traditional game theory results which require superadditivity to this class of problems.

Definition 6.4. *A* **solution configuration** *is a pair $(r, \mathcal{P})$, where r is an n-dimensional vector (called an* **imputation***) whose elements r_i $(i = 1, \ldots, n)$ represent the payoff to each player i under coalition structure $\mathcal{P}$.*

Definition 6.5. *A solution configuration $(r, \mathcal{P})$ is a* **feasible solution configuration** *if and only if $\sum_{i \in R} r_i \leq v(R, \mathcal{P})$ for all $R \in \mathcal{P}$.*

Let Θ denote the set of all solution configurations which are feasible for the hierarchical decision-making problem under consideration. We can then define the binary relation dom, as follows:

Definition 6.6. *Let $(r, \mathcal{P}_r)$, $(s, \mathcal{P}_s) \in \Theta$. Then $(r, \mathcal{P}_r)$* **dominates** *$(s, \mathcal{P}_s)$ denoted by $(r, \mathcal{P}_r)\,\mathrm{dom}\,(s, \mathcal{P}_s)$, if and only if there exists an nonempty $R \in \mathcal{P}$, such that*

$$r_i > s_i \quad \text{for all} \quad i \in R \quad \text{and} \tag{2}$$

$$\sum_{i \in R} r_i \leq v(R, \mathcal{P}_r). \tag{3}$$

Condition (2) implies that each decision maker in R prefers coalition structure $\mathcal{P}_r$ to coalition structure $\mathcal{P}_s$. Condition (3) ensures that R is a feasible coalition in $\mathcal{P}_r$. That is, R must not demand more for the imputation r than its value $v(R, \mathcal{P}_r)$.

Definition 6.7. *The* **core***, $\mathcal{C}$, of an abstract game is the set of undominated, feasible solution configurations.*

When the core is nonempty, each of its elements represents an enforceable solution configuration within the hierarchy.

6.4.3 Results

We have now defined a model of the formation of coalitions among players in a Stackelberg game. Perfect information is assumed among the players, and coalitions are allowed to form freely. No matter which coalitions form, the order of the players' actions remains the same. Each coalition earns the combined proceeds that each individual coalition member would have received in the original Stackelberg game. Therefore, a player's rational decision may now be altered because he is acting for the joint benefit of the members of his coalition.

Using the above model, several results can be obtained regarding the formation of coalitions among the players. First, the distribution of wealth to any feasible coalition cannot exceed the value of the grand coalition. This is provided by the following lemma:

Lemma 6.1. *If solution configuration* $(z, \mathcal{P}) \in \Theta$ *then*

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P}_G)) = v(G, \mathcal{P}_G) \equiv V^*.$$

Theorem 6.5. *If* $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ *then* $\sum_{i=1}^{n} z_i = V^*$.

It is also possible to construct a simple sufficient condition for the core to be empty. This is provided in Theorem 6.6.

Theorem 6.6. *The abstract game* (Θ, dom) *has* $\mathcal{C} = \emptyset$ *if there exists coalition structures* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ *and coalitions* $R_j \in \mathcal{P}_j$ $(j = 1, \ldots, m)$ *with* $R_j \cap R_k = \emptyset$ *for all* $j \neq k$ *such that*

$$\sum_{j=1}^{m} v(R_j, \mathcal{P}_j) > V^*. \tag{4}$$

Finally, we can easily show that, in any 2-person game of this type, the core is always nonempty.

Theorem 6.7. *If* $n = 2$ *then* $\mathcal{C} \neq \emptyset$.

6.4.4 Examples and Computations

We will expand on the illustration given in Section 6.4.1. Let c_{ij} represent the reward to player i if the commodity controlled by player j is placed in the container. Let C represent the matrix $[c_{ij}]$ and let x be an n-dimensional vector with x_j representing the amount of commodity j placed in the container. Note that $\sum_{j=1}^{n} x_j \leq 1$

and $x_j \geq 0$ for $j = 1, \ldots, n$. For the illustration provided in Section 6.4.1,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}.$$

Note that Cx^{T} is a vector whose components represent the earnings to each player.

Chew [14] provides a simple procedure to solve this game. The algorithm requires $c_{11} > 0$.

Step 0: Initialize i=1 and j=1. Go to *Step 1*.

Step 1: If $i = n$, stop. The solution is $\hat{x}_j = 1$ and $\hat{x}_k = 0$ for $k \neq j$. If $i \neq n$, then go to *Step 2*.

Step 2: Set $i = i + 1$. If $c_{ii} > c_{ij}$, then set $j = i$. Go to *Step 1*.

If no ties occur in *Step 2* (i.e., $c_{ii} \neq c_{ij}$) then it can be shown that the above algorithm solves the problem (see Chew [14]).

Example 6.1. Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 10 & 4 & 0 \\ 0 & 1 & 1 \\ 1 & 4 & 3 \end{bmatrix}.$$

With coalition structure $\mathcal{P}_0 = \{\{1\},\{2\},\{3\}\}$, the solution is $(x_1, x_2, x_3) = (0, 1, 0)$ and the coalition values are $v(\{1\}, \mathcal{P}_0) = 4$, $v(\{2\}, \mathcal{P}_0) = 1$ and $v(\{3\}, \mathcal{P}_0) = 4$.

Consider coalition structure $\mathcal{P} = \{\{1, 2\}, \{3\}\}$, The payoff matrix becomes

$$C_{\mathcal{P}} = \begin{bmatrix} 10 & 5 & 1 \\ 10 & 5 & 1 \\ 1 & 4 & 3 \end{bmatrix}$$

and a solution of $(0, 0, 1)$. The values of the coalitions in this case are $v(\{1, 2\}, \mathcal{P}) = 1$ and $v(\{3\}, \mathcal{P}) = 3$.

Note that coalition structure $\mathcal{P}$ is not superadditive since

$$v(\{1\}, \mathcal{P}_0) + v(\{2\}, \mathcal{P}_0) > v(\{1, 2\}, \mathcal{P}).$$

When Players 1 and 2 do not cooperate, Player 2 fills the container with a benefit of 4 to Player 3. Suppose the bottom two players form coalition $\{1, 2\}$. Then if Player 2 is given an *empty* container, the coalition will have Player 1 fill it with his commodity, earning 10 for the coalition. So, if Player 3 does not fill the container, the formation of coalition $\{1, 2\}$ reduces Player 3's benefit from 4 to 1. As a result, Player 3 fills the container himself, and earns 3. The coalition $\{1, 2\}$ only earns 1 (not 10).

Remember that Chew's model assumes that all players have full knowledge of the coalition structure that has formed. Obvious natural extensions of this simple model would incorporate secret coalitions and delayed coalition formation (i.e., changes in the coalition structure while the container is being passed).

Example 6.2. Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 0 & 3 \\ 2 & 5 & 1 \end{bmatrix}.$$

With coalition structure $\mathcal{P}_0 = \{\{1\}, \{2\}, \{3\}\}$, the solution is $(x_1, x_2, x_3) = (1, 0, 0)$ and the coalition values are $v(\{1\}, \mathcal{P}_0) = 4$, $v(\{2\}, \mathcal{P}_0) = 1$ and $v(\{3\}, \mathcal{P}_0) = 2$.

Under the formation of coalition structure $\mathcal{P} = \{\{1\}, \{2, 3\}\}$, the resources of players 2 and 3 are combined. This yields a payoff matrix of

$$C_{\mathcal{P}} = \begin{bmatrix} 4 & 1 & 4 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

and a solution of $(0, 1, 0)$. The values of the coalitions in this case are $v(\{1\}, \mathcal{P}) = 1$ and $v(\{2, 3\}, \mathcal{P}) = 5$.

Finally, if all of the players join to form the grand coalition, $\mathcal{P}_G$, the payoff matrix becomes

$$C_{\mathcal{P}_G} = \begin{bmatrix} 7 & 6 & 8 \\ 7 & 6 & 8 \\ 7 & 6 & 8 \end{bmatrix}$$

with a solution of $(0, 0, 1)$ and $v(\{1, 2, 3\}, \mathcal{P}_G) = 8$. Note that

$$v(\{1\}, \mathcal{P}_0) + v(\{2, 3\}, \mathcal{P}) > v(\{1, 2, 3\}, \mathcal{P}_G).$$

From Theorem 6.6, we know that the core for this game is empty.

6.5 BIBLIOGRAPHY

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